GLOBAL ANALYSIS OF A BUCK REGULATOR

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ABSTRACT

Liapunov's direct stability method as applied to discrete systems and the method of paired systems due to Kalman are used to obtain sufficient conditions for global stability. The global convergence of a buck regulator is also investigated.

INTRODUCTION

The phase plane approach for discrete systems is not as powerful as the techniques developed for continuous systems, but it probably is the only way of systematically dealing with the nonlinearities of a switching regulator. The trajectories of a continuous system naturally dissect the phase plane into regions which can then be classified for stability. The difference equation transforms one point of the phase plane into another and no curves are identified with it. A buck regulator utilizing a discrete control law is analysed to illustrate how the discrete phase plane of a switching regulator can be dissected.

The control law determines, for each switching cycle, the time the switch is on. The phase plane consists of two saturated regions separated by a narrow unsaturated region with the boundaries of these regions determined from the minimum and maximum value of the on-time of the switch. The idea of pairing continuous systems with discrete systems, R. E. Kalman (1), can be used to associate a trajectory with the discrete system in the saturated regions of the phase plane. Global stability is shown by finding a Liapunov function which decreases for any point in the unsaturated region and also decreases along the associated continuous trajectories after a few steps in the saturated region. The Liapunov function, in this way, is shown to decrease although it does not necessarily decrease each step.

The unsaturated region of the phase plane in some ways resembles a switching line. This resemblance is due to the fact that the system changes trajectories when crossing from one saturated region to another. In fact, the phase plane can be used to study the convergence of the system with the unsaturated region acting as a switching line. The optimal switching line found for the paired continuous systems in the saturated regions approximates the optimal switching line of the discrete system. If the maximum amount of control is being used (i.e., the switch is on or off during the entire switching period), then the paired continuous trajectories are the true trajectories of the discrete system, and the optimal switching line found from the paired systems is the optimal switching line for the discrete system (this type of control is called "bang-bang" control).

The discrete control law used in this paper can be implemented digitally, see ref. [2]. The use of digital control has been hampered in the past by its lack of speed, but it is now possible to implement digital control which takes only a microsecond for the feedback to be calculated, see ref. [3]. The digital control circuitry can be easily interfaced with a microprocessor thereby greatly increasing the control capability of the system. The use of microprocessors to accomplish sophisticated control algorithms such as optimal switching lines will occur when the microprocessors become fast enough. The popularity of digital control circuits used in conjunction with microprocessors should increase in the future, and an understanding of the discrete phase plane will be helpful in optimizing such a system.

DISCRETE CONTROL LAW

A block diagram of the buck regulator is shown in Fig. 1.

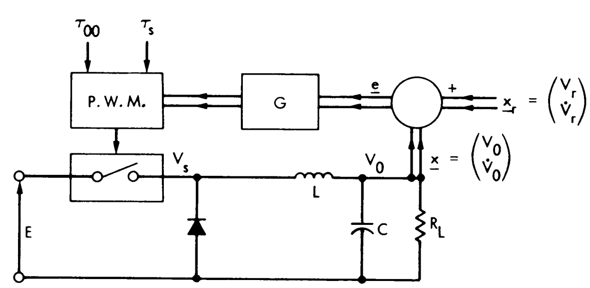


Figure 1. Buck Regulator

The P.W.M. controls the switch such that when no error exists (i.e., $\underline{e} \equiv \underline{0}$), the on-time and switching period will be $\tau_{00}$ and $\tau_s$ respectively. When the error vector is not zero, the input to the P.W.M. will be the gain matrix, G, multiplied by the error, $\underline{e}$. The on-time will be modified to decrease this error according to the following discrete control law

$$\tau_0(\underline{x}_n) = \tau_{00} + a_1(x_r - x_n) + b_1(\dot{x}_r - \dot{x}_n) \quad (1.a)$$

The coefficients $a_1$ and $b_1$ in the control law are the feedback constants.

