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YUDIN CONES AND INDUCTIVE LIMIT TOPOLOGIES

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Abstract

A cone C in a vector space has a *Yudin basis* $\{e_i\}_{i \in I}$ if every $c \in C$ can be written uniquely in the form $c = \sum_{i \in I} \lambda_i e_i$, where $\lambda_i \geq 0$ for each $i \in I$ and $\lambda_i = 0$ for all but finitely many i . A *Yudin cone* is a cone with a Yudin basis. Yudin cones arise naturally since the cone generated by an arbitrary family $\{e_i\}_{i \in I}$ of linearly independent vectors

$$C = \left\{ \sum_{i \in I} \lambda_i e_i : \lambda_i \geq 0 \text{ for each } i \text{ and } \lambda_i = 0 \text{ for all but finitely many } i \right\}$$

is always a Yudin cone having the family $\{e_i\}_{i \in I}$ as a Yudin basis. The Yudin cones possess several remarkable order and topological properties. Here is a list of some of these properties.

1. A Yudin cone C is a lattice cone in the vector subspace it generates $M = C - C$.
2. A closed generating cone in a two-dimensional vector space is always a Yudin cone.
3. If the cone of a Riesz space is a Yudin cone, then the lattice operations of the space are given pointwise relative to the Yudin basis.
4. If a Riesz space has a Yudin cone, then the inductive limit topology generated by the finite dimensional subspaces is a Hausdorff order continuous locally convex-solid topology.
5. In a Riesz space with a Yudin cone the order intervals lie in finite dimensional Riesz subspaces (and so they are all compact with respect to any Hausdorff linear topology on the space).

The notion of a Yudin basis originated in studies on the optimality and efficiency of competitive securities markets in the provision of insurance for investors against risk or price uncertainty. It is a natural extension to incomplete markets of Arrow's notion of a basis for complete markets, i.e., markets where full insurance against risk can be purchased. The obtained results have immediate applications to competitive securities markets. Especially, they are sufficient for establishing the efficiency of stock markets as a means for insuring against risk or price uncertainty.

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1 Introduction

A *cone* in a (real) vector space X is any non-empty convex subset C which is closed under multiplication by non-negative scalars and satisfies $C \cap (-C) = \{0\}$. Cones appear in many sizes and shapes. The “smallest” cones are the half-rays $C = \{\lambda x: \lambda \geq 0\}$ while the generating cones (i.e., the cones C satisfying $X = C - C$) are the “largest” ones.

Every cone C induces a (unique) linear order on X by defining $x \geq y$ if $x - y \in C$. A vector space X equipped with a cone C , i.e., the pair (X, C) , is called a *partially ordered vector space*. A *lattice cone* is a cone that induces a lattice ordering on X . Lattice cones give rise to linear orderings that in many respects resemble the properties of the real numbers. If C is a lattice cone, then the partially ordered vector space (X, C) is called a *vector lattice* or a *Riesz space*.

Any collection $\{e_i\}_{i \in I}$ of linearly independent vectors in a vector space X generates the cone

$$C = \left\{ \sum_{i \in I} \lambda_i e_i: \lambda_i \geq 0 \text{ for each } i \text{ and } \lambda_i = 0 \text{ for all but finitely many } i \right\}.$$

Any such cone C is called a *Yudin cone* and $\{e_i\}_{i \in I}$ is referred to as its *Yudin basis*. In case $\{e_i\}_{i \in I}$ is a Hamel basis the Yudin cone C is a lattice cone. Our objective is to study the algebraic and topological properties of cones with Yudin bases. Fragments of some of the results presented here have appeared elsewhere. We take this opportunity to present the properties of Yudin bases in a unified manner that will be beneficial to many scientists, especially to economists and engineers, and we hope to be useful to mathematicians in general as well.

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The notion of a Yudin basis originated in studies of the optimality and efficiency of competitive markets in the provision of insurance for investors against risk or price uncertainty. It is a natural extension to incomplete markets of Arrow's notion of a basis for complete markets, which are markets where full insurance against risk can be purchased. In the last section of this paper, we apply our theory to show a fundamental property of exchange economies when the cone of the commodity space has a Yudin basis. For more discussion regarding the economic interpretation of the results obtained here, we refer the reader to [4], [5] and [7].

2 Preliminaries

We shall describe here certain properties of linear order relations and cones in vector spaces. For detailed accounts of the theory of partially ordered vector spaces and vector lattices, we refer the reader to the monographs [2, 6, 11, 18].

An order relation \geq on a vector space X is said to be a **linear order** if, in addition to being reflexive, antisymmetric and transitive, it is also compatible with the algebraic structure of X in the sense that $x \geq y$ implies

- a. $x + z \geq y + z$ for each z , and
- b. $\alpha x \geq \alpha y$ for all $\alpha \geq 0$.

A vector space equipped with a linear order is called a **partially ordered vector space** or simply an **ordered vector space**. In a partially ordered vector space (X, \geq) any vector satisfying $x \geq 0$ is known as a **positive vector** and the collection of all positive vectors $X^+ = \{x \in X: x \geq 0\}$ is referred to as the **positive cone** of X . A linear operator $T: X \rightarrow Y$ between two partially ordered vector spaces is **positive** if $Tx \geq 0$ for each $x \geq 0$.

A subset C of a vector space X is said to be a **cone** if:

- 1. $C + C \subseteq C$,
- 2. $\lambda C \subseteq C$ for each $\lambda \geq 0$, and
- 3. $C \cap (-C) = \{0\}$.

Notice that (1) and (2) guarantee that every cone is a convex set. An arbitrary cone C of a vector space X defines a linear order on X by saying that $x \geq y$ if $x - y \in C$, in which case $X^+ = C$. On the other hand, if (X, \geq) is an ordered vector space, then X^+ is a cone in the above sense. These show that the linear order relations and cones correspond in one-to-one fashion.

A partially ordered vector space X is said to be a **vector lattice** or a **Riesz space** if it is also a lattice. That is, a partially ordered vector space X is a vector lattice if for every pair of vectors $x, y \in X$ their supremum (least upper bound) and infimum (greatest lower bound) exist in X . Any cone of a vector space that makes it a Riesz space will be referred to as a **lattice cone**. As usual, the supremum and infimum of a pair of vectors x, y in a vector lattice are denoted by $x \vee y$ and $x \wedge y$ respectively. In a vector lattice, the element $x^+ = x \vee 0$, $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$ are called the **positive**, **negative**, and **absolute value** of x . We always have the identities

$$x = x^+ - x^- \quad \text{and} \quad |x| = x^+ + x^-.$$

A vector subspace Y of a vector lattice X is said to be:

1. a **vector sublattice** if for each $x, y \in Y$ we have $x \vee y$ and $x \wedge y$ in Y ; and
2. an **ideal** if $|y| \leq |x|$ and $x \in Y$ implies $y \in Y$.

An ideal is always a vector sublattice but a vector sublattice need not be an ideal.

Definition 1 *A vector subspace Y of a partially ordered vector space X is said to be a **lattice-subspace** if Y under the induced ordering from X is a vector lattice in its own right. That is, Y is a lattice-subspace if for every $x, y \in Y$ the least upper bound and greatest lower bound of the set $\{x, y\}$ exist in Y when ordered by the cone $Y \cap X^+$.*

If X is a vector lattice, then every vector sublattice of X is automatically a lattice-subspace, but a lattice-subspace need not be a vector sublattice. For details about lattice-subspaces see [1, 12, 13, 14].

