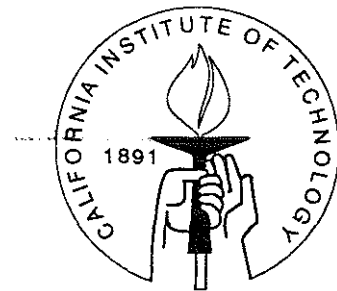DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

RELATIONAL-FUNCTIONAL VOTING OPERATORS

Fuad Aleskerov
Institute of Control Sciences, Moscow, and
California Institute of Technology

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Abstract

The operators which transform binary relations of voters to collective choice function are studied. The conditions on operators are given, the main one is the condition of locality. The explicit form of operators for special subclass of local operators is obtained. All study are made in terms of special language — list-form representation of operators.

RELATIONAL-FUNCTIONAL VOTING OPERATORS

Fuad Aleskerov*

1   Introduction

In Arrow (1963) the voting problem (also called collective choice problem) was formalized in a framework where the following (first) model was presented: The individual opinions were described in terms of binary relations and the collective choice was described in terms of choice function.

A. Sen (1970) focused on the fact that this problem, could be reinterpreted as the problem of the aggregation of binary relations. Indeed, in this first model, the choice functions mentioned above are using the information on the pairs of options (in the terminology used in Aizerman and Aleskerov (1990) , these choice functions are pair-dominant) That is why, this first model with the collective choice presented as a choice function is equivalent to the same problem but where the collective decision is presented as a collective binary relation. This first model was studied in many other papers, the most wide investigation made in Ferejohn and Fishburn (1977) and Aleskerov and Vladimirov (1986).

In Aleskerov (1984) and Aizerman and Aleskerov (1986), the collective choice problem was presented in the framework of another (second) model where both the individual opinions and the collective choice were given in terms of choice functions.

The third model, where the individual opinions are given in terms of binary relations and the collective choice is expressed as a (non-binary) choice function, was studied by few papers (see, e.g., Blair et al. (1976), Aleskerov (1984), (1985), (1991)).

Before we turn to the analysis of this third model let us mention that the operators which transform individual binary relations to the collective one are picked out in Arrow (1963) by the condition of the independence of irrelevant alternatives. The analogous condition in Aleskerov and Vladimirov (1986) was called as quasi-locality one and in Aizerman and Aleskerov (1986) under the other framework – the locality one.

Further, those operators were restricted by some other obvious and natural conditions and limitations which lead to find the explicit form of those operators such as dictatorship (when the collective decision coincides with the individual opinion of some voter), oligarchy (when the collective decision coincides with the unanimous opinion of selected group of voters), etc.

In Blair et al. (1976) the condition of locality for the third model was formalized in a very general form and lead to few concrete results and the dictatorship operators were obtained under very strong and nonobvious additional restrictions.

This paper deals with the voting problem in the framework of the third model which is introduced in the section 2. The locality condition is given in the section 3 as well as the list-form representation of local operators – the special language which is used to prove all theorems. In section 4 some normative conditions of operators are given and the section 5 will introduce them in terms of list-form representation of operators. In section 6 explicit form of local operators are given for a special class called Central Region. In section 7 the special cases of operators introduced in the previous section are studied. In section 8 range constraints on the operators from Central Region are investigated, these constraints being given as characteristic conditions on choice functions. In sections 9 and 10 the operators from Symmetrically Central Region are considered, the explicit form of these operators is given. In section 11 the main theorem is given showing the closedness of domains in the space of choice functions relative to operators from Central and Symmetrically Central Regions.

2 Framework

The following problem will be studied: Let us consider the set A ($|A| = m \geq 2$) of options and the set N ($|N| = n \geq 2$) of individuals. The options will be denoted by letters such that $a, b, x_1, x' \ldots$ and the voters will be denoted by their indices $1, 2, \ldots, n$.

Each voter $i \in N$ is independent from the other voters and expresses his/her opinion on the options with the binary relation G_i. A binary relation can also be interpreted in terms of preference relation. In this case, the pair (a, b) is in G_i if "the option a is preferred to the option b".

For simplicity, the binary relation $G_i, \quad i = 1, \ldots, n$ is assumed to satisfy the following conditions:

1. irreflexivity: $\forall a \in A, \quad (a, a) \notin G_i$;

2. transitivity: $\forall a, b \in A, \quad (a, b) \in G_i \quad (b, c) \in G_i \quad \Longrightarrow \quad (a, c) \in G_i$;

3. negative transitivity: $\forall a, b \in A, \quad (a, b) \notin G_i \quad (b, c) \notin G_i \quad \Longrightarrow \quad (a, c) \notin G_i$.

Such a binary relation is called a weak order. It admits the obvious following interpretation: for any pair of options, it is possible to establish their equity or the fact that one option is more preferable than the other.

Hereafter we consider that individual binary relations are weak orders. The totality of all weak orders is denoted by $\mathcal{W}$. $\vec{G}$ will be hereafter denoted the profile $\{G_1, \ldots, G_n\}$ of weak orders on the totality of voters.

Now let us add the condition of linearity: $\forall a, b \in A \quad (a, b) \in G_i \quad$ or $\quad (b, a) \in G_i$ to the previous 1),2), and 3) conditions. In this case the weak order is called a linear order,[1] which means that for each pair of options in G_i it is possible to establish that one option is more preferable than the other.

Now we introduce the "dominant set" for the option x which will be very useful further.

Definition 1 The dominant set $\mathcal{D}(x)$ for the option x in the binary relation G is defined as follows:
$$\mathcal{D}(x) = \{y \in A \mid \quad (y, x) \in G \quad \},$$
i.e. the dominant set for x is a set of all options which are more preferable than x.

We define now the choice function with which the collective decision is presented. Let us note that we study the collective choice problem not only on the whole set A but also on subsets from A named X.

Let $\mathcal{A}^o$ be the set of all non-empty subsets of A. The choice Y on the set X satisfies to the condition $Y \subseteq X$. The set of pairs $\{(X, Y)\}$ for all X defines the choice function $Y = C(X)$.

The totality of all choice functions on A is called the space of choice functions and is denoted by $\mathcal{C}$.

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[1] Strictly speaking, linear order can be defined by conditions 1), 2) and the condition of linearity.

Further we will not restrict somehow the set $\mathcal{C}$, i.e. all choice functions are admissible as a collective decision. Thus, the voting operator, also simply called operator, transforms one profile $\{G_i\}$ in a choice function $C(.)$; or strictly speaking transforms the domain defined by the n-tuple cartesian product $\mathcal{W}^n = \mathcal{W} \times \ldots \times \mathcal{W}$ in the space $\mathcal{C}$.

These operators will be called relational-functional voting operators.

3   Locality Condition

There are many operators which implement the mapping $\mathcal{W}^n \longrightarrow \mathcal{C}$. This is too wide space and the operators will be studied which are satisfying to a special locality condition.

Definition 2 The operator $F : \quad \mathcal{W}^n \longrightarrow \mathcal{C}$ will be called a local one if it satisfies to the following condition: Let two profiles $\vec{G}$, $\vec{G}'$ are given and suppose that for some x, X, $x \in X \in \mathcal{A}^o$ the following equality $\forall i \in N \ X \bigcap \mathcal{D}_i(x) = X \bigcap \mathcal{D}'_i(x)$ holds which means that the dominant sets for the option x in X in the relations G_i and G'_i coincide. Then $x \in C(X)$ if and only if $x \in C'(X)$ where $C(.) = F(\vec{G})$ and $C'(.) = F(\vec{G}')$.

This condition in a more general form was introduced in Aleskerov (1985).

On Figure 1 are drawn two profiles $\vec{G}$ and $\vec{G}'$ (linear orders under graph form) on the set $A = \{a, b, c, d\}$ and $N = \{1, 2, 3\}$. It can be easily seen that for the subset $X = \{a, b, c\}$ the dominant sets for option c coincide for all binary relations G_i, $i = 1, 2, 3$. Let us note, however, that for the relations G_1 and G_1' the dominant sets are such that $\mathcal{D}_1(c) \cap X = \mathcal{D}'_1(c) \cap X = \{a, b\}$ but for the option b instead of c, $\mathcal{D}_1(b) \cap X = \emptyset$ and $\mathcal{D}'_1(b) \cap X = \{a\}$. Thus the locality condition is satisfied if c is included (or not included) in $C(X)$ **and** also if c is included (respectively not included) in $C'(X)$. Below, two examples of relational-functional operators are given.

Example 1 *Let us consider the Pareto rule. This rule will be denoted by F_{Par}. For all x, X such that $x \in X \in \mathcal{A}^o$ the following definition of this rule will be considered:*

$$x \in C(X) \Longleftrightarrow (\bar{\exists} y \in X : \forall i \in N \ yG_ix)$$

It means that x is included in the choice $C(X)$ if and only if another option y is not preferred to x in all relations G_i.

It can be proved that the operator F_{Par} satisfies to the condition of locality.

