Application of the California Institute of Technology Electric Analog Computer to Nonlinear Mechanics and Servomechanisms

G. D. McCANN
MEMBER AIEE

C. H. WILTS
ASSOCIATE AIEE

B. N. LOCANTHI
NONMEMBER AIEE

Synopsis: This paper describes the non-linear elements and circuit techniques used with the California Institute of Technology electric analog computer. Their application to nonlinear mechanical vibratory systems and nonlinear servomotors is discussed in detail. These techniques have been found to be generally suitable for representing single valued nonlinear functions of a dependent variable. Nonlinear springs, spring loaded backlash, and nonlinear damping factors can be readily simulated as well as saturation effects and other single valued non-linearities in servomotors. Methods of analysis are illustrated for several typical problems including a nonlinear rotating mechanical system and an autopilot employing a solenoid-operated rate and position limited hydraulic motor. Numerous servos of this type have now been studied and correlation of computer solutions with actual servo test data have shown in every case that the mathematical equations presented here accurately describe this type of motor.

THE California Institute of Technology electric analog computer has now been used for the solution and general analysis of a sufficient number of nonlinear servomechanisms and nonlinear mechanical vibratory systems to warrant a general discussion of the electrical techniques and analogies suitable for such problems. This paper deals particularly with its application to problems involving single valued nonlinear parameters that are a function of one dependent variable. The more common types of such nonlinearity in mechanical vibratory systems are nonlinear springs, backlash, and nonlinear damping such a coulomb friction. Many types of servomechanisms have nonlinearities of sufficient importance that their effects should be considered in analyzing system performance. Some of these nonlinearities are in the control system itself; others are in the system to be controlled. In position type controllers nonlinearities are commonly in the form of velocity or position saturation effects or limits, backlash, coulomb friction, or servos of the "on-off" type. Saturation effects such as occur in motors and generators of speed controllers and voltage regulators are sometimes important. The nonlinearities existing in the systems to be controlled may be of many types. In mechanical systems they may be in the form of nonlinear damping factors or spring constants, or they may be in the form of limits to certain motions being controlled. Certain thermal processes are inherently nonlinear. Some types of autopilot problems require consideration of the nonlinearities in the aerodynamic equations.

The California Technology electric analog computer has been described in detail in previous papers.[1–4] Suitable multipliers and nonlinear function devices now have been developed to enable its use for the rapid solution of a large number of problems of this type. These are described in detail in reference 4, and are only discussed briefly here with reference to their application to circuit analogies. The basic elements now being used in this computer are listed in Figure 1. The form of symbolization shown is that used in the California Technical Analysis Laboratory for preparing analogous circuits.

Description of Nonlinear Functions Devices

The devices for simulating nonlinear functions of a dependent variable are of two general types designated here as the "type I" and "type II" nonlinear functions devices. The first of these is essentially a nonlinear resistor in that it produces voltages that are arbitrary functions of currents or vice versa. The second type produces voltages that are arbitrary functions of another voltage.

TYPE I—NONLINEAR FUNCTIONS DEVICE

Diodes which conduct freely in only one direction can be used as the basic elements of a nonlinear device. In order to avoid power supply problems and to minimize the "forward resistance," crystal diodes have been used generally instead of vacuum diodes. The "back resistance" of the former elements is sufficiently high to meet most of the requirements in practical problems. Although these diodes can be associated with general impedance elements, this is usually undesirable because the forward and back impedances are resistive in nature and are most readily combined with other resistances. Nonlinear impedances can be achieved with diodes, however, through the use of amplifiers as will be described later.

The two basic elements of the diode nonlinear resistance are the voltage clipper and the current limiter shown in Figure 2. For the voltage limiter, batteries are connected with polarity opposing the normal direction of current flow, so that until the voltage exceeds the battery voltage, the effective resistance of the circuit is very high and is equal to one-half the back resistance of one diode. After the voltage exceeds this value, the

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G. D. McCANN, C. H. WILTS, and B. N. LOCANTHI are all with the California Institute of Technology, Pasadena, Calif.

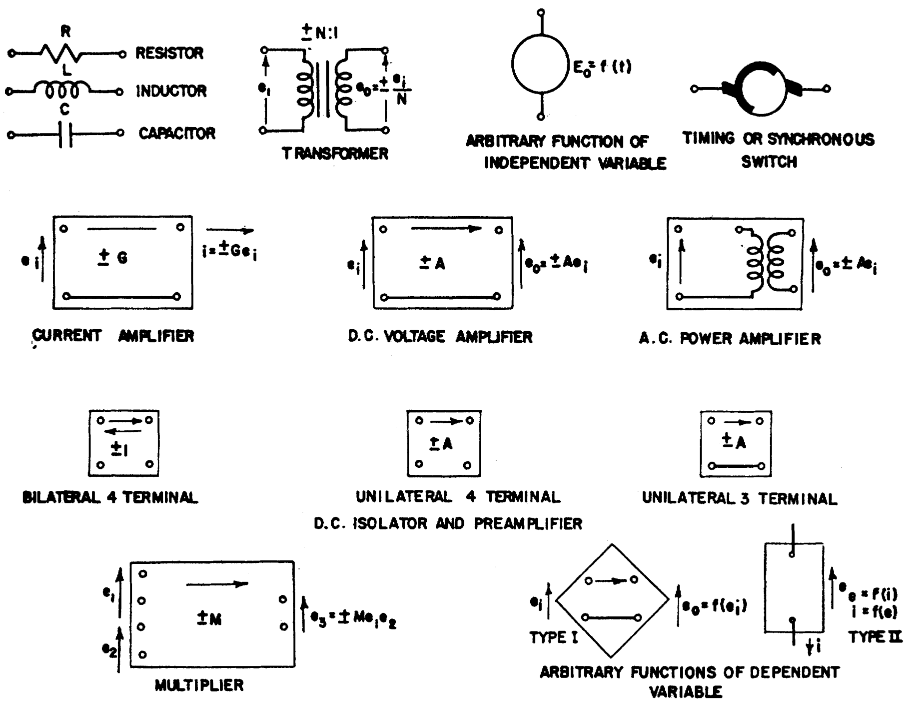

Figure 1. Symbolization of basic computer elements

effective resistance is very low, the forward resistance of one diode plus that of the battery. For a typical *1N35* crystal, the ratio of the two slopes of the voltage-current characteristic is approximately 10,000. The current limiter utilizes two large resistors, R_1, to produce circulating currents in the two diode loops. Until the current inserted at the terminal equals this circulating current, the net resistance of the network is low and twice the forward resistance of one diode, while for larger currents it becomes large, essentially equal to the resistance R_1 in parallel with the back resistance of one diode. The ratio of the slopes of the resulting characteristic is not as great as that for the voltage limiter because of the addition of the resistor R_1. However, values of 200-to-1 for this ratio can be readily achieved.