A normed space which is also a partially ordered vector space is a **partially ordered normed space**. We refer to [18] for a comprehensive introduction to the abstract theory of partially ordered normed spaces. A norm $\|\cdot\|$ on a vector lattice is said to be a **lattice norm** if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. A **normed vector lattice** is a vector lattice equipped with a lattice norm. A complete normed vector lattice is called a **Banach lattice**.

Definition 2 *Two partially ordered vector spaces are said to be **order-isomorphic** if there is a linear operator $T: X \rightarrow Y$ (called an **order-isomorphism**) such that*

- a. T is one-to-one and onto; and
- b. $x \geq 0$ holds in X if and only if $Tx \geq 0$ holds in Y .

An order-isomorphism between Riesz spaces is also called a **lattice isomorphism**.

If, in addition, X and Y are topological vector spaces and T and T^{-1} are both continuous, then X and Y are called **topologically order-isomorphic** (and T is called a **topological order-isomorphism** between X and Y).

Extending the previous definition we say that an ordered vector space X is **order-embeddable** into an ordered vector space Y if there exists a one-to-one linear operator $T: X \rightarrow Y$ such that $Tx \in Y^+$ if and only if $x \in X^+$. In this case, the subspace $T(X)$ ordered by the cone $T(X) \cap Y^+$ is referred to as a **copy** of X in Y .

The next result of B. Z. Vulikh [18, Theorem I.7.1, p. 13] presents a simple condition for a vector space to be a Riesz space. This result in connection with Theorem 11 will provide a variety of interesting results concerning lattice cones.

Lemma 3 (Vulikh) *Let $T: X \rightarrow Y$ be a linear isomorphism between two vector spaces, i.e., T is a surjective one-to-one linear operator. If Y is a Riesz space and we define the linear ordering \geq on X by letting $u \geq v$ whenever $Tu \geq Tv$, then X is a Riesz space and $T: X \rightarrow Y$ is a lattice isomorphism.*

Corollary 4 *Every copy of a vector lattice X in a partially ordered vector space Y is a lattice-subspace of Y .*

Proof: Let $T: X \rightarrow Y$ be an order-embedding, where X is a Riesz space and Y is an ordered vector space. Let $Z = T(X)$ and consider Z equipped with the induced ordering from Y . Clearly, $T: X \rightarrow Z$ is a surjective linear isomorphism. Moreover, we have $x \geq y$ in X if and only if $Tx \geq Ty$ in Z . By applying Lemma 3 to T^{-1} , Z under the induced ordering from Y is a Riesz space and $T: X \rightarrow Z$ is a lattice isomorphism. ■

A source for lattice-subspaces is described in the following result.

Theorem 5 (Polyrakis [13]) *If X is a separable Banach lattice, then there exists a closed lattice-subspace Y of $C[0, 1]$ and a surjective order-isomorphism $T: X \rightarrow Y$ satisfying*

$$\frac{1}{2}\|x\| \leq \|Tx\|_\infty \leq \|x\|$$

for each $x \in X$, where $\|\cdot\|_\infty$ denotes the sup norm of $C[0, 1]$.

This result says that $C[0, 1]$ contains all separable Banach lattices as closed lattice-subspaces and, therefore, $C[0, 1]$ can be viewed as a universal Banach lattice. Since $C[0, 1]$ can be embedded lattice isometrically in ℓ_∞ (for example, if $\{r_1, r_2, \dots\}$ is an enumeration of the rational numbers of $[0, 1]$, then the mapping $f \mapsto (f(r_1), f(r_2), \dots)$, is a lattice isometry), it follows that ℓ_∞ is also a universal Banach lattice in the sense stated in Theorem 5.

An important class of Riesz spaces that provides the basic economic models in [4] are the Riesz spaces of ϕ_I -type. If I is an arbitrary index set, then ϕ_I is the ideal of \mathbb{R}^I consisting of all “eventually zero” functions. That is,

$$\phi_I = \left\{ \theta = (\theta_i)_{i \in I} \in \mathbb{R}^I : \theta_i = 0 \text{ for all but a finite number of } i \right\}.$$

The ordering and the lattice operations in ϕ_I are the pointwise ones. Moreover, ϕ_I equipped with the sup norm, defined by

$$\|\theta\|_\infty = \sup_{i \in I} |\theta_i|,$$

is a Dedekind complete normed Riesz space. The standard cone of ϕ_I is

$$\phi_I^+ = \left\{ (\theta_i)_{i \in I} \in \phi_I : \theta_i \geq 0 \text{ for all } i \in I \right\}.$$

In case $I = \mathbb{N}$ (the set of natural numbers), we shall simply write ϕ instead of $\phi_{\mathbb{N}}$ and ϕ^+ instead of $\phi_{\mathbb{N}}^+$. The norm completion of $(\phi_I, \|\cdot\|_\infty)$ is the Banach lattice $c_0(I)$.

In essence, this paper studies the lattice and topological properties of Riesz spaces which are lattice isomorphic to ϕ_I -spaces.

3 Yudin cones

In this section, we shall discuss cones equipped with an algebraic type of a basis. Let C be a cone of a vector space X . A vector $e \in C$ is said to be a **discrete** (or an **extremal**) **element** of C if $0 \leq x \leq e$ implies that $x = \lambda e$ for some $\lambda \geq 0$, where $0 \leq x \leq e$ means $x \in C$ and $e - x \in C$. In this case, the half-line $\{\alpha e : \alpha \geq 0\}$ is called an **extremal ray** of the cone C .

Definition 6 *We shall say that:*

1. A collection $\{e_i\}_{i \in I}$ of elements of a cone C in a vector space is a **Yudin basis** of C if each $x \in C$ has a unique representation of the form $x = \sum_{i \in I} \lambda_i e_i$, where $\lambda_i \geq 0$ for each i and $\lambda_i = 0$ for all but finitely many i .
2. A cone is a **Yudin cone** if it has a Yudin basis.
3. A partially ordered vector space X has a **Yudin basis** if its positive cone X^+ has a Yudin basis.

Here are two basic properties of Yudin bases.

Lemma 7 *If $\{e_i\}_{i \in I}$ is a Yudin basis of a cone C , then:*

- i. $\{e_i\}_{i \in I}$ is a collection of linearly independent vectors which is a Hamel basis for the linear span $M = C - C$ of C , and
- ii. each e_i is a discrete element of C .

Proof: (i) Assume $\sum_{k=1}^m \lambda_k e_k = 0$ with $\lambda_k \neq 0$ for each k . Let $A = \{k: \lambda_k > 0\}$ and $B = \{k: \lambda_k < 0\}$. From the definition of a Yudin basis, we see that neither A nor B can be empty. But then, the vector $x = \sum_{k \in A} \lambda_k e_k = \sum_{k \in B} (-\lambda_k) e_k \in C$ violates the definition of the Yudin basis. Hence, $\{e_i\}_{i \in I}$ is a family of linearly independent vectors. An easy argument shows that $\{e_i\}_{i \in I}$ is also a Hamel basis for the span $M = C - C$ of C .