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Example 2 *Let us consider now another operator which will be called Borda[2] operator and denoted by F_{Borda}: Let us present a binary relation as a graph and let for each option a in binary relation G_i $r_i(a)$ is the number of arcs going from corresponding vertex. As a rule of aggregation, we consider the rule which includes in the collective choice those options which have maximum sum value $\sum\limits_{i \in N} r_i(a)$. This rule can be written in the following form:*

$$x \in C(X) \Longleftrightarrow (x \in X \mid x = arg \max_{a \in X} \sum_{i \in N} r_i(a)).$$

Let us consider now two profiles $\vec{G}$ and $\vec{G}'$ shown in Figure 2 and the set $A = \{x, y, z\}$, the set $N = \{1, 2, 3\}$, the alternative x. Let us note that $\forall i \in N \quad \mathcal{D}_i(x) \cap A = \mathcal{D}'_i(x) \cap A$ and according to condition of locality x is included (or is not included) in choice $C(A)$ if and only if x belongs (or does not belong) to $C'(A)$. Let us consider now the rule introduced above for profile $\vec{G}$. One can see that $\sum\limits_{i \in N} r_i(x) = \sum\limits_{i \in N} r_i(y) = \sum\limits_{i \in N} r_i(z) = 3$ and hence $C(A) = A$.

However, in the profile $\vec{G}'$, $\sum\limits_{i \in N} r'_i(x) = 3$, but $\sum\limits_{i \in N} r'_i(y) = 2$ and $\sum\limits_{i \in N} r'_i(z) = 4$ and thus $z \in C'(A)$ and $x \notin C'(A)$.

According to this example the operator F_{Borda} is not local.

Now let us introduce a new language for the study of relational-functional operators. The totality of sets $\{\vec{Z}\}_1^s$ where $\vec{Z} = (Z_1, \ldots, Z_n)$ and $\forall i \in N \quad Z_i \subseteq X \backslash \{x\}$ is called list for the pair (x, X), $x \in X \in \mathcal{A}^\circ$. The list for the pair (x, X) will be denoted $\Omega(x, X)$: $\Omega(x, X) = \{\vec{Z}\}_1^s$.

The relational-functional operator F under list-form representation is defined by $\Omega_F = \{\Omega(x, X)\}$ and the rule:

$$x \in C(X) \Longleftrightarrow (\exists \vec{Z} : \vec{Z} = (Z_1, \ldots, Z_n) \in \Omega(x, X) \in \Omega_F \quad \text{and} \quad \forall i \in N \ X \bigcap \mathcal{D}_i(x) = Z_i) \tag{1}$$

Theorem 1 *Each local relational-functional operator has a list-form representation. Each relational-functional operator which admits the list-form representation is a local operator.*

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[2]This operator is analogues to that studied in Borda (1781). His work was one of the first in which voting problem was stated formally.

Proof: Let us consider the whole set of profiles on weak orders $\vec{G}$ and arbitrary set $X \in \mathcal{A}^o$ and the option $x \in X$. Construct a list for local operator F according to the following algorithm. For a given profile $\vec{G}$, if $x \in C(X)$, let us include the set $\vec{Z} = (Z_1, \ldots, Z_n)$ where each component is $\forall i \in N \quad Z_i = \mathcal{D}_i(x) \cap X$, as an element in the list $\Omega(x, X)$. All such lists $\Omega(x, X)$ define the whole list Ω_F for the operator F.

Making use of this list and the rule (1) defines some operator F. Let us show that $\tilde{F} \equiv F$. The functions generated by the operators F and $\tilde{F}$ will be denoted as $C(.)$ and $\tilde{C}(\cdot)$ respectively. Consider the profile $\vec{G}$ and assume that there exist X and x such that $x \in \tilde{C}(X)$ and $x \notin C(X)$. Because of the fact that $x \in C(X)$, according to the construction of the operator $\tilde{F}$ it means that there exist a profile $\vec{G}'$ s.t. $X \cap \mathcal{D}_i(x) = Z_i$ for all i. Because of the pre-assumption that F is local, the condition $\forall i \quad \mathcal{D}_i(x) \cap X = \mathcal{D}_i'(x) \cap X$ and $x \in C'(X)$ imply that $x \in C(X)$.

Let now operator F has a list-form representation Ω_F and prove its locality. Let us consider two profiles $\vec{G}$ and $\vec{G}'$ such that $\forall i \in N \quad \mathcal{D}_i(x) \cap X = \mathcal{D}_i'(x) \cap X$. Then if $\{\mathcal{D}_i(x) \cap X\} \in \Omega_F(x, X)$ then $x \in C(X)$ and $x \in C'(X)$. On the other hand, if $\forall i \in N \quad \{\mathcal{D}_i(x) \cap X\} \notin \Omega_F(x, X)$ then $x \notin C(X)$ and $x \notin C'(X)$. The theorem is proved.   Q.E.D.

4   Characteristic Conditions on Local Operators

Let us introduce now the characteristic conditions which are added to the condition of locality.

1°. *Sovereignty*. This condition consists of two conditions.

1^o_+ *Positive Sovereignty*. For all x, X $(x \in X \in \mathcal{A}^o)$ it exists a profile $\vec{G}$ such that $x \in C(X)$.

1^o_- *Negative Sovereignty*. For all x, X $(x \in X \in \mathcal{A}^\bullet)$ it exists a profile $\vec{G}$ such that $x \notin C(X)$.

In other words, while creating the function $C(.)$, operator F takes into consideration the individual opinions: there are no x, X for which the condition that x is always in $C(X)$ or x is never in $C(X)$ had been fixed in advance (independently of profile).

$2^\bullet$. *Monotonicity*. Let us consider some profile $\vec{G}$, and for some x and X, $(x \in X \in \mathcal{A}^\bullet)$ $\mathcal{D}_i(x) \cap X$ for all i in N. Let now in the profile $\vec{G}'$: $\forall i \in N \quad \mathcal{D}_i'(x) \cap X \subseteq \mathcal{D}_i(x) \cap X$ holds. Then, $x \in C(X) \implies x \in C'(X)$.

Let us explain this condition. On the figure 1 for the profile $\vec{G}$, $\mathcal{D}_1(d) \cap A = \{a, b, c\}$, $\mathcal{D}_2(d) \cap A = \emptyset$ and $\mathcal{D}_3(d) \cap A = \{a, c\}$ and for the profile $\vec{G}'$ $\mathcal{D}_1'(d) \cap A = \{a, b\}$, $\mathcal{D}_2'(d) \cap A = \emptyset$ and $\mathcal{D}_3'(d) \cap A = \{a, c\}$. In these two profiles $\mathcal{D}_2(d) \cap A$ and $\mathcal{D}_2'(d) \cap A$ coincide, this is also the case for $\mathcal{D}_3(d) \cap A$ and $\mathcal{D}_3'(d) \cap A$. However, for G_1, the dominant set of d consists of a, b, c and for G_1' it consists of a, b. It can be interpreted in terms of preference relations: the set of preferred options for d in G_1' is narrower than in relation G_1. So if d is in $C(A)$ (collective choice) with profile $\vec{G}$ then d is to be in $C'(A)$ with the profile $\vec{G}'$.

It is obvious that the condition of monotonicity is a reinforcement of the condition of locality, i.e. the monotonic operator is local.

This condition of monotonicity were used in other terms by Muller and Satterthwaite (1977), Moulin and Peleg (1982), and Maskin (1986)

$3°$. *Neutrality to options.* This condition is divided in two conditions:

$3_a^\bullet$. *Independence of options (of x).* Let for some x and y in X $\forall i$ $X \cap \mathcal{D}_i(x) = X \cap \mathcal{D}_i(y)$. Then $y \in C(X)$ iff $x \in C'(X)$.

$3_b^\bullet$. *Independence of context (of the subset X).* Let us consider two subsets X and X' of $\mathcal{A}^\bullet$ with the condition $\forall i$ $\mathcal{D}_i(x) \cap X = \mathcal{D}_i(x) \cap X'$. Then $x \in C(X)$ if and only if $x \in C(X')$.

Conditions $3_a^\bullet$ and 3_b^o are also a reinforcement of locality condition: put $x = y$ or $X' = X$.

$4°$. *Anonymity.* Let $\eta : N \Longrightarrow N$ is a one-to-one mapping from the set N to N. Then $C(.) = C'(.)$ where $C(.) = F(G_1, \ldots, G_n)$ and $C'(.) = F(G_{\eta(1)}, \ldots, G_{\eta(n)})$.

$5°$. *Non-dominance.* This condition is divided in the two following conditions:

5_+^o. *Positive non-dominance.* For all x, X ($x \in X \in \mathcal{A}^o$), if it exists $i_o \in N$ such that $\mathcal{D}_{i_o}(x) \cap X = \emptyset$, then $x \in C(X)$.

$5_-^\bullet$. *Negative non-dominance.* For all x, X ($x \in X \in \mathcal{A}^\bullet$), if $\forall i \in N$ $\mathcal{D}_i(x) \cap X \neq \emptyset$, then $x \notin C(X)$.

$6°$. *Unanimity.* It for some x, X, ($x \in X \in \mathcal{A}^\bullet$), and for all $i \in N$ $\mathcal{D}_i(x) \cap X = \emptyset$, then $x \in C(X)$.