In both limiters, the ratio of slopes can be reduced to any desired value by insertion of resistors in parallel and in series with the diode branches. Such a modification forms the basis of the general nonlinear device shown in Figure 3. The adjustable resistors are made relatively large compared to the forward impedance of the crystals so that the variations of the latter with current do not affect the overall impedance appreciably. If the switches are all thrown to the left in Figure 3, each parallel branch will not conduct current until the terminal voltage rises above the battery voltage of that branch. The resulting current voltage characteristic is composed of a series of straight lines, with the slope of successive segments decreasing as the voltage rises. Such a characteristic is shown in curve "A" of Figure 3.

If the switches are thrown to the right, a different characteristic results. Each branch produces a circulating current which passes through R_0. The battery voltage E_0 is provided to give zero terminal voltage when no external current enters the terminals. As voltage is applied, the current which flows is determined by the effective resistance of all branches in parallel. As soon as the terminal voltage rises above the lowest branch battery voltage, the current in that branch ceases to flow, and the effective impedance of the device is raised. As the voltage exceeds the highest battery voltage, the impedance becomes equal to R_0 alone, the maximum value. The resulting current-voltage characteristic is one of constantly increasing slope, as shown in curve "B". By combination of branches of each type, a characteristic may be achieved whose slope is alternately increasing and decreasing, provided the slope always remains positive. Curve "C," Figure 3, is typical.

A nonlinear resistor with 22 parallel branches, 11 each for positive and negative voltages is shown in Figure 5. The simpler voltage and current limiter is shown in the same figure.

TYPE II—NONLINEAR FUNCTIONS DEVICE

Although the first nonlinear device described is essentially a resistance element, another nonlinear device has been constructed which produces a voltage that is an arbitrary (though necessarily single valued) function of any voltage however derived. This generator is based upon a principle developed elsewhere.[6,7] It utilizes a cathode-ray tube and photo cell for viewing the cathode-ray tube screen. The quiescent position of the electron beam is set to one side of the tube and the photo cell output is amplified and applied to the plates with a polarity which will drive the beam to the other side. If an opaque template is placed on the surface of the tube, as shown in Figure 4, the beam spot will be driven below the edge of

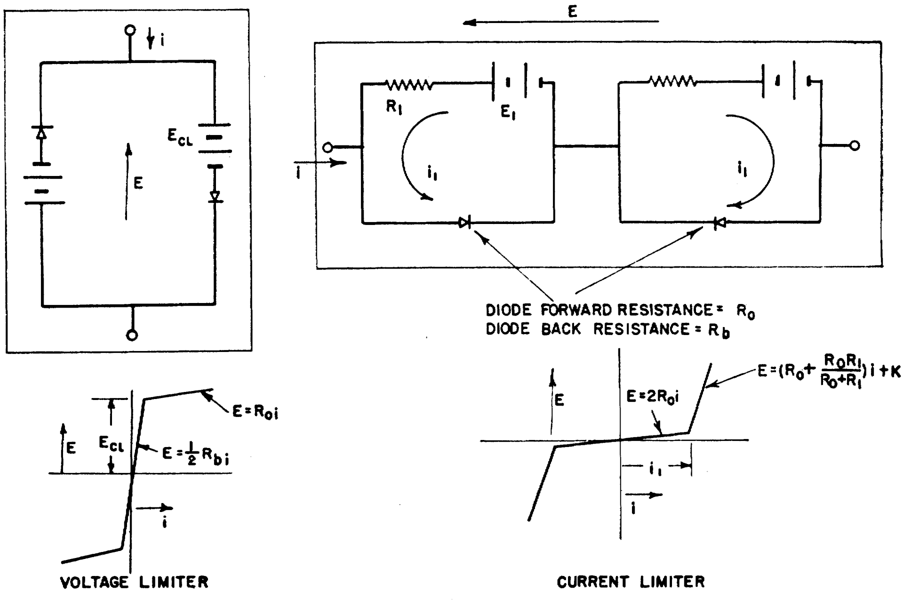

Figure 2. Schematic circuits of simple voltage and current limiters

the template until the light reaching the photo tube is just sufficient to produce a signal that will deflect the beam to the edge of the template. Since the beam will follow closely the template, and since the voltage required to deflect the beam is a linear function of the deflection, the output voltage of the device is proportional to the height of the template at the particular position of the beam. A generator of this type is shown in Figure 5. A typical characteristic is shown as viewed on an oscilloscope in Figure 4. A more detailed description of both of the nonlinear functions devices has been given elsewhere.[4]

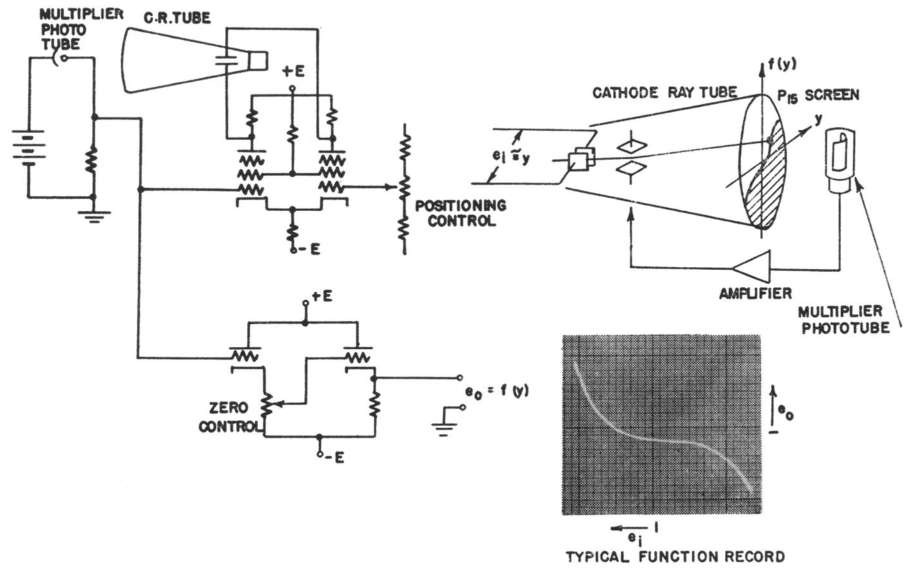

Figure 4. Schematic diagram of dependent variable device—type II

General Application of Nonlinear Devices

VOLTAGE CLIPPERS

In some physical systems a variable exhibits a linear characteristic until it reaches a limiting value which it cannot exceed. Such characteristics can be produced electrically with a voltage limiter if the voltage at some point in the electrical circuit can be made the analog of the variable in the physical system. The only important electrical requirement is that the voltage clipper be driven by a source with high impedance so that the forward resistance of the diode is very small compared with the source impedance. In typical cases the slope of the characteristic may be changed by a factor as large as 2,000.