(ii) Assume $0 \leq x \leq e_k$, i.e., suppose $x \in C$ and $e_k - x \in C$. Write $x = \sum_{i \in I} \lambda_i e_i$ and $e_k - x = \sum_{i \in I} \mu_i e_i$ with the λ_i and μ_i all non-negative and zero for all but finitely many i . So, $x = \sum_{i \in I} \lambda_i e_i = e_k - \sum_{i \in I} \mu_i e_i$. Invoking (i), we get $\lambda_k = 1 - \mu_k$ and $0 \leq \lambda_i = -\mu_i \leq 0$ for $i \neq k$. Consequently, $\lambda_i = 0$ for $i \neq k$ and so $x = \lambda_k e_k$, proving that each e_k is a discrete element of the cone C . ■

When Yudin bases exist, they are essentially unique.

Lemma 8 *A given cone C of a vector space has essentially at most one Yudin basis in the following sense: If $\{e_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ are two Yudin bases of C , then each e_i is a scalar multiple of some b_j and each b_j is a scalar multiple of some e_i .*

In other words, if $\{e_i\}_{i \in I}$ is a Yudin basis of a cone C , then every other Yudin basis of C is of the form $\{\lambda_i e_i\}_{i \in I}$, where $\lambda_i > 0$ for each i .

Proof: Fix $i \in I$ and choose indices $j_1, \dots, j_m \in J$ and positive scalars $\lambda_1, \dots, \lambda_m$ such that $e_i = \sum_{k=1}^m \lambda_k b_{j_k}$. Now there exist indices $i_1, \dots, i_n \in I$ such that for each $1 \leq k \leq m$ we have $b_{j_k} = \sum_{r=1}^n \beta_r^k e_{i_r}$, with the scalars $\beta_r^k \geq 0$. It follows that

$$e_i = \sum_{k=1}^m \lambda_k b_{j_k} = \sum_{k=1}^m \lambda_k \left(\sum_{r=1}^n \beta_r^k e_{i_r} \right) = \sum_{r=1}^n \left(\sum_{k=1}^m \lambda_k \beta_r^k \right) e_{i_r},$$

and so $\sum_{k=1}^m \lambda_k \beta_r^k = 0$ for all but one $r \in \{1, \dots, n\}$, say $r = 1$. This implies $e_i = e_{i_1}$ and $\beta_r^k = 0$ for all k and all $r \neq 1$ and $\beta_1^k > 0$ for some k . For this particular k $e_i = e_{i_1} = \frac{1}{\beta_1^k} b_{j_k}$, and the proof is finished. ■

In Figure 1 the reader will see the difference of a Yudin cone from an arbitrary cone.

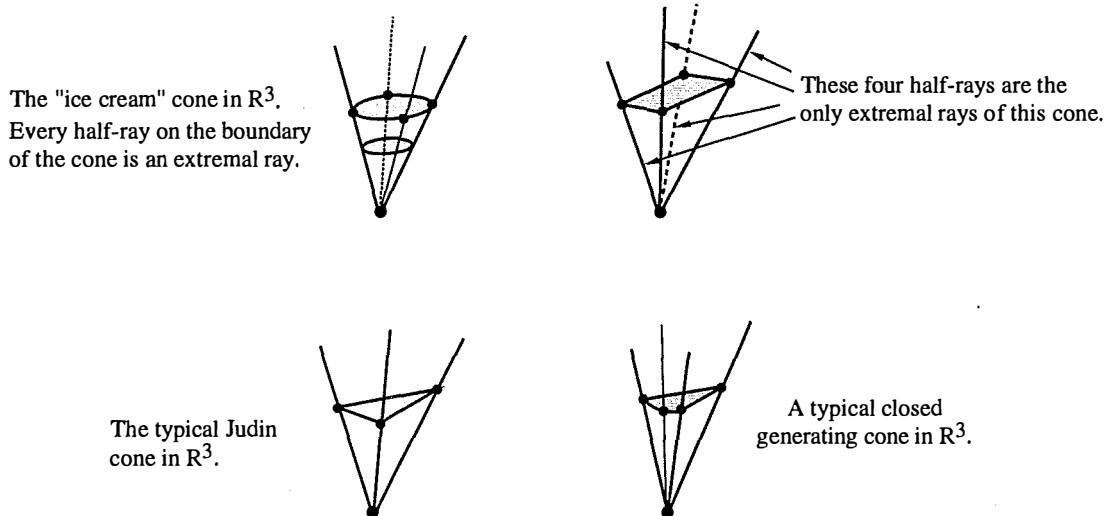


Fig. 1

An arbitrary cone need not have a Yudin basis even if it is a lattice cone. The next result provides some examples.

Theorem 9 *If an infinite dimensional partially ordered vector space has an order unit, then its positive cone does not have a Yudin basis.*¹

Proof: Let $e > 0$ be a unit in an infinite dimensional partially ordered vector space X , and assume by way of contradiction that X^+ has a Yudin basis $\{e_i\}_{i \in I}$. Pick non-negative scalars μ_1, \dots, μ_k and indices $i_1, \dots, i_k \in I$ such that $e = \sum_{j=1}^k \mu_j e_{i_j}$ and fix any index $r \in I \setminus \{i_1, \dots, i_k\}$. Since e is an order unit, there exists some $\lambda > 0$ such that $\lambda e \geq e_r$. This implies

$$(\lambda\mu_1)e_{i_1} + (\lambda\mu_2)e_{i_2} + \dots + (\lambda\mu_k)e_{i_k} - e_r = \lambda e - e_r \geq 0,$$

which contradicts the fact that $\{e_i\}_{i \in I}$ is a Yudin basis of X^+ . Hence, the cone X^+ does not have a Yudin basis. ■

Cones in two-dimensional partially ordered vector spaces quite often have Yudin bases.

Theorem 10 *Every cone in a two-dimensional vector space which is closed and generating is a Yudin (and hence a lattice) cone.*

In particular, if a partially ordered topological vector space X has a closed cone, then every two-dimensional subspace M of X whose induced cone $M^+ = M \cap X^+$ is generating is a lattice-subspace.

¹Recall that a positive vector e in a partially ordered vector space X is an **order unit** if for each $x \in X$ there exists some $\lambda > 0$ with $x \leq \lambda e$.

Proof: Let C be a closed generating cone in a two-dimensional vector space X . We denote by $\|\cdot\|$ the Euclidean norm of X relative to a fixed basis. Let $S = \{x \in X: \|x\| = 1\}$ and note that $S \cap C$ is a non-empty norm compact subset of X . Now let

$$m = \inf\{x \cdot y: x, y \in S \cap C\}.$$

Since the dot product is a jointly continuous function and $S \cap C$ is a norm compact set, there exist $e_1, e_2 \in S \cap C$ such that $e_1 \cdot e_2 = m = \cos \phi$; see Figure 2. Since C is a generating cone, it follows that $-1 < m < 1$ and this implies that $\{e_1, e_2\}$ is a Hamel basis of X . To finish the proof, we shall show that $\{e_1, e_2\}$ is a Yudin basis of C .