Let us consider the operator Pareto and check to what conditions it satisfies. The satisfaction to the condition $1°$ of non-imposedness is obvious. Show that this operator satisfies to the condition $2^\bullet$ of monotonicity. Actually, let us consider two profiles $\vec{G}$ and

$\vec{G}'$; let us put that for subset X and $x \in X$ it exists no y such that $\forall i \in N \quad y \in \mathcal{D}_i(x) \cap X$ and then $x \in C(X)$. Now let us put that $\forall i \in N \quad \mathcal{D}'_i(x) \cap X \subseteq \mathcal{D}_i(x) \cap X$ and it exists i_o such that $\mathcal{D}'_{i_o}(x) \cap X \subset \mathcal{D}_{i_o}(x) \cap X$. But then obviously $\bigcap (\mathcal{D}'_i(x) \cap X) = \emptyset$ again, and according to the Pareto rule, $x \in C(X)$. It can be shown the satisfaction to the conditions 3° and 4°. The condition 5^o_+, 6° and can be shown obviously. Then, in order to summarize, the operator Pareto satisfies to $1^o \cap 2^o \cap 3^o \cap 4^o \cap 5^o_+ \cap 6^o$.

Let us introduce some special operators which will be useful further.

Definition 3 *The "trivial operators"* **0** *and* **1** *are defined as follows:*

$$\text{Trivial operator } \mathbf{0}: \quad \forall X \in A \quad C(X) = \emptyset.$$
$$\text{Trivial operator } \mathbf{1}: \quad \forall X \in A \quad C(X) = X.$$

The operator "unanimity" is denoted "$\mathcal{U}$" and defined as follows:
$\mathcal{U} : C(X) = \{x \in X \mid \forall i \in N \; X \bigcap \mathcal{D}_i(x) = \emptyset\}$.

The operator "at least one vote" is denoted "$\mathcal{V}$" and defined as follows:
$\mathcal{V} : C(X) = \{x \in X \mid \exists i_o \in N \quad \mathcal{D}_{i_o}(x) \cap X = \emptyset\}$

Let us note that the trivial operator **0** satisfies to the conditions $1^o_- \cap 2^o \cap 3^o \cap 4^o \cap 5^o_-$, the trivial operator **1** satisfies to the conditions $1^o_+ \cap 2^o \cap 3^o \cap 4^o \cap 5^{\bullet}_+$. The operator unanimity $\mathcal{U}$ satisfies to $1^o \cap 2^o \cap 3^o \cap 4^o \cap 5^o \cap 6^o$ and finally the operator "at least one vote" $\mathcal{V}$ satisfies to the conditions $1^o \cap 2^o \cap 3^o \cap 4^o \cap 5^{\bullet}_+ \cap 5^{\bullet}_-$.

5 Characteristic Conditions of Operators in Terms of List-form Representation

The reformulation of the characteristic conditions on local operators, introduced in previous section in terms of list-form representation and the prove of their equivalency is given below.

1°. *Sovereignty.* This condition consists of two conditions.

1^o_+ *Positive sovereignty.* For all x, X $(x \in X \in \mathcal{A}^o)$ $\Omega(x, X) \neq \emptyset$.

In order to formalize the condition of negative non-imposedness it is necessary to define $T(x, X)$ as the list which contains all possible $\vec{Z}$.

1^o_- *Negative Sovereignty.* For all x, X $(x \in X \in \mathcal{A}^{\bullet})$ $\Omega(x, X) \neq T(x, X)$.

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2°. Monotonicity

Let us consider some profile $\vec{G}$, and for some x and X, $(x \in X \in \mathcal{A}^o)$ $\vec{Z} = (Z_1, \ldots, Z_n) \in \Omega(x, X)$ and $\vec{Z}'$ such that: $\forall i \in N$ $Z_i' \subseteq Z_i$. Then $Z' \in \Omega(x, X)$.

3°. Neutrality to options. $\Omega(x, X) \equiv \Omega$. This condition is divided in two conditions:

3^o_a. Independence of options (of x). $\Omega(x, X) = \Omega(X)$.

3^o_b. Independence of context (of the subset X) $\Omega(x, X) = \Omega(x)$.

4°. Anonymity. Let $\eta : \quad N \Longrightarrow N$ is a one-to-one mapping and $\vec{Z} \in \Omega(x, X)$. Then $\eta.\vec{Z} \in \Omega(x, X)$ where $\eta.\vec{Z} = (Z_{\eta(1)}, \ldots, Z_{\eta(n)})$.

5°. Non-dominance. This condition is divided in the two following conditions:

5$^\bullet_+$. Positive non-dominance. For all $\vec{Z} = (Z_1, \ldots, Z_n)$ where one of its component Z_{i_o} is such that $Z_{i_o} = \emptyset$ $\vec{Z} \in \Omega(x, X)$ holds.

5$^\bullet_-$. Negative non-dominance. For all $\vec{Z} = (Z_1, \ldots, Z_n)$ s.t. $\forall i \in N$ $Z_i \neq \emptyset$ $\vec{Z} \notin \Omega(x, X)$.

6°. Unanimity. $(\emptyset, \ldots, \emptyset)$ belongs to $\Omega(x, X)$.

Let us study now using list form representation how the classes of operators isolated by the conditions introduced above are related.

Theorem 2 *The following relations between the conditions* $1^\bullet_-, \ldots, 6^o$ *hold:* $5^o_+ \Rightarrow 1^o_+$; $5^o_- \Rightarrow 1^\bullet_-$; $6^o \Rightarrow 1^o_+$; $5^o_+ \Rightarrow 6^o$; $\overline{1^o_+} \cap 3^o \Rightarrow 0$; $\overline{1^o_-} \cap 3^o \Rightarrow 1$; $\overline{1^o_+} \cap 3^o \Rightarrow 2^o$; $\overline{1^o_-} \cap 3^o \Rightarrow 2^o$; $\overline{1^\bullet_-} \cap 3^o \Rightarrow 5^o_+$; $\overline{1^o_-} \cap 3^o \Rightarrow 6^o$; $\overline{1^o_+} \cap \Rightarrow 5^o_-$; $\overline{1^o_-} \cap \overline{1^o_+} \cap 3^o = \emptyset$; $\overline{5^o_-} \cap 2^o \cap 3^o \Rightarrow 5^o_+$; $\overline{5^o_-} \cap 2^o \cap 3^o \Rightarrow 6^o$; $\overline{1^\bullet_-} \cap 3^o \Rightarrow 4^o$; $\overline{1^o_+} \cap 3^o \Rightarrow 4^o$; $\overline{1^o_+} \cap 3^o \Rightarrow \overline{6^o}$; $\overline{1^o_+} \cap 3^o \Rightarrow \overline{5^o_+}$.

Proof: It follows from the definitions of corresponding conditions. As an example we prove here only one correlation $\overline{5^\bullet_-} \cap 2^o \cap 3^o \Rightarrow 6^o$. Since 3° holds, then $\Omega(x, X) \equiv \Omega$ for all x and X, and because of $\overline{5^o_-}$, Ω contains one $\vec{Z} = (Z_1, \ldots, Z_n)$ s.t. $\forall j = 1, \ldots, n$ $Z_j \neq \emptyset$, and monotonicity implies that $\vec{Z}' = (\emptyset, \ldots, \emptyset)$ will belong to Ω. Q.E.D.

The class of operators satisfying respectively to the conditions of a) sovereignty, b) monotonicity, c) neutrality to options, d) anonymity will be denoted as a) Λ^S, b) Λ^M, c) Λ^N, d) Λ^A.

Two special classes in the space (set) $\mathcal{L}$ of all local operators are of a special interest: that one which isolated by conditions of sovereignty, monotonicity, and neutrality to

options - this class will be denoted as Λ^{SMN} and will be called as Central Region in $\mathcal{L}$; and the other one which isolated inside the Central Region by Anonymity condition — the latter one will be denoted as Λ^{SMNA} and called as Symmetrically-Central Region in $\mathcal{L}$. The operators from Symmetrically-Central Region Λ^{SMNA} satisfy the conditions which considered as necessary for each voting system.

6 Explicit Form of Local Operators

Several operators in explicit form are introduced below. Using the words the "explicit form" means some explicitly given rule (or algorithm) which allows to construct the collective choice function $C(.)$ according to the given profile $\{G_1, \ldots, G_n\}$ of weak orders.

Let us note that the operators $\mathbf{0}$, $\mathbf{1}$, $\mathcal{U}$ and $\mathcal{V}$ are the examples of operators given in an explicit form.

Because the variety of the operators which satisfy to the conditions introduced above is very large, we first introduce some particular cases of operators in an explicit form and then give their generalization. Consider now the following operator:

$$F_{\cap}(N,0) : \forall x, X \ x \in C(X) \Leftrightarrow \mid \bigcap_{i \in N} (X \bigcap \mathcal{D}_i(x)) \mid = 0,$$

so x belongs to the collective choice $C(X)$ iff the number of options which are more preferable than x in each G_i is null, or otherwise speaking there is no option y which is more preferable than x for each voter. One can see that this operator is exactly the operator F_{Par} introduced above, i.e. $F_{Par} = F(N,0)$.

The following generalization of this operator can be defined not for all set N but for some coalition $I \subset N$ and will be denoted by $F_{\cap}(I,0)$, i.e. choosing option in Pareto-optimal for some coalition I. This can be called as partial Pareto operator.