As discussed later this principle can be used directly to simulate position and velocity stops in servomechanisms, coulomb friction, and backlash.

CURRENT LIMITERS

A current limiter may be used in those cases where the current is analogous to a physical variable which is limited in its maximum value. Its application is discussed later with reference to rate limited hydraulic servos.

NONLINEAR GAIN CHARACTERISTICS

The simulation of such nonlinear amplification characteristics as saturation effects in electronic amplifiers or rotating machines can be accomplished readily with either type of nonlinear functions device. The type II device need merely be inserted in cascade with linear amplifiers of the computer. The type I device also could be used in the same manner as illustrated later for simulating nonlinear springs.

However, it has usually been found more suitable to use it as a nonlinear feed-back resistor as shown in Figure 6. This type of circuit also is used where the resistance range provided by the type I device alone is too high.

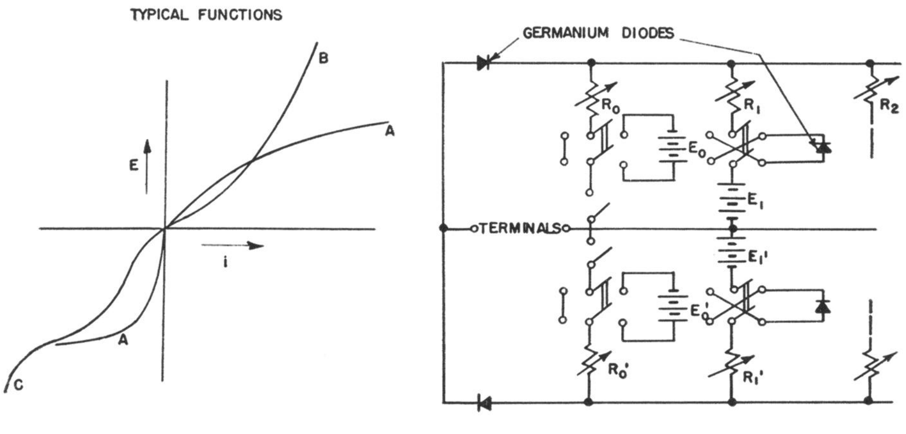

Figure 3. Schematic diagram of dependent variable device—type I

NONLINEAR IMPEDANCES

The nonlinear terms arising in many physical or mathematical problems may be interpreted in the analog as nonlinear self or mutual impedances. It will be sufficient to describe methods of achieving nonlinear self impedances. The type I device must be used in conjunction with resistors to avoid unpredictable phase shift characteristics. In order to use this element, therefore, the voltage from a small impedance of the type specified must be applied to the nonlinear resistor by means of an amplifier, and the resulting nonlinear voltage reinserted in the circuit by means of another amplifier. Such a circuit and the resulting characteristic is shown in Figure 7. The use of the type II nonlinear device is shown in the same figure and requires no additional explanation.

As an example, suppose the charge flowing in a circuit is equal to a variable x, and a term $f(x)$ appears in the differential equation to be solved, then the impedance Z_0 would become a large capacitor and either nonlinear device would be adjusted to give an output voltage equal to $f(x)$. A correction would be applied if the voltage across the capacitor were appreciable in the circuit.

The accuracy obtained with the foregoing circuit techniques depends primarily upon how accurately it is attempted to simulate a given function. The type I device will simulate many functions with an error ranging from one to five per cent of the maximum value of the function. The type II device will simulate most functions to a somewhat greater accuracy, and is applicable to a wider variety of functions.

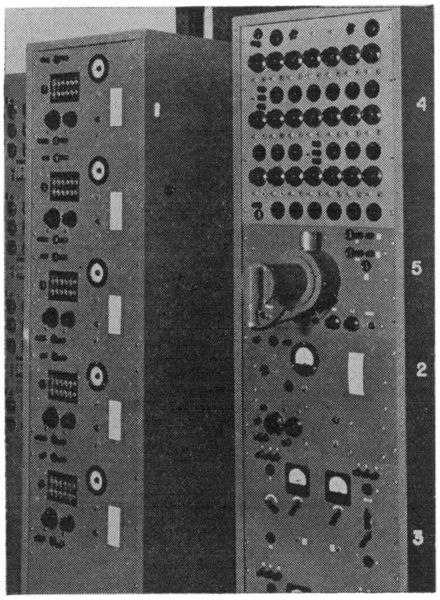

Figure 5. Nonlinear functions devices and other computer elements

1. Negative gain servo amplifiers
2. Positive gain d-c amplifiers
3. Voltage and current limiter
4. Arbitrary functions device—type I
5. Arbitrary functions device—type II

Nonlinearities in Mechanical Vibratory Systems

The preceding discussion makes it readily apparent that the nonlinear functions devices can be used to simulate most types of single valued nonlinearities occurring in mechanical systems. These are generally either nonlinear springs or damping factors. With the use of the force-voltage analogy, in which current is analogous to velocity and charge to position, it is evident that the nonlinear resistance devices can be used directly to simulate nonlinear mechanical damping. Their application to the analysis of problems involving coulomb friction has been discussed previously.[5]

Nonlinear Springs and Backlash

The application of the nonlinear impedance circuits to mechanical systems with nonlinear springs and spring-loaded backlash is illustrated by the system of Figure 8. This is the rotating mechanical system of a radial aircraft engine that was studied quite extensively with the computer.

In this system all three shafts are geared and have appreciable spring-loaded backlash. In addition, shaft number 3 is nonlinear as is shown in Figure 9.

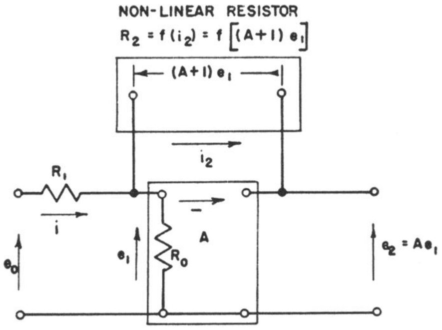

Figure 6. Circuit for nonlinear amplification factor or reduced impedance nonlinear resistor

e_0=input voltage from source of negligible impedance

$$e_1=\frac{R_2R_0e_0}{R_1R_2+R_2R_0+(A+1)R_1R_0}$$

if $R_1<<R_0$ and $A>>1$

$$e_1=\frac{R_2e_0}{R_2+(A+1)R_1}=\frac{e_0}{1+\frac{R_1}{R_2}(A+1)}$$

or if $\frac{R_1}{R_2}(A+1)>>1$

$$e_1=\frac{R_2e_0}{R_1(A+1)};\quad e_2=\frac{-AR_2e_0}{R_1(A+1)}$$

The following equations specify the system

$$\left.\begin{aligned}0&=Ip^2\theta_1+K_1(\theta_1-\theta_2)\\0&=I_2p^2\theta_2+K_1(\theta_2-\theta_1)+K_2(\theta_2-\theta_3)\\T&=I_3p^2\theta_3+K_2(\theta_3-\theta_2)+K_3(\theta_3-\theta_4)\\0&=I_4p^2\theta_2+K_3(\theta_4-\theta_3)\end{aligned}\right\}\quad(1)$$

