Gaussian Hypothesis Testing and Quantum Illumination

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Quantum hypothesis testing is one of the most basic tasks in quantum information theory and has fundamental links with quantum communication and estimation theory. In this paper, we establish a formula that characterizes the decay rate of the minimal type-II error probability in a quantum hypothesis test of two Gaussian states given a fixed constraint on the type-I error probability. This formula is a direct function of the mean vectors and covariance matrices of the quantum Gaussian states in question. We give an application to quantum illumination, which is the task of determining whether there is a low-reflectivity object embedded in a target region with a bright thermal-noise bath. For the asymmetric-error setting, we find that a quantum illumination transmitter can achieve an error probability exponent stronger than a coherent-state transmitter of the same mean photon number, and furthermore, that it requires far fewer trials to do so. This occurs when the background thermal noise is either low or bright, which means that a quantum advantage is even easier to witness than in the symmetric-error setting because it occurs for a larger range of parameters. Going forward from here, we expect our formula to have applications in settings well beyond those considered in this paper, especially to quantum communication tasks involving quantum Gaussian channels.

Introduction.—Hypothesis testing is critical for the scientific method [1], underlying our ability to distinguish various models of reality and draw conclusions accordingly. It also has fundamental links with both communication [2] and estimation theory [3]. By increasing the number of independent samples observed in a given experimental setup, one can reduce the probability of making an incorrect inference, thus increasing the confidence in conclusions drawn from the experiment.

In the most basic setting of binary hypothesis testing the goal is to distinguish two hypotheses (null and alternative). There are two ways that one can err: a type-I error ("false alarm") occurs when rejecting the null hypothesis when it is in fact true, and analogously a type-II error ("false negative") occurs when incorrectly rejecting the alternative hypothesis. If it is possible to obtain many independent samples, one can study how error probabilities decay as a function of the number of samples for an optimal sequence of tests. Most prominently, the Chernoff bound [4] tells us that both error probabilities decay exponentially fast (in the number of samples) for an appropriately chosen sequence of tests. Beyond this, it is often desirable to treat the two types of errors asymmetrically. For example, the experimenter may only require a fixed bound on the false alarm probability and then seek to minimize the false negative probability subject to this constraint. The well-known result

here is the Chernoff-Stein lemma (sometimes called Stein's lemma) [4], which establishes how fast the false negative probability decays in this setting.

Since the rise of quantum information science, researchers have generalized these notions to the fully quantum setup, which is arguably more fundamental than the classical settings discussed above. Here the basic setting involves determining whether $M \geq 1$ quantum systems are described by the density operator $\rho^{\otimes M}$ or another density operator $\sigma^{\otimes M}$, and the experimenter is allowed to perform a collective quantum measurement on all M systems in order to guess which is the case. The fundamental results are the quantum Chernoff bound [5,6], which states that the quantum Chernoff information is the optimal decay rate when minimizing both error probabilities simultaneously, and the quantum Stein's lemma [7,8], which states that the quantum relative entropy between ρ and σ is the optimal decay rate for the type-II error probability given a fixed (independent of M) constraint on the type-I error probability. In more recent years, we have seen strong refinements of quantum Stein's lemma [9–13] that characterize the decay in higher orders of M and are crucial for a finite-size analysis.

One of the major applications of the results of quantum hypothesis testing is quantum illumination [14]. In the setting of quantum illumination, a source emits photons entangled in signal and idler beams, and the signal beam is subsequently

subjected to a modulation, loss, and environmental noise. A quantum receiver then makes a collective measurement on both the returned signal and idler beams in order to determine which modulation was applied. The typical task considered in previous work is to determine whether a target region containing a bright thermal-noise bath has a low-reflectivity object embedded [14,15]. Alternatively, one could also use the quantum illumination setup as a secure communication system, as proposed in [16]. After the original proposal of quantum illumination [14], a full Gaussian state treatment appeared [15] and strengthened the predictions of [14]. The upshot is that quantum illumination can offer a significant performance advantage over a classical coherent-state transmitter of the same average photon number, when considering the sensing application mentioned above. To date, several experiments have been conducted that demonstrate the advantage quantum illumination offers [17–20].