Let now the totality $\mathcal{I}$ of coalitions be given, and define the following two operators a) $\bigcap_{I \in \mathcal{I}} F_{\cap}(I,0)$ and b) $\bigcup_{I \in \mathcal{I}} F_{\cap}(I,0)$.

In case of a) choosing option has to be Pareto-optimal for all coalitions from $\mathcal{I}$; if $\mathcal{I}$ contains all single-element coalitions then the choosing option can be called as Cournot-optimal one, if $\mathcal{I}$ contains all non-empty coalitions then x can be called as Edgeworth-optimal option.

In case of b) the choosing option has to be Pareto-optimal in at least one coalition I from $\mathcal{I}$.

Generalizing these definitions let us introduce the following operator

$$F_\cap(I, q^I) : x \in C(X) \Leftrightarrow |\bigcap_{i \in I} (X \cap \mathcal{D}_i(x))| \leq q^I,$$

i.e. the option x is chosen even if there are q^i options which are more preferable than x for every number of coalition I.

This operator can be called as partial q-Pareto one. Analogously we can introduce a partial q-Pareto operator for a totality of coalitions and a partial q-Pareto one for at least one coalition in $\mathcal{I}$. The option chosen by operator $F_\cap(I, q^I)$ will be called as q-Pareto optimal in I.

Consider now some other operator

$$F_\cup(N, 0) : x \in C(X) \Leftrightarrow |\bigcup_{i \in N} (X \cap \mathcal{D}_i(x))| = 0,$$

i.e. x is chosen if there is no some y which is preferred to x at least by one voter from N. One can see that by this operator the Condorcet winner is defined, so this operator will be called as Condorcet operator. Partial Condorcet operator will be defined as $F_\cup(J, 0)$ where $J \in 2^N \setminus \{\emptyset\}$ is some non-empty coalition from N.

Again we can define partial Condorcet operator for a totality of coalitions $\mathcal{J}$ as $\bigcap_{I \in \mathcal{J}} F_\cup(J, 0)$, and partial Condorcet operator for at least one coalition J in $\mathcal{J}$, i.e. $\bigcup_{J \in \mathcal{J}} F_\cup(J, 0)$.

The following generalization is important: we introduce the partial p-Condorcet operator, i.e.

$$F_\cup(J, p^J) : x \in C(X) \Leftrightarrow |\bigcup_{i \in J} (X \cap \mathcal{D}_i(x))| \leq p^J,$$

i.e. x is chosen if there are no more than p^J options which are preferred to x at least by one voter from J. The option chosen by operator $F_\cup(J, q^J)$ will be called as q-Condorcet winner in J.

Analogously we introduce the operators partial p-Condorcet for a totality of coalitions $\bigcap_{J \in \mathcal{J}} F_\cup(J, p^J)$ and partial p-Condorcet operator for at least one coalition from $\mathcal{J}$, i.e. $\bigcup_{J \in \mathcal{J}} F_\cup(J, p^J)$.

Consider now one special form of combination of these operators, namely $F_\cup(J, p^J) \cap F_\cap(I, q^I)$. The option choosing by this operator has to be "p-Condorcet winner" in coalition J and q-Pareto-optimal in coalition I.

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Let us note now that if in the definition of the operator $F_\cap(I, q^I)$ the number q^I is greater or equal than $|A| - 1$, then by this operator the trivial one, namely $\mathbf{1}$, is defined. If we redefine this operator for the case when $q^I < 0$, then the other trivial operator $\mathbf{0}$ will be obtained. The analogous situation takes place for the operator $F_\cup(J, p^J)$, so hereafter we restrict these values by $0 < p^J, q^I < m - 1$, where $m = |A|$.

Let us mention finally that if in the definition of operator $F_\cup(J, p^J)$ the coalition J contains only one element i, then this operator will be denoted as $F(i, q^i)$, and according to this operator the option x is chosen if there are no more than p options which are better than x in the weak order G_i. Analogous situation takes place for operator $F_\cap(I, q^I)$ if $|I| = 1$.

These operators are local and satisfy additionally to the conditions of sovereignty (if $0 < p, q < m - 1$), monotonicity, neutrality, and positive non-dominance, which can be checked directly. So these operators belongs to the Central Region Λ^{SMN}. Varying the numbers p and q in the definition of these operators one can obtain the class of operators, depending of p and q, i.e. p, q are parameters in the definition of these classes.[3]

The importance of the operators introduced above are explicated in the following.

Theorem 3 *The operator F belongs to the Central Region Λ^{CR} iff there exist $S \geq 0$, two sets of totalities of coalitions $\{\mathcal{I}_\ell\}_1^s$ and $\{\mathcal{J}_\ell\}_1^s$, the numbers $\{p_\ell^J\}_{J \in \mathcal{J}_\ell}$ and $\{q_\ell^I\}_{I \in \mathcal{I}_\ell}$, s.t. $\forall J \ 0 < p_\ell^J < m - 1$, $\forall I \ 0 < q_\ell^I < m - 1$ and*

$$F = \bigcup_{\ell = 1}^{s} \left[\left(\bigcap_{J \in \mathcal{J}_\ell} F_\cup(J, p_\ell^J) \right) \bigcap \left(\bigcap_{I \in \mathcal{I}_\ell} F_\cap(I, q_\ell^I) \right) \right] \tag{2}$$

Proof: Let F belong to the Central Region, this means that F can be represented in list-form $\Omega = \{\vec{Z}_j\}_1^s$, where $\vec{Z}_j = (Z_1^j, \ldots, Z_n^j)$, $Z_i^j \subseteq A$, $i = 1, \ldots, n$.

Consider in sequence according to each $\vec{Z} \in \Omega$ the following operators and define $\forall i \in N \ F(i, q_i) \equiv F(i, |Z_i|)$; then for all $I = \{i, k\}$ define

$$F_\cup(I, q^I) \equiv F_\cup(I, |Z_i \bigcup Z_k|), \quad F_\cap(I, q^I) \equiv F_\cap(I, |Z_i \bigcap Z_k|), \quad \text{etc.}$$

i.e.

$$\forall I, |I| > 1, \ F_\cup(I, p^I) \equiv F_\cup(I, |\bigcup_{i \in I} Z_i|),$$

$$F_\cap(I, q^I) \equiv F_\cap(I, |\bigcap_{i \in I} Z_i|),$$

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[3]The correlation between these classes of operators were studied by Arnaud Taddei in his diploma at the Institute of Control Sciences in 1992.

Construct the following operator

$$\tilde{F} = \bigcup_{\vec{Z} \in \Omega} \left[\left(\bigcap_{i \in N} F(i, q_i) \right) \cap \left(\bigcap_{I \in 2^N \setminus \{\emptyset\}} F_\cup(I, p^I) \right) \cap \right.$$
$$\left. \cap \left(\bigcap_{I \in 2^N \setminus \{\emptyset\}} F_\cap(I, q_\cap^I) \right) \right]. \tag{3}$$

Let $C(\cdot)$ denote collective choice function corresponding to the operator F, and $\tilde{C}(\cdot)$ denote the choice function generating by $\tilde{F}$, where $\tilde{F}$ is the operator in the form (3). Show that $C(\cdot) = \tilde{C}(\cdot)$. Let $x \in C(x)$, then $x \in \tilde{C}(x)$. Suppose on the contrary that $x \notin \tilde{C}(x)$. Q.E.D.

Lemma 1 *Let Ω be a list-form for some operator $F \in \Lambda^{CR}$ and $\vec{Z} \in \Omega$. Then the set $\vec{Z}' = (Z_1 \cap B_1, \ldots, Z_n \cap B_n)$, where $B_i \subseteq A$, $i = 1, \ldots, n$, belongs to Ω.*

Proof: It can be obtained immediately from the monotonicity of F. Q.E.D.

Lemma 2 *Let F_1 and F_2 be some operators from the Central Region Λ^{CR} and Ω_1 and Ω_2 are their list-forms. Then the list form Ω of operator $F = F_1 \bigcup F_2$ is just a union of Ω_1 and Ω_2, i.e.*
$$\Omega = \{\vec{Z} | \vec{Z} \in \Omega_1 \quad or \quad \vec{Z} \in \Omega_2\}.$$

Proof: It follows immediately from the definition of the list-form representation. Q.E.D.

Lemma 3 *Let F_1 and F_2 be some operators from Λ^{CR} and Ω_1 and Ω_2 are their list-forms. Then the list-form Ω of operator $F = F_1 \bigcap F_2$ is equal to $\Omega = \{\vec{Z} | \vec{Z} = \vec{Z}_i \cap \vec{Z}_j, \vec{Z}_i \in \Omega_1, \vec{Z}_j \in \Omega_2, \vec{Z}_i \cap \vec{Z}_j = (Z_1^i \cap Z_1^j, \ldots, Z_n^i \cap Z_n^j)\}$*

Proof: Let $\vec{Z}_1 \in \Omega_1$ and $\vec{Z}_2 \in \Omega_2$, then according to Lemma 1, the set $\vec{Z} = \vec{Z}_1 \cap \vec{Z}_2$ belongs both to Ω_1 and Ω_2. Now $x \in F_1(X, \vec{G}) \bigcap F_2(X, \vec{G})$ iff there exist $\vec{Z}_1 \in \Omega_1$ and $\vec{Z}_2 \in \Omega_2$ s.t. $\forall i \ X \cap \mathcal{D}_i(x) = Z_i^1 \in \vec{Z}_1$ and $X \cap \mathcal{D}_i(x) = Z_i^2 \in \vec{Z}_2$, i.e. $Z_i^1 = Z_i^2$ for all i, which implies $\vec{Z}_1 = \vec{Z}_2$. This equality holds only for such element $\vec{Z}$ of Ω_1 and Ω_2 which can be represented in the form $\vec{Z} = \vec{Z}_1 \cap \vec{Z}_2$, $\vec{Z}_1 \in \Omega_1$, $\vec{Z}_2 \in \Omega_2$. Q.E.D.