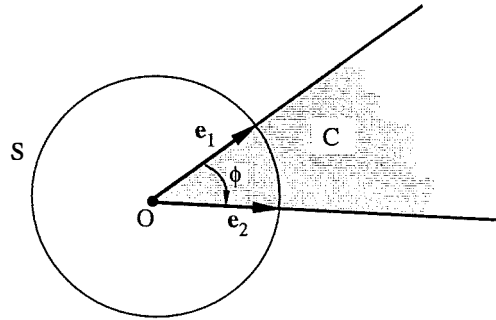


Fig. 2

To this end, let $x \in S \cap C$ and write $x = \lambda_1 e_1 + \lambda_2 e_2$. We can assume $\lambda_1, \lambda_2 \neq 0$. Also, notice that we cannot have $\lambda_1 < 0$ and $\lambda_2 < 0$. Otherwise, $\lambda_2 e_2 = x + (-\lambda_1) e_1 \in C$ and $-\lambda_2 e_2 = (-\lambda_2) e_2 \in C$ imply $\lambda_2 e_2 = 0$ or $\lambda_2 = 0$. So, assume by way of contradiction that $\lambda_1 > 0$ and $\lambda_2 < 0$. This implies

$$e_1 = \alpha_1 x + \beta_1 e_2, \quad (\star)$$

where $\alpha_1 = \frac{1}{\lambda_1} > 0$ and $\beta_1 = -\frac{\lambda_2}{\lambda_1} > 0$.

Put $\mu = e_2 \cdot x$ and note that $|\mu| \leq 1$. In addition, $|\mu| = 1$ cannot happen. If $|\mu| = 1$, then the angle between x and e_2 is either 0° or 180° and so $x = \pm e_2$, which in connection with (\star) shows that e_1 and e_2 are linearly dependent, a contradiction. Now consider the function $f: \mathbb{R}_+^2 \rightarrow \mathbb{R}$ defined by

$$f(\alpha, \beta) = e_2 \cdot (\alpha x + \beta e_2) = \mu \alpha + \beta.$$

Next, notice that the set

$$A = \left\{ (\alpha, \beta) \in \mathbb{R}_+^2: \|\alpha x + \beta e_2\|^2 = \alpha^2 + \beta^2 + 2\alpha\beta\mu = (\alpha + \mu\beta)^2 + (1 - \mu^2)\beta^2 = 1 \right\},$$

is a compact subset of \mathbb{R}_+^2 . (The condition $|\mu| < 1$ implies that β as a parameter of the set A is bounded and this guarantees that α is also bounded.) So, f attains a maximum and a minimum over A . If any one of these extrema takes place at some (α, β) with $\alpha > 0$ and $\beta > 0$, then the Lagrange Multiplier Method applies and yields

$$\mu = \frac{\partial f}{\partial \alpha} = \lambda(\alpha + \mu\beta) \quad \text{and} \quad 1 = \frac{\partial f}{\partial \beta} = \lambda(\beta + \mu\alpha).$$

This implies $\alpha\mu^2 = \alpha$, which means either $\alpha = 0$ or $\mu = \pm 1$, a contradiction. So, the extrema of f over A take place only at the points $(0, 1)$ and $(1, 0)$. It follows that the minimum of f over A is $f(1, 0) = \mu = e_2 \cdot x$. From (\star) and the definition of the function f , we get

$$m = e_2 \cdot e_1 = f(\alpha_1, \beta_1) > \mu = e_2 \cdot x \geq m,$$

a contradiction. This contradiction establishes that $\{e_1, e_2\}$ is a Yudin basis of C . \blacksquare

And now we present a result that describes the lattice structure of the vector space generated by a Yudin cone.

Theorem 11 *Let C be a Yudin cone in a vector space X with a Yudin basis $\{e_i\}_{i \in I}$ and let $M = C - C$ be the linear span of C . For each finite subset \mathcal{F} of I let $M_{\mathcal{F}}$ denote the vector subspace generated by the finite set $\{e_i; i \in \mathcal{F}\}$. That is,*

$$M_{\mathcal{F}} = \left\{ x \in X : \exists \text{ scalars } \{\lambda_i\}_{i \in \mathcal{F}} \text{ such that } x = \sum_{i \in \mathcal{F}} \lambda_i e_i \right\}.$$

Then we have the following properties.

1. *The partially ordered vector space (M, C) is a Dedekind complete Riesz space. If $x = \sum_{i \in I} \lambda_i e_i$ and $y = \sum_{i \in I} \mu_i e_i$ are arbitrary elements of M , then $x \geq y$ (i.e., $x - y \in C$) is equivalent to $\lambda_i \geq \mu_i$ for each $i \in I$. The lattice operations of (M, C) are given by*

$$x \vee y = \sum_{i \in I} (\lambda_i \vee \mu_i) e_i \quad \text{and} \quad x \wedge y = \sum_{i \in I} (\lambda_i \wedge \mu_i) e_i.$$

2. *The vector space M is also a normed Riesz space under the lattice norm*

$$\|x\|_M = \left\| \sum_{i \in I} \theta_i e_i \right\| := \max_{i \in I} |\theta_i|, \quad x = \sum_{i \in I} \theta_i e_i.$$

Moreover, the operator $R: \phi_I \rightarrow M$, defined by $R(\theta) = \sum_{i \in I} \theta_i e_i$, is an onto lattice isometry.

3. *Each subspace $M_{\mathcal{F}}$ is a finite dimensional ideal of M (and also of each $M_{\mathcal{G}}$ for $\mathcal{F} \subseteq \mathcal{G}$) and a Dedekind complete Banach lattice under the $\|\cdot\|_M$ -norm.*
4. *The order intervals of M lie in finite dimensional subspaces—and hence they are norm compact.*

Proof: The reader should observe that everything follows from (1). To see (1), notice that if $x = \sum_{i \in I} \lambda_i e_i$ and $y = \sum_{i \in I} \mu_i e_i$, then $x - y = \sum_{i \in I} (\lambda_i - \mu_i) e_i$ belongs to C if and only if $\lambda_i \geq \mu_i$ for each i . \blacksquare

Basically, Theorem 11 guarantees that every partially ordered vector space with a Yudin cone is a copy of some ϕ_I Riesz space—and, of course, the cone of every ϕ_I -space is a Yudin cone. In another direction, A. C. M. Rooij [15], has established that an Archimedean Riesz space E is lattice isomorphic to some ϕ_I -space if and only for each Archimedean Riesz space F the partially ordered vector space $\mathcal{L}_r(E, F)$ of all regular operators² from E into F is itself a Riesz space.

We are now ready to state several consequences of Lemma 3 and Theorem 11.

Corollary 12 *Let M be a vector subspace of a partially ordered vector space X . If the cone $M^+ = M \cap X^+$ is a Yudin cone generating M (i.e., $M = M^+ - M^+$), then M is a lattice-subspace of X .*

Although an arbitrary cone need not have a Yudin basis, it always includes a Hamel basis for the vector space it generates.

Lemma 13 *Let C be a cone of a vector space and let $M = C - C$ be the linear span of C . Then M has a Hamel basis consisting of vectors of C .*

Proof: By Zorn's Lemma there exists a maximal linearly independent subset H of the cone C . We claim that H is a Hamel basis for M .

Indeed, if $H \cup \{x\}$ were an independent set for some $x \in M$, then we would write $x = c_1 - c_2$ with $c_1, c_2 \in C \setminus \{0\}$, and so either $H \cup \{c_1\}$ or $H \cup \{c_2\}$ would be an independent set, contrary to the choice of H . ■

When is a Hamel basis of a cone also a Yudin basis? The next result provides the answer.