where $p=\frac{d}{dt}$

If the spring characteristics were perfectly linear the analogous electric circuit would be that given in Figure 8(B) with simple capacitors representing the spring constants. The electric circuit equations would be the following

$$\left.\begin{aligned}0&=L_1p'^2q_1+\frac{1}{C_1}(q_1-q_2)\\0&=L_2p'^2q_2+\frac{1}{C_1}(q_2-q_1)+\frac{1}{C_2}(q_2-q_3)\\E_0&=L_3p'^2q_3+\frac{1}{C_2}(q_3-q_2)+\frac{1}{C_3}(q_3-q_4)\\0&=L_4p'^2q_4+\frac{1}{C_3}(q_4-q_3)\\p'&=\frac{d}{dt'}\end{aligned}\right\}\quad(2)$$

By establishing values for an impedance base factor (a) a time base change factor $\left(n=\frac{t}{t'}\text{ or }\frac{d}{dt'}=n\frac{d}{dt}\right)$ and a forcing functions ratio $r=\frac{T_0}{E_0}$ the following electrical parameters can be determined

$$L_1\ldots\ldots=I_1\frac{a}{n^2};\ \frac{1}{C_1}\ldots\ldots=aK_1;$$

$$\theta=arq;\ T=\frac{E}{r}\ \ldots\ldots\quad(3)$$

In the analysis of nonlinear problems of this sort, the proper representation of initial conditions is more difficult than for linear systems. This can be best illustrated by considering a specific type of disturbance, namely the torques produced by a cylinder misfire under a given load condition.

To simplify the initial conditions and the representation of the nonlinear springs for such a case a change can be made in the co-ordinate specifying the system such that the steady-state values of all functions are eliminated. Thus

$$T_0'=T_0-T_{0_{ss}};\ T_1'=T_1-T_{1_{ss}};$$

$$\theta_1'=\theta_1-\theta_{1_{ss}};\ \ldots\ldots\quad(4)$$

This changes the stress-strain curves representing the nonlinear springs in the manner illustrated by Figure 9(B). As shown in Figure 8(B) voltage clippers were used in the circuits of K_1 and K_2 to simulate the backlash. For shaft number 3 both the backlash and change of spring constant were simulated with the circuit of the type I functions device. However, in Figure 8(C) only those elements required are illustrated to clarify a detailed description of the analogy.

Referring to Figure 9(C) the charge flowing through the spring analogy circuit represents the strain and the voltage across it represents the stress. The analogous circuits of Figure 8 are actually positive self-impedance circuits that alter the effective capacities of the condensers C_1 and so forth, by a variable gain which is a function of (q_3–q_4) and is adjusted to properly fit the desired stress-strain curve. When the voltage E_1 exceeds the clipper voltage E_{c1} and the voltage E_2 is less than clipper voltage E_{c2}, Figure 8(C), the amplifier circuit has its maximum gain and the analogous spring factor for K_3 is a maximum given by the equation

$$aK_{3\max}=\frac{1}{C_{\min}}=\frac{1+kA_1A_2}{C_3}\quad(5)$$

where

$$k=\frac{R_1}{R_0+R_1}$$

Referring again to Figure 8(C), the backlash characteristic is simulated with the diode voltage clipper (E_{c1}). Whenever the voltage A_1E_0 plus E_1 is less than E_{c1} the clipper acts like an open circuit and amplifier A_2 puts out no voltage. Under these conditions the spring constant is

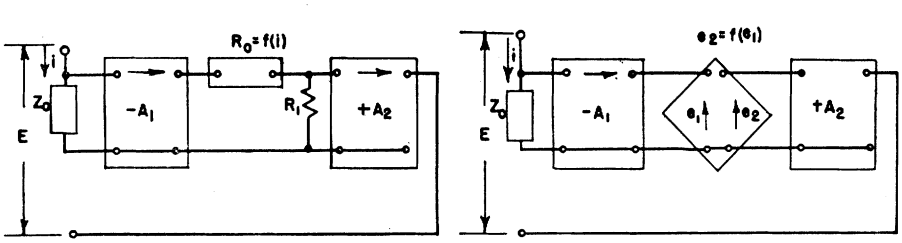

Figure 7. Circuits for obtaining nonlinear impedances with arbitrary functions devices

(Left) Using type I functions device

$$E=\left(Z_0+\frac{A_1A_2R_1}{R_1+R_0}Z_0\right);$$

$$Z=Z_0\left(1+\frac{A_1A_2R_1}{R_1+R_0}\right)$$

$$\text{if } R_0>>R_1 \text{ and } \frac{A_1A_2R_1}{R_0}>>1, Z\cong\frac{KZ_0}{R_0}$$

Interchanging R_1 and R_0 in circuit

$$Z=Z_0\left(1+\frac{A_1A_2R_0}{R_1+R_0}\right)$$

$$\text{if } R_1>>R_0;\ \frac{A_1A_2R_0}{R_1}>>1\quad Z\cong KR_0Z_0$$

(Right) Using type II functions device

$$E=Z_0i+A_2f(A_1Z_0i)$$
$$E=Z(i)i$$

$$Z(i)=\left(1+\frac{A_2f(A_1Z_0i)}{i}\right)$$

very low and equal to the following

$$aK_{3\min}=\frac{1}{C_3} \tag{6}$$

It will have this characteristic over a range of q_0 given by the following equation

$$\frac{q_0}{C_3}A_1=2E_{c1};\ \Delta E=\frac{2E_{c1}}{A_1} \tag{7}$$

This slope can be made negligible by using a large value for A_1A_2 and C_3.

The stress-strain curve is shifted to the proper zero position by adjusting E_1 to satisfy the relation

$$A_2(E_1-E_{c1})=\frac{T_a}{r} \tag{8}$$

and then the actual circuit voltage is set to zero by adjusting E_3 until

$$E_3=-A_2(E_1-E_{c1}) \tag{9}$$

Additional changes in the spring constants are accomplished by additional voltage clippers such as E_{c2}. The spring constant above this level is given by the equation

$$aK=\frac{1+\frac{R_a}{k}A_1A_2}{C_3} \tag{10}$$

where

$$R_a=\frac{R_1R_2}{R_1+R_2}$$

Smooth variations in the spring factor can be accomplished by replacing the second voltage clipper with the arbitrary functions device type II.