Lemma 4 *For an arbitrary operator $F_\cup(I, p^I)$ constructed according to some $\vec{Z} = (Z_1, \ldots, Z_n)$ from Ω in $\Omega_\cup^I$ there exist $\vec{Z}'$ such that $\forall i \in I \ Z_i = \bigcup_{j \in I} Z_j, \forall i \in N \setminus I \ Z_i = A \setminus \{x\}$, where $\Omega_\cup^I$ is the list form for the operator $F_\cup(I, q^I)$.*

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Proof: Just because of definition of operator $F_{\cup}(I, p^I)$ the list-form representation for this operator has to contain $\forall i \in N \setminus I$ the set $Z_i^{\cup} = A \setminus \{x\}$, and because of neutrality in different $\vec{Z}^{\cup}$ all sets $Z_i^{\cup} = A \setminus \{x\}$ for all $x \in A$. According to the construction of this operator for all $i \in I$ $Z_i^{\cup} = \bigcup_{j \in I} Z_j$, where $Z_i^{\cup}$ is the element in $\vec{Z}^{\cup}$, and Z_j are elements of $\vec{Z}$ according to which the operator $F_{\cup}(I, p^I)$ has been constructed. $\qquad$ Q.E.D.

Lemma 5 *Let the operator $F_{\cap}(I, q_{\cap}^I)$ is constructed according to some $\vec{Z} = (Z_1, \ldots, Z_n)$ from Ω. Then in $\Omega_I^{\cap}$ there exist some $\vec{Z}'$ such that $\forall i \in I$ $Z_i' = Z_i$, $\forall i \in N \setminus I$ $Z_i' = A \setminus \{x\}$, where $\Omega_I^{\cap}$ is a list form for $F_{\cap}(I, q_{\cap}^I)$.*

Proof: Analogous to the proof of Lemma 10. $\qquad$ Q.E.D.

Lemma 6 *Let Ω^i be a list-form of operator $F(i, q_i)$ constructed according to some $\vec{Z} = (Z_1, \ldots, Z_n)$ from Ω. Then there exist $\vec{Z}' \in \Omega^i$ such that $Z_i' = Z_i$ and $\forall j \neq i$ $Z_j = A \setminus \{x\}$.*

Proof: Analogous to the proof of Lemma 10.

Let $\vec{Z} \in \Omega$. Show that $\vec{Z} \in \dot{\Omega}$, where $\dot{\Omega}$ is a list-form for the operator $\tilde{F}$. Consider some $Z_i \in \vec{Z}$. According to Lemmas 4-6 for each component of $\tilde{F}$ contains $\vec{Z}'$ such that $\forall i$ $Z_i' \supseteq Z_i$, and there exist at least one component $\vec{Z}''$ such that $\forall i$ $Z_i'' \supseteq Z_i$ and for $i_\bullet$ $Z_{i_o}'' = Z_{i_o}$. Hence according to Lemma 3 $\dot{\Omega}$ contains $\dot{Z}$ in which $Z_{i_o} = Z_i$, i.e. $x \in \dot{C}(X)$.

Let now $x \notin C(X)$ and $x \in \dot{C}(X)$.

Because $\tilde{F} \in \Lambda^{CR}$, then $\tilde{F}$ has a list-form representation $\tilde{\Omega} = \{\tilde{Z}_j\}_1^S$, and the fact that $x \in C(X)$, and $x \notin \dot{C}(X)$ means that there exist $\vec{Z}_j$ and $\dot{Z}_j$ such that $\forall i \neq i_o$ $Z_i = \tilde{Z}_i$, $Z_i \in Z_j$, $\tilde{Z}_i \in \tilde{Z}_j$, and $Z_{i_o} \neq \tilde{Z}_{i_\bullet}$.

Consider the following cases: a) $Z_{i_o} \supset \dot{Z}_{i_o}$; b) $Z_{i_o} \subset \dot{Z}_{i_o}$; c) $Z_{i_o} \cap \dot{Z}_{i_o} \neq \emptyset$, $Z_{i_o} \not\subseteq \dot{Z}_{i_o}$, $Z_{i_o} \not\supseteq \dot{Z}_{i_o}$; c) $Z_{i_o} \cap \dot{Z}_{i_o} = \emptyset$.

In the case a) according to monotonicity of F if $\vec{Z} \in \Omega$ and $\vec{Z} = (Z_1, \ldots, Z_n)$ with Z_{i_o}, then $\vec{Z}'$ with $\dot{Z}_{i_\bullet}$ also belongs to Ω and hence $x \in C(X)$.

In the case b) we obtain the contradiction with the construction of $\tilde{F}$, because $F(i_\bullet, |\dot{Z}_{i_o}|)$ can not be the component in the definition of $\tilde{F}$.

In the case c) let us consider without loss of generality the following situation $Z_{i_o} = \{a, c\}$ and $\dot{Z}_{i_o} = \{b, c\}$. Consider other Z_i, and the following two cases

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c_1) $\forall i \neq i_\bullet$ $Z_i = \emptyset$. Then because of neutrality there exist $\vec{Z}_j'$ s.t. $\vec{Z}_j' = (\emptyset, \ldots, \tilde{Z}_{i_o}, \emptyset, \ldots, \emptyset)$ and $x \in C(X), x \in \hat{C}(X)$.

c_2) There exist Z_i, $i \neq i_o$, s.t. $Z_i \neq \emptyset$, then c_{2-1}) $Z_i \cap Z_{i_o} = \emptyset$ or $Z_i \cap Z_{i_o} \not\ni a$, or c_{2-2}) $Z_i \cap Z_{i_o} \ni a$. The case C_{2-1}) is admissible because of neutrality: with transformation $\gamma(a) = b$ we obtain the set $\vec{Z}_j'$ which belongs to Ω; the case c_{2-2}) is not: it contradicts to the construction of the operator $F_\cup(\{i, i_o\}, p^{\{i, i_o\}})$, because $p^{\{i, i_o\}} = |Z_i \cup Z_{i_o}|$. The last case d) can be reduced to the case c). It is obvious that the operator $\hat{F}$ given in the form (3) can be reduced to the form (2).

The theorem is completely proved. $\hfill$ Q.E.D.

7 Explicit Form of Operators with Extremal Values of Parameter q

Let us investigate the introduced operators with values of parameter q equal to $0, m-1$ and $q < 0$.

The following relations are obviously holding: $\forall i, I$

$$F(i, m-1) = F_\cup(I, m-1) = F_\cap(I, m-1) = 1$$

Additionally we can redefine these operators for the case when q is less than 0:

$$\forall i, I \text{ and } q < 0 \quad F(i, q) = F_\cup(I, q) = F_\cap(I, q) = 0.$$

Let us introduce now the choice function $C_{G_i}(\cdot)$ such that

$$C_{G_i}(X) = \{y \in X | \exists x \in X \text{ s.t. } |x G_i y\},$$

i.e. the function $C_{G_i}(\cdot)$ is a pair-dominant on the relation G_i.

Theorem 4 *For all I the operator $F_\cup(I, 0)$ is such that the collective choice function $C(\cdot)$ is constructed by the following way*

$$C(X) = \bigcap_{i \in I} C_{G_i}(X)$$

On the other hand this operator is equal to

$$F_\cup(I, \bullet) = \bigcap_{i \in I} F(i, 0)$$

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Proof: can be obtained from the definition of operator $F_\cup(I, p)$.   Q.E.D.

Consider now the operator $\bigcup_{i \in I} F(i, 0)$. Using the representation of collective choice function through the functions $C_{G_i}(\cdot)$, one can obtain that

$$\bigcup_{i \in I} F(i, 0) : C(X) = \bigcup_{i \in I} C_{G_i}(X),$$

i.e. by this operator is implemented the partial (on the set I) unified-dominant mechanism of choice (see Aizerman and Aleskerov (1990)).

Let us obtain some other correlations which follow just from the definition of these operators

$$\begin{aligned}
F_\cup(\{i\}, p^i) &= F(i, p^i); \\
F_\cap(\{i\}, q_\cap^i) &= F(i, q^i).
\end{aligned}$$

Theorem 5 $F_\cup(I', q) \subseteq F_\cup(I, q)$ *for all* I *and* I' *s.t.* $I \subseteq I'$. *Analogously,* $F_\cap(I, q) \subseteq F_\cap(I', q)$ *for all* I, I' *s.t.* $I \subseteq I'$.