Hitherto quantum illumination has mostly been considered in the symmetric-error setting [15,20], and as such, one of the main technical tools employed in the analysis of quantum illumination is the quantum Chernoff bound. However, there are many scenarios where one is interested in the performance of quantum illumination in the asymmetric-error setting. Indeed, one might be willing to accept a particular type-I error (false alarm) probability (the error being to declare a target present when in fact it is not), and then minimize the type-II error (false negative) probability subject to this constraint.

In this paper, we determine the second-order refinement of quantum Stein's lemma in Gaussian quantum hypothesis testing. As our main result we derive an analytical formula that expresses the second-order behavior for any two Gaussian states as a function of their vector means and covariance matrices. Our result has applications to quantum illumination in the asymmetric-error setting, where we find that there are significant gains over a classical coherent-state emitter. Notably, we find that the quantum advantage is even easier to witness than in the symmetric-error setting because it occurs for a larger range of parameters.

We expect our formula to have applications well beyond the setting considered here, to various tasks in quantum communication theory. In fact, it is the basis for the strongest known upper bounds on quantum key distribution protocols conducted over quantum Gaussian channels [21,22]. In light of this, we expect our result to be useful in establishing sharp refinements of various capacities of quantum Gaussian communication channels, when combined with generalizations of the methods from [23–27].

To elaborate on our main result, if the task is to distinguish $\rho^{\otimes M}$ from $\sigma^{\otimes M}$ and the type-I error cannot exceed $\varepsilon \in (0, 1)$, then the optimal type-II error probability β takes the exponential form

$$\exp\left[-\left(Ma + \sqrt{M}b\,\Phi^{-1}(\varepsilon) + O(\ln M)\right)\right]. \quad (1)$$

The optimal constant $a \geq 0$ was identified in [7,8] to be the quantum relative entropy [28], defined as $a = D(\rho\|\sigma) \equiv \langle \ln \rho - \ln \sigma \rangle_\rho$ for faithful σ where we used the convention $\langle \cdot \rangle_\rho \equiv \mathrm{Tr}\{\rho \cdot\}$. The optimal constant $b \geq 0$ was identified in [9,10,13] to be the quantum relative entropy variance, defined in terms of the variance of the operator $\ln \rho - \ln \sigma$,

$$b = V(\rho\|\sigma) \equiv \langle [\ln \rho - \ln \sigma - D(\rho\|\sigma)]^2 \rangle_\rho. \quad (2)$$

In the above, we have also used the cumulative distribution function for a standard normal random variable: $\Phi(y) \equiv (1/\sqrt{2\pi}) \int_{-\infty}^{y} dx \exp(-x^2/2)$. The derivation of [13] also applies to particular states on separable Hilbert spaces [29], of which Gaussian states are examples.

An explicit formula for the quantum relative entropy between any two Gaussian states, as a function of their mean vectors and covariance matrices, was given in [30] and refined in [31]. Here we derive an explicit formula for the quantum relative entropy variance of two Gaussian states, given as a function of their mean vectors and covariance matrices. The formula allows for a deeper understanding of quantum hypothesis testing of Gaussian states. We state our result after a brief recollection of the Gaussian state formalism (see [32,33] for detailed reviews), and provide a detailed proof in the Supplemental Material. Finally, we apply our formula in the context of quantum illumination, giving a characterization of its performance in the asymmetric-error setting.

Related work.—The authors of [34] considered asymmetric hypothesis testing of quantum Gaussian states, deriving a formula for the quantum Hoeffding bound [6,35–37] in the context of Gaussian state discrimination. However, the setting of the quantum Hoeffding bound is conceptually different from what we consider here.

Gaussian state formalism.—We begin by reviewing some background on Gaussian states and then review a formula for quantum relative entropy from [30,31] (see [31,32] for more details on the conventions used). Our development applies to n-mode Gaussian states, where n is some fixed positive integer. Let $\hat{x}_j$ denote each quadrature operator ($2n$ of them for an n-mode state), and let $\hat{x} \equiv [\hat{q}_1, ..., \hat{q}_n, \hat{p}_1, ..., \hat{p}_n] \equiv [\hat{x}_1, ..., \hat{x}_{2n}]$ denote the vector of quadrature operators, so that the first n entries correspond to position-quadrature operators and the last n to momentum-quadrature operators. The quadrature operators satisfy the commutation relations

$$[\hat{x}_j, \hat{x}_k] = i\Omega_{j,k}, \quad (3)$$

where $\Omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes I_n$ and I_n is the $n \times n$ identity matrix. We also take the annihilation operator $\hat{a} = (\hat{q} + i\hat{p})/\sqrt{2}$. Let ρ be a Gaussian state, with the mean-vector entries $\langle \hat{x}_j \rangle_\rho = \mu_j^\rho$, and let μ^ρ denote the mean vector. The entries of the Wigner function covariance matrix V^ρ of ρ are given by

$$V_{j,k}^\rho \equiv \frac{1}{2} \langle \{\hat{x}_j - \mu_j^\rho, \hat{x}_k - \mu_k^\rho\} \rangle_\rho. \quad (4)$$