The resistance R_n could be inserted to add enough negative resistance to the circuit to cancel the effect of the resistance at the output terminals of amplifier A_2 should it be appreciable. However, the positive gain amplifiers used in the computer have an impedance of only one ohm so that such compensation is rarely necessary. The voltages E_1 and E_2 are obtained by adjustment of the d-c amplifier zero-control. Figure 10 presents typical solutions for the transient torque pulse resulting from a single cylinder misfiring. As mentioned before, this was simulated as an applied voltage pulse having no steady-state value.

Another nonlinear spring problem studied with the computer is illustrated in Figure 11. This represents a one degree of freedom mechanical system (having a nonlinear spring) that is controlled with a servo using rate and position feedback. Figure 11 shows typical solutions for both the controlled and uncontrolled system under the influence of a suddenly applied constant disturbance torque. Linear feedback only is shown. However, this system was studied with nonlinear position feedback. It is readily apparent that if the servo time delays are not too great, the system can be linearized by the use of a nonlinear position feed-back characteristic that just counteracts the nonlinearity of the spring.

Nonlinear Servomotors

During the past year and a half the California Technology computer has been used extensively for the design analysis of guided missile control systems. This work has all been of a restricted nature and the studies cannot be described in detail. However, the analog techniques themselves are of sufficient interest to merit discussion. In most of these studies the aerodynamical equations have been linearized but some have included the effects of angle of attack and interaction between the three degrees of freedom of the missile. The latter type have included studies of "stabilized" platforms.

RATE AND POSITION LIMITED HYDRAULIC SERVO SYSTEMS

Several types of servomotors have been analyzed. One of the most important of the nonlinear motors studied extensively is the "rate limited" hydraulic motor. In

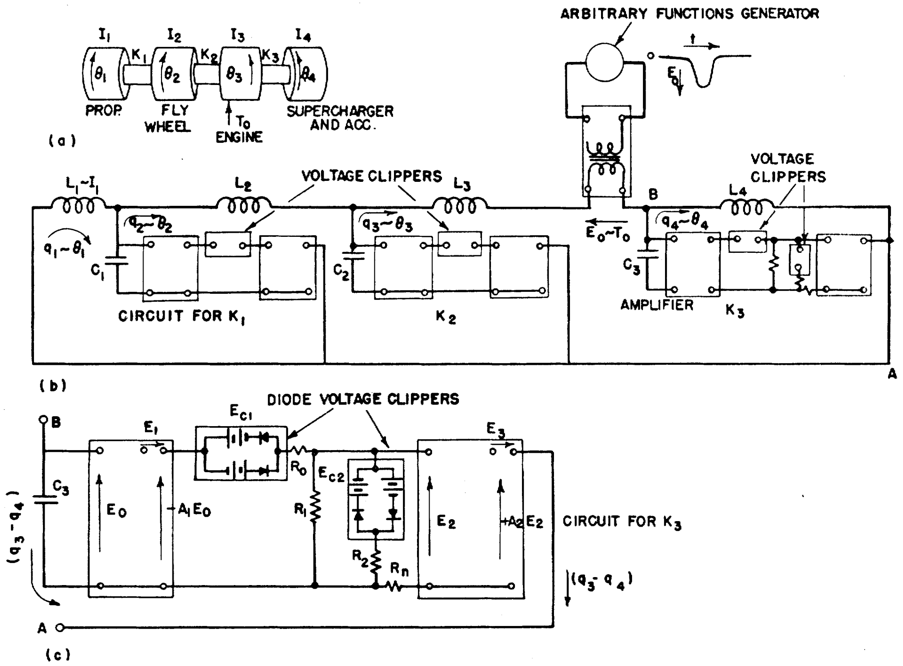

Figure 8. Analogy for radial aircraft engine with nonlinear spring constants and backlash

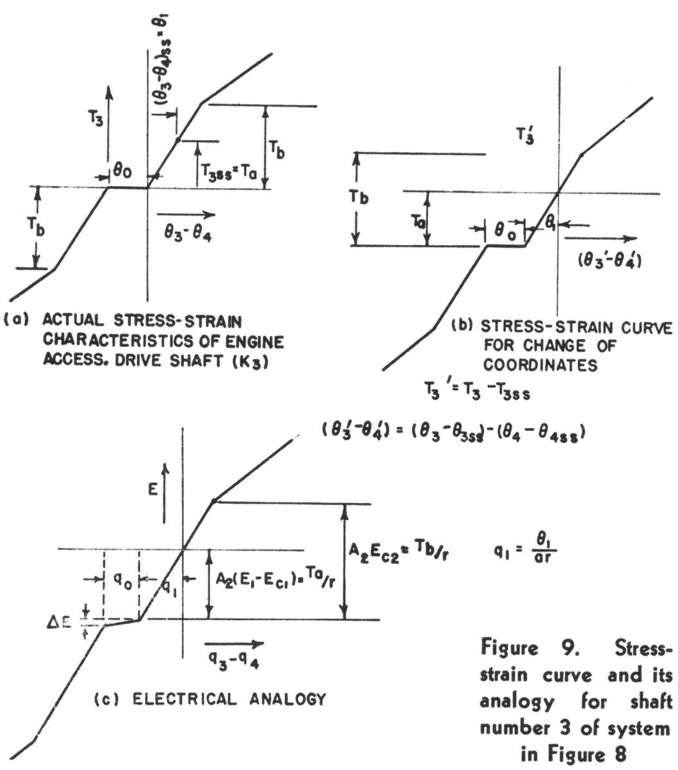

Figure 9. Stress-strain curve and its analogy for shaft number 3 of system in Figure 8

this a hydraulic valve is actuated, usually by an electromagnetic solenoid circuit. The valve controls the flow of the hydraulic line which will drive the missile control surfaces in either direction at a velocity proportional to the valve opening. However, the maximum control surface velocity is limited to a value corresponding to the full open position of