If the origin of the phase plane is taken to be the reference value, $\underline{x}_r$, as shown in fig. 2, then the control can be written as

$$\tau_0(\zeta_n) = \tau_{00} - a_1 \zeta_n - b_1 \dot{\zeta}_n \quad (1.b)$$

let

$$\eta_n = \zeta_n + \frac{b_1}{a_1} \dot{\zeta}_n$$

so

$$\tau_0(\eta_n) = \tau_{00} - a_1 \eta_n \quad (1.c)$$

The control law is only a function of one variable $\eta_n$. Since the on-time of the controller is limited to a minimum and maximum value; a lower and higher limit, $\eta_\ell$ and $\eta_h$ respectively, exists on the independent variable. In Fig. 2 the on-time will be two-thirds of the switching period for the region left of the line marked $\eta_\ell$, and it will be zero for the region right of the line marked $\eta_h$. The above mentioned regions are the two saturated regions of the phase plane. The reason for not allowing the on-time to equal the switching period is discussed later.

## PAIRED SYSTEMS AND CENTERS

The recursion formula for a buck regulator is

$$\underline{x}_{n+1} = Y(\tau_s)\underline{x}_n + \underline{f}(\tau_0, \tau_s) \quad (2)$$

where

$$\underline{f}(\tau_0, \tau_s) = E \begin{pmatrix} y_{11}(\tau_s - \tau_0) - y_{11}(\tau_s) \\ y_{12}(\tau_s) - y_{12}(\tau_s - \tau_0) \end{pmatrix}$$

and

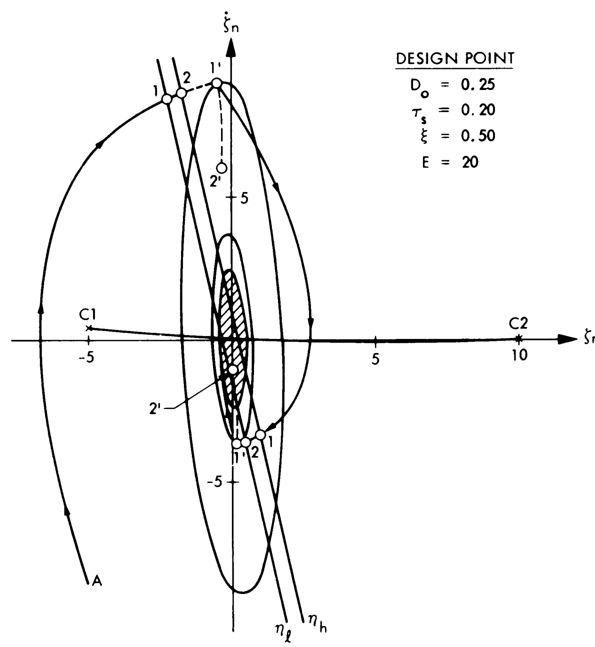


Figure 2. Phase Plane

$$Y(\tau) = \begin{pmatrix} y_{11}(\tau) & y_{12}(\tau) \\ y_{21}(\tau) & y_{22}(\tau) \end{pmatrix} \text{ - principal matrix solution}$$

The elements of the principal matrix solution for values of the damping factor, $\xi$, less than one are

$$y_{11}(\tau) = e^{-\xi\tau}\left(\cos \omega_d \tau + \frac{\xi}{\sqrt{1 - \xi^2}} \sin \omega_d \tau\right)$$

$$y_{12}(\tau) = e^{-\xi\tau} \frac{\sin \omega_d \tau}{\sqrt{1 - \xi^2}}$$

$$y_{21}(\tau) = -y_{12}(\tau)$$

$$y_{22}(\tau) = e^{-\xi\tau}\left(\cos \omega_d \tau - \frac{\xi}{\sqrt{1 - \xi^2}} \sin \omega_d \tau\right)$$

where

$$\omega_K^2 = \frac{1}{LC}$$

$$\tau = \omega_K t \quad \text{(non-dimensional time)}$$

$$\xi = \frac{1}{2R_L} \sqrt{\frac{L}{C}}$$

and

$$\omega_d = \sqrt{1 - \xi^2}$$

The recursion formula along with the control law is all that is needed to completely describe the system.