Proof: We prove only the first statement of the theorem. Let $C'(\cdot)$ and $C(\cdot)$ be a choice function generated by operators $F_\cup(I', q)$ and $F_\cup(I, q)$. Show that if $x \in C'(X)$, then $x \in C(X)$. If $x \in C'(X)$ implies that $|\bigcup_{i \in I'} X \cap \mathcal{D}_i(x)| \leq q$. Then $|\bigcup_{i \in I} X \cap \mathcal{D}_i(x)| \leq q$ and $x \in C(X)$.   Q.E.D.

8   Range Constraints on Operator F

As it was defined before, all over the paper that the domain of operator F was assumed to be n-tuple product of the set of all weak orders. Because the range of operator F is the space $\mathcal{C}$ of all choice functions let us define some subclasses of this space which will be used as range constraints for operator F.

Definition 4 *Choice function* $C(\cdot)$ *is said to satisfy the condition of*

a)	*Heritage* (**H**) *if*	$\forall X, X'$ s.t. $X' \subseteq X \Rightarrow C(X') \supseteq C(X) \cap X'$;
b)	*Concordance* (**C**) *if*	$\forall X', X'' \Rightarrow C(X') \cap C(X'') \subseteq C(X' \cup X'')$;
c)	*Independence of outcast*	
	options (**O**) *if*	$\forall X, X'$ s.t. $X' \subseteq X \setminus C(X) \Rightarrow C(X \setminus X') = C(X)$;
d)	*Constancy* (**K**)	$\forall X, X'$ s.t. $C(X) \cap X' \neq \emptyset \Rightarrow C(X') = C(X) \cap X'$.

The classes in $\mathcal{C}$ isolated with these conditions will be denoted with the same letters as the conditions itself.

Definition 5 *The domain D of choice functions will be called as closed relative to relational-functional operator F iff $\forall \vec{G} \; F(\vec{G}) \in D$ holds. The domain D will be called as closed according to the class $\mathcal{F}$ of operators F if D is closed relative to each $F \in \mathcal{F}$. The maximal in set-theoretic sense class of operators $\mathcal{F}$ according to which the domain D is closed will be called as a complete class of operator closedness for the domain D and will be denoted as $\Lambda(D)$.*

Theorem 6 *Operator $F(i, q_i)$ generates the choice function $C(\cdot)$ which belongs in general to the domain $\mathbf{H} \cap \mathbf{O}$ and only in the case $q_i = 0$ function $C(\cdot)$ which belongs to the domain $\mathbf{K}$.*

Proof: The case $q_i = 0$ is obvious. Let us consider the case when $q_i > 0$. Consider an arbitrary $X \in \mathcal{A}^o$, and $x \in C(X)$. Let $X' \subset X$, $x \in X'$. Show that $x \in C(X')$. Because of $x \in C(X)$ and hence $|X \cap \mathcal{D}_i(x)| \leq q_i$, then $|X' \cap \mathcal{D}_i(x)| \leq q_i$ and $x \in C(X')$. Hence the condition H is satisfied.

Let us show that the condition $\mathbf{O}$ is satisfied. Let X be an arbitrary set and $C(X) \subset X$. It implies that if $x \in C(X)$ and $z \notin C(X)$, $|X \cap \mathcal{D}_i(x)| \leq q_i$ and $|X \cap \mathcal{D}_i(z)| > q_i$. Then it is obvious that $|(X \setminus \{z\}) \cap \mathcal{D}_i(x)| \leq q_i$ and $x \in C(X \setminus \{z\})$. If $y \notin C(X)$, then for the same reason $|(X \setminus \{z\}) \cap \mathcal{D}_i(y)| > q_i$ and $y \notin C(X \setminus \{z\})$. We used the outcast condition in its equivalent form (see, e.g. Aizerman and Aleskerov (1990)). The theorem is proved. Q.E.D.

Theorem 7 *Operator $F_{\cup}(I, p^I)$ generates a choice function $C(\cdot)$ which belongs in general to the domain $\mathbf{H}$ and only in the case $p^I = 0$ a function $C(\cdot)$ belongs to the domain $\mathbf{H} \cap \mathbf{C}$.*

Proof: Consider first the case $p^I = 0$. In this case according to the definition of operator $F_{\cup}(I, p^I)$ x belongs to $C(X)$ if $|\bigcup\limits_{i \in I} X \cup \mathcal{D}_i(x)| = 0$, i.e. x is undominated option in all G_i/X, $i \in I$.

Let us construct the choice functions $C_i(\cdot)$ s.t.

$$C_i(X) = \{x \in X | \bar{\exists} y \in X \text{ s.t. } y G_i x\}.$$

It is obvious that because G_i are weak orders, then $C_i(\cdot) \in \mathbf{K}$, and $F_\cup(I,0) = \bigcap\limits_{i \in I} C_i(\cdot)$, but intersection of the functions from $\mathbf{K}$ is in $\mathbf{H} \bigcap \mathbf{C}$ (see Aizerman and Aleskerov (1990)).

Now consider general case $0 < p^I < m - 1$. Let $x \in C(X)$, and $x \in X' \subset X$. This means that $p^I \geq |\bigcup\limits_{i \in I} X \bigcap \mathcal{D}_i(x)| \geq |\bigcup X' \bigcap \mathcal{D}_i(x)|$. Hence $x \in C(X')$, so the $\mathbf{H}$ condition is obeyed. Show that condition $\mathbf{O}$ is not satisfied. Let us consider $I = \{1, 2\}$, operator $F_\cup(I, 1)$ and the binary relations G_1, G_2 shown in Figure 3. For this profile $x \in C(A)$; $y, z \notin C(A)$ because $|\bigcup\limits_{i \in I} A \bigcap \mathcal{D}_i(y)| = |\bigcup\limits_{i \in I} A \bigcap \mathcal{D}_i(z)| = 2$. But $|\bigcup\limits_{i \in I} A \setminus \{z\}) \bigcap \mathcal{D}_i(y)| = 1$ and y belongs to $C(X \setminus \{z\})$.

Consider now the $\mathbf{C}$ condition. At the same binary relations G_1 and G_2 and with the same operator $F_\cup(I, 1)$ it is obvious that $z \in C(\{x, z\})$ and $z \in C(\{y, z\})$ but $z \notin C(A)$, so the condition $\mathbf{C}$ is violated.

It is obvious that this example can be extended on arbitrary number of options in A and arbitrary set I. Q.E.D.

Theorem 8 *Operator $F_\cap(I, q^I)$ generates a choice function $C(\cdot)$ which belongs in general to the domain $\mathbf{H} \bigcap \mathbf{O}$ and only in the case $q^I = 0$ a function $C(\cdot)$ belongs to the domain $\mathbf{H} \bigcap \mathbf{C} \bigcap \mathbf{O}$.*

Proof: Consider first the case $q^I = 0$. Rewriting the operator $F_\cap(I, q^I)$ in the form

$$C(X) = \{y \in X | \overline{\exists} x \in X \text{ s.t. } \forall i \in I \ x G_i y\}$$

we obtain that it is exactly the Pareto rule which generates the functions from the domain $\mathbf{H} \bigcap \mathbf{C} \bigcap \mathbf{O}$.

Consider now general case $0 < q^I < m - 1$. Let $x \in C(X)$ and $x \in X' \subset X$. $q^I \leq |\bigcap\limits_{i \in I} X \bigcap \mathcal{D}_i(x)| \leq |\bigcap\limits_{i \in I} X' \bigcap \mathcal{D}_i(x)|$, hence $x \in C(X')$ and condition $\mathbf{H}$ is obeyed.

Consider now two binary relations G_1 and G_2 shown in Figure 4, and the operator $F_\cap(\{1, 2\}, 1)$. For this situation we obtain $|\bigcap\limits_{i \in I} \{x, z\} \bigcap \mathcal{D}_i(x)| \leq 1$ and $|\bigcap\limits_{i \in I} \{x, y\} \bigcap \mathcal{D}_i(x)| = 1$, so $x \in C(\{x, y\}) \bigcap C(\{x, z\})$. But $x \notin C(A)$ because $|\bigcap\limits_{i \in I} A \bigcap \mathcal{D}_i(x)| = 2$. Hence the condition $\mathbf{C}$ is violated.

Consider the condition O. Suppose $x \in C(X)$; $y, z \notin C(X)$. This means that $|\bigcap\limits_{i \in I} X \bigcap \mathcal{D}_i(x)| \leq q^I$, $|\bigcap\limits_{i \in I} X \bigcap \mathcal{D}_i(y)| > q^I$ and $|\bigcap\limits_{i \in I} X \bigcap \mathcal{D}_i(z)| > q^I$. Let us consider

the set $X' = X \setminus \{z\}$ and show that $x \in C(X')$. Let on the contrary $x \notin C(X')$, i.e. $|\bigcap_{i \in I} X' \cap \mathcal{D}_i(x)| > q^I$, so there exist $z' \notin \bigcap_{i \in I} (X \cap \mathcal{D}_i(x))$ and $z' \in \bigcap_{i \in I} (X' \cap \mathcal{D}_i(x))$.