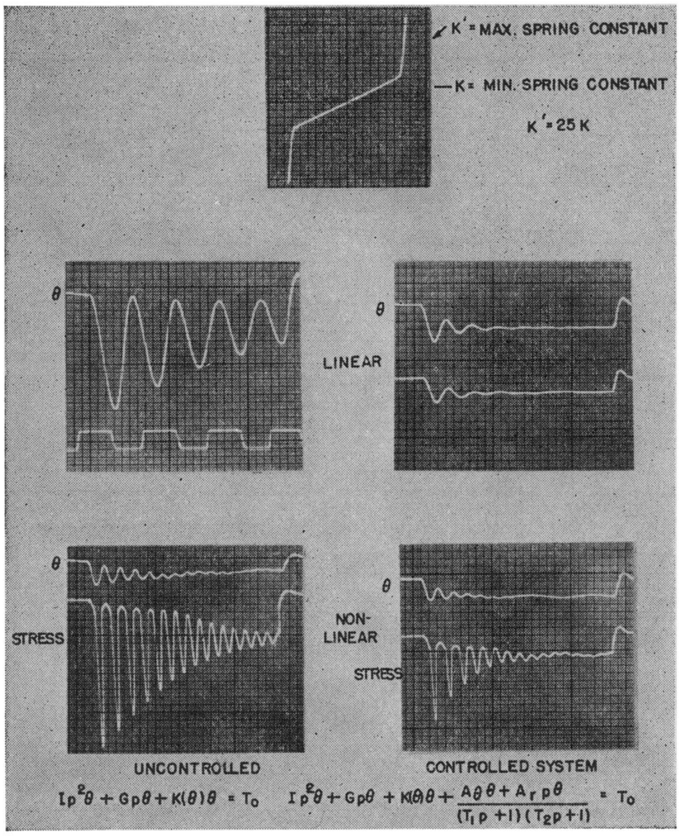

Figure 11. Computer solutions to servo controlled system with nonlinear spring

$$\omega_0=\frac{K}{I};\ \omega_0'=\frac{K+A_\theta}{I};\ \omega_0'=1.72\omega_0$$

T_0 = suddenly applied constant torque $=3.3T\alpha$

$$\frac{A_r}{A_\theta}\omega_0=0.49;\ T_1\omega_0=0.098;\ T_1\omega_0=0.196$$

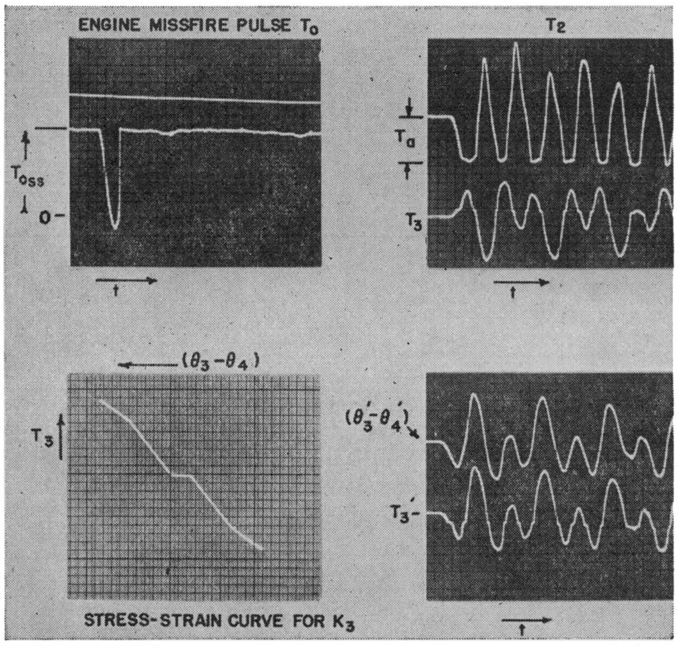

Figure 10. Typical solutions for system of Figure 8

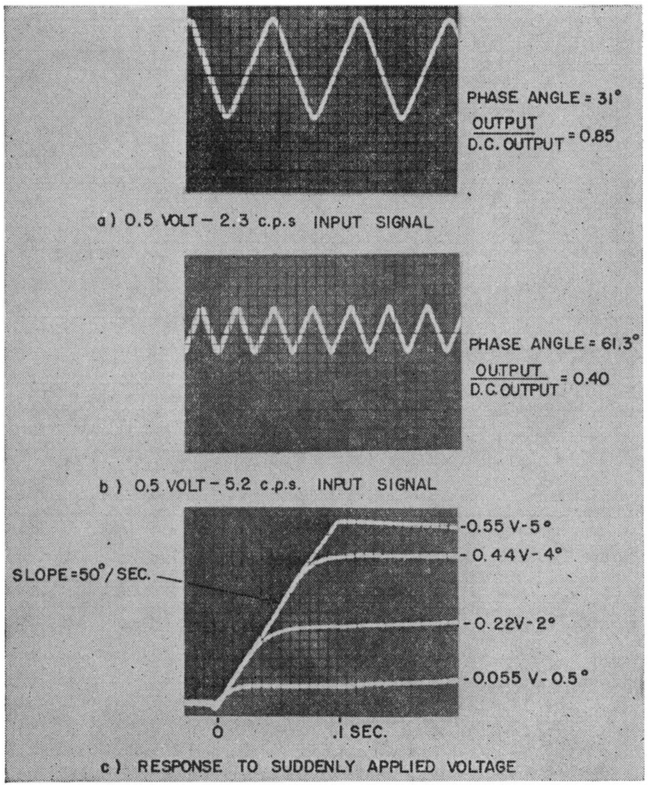

Figure 12 (right). Computer solutions for steady-state sinusoidal response and impulse response characteristics of rate and position limited hydraulic servomotor

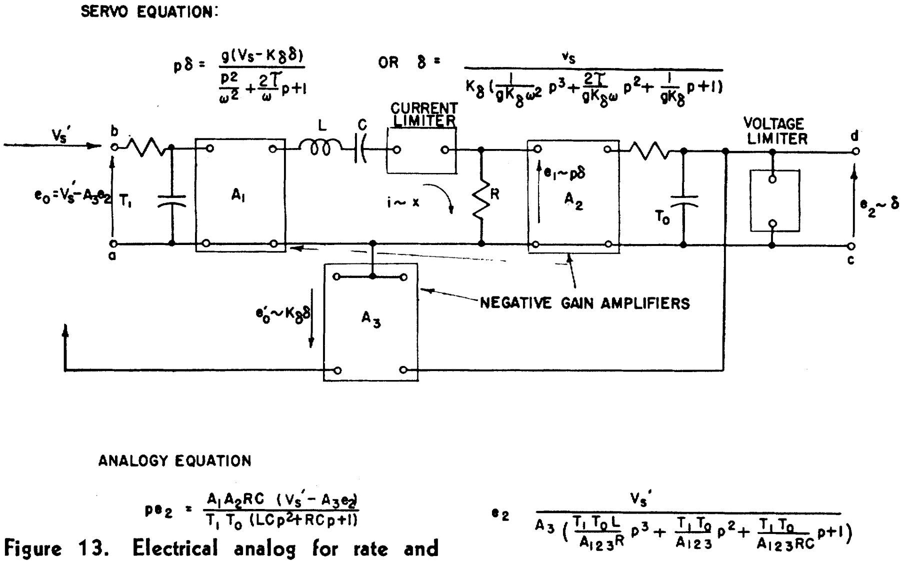

Figure 13. Electrical analog for rate and position limited hydraulic servomotors

the valve. Such a motor can generally be quite accurately specified by the mathematical equations of Figure 13. These equations are derived in the Appendix together with the electrical analogy. There both the electrical and mechanical elements of the solenoid are represented by one lumped transfer function. This usually provides a sufficiently accurate analogy. However, if desired the complete analogy for both the electrical and mechanical elements of the solenoid can be used. The nonlinearities in motors of this type can usually be accurately specified by assigning a definite maximum value to the rudder velocity and assuming the system to be linear below these limits. Rudder position stops can be treated in the same way.