Corollary 14 *For a cone C of a vector space X and a Hamel basis $\{e_i\}_{i \in I} \subseteq C$ of $M = C - C$, the following statements are equivalent.*

1. *The Hamel basis $\{e_i\}_{i \in I}$ is a Yudin basis of C .*
2. *The cone C is a lattice cone, and for each finite subset \mathcal{F} of the index set I the linear span $M_{\mathcal{F}}$ of the set $\{e_i : i \in \mathcal{F}\}$ is a Riesz subspace of (M, C) whose cone $M_{\mathcal{F}}^+$ is a Yudin cone having $\{e_i : i \in \mathcal{F}\}$ as a Yudin basis.*
3. *There exists a family $\{f_i\}_{i \in I}$ of positive linear functionals on the partially ordered vector space (M, C) satisfying $f_i(e_j) = \delta_{ij}$ for each $i, j \in I$.*

²A regular operator between two Riesz spaces is any operator that can be written as a difference of two positive (linear) operators.

Proof: (1) \implies (2) This is part (3) of Theorem 11.

(2) \implies (3) We know that every linear functional on M is determined completely by its action on the vectors of the Hamel basis $\{e_i: i \in I\}$. Now for each $k \in I$ define the linear functional $f_k: M \rightarrow \mathbb{R}$ by letting $f_k(e_i) = \delta_{ki}$.

We claim that each f_k is positive on (M, C) . To see this, let $0 \leq x \in M$, i.e., let $x \in C$. Since $\{e_i\}_{i \in I}$ is a Hamel basis, there exists a finite subset \mathcal{F} of I and scalars $\{\lambda_i: i \in \mathcal{F}\}$ such that $x = \sum_{i \in \mathcal{F}} \lambda_i e_i$. By our hypothesis, we have $\lambda_i \geq 0$ for each $i \in \mathcal{F}$. Therefore, $f_k(x) = \sum_{i \in \mathcal{F}} \lambda_i f_k(e_i) = \sum_{i \in \mathcal{F}} \lambda_i \delta_{ki} \geq 0$, and so each f_k is positive.

(3) \implies (1) Assume $x = \sum_{i \in I} \lambda_i e_i \geq 0$, i.e., assume $x \in C$. Then $\lambda_k = f_k(x) \geq 0$ for each $k \in I$ and this guarantees that $\{e_i: i \in I\}$ is a Yudin basis of C . \blacksquare

Corollary 15 *Let $\{x_i\}_{i \in I}$ be a family of linearly independent vectors in a vector space X and let M be its linear span. Then the cone*

$$C = \left\{ \sum_{i \in I} \lambda_i x_i: \lambda_i \geq 0 \text{ for each } i \text{ and } \lambda_i = 0 \text{ for all but finitely many } i \right\}$$

is a lattice cone of M . Moreover, (M, C) is a Dedekind complete Riesz space and $\{x_i\}_{i \in I}$ is a Yudin basis for C .

Proof: Consider the linear isomorphism $R: \phi_I \rightarrow M$ defined by $R(\theta) = \sum_{i \in I} \theta_i x_i$ and then apply Lemma 3. \blacksquare

The finite dimensional analogue of the preceding corollary is also very interesting.

Corollary 16 *For a finite dimensional vector space X of dimension n , we have:*

1. *If x_1, \dots, x_n are linearly independent vectors in X , then the cone*

$$C = \left\{ \sum_{i=1}^n \lambda_i x_i: \lambda_i \geq 0 \text{ for all } i \right\}$$

is a lattice cone of X . Moreover:

1. *(X, C) is a Dedekind complete Riesz space,*
2. *C is a Yudin cone having $\{x_1, \dots, x_n\}$ as a Yudin basis, and*
3. *the Euclidean topology on (X, C) is locally solid and is generated by a C -lattice norm.*

2. *Conversely, if X under a cone C is an Archimedean Riesz space, then C is a Yudin cone.*

Proof: Part (2) is a famous result due to A. I. Yudin [10]; see also [11, Theorem 26.11, p. 152]. This part justifies the name “Yudin” employed to describe the above cone properties. For part (c) of (1), notice that the norm $\|\cdot\|$ on X defined by $\|\sum_{i=1}^n \lambda_i x_i\| = \sum_{i=1}^n |\lambda_i|$ is a C -lattice norm. ■

Corollary 17 *Every generating cone in a vector space contains a lattice subcone.*

Proof: Assume that a cone C in a vector space X satisfies $X = C - C$. By Lemma 13, there exists a Hamel basis $\{e_i\}_{i \in I} \subseteq C$ of X . If

$$K = \left\{ \sum_{i \in I} \lambda_i e_i : \lambda_i \geq 0 \text{ for each } i \text{ and } \lambda_i = 0 \text{ for all but finitely many } i \right\},$$

then, by Corollary 15, K is a lattice subcone of C . ■

Corollary 18 *Every vector space X can be ordered to become a Dedekind complete Riesz space. Specifically, if $\{e_i\}_{i \in I}$ is a Hamel basis of X , then:*

1. X is a Dedekind complete Riesz space under the Yudin cone

$$C = \left\{ \sum_{i \in I} \lambda_i e_i : \lambda_i \geq 0 \text{ for each } i \text{ and } \lambda_i = 0 \text{ for all but finitely many } i \right\},$$

2. the family $\{e_i\}_{i \in I}$ is a Yudin basis of C , and

3. the operator $R: \phi_I \rightarrow X$, defined by $R(\theta) = \sum_{i \in I} \theta_i e_i$, is an onto lattice isometry, i.e., X can be viewed as a lattice isometric copy of some ϕ_I .

Corollary 19 *Let X be a partially ordered vector space and let $\{e_i\}_{i \in I}$ be a family of linearly independent positive vectors of X with linear span M . Put $M^+ = M \cap X^+$ and consider the operator $R: \phi_I \rightarrow M$ defined by*

$$R(\theta) = \sum_{i \in I} \theta_i e_i.$$

Then we have the following properties.

1. The set

$$\phi_{I,R}^+ = R^{-1}(M^+) = \{\theta \in \phi_I : R(\theta) \in M^+\}$$

is a cone of ϕ_I .

2. *If M is a lattice-subspace of X , then $\phi_{I,R}^+$ is a lattice cone of ϕ_I .*

3. *If M^+ has a Yudin basis, then*

1. $\phi_{I,R}^+$ also has a Yudin basis, and

2. $(\phi_I, \phi_{I,R}^+)$ is a Dedekind complete Riesz space whose order intervals lie in finite dimensional Riesz subspaces of $(\phi_I, \phi_{I,R}^+)$.

4 Inductive limit topologies on Riesz spaces with Yudin cones

The purpose of this section is to discuss the basic properties of the inductive limit topology generated by the family of the finite dimensional vector subspaces of a Riesz space with a Yudin cone. First, we recall the definition of the inductive limit topology on a vector space generated by the family of its finite dimensional vector subspaces. For details about inductive limit topologies, we refer the reader to the books [8, 9, 16]. A general study of inductive limit topologies can also be found in the work of L. Tsitsas [17].

Let M be an arbitrary vector space and let $\{M_\alpha\}_{\alpha \in A}$ denote the family of all finite dimensional subspaces of M . As a finite dimensional space, each M_α admits a unique Hausdorff linear topology (the Euclidean one), say τ_α . In particular, if $M_\alpha \subseteq M_\beta$, then τ_β induces τ_α on M_α . Now the **inductive limit topology** ξ_M on M is defined as the finest locally convex topology on M for which all natural embeddings $i_\alpha: (M_\alpha, \tau_\alpha) \hookrightarrow (M, \xi_M)$ are continuous. The reader should also notice the following simple fact.