A $2n \times 2n$ matrix S is symplectic if it preserves the symplectic form $S\Omega S^T = \Omega$. According to Williamson's theorem [38], there is a diagonalization of the covariance matrix V^ρ of the form $V^\rho = S^\rho(D^\rho \oplus D^\rho)(S^\rho)^T$, where S^ρ is a symplectic matrix and $D^\rho \equiv \mathrm{diag}(\nu_1, ..., \nu_n)$ is a diagonal matrix of symplectic eigenvalues such that $\nu_i \geq 1/2$ for all $i \in \{1, ..., n\}$. We can write the density operator ρ in the exponential form [30,39–41],

$$\rho = Z_\rho^{-1/2} \exp\left[-\frac{1}{2}(\hat{x} - \mu^\rho)^T G_\rho(\hat{x} - \mu^\rho)\right], \quad (5)$$

$$\text{with} \quad G_\rho \equiv -2\Omega S^\rho[\mathrm{arcoth}(2D^\rho)]^{\oplus 2}(S^\rho)^T\Omega, \quad (6)$$

and $Z_\rho \equiv \det(V^\rho + i\Omega/2)$, where $\mathrm{arcoth}(x) \equiv \frac{1}{2}\ln(x + 1/x - 1)$ with domain $(-\infty, -1) \cup (1, +\infty)$. Note that we can also write $G_\rho = 2i\Omega \mathrm{arcoth}(2iV^\rho\Omega)$, so that G_ρ is represented directly in terms of the covariance matrix V^ρ (see Supplemental Material [42] on how to compute the symplectic decomposition of V^ρ). By inspection, the G and V matrices are symmetric, which is critical in our analysis below. As a result, $\mathrm{Tr}\{G\Omega\} = \mathrm{Tr}\{V\Omega\} = 0$ because G and V are symmetric while Ω is antisymmetric. In what follows, we adopt the same notation for quantities associated with a density operator σ, such as μ^σ, V^σ, S^σ, D^σ, Z_σ, and G_σ.

Relative entropy for Gaussian states.—We first revisit the relative entropy calculation from [30], but following the particular aspects of [31]. Suppose for simplicity that ρ and σ are zero-mean Gaussian states. By employing the exponential form in (5), we see that

$$\langle \ln\rho - \ln\sigma \rangle_\rho = \frac{1}{2}[\ln Z_\sigma - \ln Z_\rho - \langle \hat{x}^T\Gamma\hat{x} \rangle_\rho], \quad (7)$$

where $\Gamma \equiv G_\rho - G_\sigma$ is symmetric. To evaluate the expectation $\langle \hat{x}^T\Gamma\hat{x} \rangle_\rho$, we can use that $\hat{x}_k\hat{x}_l = \frac{1}{2}(\{\hat{x}_l, \hat{x}_k\} - [\hat{x}_l, \hat{x}_k]) = \frac{1}{2}(\{\hat{x}_l, \hat{x}_k\} - i\Omega_{l,k})$ and write $\langle \hat{x}^T\Gamma\hat{x} \rangle_\rho$ as

$$\sum_{k,l} \Gamma_{k,l}\langle \hat{x}_k\hat{x}_l \rangle_\rho = \sum_{k,l} \Gamma_{k,l}V^\rho_{l,k} = \mathrm{Tr}\{\Gamma V^\rho\}, \quad (8)$$

implying that $D(\rho\|\sigma) = [\ln(Z_\sigma/Z_\rho) - \mathrm{Tr}\{\Gamma V^\rho\}]/2$. For states ρ and σ that are not zero mean, one can incorporate a shift into the above calculation to find that

$$D(\rho\|\sigma) = [\ln(Z_\sigma/Z_\rho) - \mathrm{Tr}\{\Gamma V^\rho\} + \gamma^T G_\sigma\gamma]/2, \quad (9)$$

where $\gamma \equiv \mu^\rho - \mu^\sigma$. Alternatively, one can write the formula for relative entropy as $D(\rho\|\sigma) = [\ln(Z_\sigma) + \mathrm{Tr}\{G_\sigma V^\rho\} + \gamma^T G_\sigma\gamma]/2 - \sum_{i=1}^n g(\nu_i^\rho - 1/2)$, where $\{\nu_i^\rho\}_i$ are the symplectic eigenvalues of ρ and $g(x) \equiv (x+1)\ln(x+1) - x\ln x$ [43].