But it implies that there exist i_o s.t. $z' \notin \mathcal{D}_{i_o}(x) \cap X$, then exclusion from X the option z does not change the dominance set for x in X', i.e. $z' \notin \bigcap_{i \in I} (X' \cap \mathcal{D}_i(x))$.

On the other hand if $y \notin C(X)$ and $y \in C(X')$ this means that $|\bigcap_{i \in I} X' \cap \mathcal{D}_i(y)| \le q^I$ but $|\bigcap_{i \in I} X \cap \mathcal{D}_i(y)| > q^I$. Hence $z \in \bigcap_{i \in I} \mathcal{D}_i(y) \cap X$, but in this case $|\bigcap_{i \in I} X \cap \mathcal{D}_i(z)| \le q^I$ and $z \in C(X)$ on the contrast to supposition. The theorem is proved.   Q.E.D.

So, according to the Theorems 6-8 in the Central Region Λ^{SMN} the class of operators $\Lambda(F(i, q_i))$ is a class of operator closedness for the domain $\mathbf{H} \cap \mathbf{0}$, $\Lambda(F_\cup(I, p^I))$ is a class of operator closedness for the domain $\mathbf{H}$ and the class $\Lambda(F_\cap(I, q^I))$ is that one for $\mathbf{H} \cap \mathbf{0}$.

9 Special Subclasses of the Class Λ^{CR}

Consider now special classes for the operators from the Central Region.

Definition 6 *Operator F from the Central Region will be called as*

a) *q-Pareto for at least one coalition I;*

b) *q-Pareto for a totality of coalitions;*

c) *p-Condorcet for at least one coalition I;*

d) *p-Condorcet for a totality of coalitions iff it is representable in the form: there exist a totality $\mathcal{I}$ (correspondingly. $\mathcal{J}$) of coalitions I (J) s.t.*

a) *$F = \bigcup_{I \in \mathcal{I}} F_\cap(I, q^I)$;*

b) *$F = \bigcap_{I \in \mathcal{I}} F_\cap(I, q^I)$;*

c) *$F = \bigcup_{J \in \mathcal{J}} F_\cup(J, p^{\mathcal{J}})$;*

d) *$F = \bigcap_{J \in \mathcal{J}} F_\cup(J, p^J)$, respectively.*

Consider now some particular cases of these operators. In case $q = 0$, the operator $F_\cap(I, 0)$ defines the choice of Pareto-optimal elements for the coalition I; correspondingly according to the operator 0-Pareto for at least one coalition I provides that the collective choice will consist of Pareto-optimal elements for different coalitions from $\mathcal{I}$. The operator 0-Pareto for a totality of coalitions guarantees that in the collective choice the options have to be included which are Pareto-optimal for each coalition from $\mathcal{I}$. If $\mathcal{I}$ contains all non-empty coalitions from N, then the option chosen by such operator can be called as Edgeworth-optimal option, and if $\mathcal{I}$ contains only all single-element coalitions, then the chosen option can be called as Cournot-optimal one. Finally, if $\mathcal{I}$ contains only one coalition N, then the operators 0-Pareto for at least one coalition and 0-Pareto for a totality of coalitions coincide and define one operator $F_\cap(N, 0)$ which chooses Pareto-optimal elements from X. Let us note that this operator satisfies the following property

$$\mathcal{U} \subset F_\cap(N, 0) \subset \mathcal{V}.$$

Let us study now the operators $F_\cup(I, 0)$. According to the definition of this operator if we introduce the function

$$C_{G_i}(X) = \{y \in X | \overline{\exists} x \in X \text{ s.t. } x G_i y\},$$

then $F_\cup(I, 0)$ is such that the function $C(\cdot)$ generated by this very operator can be represented in the form: $\forall X \in \mathcal{A}^o$

$$C(X) = \bigcap_{i \in I} C_{G_i}(X),$$

i.e. $C(\cdot)$ is a collection of undominated elements (Condorcet winners) for all i from I. Naturally, such choice function very often will be empty.

So the operator "0-Condorcet for at least one coalition I" defines the collective choice as the options which are undominated for at least one coalition $I \in \mathcal{I}$. In the case when $\mathcal{I}$ contains only one coalition N, i.e. the set of all voters, then this operator chooses the options which are undominated in all binary relations G_i, i.e. the Condorcet winners for N.

10 Operators from Symmetrically-Central Region

Several definitions on the operators from Symmetrically-Central Region are given below.

Definition 7 *The operator $F \in \Lambda^{CR}$ will be called*

a) *weak k-majority q-Pareto one;*

b) *strong k-majority q-Pareto one;*

c) *weak k-majority q-Condorcet one;*

d) *strong k-majority q-Condorcet one iff it is representable in the form*

a) $F = \bigcup\limits_{\ell=1}^{s} F_{\cap}(I_\ell, q)$, *where* $|I_\ell| = k$, $s = C_n^k$ *is equal to all combinations from n to k;*

b) $F = \bigcap\limits_{\ell=1}^{s} F_{\cap}(I_\ell, \blacklozenge)$, $|I_\ell| = k$, $s = C_n^k$;

c) $F = \bigcup\limits_{\ell=1}^{s} F_{\cup}(J_\ell, p)$, $|J_\ell| = k$, $s = C_n^k$;

d) $F = \bigcap\limits_{\ell=1}^{s} F_{\cup}(J_\ell, p)$, $|J_\ell| = k$, $s = C_n^k$, *respectively.*

Let us discuss these operators. In the case a) the chosen option x has to be q-Pareto-optimal for at least one coalition comprising of k voters; in case b) it has to be q-Pareto-optimal for all coalitions of the size k.

Example 3 *Let us show that the request that* x *is a Pareto-optimal (q-Pareto-optimal) in a strong sense, i.e.* x *is chosen according to k-partial strong q-Pareto rule, is stronger that usual Pareto optimality. The profile is given (see Figure 5). Then* x *is Pareto-optimal for* $k = n$, $q = 0$, *but it is not Pareto-optimal for any coalition consisting of two elements, i.e. for* $k = 2$ *and* $q = 0$.

In the case c) the chosen option x is p-Condorcet winner for at least one coalition of the size k; in the case d) it is p-Condorcet winner for all coalitions of the size k.

Let us consider now special case when the coalitions in the definitions of operators strong and weak k-majority q-Pareto and strong and weak k-majority q-Condorcet contain only one element, i.e. k is equal to 1.

In the cases of weak majorities (cases a) and c)) these operators coincide and define the operator which can be called q-one vote. In the cases b) and d) they also coincide

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and if $q = 0$ then such operator chooses classic Condorcet winner, or Cournot-optimal element.

Let us now generalize these operators and introduce two new operators.

Definition 8 *The operator $F \in \Lambda^{CR}$ will be called as*

a) τ_p-*family of strong k-majority q-Pareto one, and*

b) τ_c-*family of strong k-majority q-Condorcet one iff it is representable in the form*

a) $F = \bigcup\limits_{\alpha=1}^{\tau_p} \bigcap\limits_{\ell=1}^{s_\alpha} F_\cap(I_\ell^\bullet, q^\alpha), \ |I_\ell^\alpha| = k_\alpha, \ s_\alpha = \begin{pmatrix} k_\alpha \\ n \end{pmatrix}, \ and$

b) $F = \bigcup\limits_{\alpha=1}^{\tau_c} \bigcap\limits_{\ell=1}^{s_\alpha} F_\cup(I_\ell^\alpha, p^\alpha), \ |I_\ell^\alpha| = k_\alpha, \ \dot{s}_\alpha = \begin{pmatrix} k_\alpha \\ n \end{pmatrix}, \ respectively.$

Theorem 9 *An operator F belongs to the Symmetrically-Central Region iff it is representable as on intersection of τ_p-family of strong k-majorities q-Pareto operators and τ_c-family of strong k-majority q-Condorcet operators, i.e.*

$$F = \left[\bigcup\limits_{\alpha=1}^{\tau_p} \bigcap\limits_{\ell=1}^{s_\alpha} F_\cap(I_\ell^\alpha, q^\alpha) \right] \bigcap \left[\bigcup\limits_{\alpha=1}^{\tau_c} \bigcap\limits_{\ell=1}^{s_\alpha} F_\cup(I_\ell^\alpha, p^\alpha) \right] \tag{4}$$

Proof: Let us consider the representation in the form (2) for the operator from Central Region. The restriction of this representation with Anonymity condition leads to the fact that this sets of coalition totalities have to contain the coalitions with different signs $k_1, \ldots, k_\tau$. The form (4) can be obtained from (2) because of distributivity of operations $\bigcup$ and $\bigcap$. Q.E.D.

Let us call this operator for brevity as "τ-families of k majorities".

11 Closedness of the Domains in $\mathcal{C}$ Relative to Operators from Central and Symmetrically-Central Region

This section begins with the theorem which shows what kind of choice functions is generated by operators from Central Region.

Theorem 10 *The Central Region is disposed inside the complete class of operator closedness* $\Lambda_{\mathbf{H}}$ *for the domain* $\mathbf{H}$, *i.e.* $\Lambda^{SMN} \subset \Lambda_{\mathbf{H}}$.