- *If $\{M_\lambda\}_{\lambda \in \Lambda}$ is a family of finite dimensional vector subspaces of M such that for each $\alpha \in A$ there exists some $\lambda \in \Lambda$ such that $M_\alpha \subseteq M_\lambda$, then ξ_M is also the finest locally convex topology on M for which each natural embedding $i_\lambda: (M_\lambda, \tau_\lambda) \hookrightarrow (M, \xi_M)$ is continuous.*

Such a family $\{M_\lambda\}_{\lambda \in \Lambda}$ of finite dimensional subspaces is referred to as a **generating family** for ξ_M . For instance, if $\{x_i\}_{i \in I}$ is a Hamel basis of M , then the family of vector subspaces generated by the finite subsets of $\{x_i\}_{i \in I}$ is a generating family for ξ_M . In particular, if M has a countable Hamel basis, then ξ_M is generated by a countable family of finite dimensional vector subspaces.

If $\{M_\lambda\}_{\lambda \in \Lambda}$ is a generating family for ξ_M , then a base at zero for the inductive limit topology ξ_M consists of all convex and balanced subsets V of M such that $V \cap M_\lambda$ is a τ_λ -neighborhood of zero in M_λ for each λ . Equivalently, a base at zero for ξ_M consists of all convex, absorbing and balanced sets of the form $V = \text{co}(\cup_{\lambda \in \Lambda} V_\lambda)$, where each V_λ is a convex and balanced neighborhood of zero in M_λ .

Theorem 20 *Regarding the inductive limit topology ξ_M , we have:*

1. *Every operator from (M, ξ_M) to any locally convex space is continuous. In particular, the topological dual of (M, ξ_M) coincides with the algebraic dual M^* of M .*
2. *If M has a countable Hamel basis $\{e_1, e_2, \dots\}$ and M_n denotes the linear span of the set $\{e_1, \dots, e_n\}$, then the inductive limit topology of the sequence $\{(M_n, \tau_n)\}$ on the vector space M coincides with ξ_M .*

Proof: (1) Let $T: (M, \xi_M) \rightarrow (Z, \tau)$ be a linear operator, where (Z, τ) is a locally convex space. Then it is easy to see that T is continuous if and only if $T: (M_\alpha, \tau_\alpha) \rightarrow (Z, \tau)$ is

continuous for each α ; see also [9, Proposition 1, p. 159] or [3, Theorem 5.2.4, p. 240]. But since each M_α is finite dimensional this is always the case, and our conclusion follows.

(2) Let η denote the—in this case strict—inductive limit topology on M induced by the sequence $\{(M_n, \tau_n)\}$. Then by part (1), the identity operator $I: (M, \xi_M) \rightarrow (M, \eta)$ is continuous. On the other hand, the same argument, shows that $I: (M, \eta) \rightarrow (M, \xi)$ is also continuous, and hence I is a linear homeomorphism. This implies $\xi_M = \eta$. ■

Corollary 21 *If M^* denotes the algebraic dual of M , then $\langle M, M^* \rangle$ is a dual system and the inductive limit topology ξ_M coincides with the Mackey topology on M , that is, $\xi_M = \tau(M, M^*)$.*

Proof: By Theorem 20, every linear functional on M is ξ_M -continuous, and therefore $\xi_M \subseteq \tau(M, M^*)$. On the other hand, it is easily seen (since $\tau(M, M^*)$ induces the Euclidean topology τ_α on M_α) that each natural embedding $i_\alpha: (M_\alpha, \tau(M, M^*)) \hookrightarrow (M, \xi_M)$ is also continuous, and thus $\tau(M, M^*) \subseteq \xi_M$ also holds true. ■

We now turn our attention to inductive limit topologies on Riesz spaces with Yudin cones. As shown in the previous section, the canonical model for such a Riesz space is ϕ_I . For simplicity, the inductive limit topology ξ_{ϕ_I} will be denoted by ξ_I and ξ will denote the inductive limit topology on $\phi = \phi_{\mathbb{N}}$. Note that ξ is also the strict inductive limit of a countable family of finite dimensional vector subspaces. We shall write $\phi'_{\mathbb{N}} = \mathbb{R}^{\mathbb{N}} = \mathbb{R}^\infty$.

It is easy to see that a base at zero for the inductive limit topology ξ_I consists of all convex and solid sets of the form

$$W_y = \{\theta \in \phi_I: |\theta_i| < y_i \text{ for all } i \in I\},$$

where $y = (y_i)_{i \in I} \in \mathbb{R}^I$ is fixed and satisfies $y_i > 0$ for each $i \in I$. When I is an infinite set, the inductive limit topology ξ_I on ϕ_I is strictly finer than the norm and pointwise topologies on ϕ_I .

The following result describes the basic properties of the inductive limit topology ξ_I .

Theorem 22 *For an arbitrary index set I we have:*

1. ξ_I is an order continuous Hausdorff locally convex-solid topology on ϕ_I .
2. The topological dual ϕ'_I of (ϕ_I, ξ_I) is \mathbb{R}^I , where each vector $y = (y_i)_{i \in I} \in \mathbb{R}^I$ defines a linear functional on ϕ_I via the formula $y(x) = \langle x, y \rangle = \sum_{i \in I} x_i y_i$.
3. The order intervals of ϕ_I lie in finite dimensional subspaces—and so they are all ξ_I -compact.
4. The Riesz dual system $\langle \phi_I, \mathbb{R}^I \rangle$ is symmetric and $\xi_M = \tau(\phi_I, \mathbb{R}^I)$.

Proof: By Theorem 11, we can suppose that ξ_I is generated by a family $\{M_\lambda\}_{\lambda \in \Lambda}$ of finite dimensional ideals of M (and hence each V_λ is a solid subset of ϕ_I ; see the discussion a preceding Theorem 20). A basic ξ_I -neighborhood of zero is of the form $V = \text{co}(\bigcup_{\lambda \in \Lambda} V_\lambda)$, where each V_λ is a convex and solid τ_λ -neighborhood of zero in M_λ . To see that V is a solid subset of ϕ_I , assume $|x| \leq |y|$ in ϕ_I with $y \in V$. There exist indices $\lambda_1, \dots, \lambda_n \in \Lambda$, positive scalars β_1, \dots, β_n with $\sum_{i=1}^n \beta_i = 1$ and y_1, \dots, y_n with $y_i \in V_{\lambda_i}$ for each i such that $y = \sum_{i=1}^n \beta_i y_i$. By the Riesz Decomposition Property (see [6, Theorem 1.2, p. 3]) we can write $x = \sum_{i=1}^n x_i$ with $|x_i| \leq \beta_i |y_i|$ for each i . If $z_i = \frac{x_i}{\beta_i}$, then $|z_i| \leq |y_i|$ for each i . This implies $z_i \in V_{\lambda_i}$ for each i and $x = \sum_{i=1}^n \beta_i z_i \in V$. Thus, V is a solid subset of ϕ_I and so ξ is a locally convex-solid topology.