Relative entropy variance for Gaussian states.—The following theorem is our main result.

Theorem 1. For Gaussian states ρ and σ, the relative entropy variance from (2) is given by

$$V(\rho\|\sigma) = \frac{\mathrm{Tr}\{(\Gamma V^\rho)^2\}}{2} + \frac{\mathrm{Tr}\{(\Gamma\Omega)^2\}}{8} + \gamma^T G_\sigma V^\rho G_\sigma\gamma, \quad (10)$$

where $\Gamma \equiv G_\rho - G_\sigma$, G_ρ and G_σ are defined from (6), Ω is defined in (3), V^ρ is defined in (4), and $\gamma \equiv \mu^\rho - \mu^\sigma$.

To begin with, let us suppose that the states ρ and σ have zero mean. The calculation then begins with the definition of the relative entropy variance and proceeds through a few steps,

$$V(\rho\|\sigma) = \left\langle \left(-\frac{1}{2}\hat{x}^T\Gamma\hat{x} + \frac{1}{2}\langle \hat{x}^T\Gamma\hat{x} \rangle_\rho\right)^2 \right\rangle_\rho \quad (11)$$

$$= \frac{1}{4}[\langle (\hat{x}^T\Gamma\hat{x})^2 \rangle_\rho - \langle \hat{x}^T\Gamma\hat{x} \rangle_\rho^2] \quad (12)$$

$$= \frac{1}{4}[\langle (\hat{x}^T\Gamma\hat{x})^2 \rangle_\rho - [\mathrm{Tr}\{\Gamma V^\rho\}]^2], \quad (13)$$

where the last line follows from (8). At this point, it remains to calculate $\langle (\hat{x}^T\Gamma\hat{x})^2 \rangle_\rho$, which we do in Supplemental Material [42]. To summarize the calculation, one needs to expand the operator $(\hat{x}^T\Gamma\hat{x})^2$, leading to an expression of order 4 in the quadrature operators. After employing commutators and anticommutators to bring this operator into Weyl symmetric form [44] and at the same time employing symmetries of the dihedral subgroup of the symmetric group S_4, we can invoke Isserlis' theorem [45] regarding higher moments of Gaussians to evaluate it. We find that $\frac{1}{4}\langle (\hat{x}^T\Gamma\hat{x})^2 \rangle_\rho$ is equal to

$$\frac{1}{4}[\mathrm{Tr}\{\Gamma V^\rho\}]^2 + \frac{1}{2}\mathrm{Tr}\{\Gamma V^\rho\Gamma V^\rho\} + \frac{1}{8}\mathrm{Tr}\{\Gamma\Omega\Gamma\Omega\}, \quad (14)$$

which, after combining with (13), leads to the formula in (10) for zero-mean states. Incorporating a shift then leads to the full formula in (10). We provide full details of the calculation described above and generalize it to arbitrary Gaussian states in Supplemental Material [42]. Supplemental Material also argues how the formula is well defined even if ρ does not have full support and provides a further simplification of the formula for two-mode Gaussian states with covariance matrices in standard form.

Application to quantum illumination.—In the setting of quantum illumination a transmitter irradiates a target region basked in thermal noise in which a low-reflectivity object may be embedded. Let $\hat{a}_S$ denote the field-mode annihilation operator for the signal mode that is transmitted. We take the null hypothesis to be that the object is not there, and if this is the case, the annihilation operator for the return signal is $\hat{a}_R = \hat{a}_B$, where $\hat{a}_B$ represents an annihilation operator for a bath mode in a thermal state $\theta(N_B)$ of mean photon number $N_B > 0$. We take the alternative hypothesis to be that the object is there, and in this case, $\hat{a}_R = \sqrt{\eta}\hat{a}_S + \sqrt{1-\eta}\hat{a}_B$, where $\eta \in (0, 1)$ is related to the reflectivity of the object and $\hat{a}_B$ is now in a thermal state of mean photon number $N_B/(1 - \eta)$ [46].