Referring to the Appendix and Figure 13, these limits are accomplished with voltage and current limiters. It was found that the velocity limitations for some motors (although sharply defined) had a saturation characteristic which was not perfectly flat. This type of characteristic was studied by using an arbitrary functions device instead of the simple voltage clipper. The results of the study showed that sufficiently accurate results could always be obtained with the simple voltage clipper.

Typical impulse and steady-state sinusoidal response characteristics of a servomotor are shown in the computer solutions of Figure 12. Analog computer analyses have been made for systems using this type of motor in which the numerical parameters were obtained in several ways. In some cases the servo

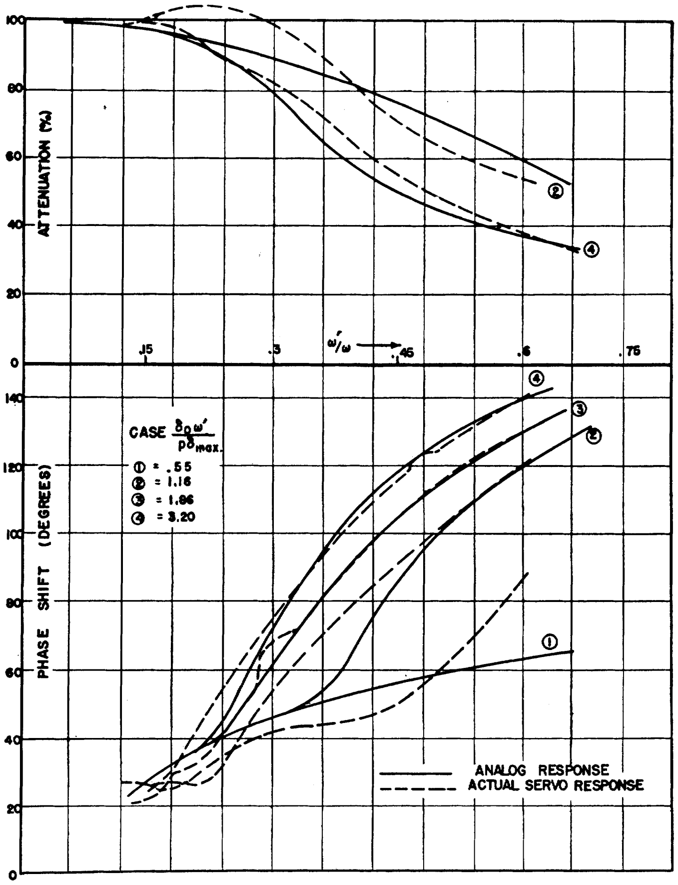

Figure 14. Frequency response curves comparing actual servomotor with electrical analogy

Motor parameters used with computer are the following (See Figure 12)

$$\frac{2}{\omega}=0.006;\quad \frac{\omega}{gK_\delta}=2.17$$

δ_0 is steady-state rudder angle without position limitation

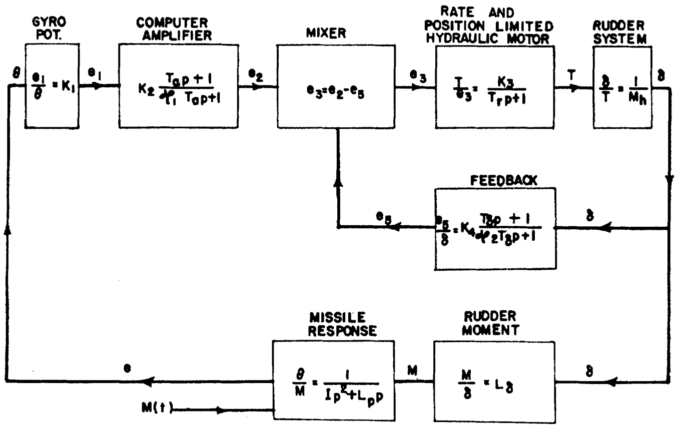

Figure 15. Block diagram—rate and position limited hydraulic autopilot

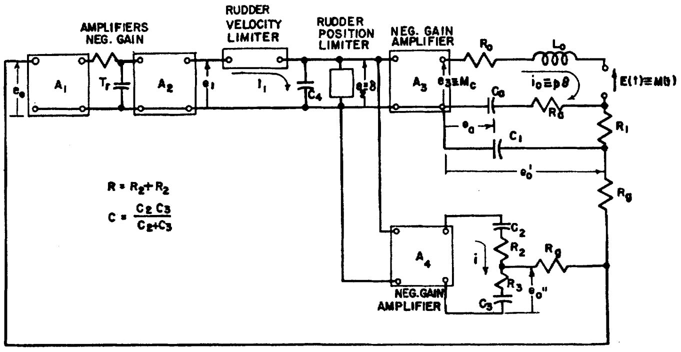

Figure 16. Electrical analog—rate and position limited hydraulic autopilot

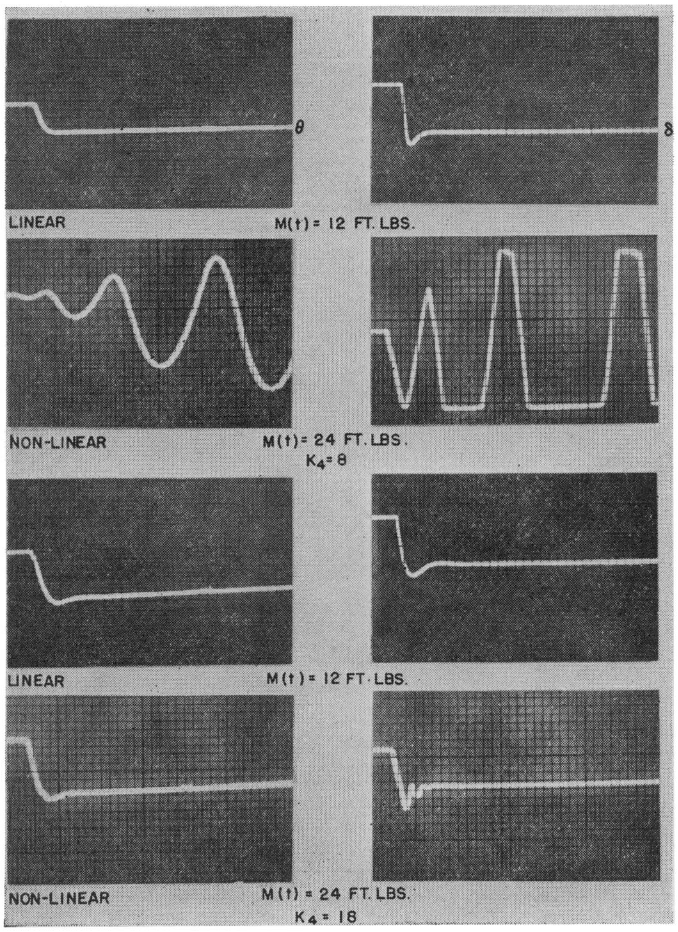

Figure 17. Response characteristics of autopilot system of Figure 15 instability induced by an applied torque exceeding the linear range of the control system, and corrected by an increase in K_4

$T_\alpha = 0.06$ second; $T_\delta = 0$; $K_2 = 50$

parameters given in Figure 12 were obtained directly by test and used directly for specifying the proper analog parameters.