In the saturated regions of the phase plane, the recursion formula takes the form (see ref. [4])

$$\underline{\zeta}_{n+1}^{1} = Y(\tau_s)\ \underline{\zeta}_n + \underline{b}$$

where

$$\underline{b} = \begin{cases} \underline{b}_1 & \tau_0(\underline{\zeta}_n) = 0 \\ \underline{b}_2 & \tau_0(\underline{\zeta}_n) = \frac{2}{3}\tau_s \end{cases}$$

Besides the origin, the system will have two equilibrium points which result because of the forcing vector saturating. The equilibrium points are

$$\underline{\zeta}_e = (I - Y(\tau_s))^{-1}\underline{b}$$

If the origin is translated to one of these equilibrium points, the recursion formula is

$$\underline{u}_{n+1} = \underline{\zeta}_{n+1} - \underline{\zeta}_e = Y(\tau_s)\underline{\zeta}_n + \underline{b} - [Y(\tau_s)\underline{\zeta}_e - \underline{b}]$$

or

$$\underline{u}_{n+1} = Y(\tau_s)\underline{u}_n \qquad (3)$$

when the forcing vector is saturated. The matrix $Y(\tau_s)$ is a constant matrix with complex eigenvalues, and it is therefore possible to pair a continuous system to the discrete system. The recursion formula in the saturated regions is given by eqn. (3) where the origin of the new coordinate system is at the equilibrium point $\underline{\zeta}_e$ of the old. The origins for the saturated regions will be called centers. The center of the saturated region located to the left of the origin in the phase plane shown in Fig. 2 is C2, and the center of the saturated region located to the right is C1. In these regions a continuous system whose origin is at the proper center can be paired to the discrete system. Trajectories can then be drawn in these regions. It is possible to find a center for every forcing vector of the system. The line connecting the two centers C1 and C2 is called the line of centers. Every point in the phase plane has a center on the line of centers where its recursion formula is given by eqn. (3).

1 - The $\zeta_n$ are the coordinates of the phase plane shown in Fig. (2).

## LIAPUNOV FUNCTION

The unsaturated region of the phase plane is contained in the thin strip between the saturated regions. The strategy of this analysis is to first find a Liapunov function which will decrease for any point in the unsaturated region; then it is only necessary to show that the Liapunov function decreases in the saturated region. The Liapunov matrix chosen is

$$L = (T^{-1})^* \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} T^{-1} \qquad (4)$$

where

$$J = T^{-1}PT - \text{Jordan form}$$

The T matrix is composed of the generalized eigenvectors of the linear perturbation matrix P. The relationship between the linear perturbation matrix and the recursion formula, eqn. (2), is

$$\underline{\zeta}_{n+1} = Y(\tau_s)\zeta_n + \underline{f}(\underline{\zeta}_n)$$

$$= P_n\ \underline{\zeta}_n$$

now

$$P_n = P + \Delta P_n$$

where

P = constant matrix (linear part of the recursion formula)

The feedback constants for the example were chosen so that the eigenvalues of the linear perturbation matrix are zero, see ref. [4] for details. This choice of feedback constants gives the most rapid convergence, two steps, for the linearized regulator.

The change in the Liapunov function for the nonlinear system is

$$\Delta V_n = \underline{\zeta}_n^*(P_n^*\ LP_n - L)\underline{\zeta}_n$$

The system will decrease at $\underline{\zeta}_n$ relative to the given Liapunov norm if the matrix $Q_n$, which is defined below, is positive definite, see ref. [5].

$$Q_n = L - P_n^* L P_n$$

The nonlinear perturbation matrix, $P_n$, is a function of only one variable, $\eta_n$, and if the Liapunov function decreases for those values of $\eta_n$ between $\eta_\ell$ and $\eta_h$, then the norm of all points in the unsaturated region will decrease.