If we prepare the signal mode in the coherent state $|\sqrt{N_S}\rangle$ of mean photon number $N_S > 0$, then the null

hypothesis state ρ_{coh} is a thermal state $\theta(N_B)$ with mean vector $(0,0)$ and covariance matrix $(N_B + 1/2)I_2$, and the alternative hypothesis state σ_{coh} is a displaced thermal state, with mean vector $(\sqrt{2\eta N_S}, 0)$ and covariance matrix $(N_B + 1/2)I_2$. It is also easy to check that the G matrix from (6) for both of these states is equal to $2\text{arcoth}(2N_B + 1)I_2$.

Plugging into the formula for relative entropy and relative entropy variance, we find that these quantities simplify as follows for the coherent-state transmitter,

$$D(\rho_{\text{coh}}\|\sigma_{\text{coh}}) = \eta N_S \ln(1 + 1/N_B), \tag{15}$$

$$V(\rho_{\text{coh}}\|\sigma_{\text{coh}}) = \eta N_S(2N_B + 1)\ln^2(1 + 1/N_B). \tag{16}$$

In calculating the above, note that the covariance matrices for ρ_{coh} and σ_{coh} are the same, so that $\Gamma = 0$ in this case, and we only need to calculate the terms involving γ in (9) and (10). What we see is that as the signal photon number N_S increases, so does the first-order term $MD(\rho_{\text{coh}}\|\sigma_{\text{coh}})$ in the type-II error probability exponent, indicating a more rapid convergence to 0. However, the second-order term $\sqrt{Mb}\Phi^{-1}(\varepsilon)$ is actually decreasing for all $\varepsilon \in (0, 1/2)$ as N_S increases, due to the fact that $\Phi^{-1}(\varepsilon) < 0$ for this range of ε.

Now if the transmitter has a quantum memory available, then it can store an idler mode entangled with the signal mode and conduct a quantum illumination strategy. The state we consider is the two-mode squeezed vacuum, with the reduced state of the signal mode having mean photon number N_S. This state has mean vector equal to 0 and covariance matrix given by $\left[\begin{smallmatrix} \mu & c \\ c & \mu \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} \mu & -c \\ -c & \mu \end{smallmatrix}\right]$, where $\mu = N_S + 1/2$ and $c = \sqrt{\mu^2 - 1/4}$. The null hypothesis state ρ_{QI} for this setup has mean vector equal to 0 and the covariance matrix $\left[\begin{smallmatrix} N_B+1/2 & 0 \\ 0 & \mu \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} N_B+1/2 & 0 \\ 0 & \mu \end{smallmatrix}\right]$, implying that the return and idler modes are in a product state.

The alternative hypothesis state σ_{QI} has mean vector equal to 0 and the covariance matrix $\left[\begin{smallmatrix} \gamma & \sqrt{\eta}c \\ \sqrt{\eta}c & \mu \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} \gamma & -\sqrt{\eta}c \\ -\sqrt{\eta}c & \mu \end{smallmatrix}\right]$, where $\gamma \equiv \eta N_S + N_B + 1/2$.

While the expressions for relative entropy and relative entropy variance for the quantum illumination transmitter are too long to report here, we can evaluate them to first and second order in N_S (an asymptotic expansion about $N_S = \infty$ while keeping N_B fixed), respectively,

$$D(\rho_{\text{QI}}\|\sigma_{\text{QI}}) = \frac{\eta N_S}{1-\eta}\ln\left(1 + \frac{1-\eta}{N_B}\right) + O(1), \tag{17}$$

$$V(\rho_{\text{QI}}\|\sigma_{\text{QI}}) = \left[\frac{\eta N_S}{1-\eta}\ln\left(1 + \frac{1-\eta}{N_B}\right)\right]^2 + O(N_S). \tag{18}$$