In other cases it was required to determine certain of the parameters by fitting frequency response curves as shown in Figure 14. Such curves can be run off very rapidly with a computer of this type so that it only requires a few hours to determine by trial and error the parameters that best simulate test data.

Figures 15 and 16 are typical examples of an autopilot employing a rate and position limited servomotor. This system was used for roll stabilization and the simplified form for the aerodynamic equations is shown. In the analogy, Figure 16, is shown a modified version of the motor analogy which can be used if the motor transfer function, Figure 13, can be expressed as a quadratic instead of a cubic equation. This analogy can be derived readily. In it the current i_1, Figure 16 is analogous to rudder velocity so that a current limiter is used for velocity limitation.

Computer solutions for the transient response characteristic of this type of servo system are given in Figure 17. These illustrate a condition found in several cases with the parameters of an actual proposed missile at certain points in its flight program. If optimum servo parameters were chosen from a perfectly linear analysis, well damped solutions such as given at the top of the figure result.

These solutions would correspond to those resulting from small disturbance torques in the actual systems. However, if the disturbance torque is sufficiently large the two nonlinearities can produce complete instability as illustrated by the second set of solutions. For this particular type of system the instability can be eliminated by increasing the rudder feed-back parameter K_4 as shown in the two bottom pairs of computer solutions, Figure 17.

With a computer of the type described here solutions can be obtained so rapidly that it is a relatively quick and simple matter to study the performance of a guided missile or autopilot over its complete range of flight conditions. Optimum servo parameters can thus be quickly determined.

In some cases a single set of servo parameters may be suitable. In others a schedule for varying the parameters may be desirable.

Appendix I. Derivation of the Equations and Analogy for Rate and Position Limited Hydraulic Servomotors

Let

V_s = control signal voltage
δ = rudder position angle
K_δ = feed-back constant from rudder position
e_0 = voltage applied to motor solenoid
$e_0 = V_s - K_\delta\delta$
M = mass of solenoid valve system
K = its spring constant
G = its damping factor
m = constant relating solenoid force to applied voltage e_c
k = constant relating rudder velocity to valve opening

$$\frac{d\delta}{dt} = p\delta = \frac{mk}{K} \cdot \frac{(V_s - K^\delta\delta)}{\frac{M}{K}p^2 + \frac{G}{K}p + 1} \tag{11}$$

δ and $p\delta$ are limited to either a specified maximum value or to a sharply defined saturation characteristic. Equation 11 can be rewritten as follows

$$p\delta = \frac{g(V_s - K\ \delta)}{\frac{p^2}{\omega^2} + \frac{2\tau}{\omega}p + 1} \tag{12}$$

where

$$g = \frac{mk}{K};\ \omega = \sqrt{\frac{K}{M}};\ = \frac{G}{2\sqrt{KM}}$$

Solving for δ gives the following equation

$$\delta = \frac{V_s}{K^{\upsilon}\left(\frac{1}{gK_{\delta\omega^2}}p^3 + \frac{2\tau}{gK\delta\omega}p^2 + \frac{1}{gK\delta}p + 1\right)} \tag{13}$$

Electrical Analogy. Referring to Figure 12, it can be seen that if V_s' is the analogous control signal voltage the input voltage e_0 is:

$$e_0 = V_s' - A_3 e_2 \tag{14}$$

where e_2 is analogous to δ. The voltage e_1 is analogous to $p\delta$ and given by the equation

$$e_1 = \frac{-A_1 RC(V_s' - A_3 e_2)}{T_1(LCp^2 + RCp + 1)} \tag{15}$$

where T_0 is time constant of first integrating circuit.
As the relation between e_1 and e_2 we have the equation

$$e_2 = \frac{-A_2 e_1}{T_0 p} \tag{16}$$

where T_0 is the RC time constant of the second integrating circuit. Combining equations 14, 15, and 16

$$pe_2 = \frac{A_1 A_2 RC(V_s' - A_3 e_2)}{T_1 T_0(LCp^2 + RCp + 1)} \tag{17}$$

and solving for e_2

$$e_2 = \frac{V_s'}{A_3\left(\frac{T_1 T_0 L}{A_{123} R}p^3 + \frac{T_1 T_0}{A_{123}}p^2 + \frac{T_1 T_0}{A_{123} RC}p + 1\right)} \tag{18}$$

giving the transfer function analogous to that for the actual servo, equation 13. In this analogy the current in the *RLC* circuit is truly analogous to the solenoid position x which is proportional to $(p\delta)$. Thus by limiting this current with a current limiter the proper nonlinear conditions are imposed.

If a time base $\left(n=\dfrac{t}{t'}\right)$ is used where t' is computer time base and t actual time base the following relationships must be satisfied by the analogy

$$\left.\begin{aligned} &A_3=K_\delta && \frac{T_1T_0}{A_{12}}=\frac{2}{2g\omega} \\ &\frac{T_1T_0L}{A_{12}R}=\frac{1}{n^3g\omega^2} && \frac{T_1T_0}{A_{12}RC}=\frac{1}{ng} \end{aligned}\right\} \tag{19}$$

The following relationships establish the clipping current and voltages necessary to simulate the maximum permissible values for rudder velocity and position.

Let $e_2=r\delta$ then $e_{2\max}=r\delta_{\max}$ (20)

when e_2 and δ are expressed in volts and radians respectively

$$pe_{2\max}=nrp\delta_{\max}$$

or from equation 6

$$e_{1\max}i_{\max}R=\frac{T_0nr}{A_2}\,p\delta_{\max} \tag{21}$$

It should be noted that the rudder position and velocity limitations can be made different for the two directions. Equations 20 and 21 can be combined to give the following required relation between the two clipping voltages

$$\frac{e_1}{e_2}=\frac{T_0n}{A_2}\,\frac{(p\delta_{\max})}{(\delta_{\max})} \tag{22}$$

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No Discussion