Proof: Let us consider the following condition on list-form representation of operators:

$$\forall x, X, X' : x \in X' \subset X, \quad \vec{Z} \in \Omega(x, X) \Rightarrow \vec{Z}' = \vec{Z} \bigcap X' \in \Omega(x, X') \qquad (5)$$

$$\text{Q.E.D.}$$

Lemma 7 *Condition (5) is necessary and sufficient one which isolates operators from* $\Lambda_{\mathbf{H}}$.

Proof: Let $F \in \Lambda_{\mathbf{H}}$. Show that condition (4) is satisfied. Suppose not, i.e. $\exists x, X, X', \vec{Z}, \vec{Z}'$ s.t. $\vec{Z} \in \Omega(x, X)$, but $\vec{Z}' \notin \Omega(x, X')$. But one can construct the profile s.t. $X \bigcap \mathcal{D}_i(x) = Z_i$ for all i, and $X'\mathcal{D}_i(x)$ will be equal to $X' \bigcap Z_i$, so if $\vec{Z}' \notin \Omega(x, X')$ then $x \notin C(X')$ in contradiction with assumption that $F \in \Lambda_{\mathbf{H}}$. Assume now that condition (5) is satisfied. Then for all X' it is obvious that $x \in C(X')$, so $F \in \Lambda_{\mathbf{H}}$. Lemma is proved.

Because condition (5) is satisfied for all $X' \subseteq X$, then for the operators which obey to neutrality condition $3^{\bullet}$, it is obvious that Ω will satisfy to monotonicity condition. This implies that $\Lambda_{\mathbf{H}} \supset \Lambda^{SMN}$. Theorem is completely proved. Q.E.D.

This result means that all operators from the Central Region generate functions from domain $\mathbf{H}$, so we always even restricting the operators from the Central Region can obtain function which satisfy to the condition $\mathbf{H}$ and probably some other conditions $\mathbf{C}$, $\mathbf{O}$, or $\mathbf{K}$.

The main result of the paper can be summarized in the following theorem.

Theorem 11 *The intersections of Central and Symmetrically-Central Regions with the complete classes of operator closedness for the domains* $\mathbf{H}, \mathbf{H} \bigcap \mathbf{O}, \mathbf{H} \bigcap \mathbf{C}, \mathbf{H} \bigcap \mathbf{C} \bigcap \mathbf{O}$ *and* $\mathbf{K}$ *contain the operators presented in corresponding cells of Table 1 and only them*

	$\Lambda_{\mathbf{H}}$	$\Lambda_{\mathbf{H} \cap \mathbf{O}}$	$\Lambda_{\mathbf{H} \cap \mathbf{C}}$	$\Lambda_{\mathbf{H} \cap \mathbf{C} \cap \mathbf{O}}$	$\Lambda_{\mathbf{K}}$
Λ^{SMN}	federation	partial q-Pareto for at least one coalition	partial Condorcet operator for all coalitions	partial Pareto operator	dictator $F(i, 0)$
Λ^{SMNA}	τ-families of k-majorities	weal k-majority q-Pareto	Condorcet operator $F_{\cup}(N, 0)$	Pareto operator	$\emptyset$

Proof: Let us prove consequently all the statements of Theorem 11.

1. The result about the intersection $\Lambda^{SNM} \cap \Lambda_{\mathbf{H}}$ is direct corollary of Theorems 3, 9, and 10.

2. The result about $\Lambda^{SNMA} \cap \Lambda_{\mathbf{H}}$ is direct corollary of Theorem 3.

3. The intersection $\Lambda^{SNMA} \cap \Lambda_{\mathbf{H} \cap \mathbf{0}}$ contains only operators $\bigcup\limits_{I \in \mathcal{I}} F_{\cap}(I, q^I)$.

4. Because $F_{\cap}(I, q^I) \in \Lambda_{\mathbf{H} \cap \mathbf{O}}$ and as it is proved in Aizerman and Aleskerov (1990) the domain $\mathbf{H} \cap \mathbf{O}$ is closed relative to $\bigcup$ operation, but not $\cap$.

5. Can be obtained from 3) with Anonymity restriction.

6. The intersection $\Lambda^{SMN} \cap \Lambda_{\mathbf{H} \cap \mathbf{C}}$ contains only operators $\bigcap\limits_{J \in \mathcal{J}} F_{\cup}(J, 0)$ because $F_{\cup}(J, 0)$ belongs to $\Lambda_{\mathbf{H} \cap \mathbf{C}}$ and as it is proved in Aizerman and Aleskerov (1990) the domain $\mathbf{H} \cap \mathbf{C}$ is closed relative to $\cap$ operation, but not $\cup$.

7. $\Lambda^{SMNA} \cap \Lambda_{\mathbf{H} \cap \mathbf{C}}$ can be obtained from 5) with Anomity restriction.

8. $\Lambda^{SMN} \cap \Lambda_{\mathbf{H} \cap \mathbf{C} \cap \mathbf{O}}$ can be obtained from Theorem 8 and the fact that this domain is closed neither relative to $\bigcup$ operation, nor $\cap$ one.

9. The result about $\Lambda^{SMNA} \cap \Lambda_{\mathbf{H} \cap \mathbf{C} \cap \mathbf{O}}$ can be obtained from 7) under Anonymity restriction.

10. $\Lambda^{SMN} \cap \Lambda_{\mathbf{K}}$. This result is direct corollary from Theorems 3 and 6.

11. Obvious taking into account 9). Theorem is completely proved.   Q.E.D.

The result that $\Lambda^{SMN} \cap \Lambda_{\mathbf{K}} = \{F(i, 0)\}$ is exactly the analog of famous Arrow's Possibility theorem, however obtained under the other than Independence of Irrelevant Alternatives Condition.
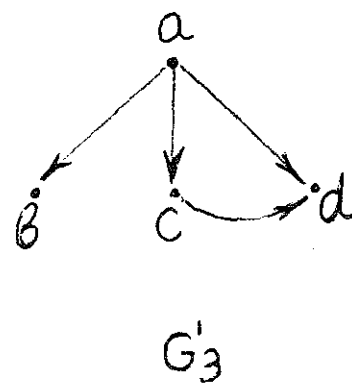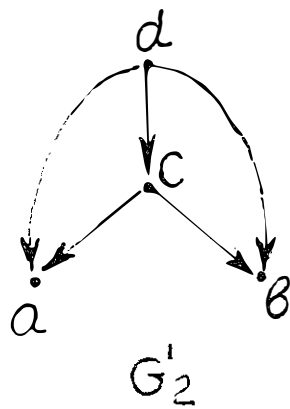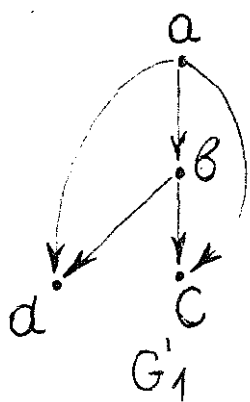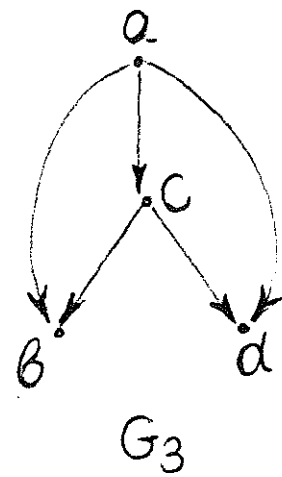
Figure 1

Figure 2

G_1 G_2

Figure 3

G_1 G_2

Figure 4

G_1 G_2 G_3
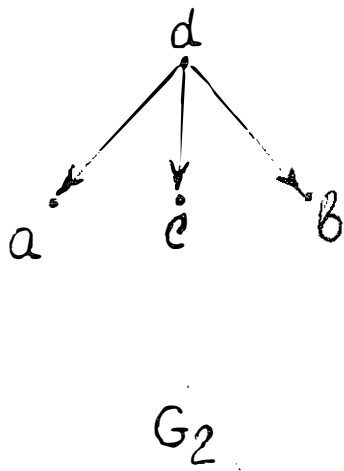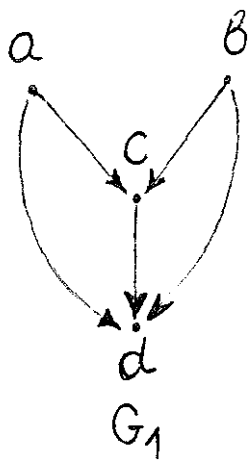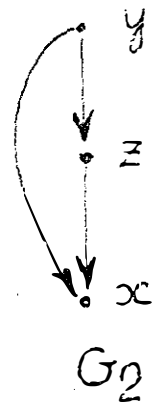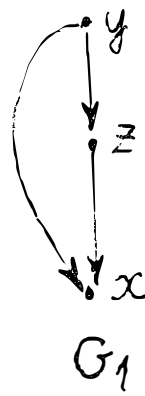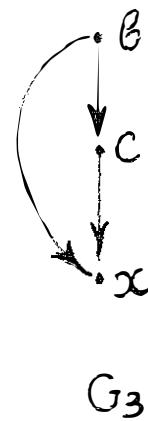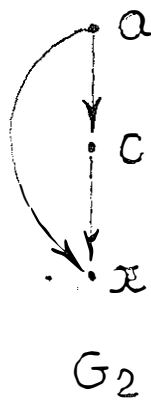
Figure 5

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