In the example chosen, a Liapunov function was not found which showed that the norm of all points in the nonlinear region decreased. It was necessary to limit the amount of time the switch was on thereby decreasing the magnitude of the maximum forcing vector, $\underline{f}$. When the on-time is limited to two-thirds the switching period, a Liapunov function is found which gives stability for the shaded region shown in Fig. 2. The Liapunov function used for this example is

$$V_n = \underline{\zeta}_n^* \begin{pmatrix} 28.1 & 1.123 \\ 1.123 & 1.055 \end{pmatrix} \underline{\zeta}_n$$

The Liapunov matrix used above is that given by eqn. (4). However, the linear perturbation matrix from which the Liapunov matrix was calculated is dimensional in time. It was found that the domain of stability could be greatly increased by varying the frequency $\omega_k$ and switching period $T_s$ while maintaining $\omega_k T_s = \tau_s$. Stability depends only on the parameter $\tau_s$, but the Liapunov matrix generated by eqn. (4) depends on $\omega_k$ and $T_s$ individually.

The phase plane of Fig. 2 is divided into three regions, the unsaturated region, which is the center region located between the two straight lines, and the saturated regions, with the Liapunov curves superimposed on these regions. The continuous system whose trajectories are associated with one of the saturated centers, C1 or C2, is paired to the discrete system $\eta_{n+1} = Y(\tau_s)\eta_n$. The system, in one step, will decrease relative to the Liapunov norm from any point in the unsaturated region. If the trajectories in the saturated regions are followed, the system can also be shown to decrease relative to the Liapunov norm from any point in the saturated region. Global stability is therefore guaranteed.

In Fig. 2, a trajectory is followed into the shaded region of the phase plane from the initial point (A). The trajectory is easily followed until it enters or jumps across the unsaturated region. The unsaturated region acts like a switching line of a continuous system in as much as the system switches from one set of trajectories to another. Since there are identified with one continuous trajectory a number of discrete trajectories depending on the initial conditions, the exact point and manner in which the switch is made is not clear.

If the system jumps across the unsaturated region, the new trajectory will begin on the dashed line whose ends are marked (2) and (1') . If the point trajectory lands in the unsaturated region, the new trajectory will begin somewhere on the line segment (1') to (2') . The worst trajectory as far as the stability analysis is concerned occurs when the system lands at (1') . The worst trajectory is always used as the new trajectory, and in this way the worst possible trajectory is obtained for the system. As can be seen in Fig. 2, each time the system crosses the unsaturated region the new trajectory is always closer to the origin than the trajectory of the previous cycle. If this were not the case, the system could be unstable. The system can only be shown to be unstable if the best trajectory for stability, not the worst one, is found to be farther away from the origin than the previous trajectory. If neither of these conditions hold, the system could be either stable or unstable. In ref. [1] R. E. Kalman notes that even though the discrete systems are completely deterministic, it is sometimes necessary to use a probabilistic approach to deal with the nonlinearities.

The two most important parameters in the global stability analysis are the switching period, $\tau_s$, and the damping factor, $\xi$. The damping factor is important because it determines the shape of the trajectories in the saturated regions. If there is a lot of damping, the trajectories will decrease rapidly relative to the centers while if there is no damping, the trajectories will be circular. The global stability is improved for large damping factors. The switching period can be viewed as the step size of the system. The larger the switching period the larger will be the distance between successive points in the discrete trajectory. The worst trajectory occurs when the system steps from (1) to (1') in Fig. 2. If the switching period is small, then the worst trajectory will be close to the origin, and the stability will be improved.

## CONVERGENCE

The example illustrates the resemblance of the unsaturated region of the phase plane to a switching line. The resemblance results from the fact that the trajectory of the paired continuous system changes as the unsaturated region is crossed. It is possible to monitor the discrete system continuously so that it always switches exactly at a switching line. Two levels of control could be used with the system being brought close to the origin by using a switching line where it would then revert to the usual discrete regulator.