Alternatively, we can evaluate them to first order in N_B (an asymptotic expansion about $N_B = \infty$ while keeping N_S fixed),

$$D(\rho_{\text{QI}}\|\sigma_{\text{QI}}) = \frac{\eta N_S(N_S + 1)}{N_B}\ln\left(1 + \frac{1}{N_S}\right) + O\left(\frac{1}{N_B^2}\right), \tag{19}$$
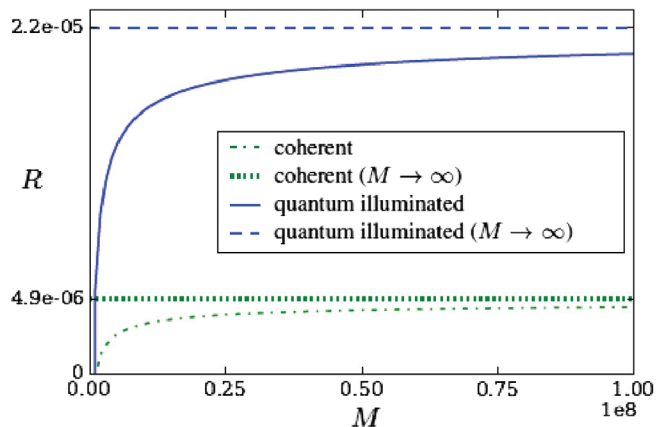
$$V(\rho_{\text{QI}}\|\sigma_{\text{QI}}) = \frac{\eta N_S(N_S + 1)(2N_S + 1)}{N_B}\ln^2\left(1 + \frac{1}{N_S}\right)$$
$$+ O\left(\frac{1}{N_B^2}\right). \tag{20}$$

Details about the derivation are in Supplemental Material [42].

There are several regimes in which the quantum illumination transmitter outperforms the coherent-state transmitter. We can consider the regime of low background thermal noise, where $N_S \gg 1$ and $N_B \ll 1$, and also the regime $N_S \ll 1$ and $N_B \gg 1$ as considered in [15]. Figures 1(a) and 1(b) compare
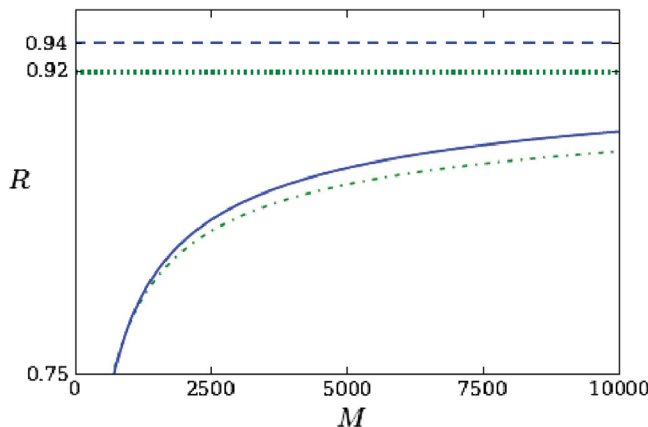
(a) Parameters: $N_S = 20$, $\eta = 0.01$, $N_B = 0.01$, and $\varepsilon = 0.001$.   (b) Parameters: $N_S = 0.01$, $\eta = 0.01$, $N_B = 20$, and $\varepsilon = 0.01$.

FIG. 1. Comparison of type-II error probability exponent, $R = -\ln\beta/M$, for the quantum illumination transmitter and the coherent-state transmitter with different parameters. In both cases, not only does the quantum illumination transmitter achieve a higher error exponent, but the Gaussian approximation suggests that far fewer trials are needed to approach this error exponent. The quantum advantage is easier to witness compared to the symmetric-error setting because it occurs for a larger parameter range.

the type-II error probability exponents of the quantum illumination transmitter and the coherent-state transmitter for a type-I error probability $\varepsilon = 0.001$ and $\varepsilon = 0.01$, respectively, showing both the first-order terms and the Gaussian approximations from (1). Not only does the quantum illumination transmitter outperform the coherent-state transmitter in exponent, but the Gaussian approximation indicates that far fewer trials are required to achieve this gain. Moreover, when compared to the symmetric-error setting, the quantum advantage is even easier to witness because it occurs for a larger range of parameters.

Discussion.—We have characterized the type-II error probability exponent of hypothesis testing of Gaussian states in terms of the relative entropy and the relative entropy variance of two Gaussian states. Our formula for the relative entropy variance should find applications well beyond the settings considered here, especially to communication tasks for quantum Gaussian channels. As an application of our result, we find that not only does a quantum illumination strategy outperform a coherent-state transmitter with respect to error probability exponent, but in some cases it requires far fewer trials in order to achieve the optimal error probability exponent.

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