To see that ξ_I is order continuous, assume $x_\epsilon \downarrow 0$ in ϕ_I ; this is equivalent to $x_\epsilon(i) \downarrow 0$ in \mathbb{R} for each $i \in I$. Now let $V = \text{co}(\bigcup_{\lambda \in \Lambda} V_\lambda)$ be a basic ξ_I -neighborhood of zero. Since $x_\epsilon \leq x_\delta$ for $\epsilon \geq \delta$, we can suppose that there exists some λ such that $x_\epsilon \in M_\lambda$ for each ϵ . Since τ_λ is order continuous and $x_\epsilon \downarrow 0$ in M_λ , there exists some ϵ_0 such that $x_\epsilon \in V_\lambda \subset V$ for all $\epsilon \geq \epsilon_0$. This shows that $\{x_\epsilon\}$ converges to zero for ξ_I and so ξ_I is order continuous.

The other properties now follow easily from Theorem 11. ■

When M is a Riesz space having a cone with a countable Yudin basis, then, besides the properties listed in Theorem 22, the inductive limit topology has a few more remarkable properties. For proofs of the claims in the next theorem, see [3, Section 5.2].

Theorem 23 *If M is a Riesz space having a cone with a countable Yudin basis, then:*

1. *The Hausdorff locally convex-solid Riesz space (M, ξ_M) is:*
 1. *topologically complete,*
 2. *non-metrizable,*
 3. *Mackey, barrelled and bornological, and*
 4. *has the Dunford–Pettis property.*
2. *The algebraic dual M^* of M (which coincides with the order dual M^\sim) equipped with the strong topology $\beta(M^*, M)$ is an order continuous Fréchet lattice.*

And now we state a topological version of Corollary 19.

Theorem 24 *Let $\{x_n\}$ be a sequence of linearly independent positive vectors in a partially ordered vector space X and let M denote the span of $\{x_n\}$. Also, let $R: \phi \rightarrow M$ be the operator defined by $R(\theta) = \sum_{n=1}^{\infty} \theta_n x_n$. If the cone $M^+ = M \cap X^+$ has a Yudin basis (which must be countable), then we have the following.*

1. *The vector space ϕ equipped with the cone*

$$\phi_R^+ = R^{-1}(M^+) = \{\theta \in \phi: R(\theta) \geq 0\}$$

is a Dedekind complete Riesz space.

2. The cone ϕ_R^+ is a Yudin cone.
3. Each order interval of (ϕ, ϕ_R^+) is ξ -compact and lies in a finite dimensional vector subspace.
4. The inductive limit topology ξ on the Riesz space (ϕ, ϕ_R^+) is Hausdorff, locally convex-solid and order continuous, and the operator $R: (\phi, \phi_R^+, \xi) \rightarrow (M, M^+, \xi_M)$ is a (surjective) topological lattice isomorphism.
5. The Riesz space \mathbb{R}^∞ coincides with the topological, algebraic and order dual of (ϕ, ϕ_R^+, ξ) . Moreover, \mathbb{R}^∞ equipped with the dual cone

$$(\phi_R^+)' = \left\{ y = (y_1, y_2, \dots) \in \mathbb{R}^\infty : \langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n \geq 0 \ \forall x = (x_1, x_2, \dots) \in \phi_R^+ \right\}$$

is a Dedekind complete Riesz space.

6. If $E = (\phi, \phi_R^+)$ and $E' = (\mathbb{R}^\infty, (\phi_R^+)')$, then $\langle E, E' \rangle$ is a symmetric Riesz dual system.
7. In case $\{x_n\}$ is itself a Yudin basis, then $\phi_R^+ = \phi^+$ (the standard positive cone of ϕ) and R is essentially the identity operator.

Proof: (1) By Corollary 12, M is a lattice-subspace of X and the conclusion follows from Corollary 19(3).

(2) This is obvious.

(3) This is a special case of Theorem 11(4).

(4) By Theorem 11, the inductive limit topology ξ is also the inductive limit topology of the family of all finite dimensional ϕ_R^+ -ideals. Hence, as before, ξ must be locally solid with respect to the Yudin cone ϕ_R^+ . Moreover, as in the proof of Theorem 22, we see that ξ is order continuous on (ϕ, ϕ_R^+) . The rest of the proof of this part follows immediately from Lemma 3 and Theorem 20(1).

(5) To see this, use the fact that the topological dual of a locally solid Riesz space is a Dedekind complete Riesz space and an ideal in its order dual; see [6, Theorem 5.7, p. 36].

(6) This follows immediately from (3).

(7) Use the fact that $R(\theta) = \sum_{i \in I} \theta_i x_i \geq 0$ if and only if $\theta_i \geq 0$ for each i (i.e., if and only if $\theta \in \phi^+$). ■

There is another remarkable connection between a Yudin basis and a Hamel basis. The proof of the next result is an immediate consequence of Lemma 3 and is omitted.

Theorem 25 Let $\{x_n\}$ be a sequence of linearly independent positive vectors of a partially ordered vector space X , let $M = M^+ - M^+$ (the span of $\{x_n\}$) and assume that M^+ has a Yudin basis $\{e_n\}$. Then for the operator $\Pi: M \rightarrow M$, defined by

$$\Pi\left(\sum_{n=1}^{\infty} \theta_n e_n\right) = \sum_{n=1}^{\infty} \theta_n x_n, \quad (\theta_1, \theta_2, \dots) \in \phi^+,$$

we have the following.

1. The cone

$$C = \Pi(M^+) = \left\{ \sum_{n=1}^{\infty} \theta_n x_n : (\theta_1, \theta_2, \dots) \in \phi^+ \right\}$$

is a lattice cone of M .

2. The operator $\Pi: (M, M^+, \xi_M) \rightarrow (M, C, \xi_M)$ is a (surjective) linear, topological and lattice isomorphism between locally convex-solid Riesz spaces.

Let us illustrate the above theorem with the diagram shown in Figure 3. Assume that $\{e_n\}$ is a Yudin basis for M . If $R_0: \phi \rightarrow M$ is defined via $R_0(\theta) = \sum_{n=1}^{\infty} \theta_n e_n$, then the operator $R_0: (\phi, \phi^+, \xi) \rightarrow (M, M^+, \xi_M)$ is a (surjective) linear, topological and lattice isomorphism. Similarly, if we define $R: \phi \rightarrow M$ via $R(\theta) = \sum_{n=1}^{\infty} \theta_n x_n$, then $R: (\phi, \phi_R^+, \xi) \rightarrow (M, M^+, \xi_M)$ is a (surjective) linear, topological and lattice isomorphism.

$$\begin{array}{ccccc} (\phi, \phi^+, \xi) & \xrightarrow{R_0} & (M, M^+, \xi_M) & \xrightarrow{\Pi} & (M, C, \xi_M) \\ \downarrow I & & \downarrow \Pi & & \downarrow I \\ (\phi, \phi_R^+, \xi) & \xrightarrow{R} & (M, M^+, \xi_M) & \xrightarrow{\Pi} & (M, C, \xi_M) \end{array}$$

Fig. 3

Now consider the linear operator $\Pi: M \rightarrow M$ defined via the formula

$$\Pi\left(\sum_{n=1}^{\infty} \theta_n e_n\right) = \sum_{n=1}^{\infty} \theta_n x_n, \quad (\theta_1, \theta_2, \dots) \in \phi.$$

Then $\Pi: (M, M^+, \xi_M) \rightarrow (M, M^+, \xi_M)$ is a (surjective) linear topological isomorphism which is also positive. However, it fails to be a lattice isomorphism since the (lattice) cone

$$C = \Pi(M^+) = \left\{ x \in M^+ : \exists \theta \in \phi^+ \text{ such that } x = \sum_{n=1}^{\infty} \theta_n x_n \right\},$$

is in general a proper subcone of M^+ . The operator $\Pi: (M, M^+, \xi_M) \rightarrow (M, C, \xi_M)$ is a (surjective) linear, topological and lattice isomorphism. Note that the operator

$\Pi: (M, M^+, \xi_M) \rightarrow (M, M^+, \xi_M)$ is a (surjective) linear, topological and lattice isomorphism if and only if $\{x_n\}$ is itself a Yudin basis. The operators I represent the formal identities between the spaces involved. From the diagram it should be also obvious that (ϕ, ϕ^+, ξ) and (ϕ, ϕ_R^+, ξ) are linearly, topologically and lattice isomorphic.

5 Yudin exchange economies

In this section, we shall present a model of an exchange economy having a plethora of weakly Pareto optimal allocations. For details about the economic concepts mentioned below, we refer the reader to [3].

The commodity-price duality of this model economy is a dual system $\langle M, M' \rangle$, where M is a partially ordered vector space whose cone M^+ is Yudin and generating. We assume that M' is simply a vector subspace of the algebraic dual M^* that separates the points of the vector space M .

There are m consumers indexed by i . Each consumer i has an initial endowment $\omega_i \in M^+$ and a utility function $u_i: M^+ \rightarrow \mathbb{R}^+$ which is monotone (i.e., $x \geq y \geq 0$ implies $u_i(x) \geq u_i(y)$) and τ_i -continuous for a Hausdorff linear topology τ_i on M . We assume that $u_i(0) = 0$ for each i . As usual, the total endowment is the vector $\omega = \sum_{i=1}^m \omega_i$. In accordance with the discussion in this paper, we call this pure exchange economy a **Yudin exchange economy**.

Recall that an **allocation** is an m -tuple (x_1, \dots, x_m) such that $x_i \in M^+$ for each i and $\sum_{i=1}^m x_i = \omega$. An allocation (x_1, \dots, x_m) is said to be **weakly Pareto optimal** if there is no other allocation (y_1, \dots, y_m) satisfying $u_i(y_i) > u_i(x_i)$ for each i .

A **feasible allocation** is an m -tuple (x_1, \dots, x_m) such that $x_i \in M^+$ for each i and $\sum_{i=1}^m x_i \leq \omega$. A **utility allocation** is any vector in \mathbb{R}_+^m of the form $(u_1(x_1), \dots, u_m(x_m))$, where (x_1, \dots, x_m) is a feasible allocation. The set of all utility allocations is called the **utility space** of the economy and is denoted by U . That is,

$$U = \{(u_1(x_1), \dots, u_m(x_m)) : (x_1, \dots, x_m) \text{ is a feasible allocation}\}.$$

If $x = (x_1, \dots, x_m)$ is a feasible allocation, then we shall denote $(u_1(x_1), \dots, u_m(x_m))$ for brevity by $U(x)$. The utility space is a bounded set and is always comprehensive from below in the sense that $0 \leq v \leq u \in U$ implies $v \in U$.

In terms of the utility space the weakly Pareto optimal allocations are characterized as follows.

Lemma 26 *An allocation $x = (x_1, \dots, x_m)$ is weakly Pareto optimal if and only if its utility allocation $U(x)$ is a boundary point of U relative to \mathbb{R}_+^m .*

Proof: Let $x = (x_1, \dots, x_m)$ be an allocation. Assume first that $U(x)$ is a boundary point of U relative to \mathbb{R}_+^m . If x is not weakly Pareto optimal, then there exists another allocation $y = (y_1, \dots, y_m)$ such that $u_i(y_i) > u_i(x_i)$ for each i . But then, this readily implies that $U(x)$ is an interior point of U relative to \mathbb{R}_+^m , a contradiction.

For the converse, assume that x is a weakly Pareto optimal allocation. If $U(x)$ is an interior point of U relative to \mathbb{R}_+^m , then there exists an open ball $B(U(x), r)$ of $U(x)$ such that $B(U(x), r) \cap \mathbb{R}_+^m \subseteq U$. This implies that there exists some $\epsilon > 0$ such that the vector $v = (u_1(x_1) + \epsilon, \dots, u_m(x_m) + \epsilon)$ satisfies $v \in U$. From this, we see that x is not weakly Pareto optimal. This contradiction establishes that $U(x)$ must be a boundary point of U relative to \mathbb{R}_+^m . ■

In view of Lemma 26, the boundary of U relative to \mathbb{R}_+^m is also known as the **weakly Pareto optimal frontier**. Since equilibria and optimal allocations are usually weakly Pareto optimal, the equilibria and optimality notions of an economy are closely connected with its weakly Pareto optimal frontier. An economy is said to be **closed** if its utility space is a closed set. From the above, it should be obvious that every closed economy has “lots” of weakly Pareto optimal allocations. An example of a closed utility space is shown in Figure 4; the vector \bar{u} is the initial utility allocation: $\bar{u} = (u_1(\omega_1), \dots, u_m(\omega_m))$.

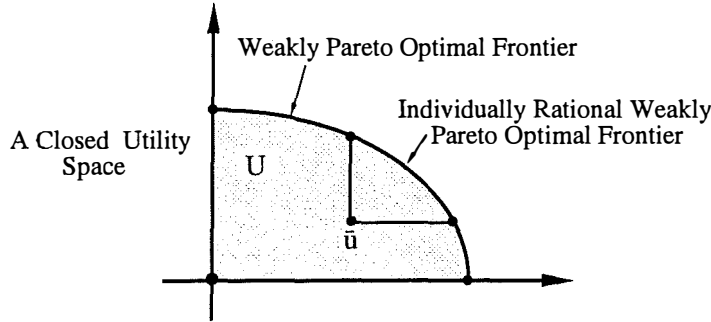


Fig. 4

It turns out that Yudin exchange economies are closed.

Theorem 27 *Every Yudin exchange economy is closed.*

Proof: By Theorem 11(4), the order interval $[0, \omega]$ lies in a finite dimensional subspace of M and so it is norm compact. This implies that $[0, \omega]^m$ is also norm compact.

Now assume that a sequence $\{v_n\} \subseteq U$ satisfies $v_n \rightarrow v$ in \mathbb{R}^m . For each n pick a feasible allocation $x_n \in [0, \omega]^m$ such that $U(x_n) = v_n$. The compactness of $[0, \omega]^m$ guarantees the existence of a subsequence $\{y_n\}$ of $\{x_n\}$ such that $y_n \rightarrow x \in [0, \omega]^m$. It easily follows that x is a feasible allocation. Moreover, since each Hausdorff linear topology τ_i induces the norm topology on $[0, \omega]$, each u_i is norm continuous on $[0, \omega]$. This implies.

$$v = \lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} U(x_n) = U(x).$$

Therefore, $v \in U$ and so U is a closed subset of \mathbb{R}^m . ■

The proofs of the existence of a competitive equilibrium and the validity of the welfare theorems in Yudin exchange economies follow from Theorem 27 and can be found in [4].

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