A desirable characteristic of any regulator is rapid convergence. For a buck regulator where the on-time is allowed to vary between zero and the switching period, an optimal switching line can be found. D. W. Bushaw, ref. [6], solves the problem of determining the control which gives the most rapid convergence for a system described by a linear, second order differential equation with a discontinuous forcing function. For the particular case mentioned, the trajectory of the associated continuous system and the discrete system are exactly the same. This fortuitous situation allows the analysis of the discrete system to be carried out exactly as a continuous system. The optimal control strategy for a linear system is to use the maximum control possible. This type of control is called "bang-bang" control.

When the trajectories of the discrete and associated continuous systems do not coincide, the analysis of the switching strategy for the associated continuous system, which can usually be solved for, can be used for the discrete system. The idea for doing this is that the optimal switching strategy for the associated system should, in some sense, approximate that of the discrete system. In fact, even when the exact switching curve is known, it is usually necessary to approximate it with a polynomial so that the implementation of the control is simplified.

The problem of finding the optimal switching line for the continuous system associated with the discrete system of Fig. 2 can be solved. It is not clear, however, that the switching line associated with the continuous system is the optimal switching line of the discrete system. Since with linear systems the optimal strategy is to use the maximum control available, the maximum use of the voltage, even if it is only for two-thirds of the switching period, appears to be a plausible control scheme. The real difficulty in the analysis is that the continuous and discrete trajectories only correspond at the sampling instants.

In Fig. 3 the optimal switching line for the paired continuous systems of the example is found when the damping factor is zero. When the damping factor is zero, the optimal switching lines are circular arcs whose centers are at C1 and C2 of Fig. 2 and 3. The discrete trajectories to the right of the switching line, which corresponds to the switch being off for the entire switching period, have the same trajectories as the associated paired continuous system. However, the discrete trajectories which correspond to the switch being on only two-thirds of the switching period are not the same as those of the paired continuous system.

## CONCLUSIONS

Sufficient conditions for global stability of a buck regulator utilizing a discrete control law are found. The method of paired systems and Liapunov functions are used to establish global stability and to study the convergence of the regulator. A heuristic argument is given that the optimal switching curves associated with the paired continuous systems approximate the optimal switching curves of the discrete system.

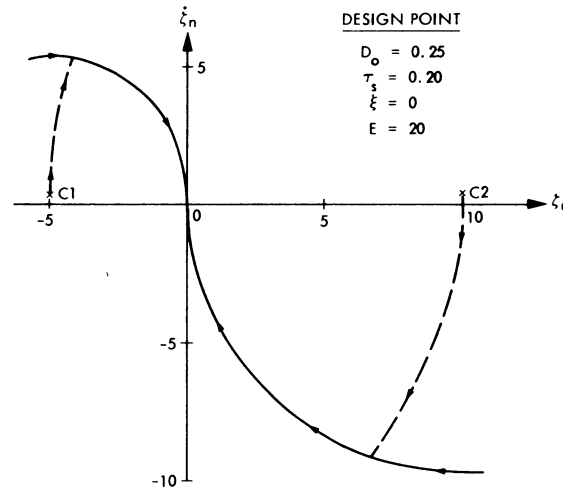


Figure 3. Switching Line

## REFERENCES

(1) R. E. Kalman, "Nonlinear Aspects of Sampled-Data Control Systems," Proceedings of the Second Symposium on Nonlinear Circuit Analysis, Polytechnic Institute of Brooklyn, April 25-27, 1956.

(2) V. B. Boros, "A Digital Proportional Integral, and Derivative Feedback Controller for Power Conditioning Equipment," PESC Record, pg. 235-141; June, 1977.

(3) N. R. Miller, "A Digitally Controlled Switching Regulator," PESC Record, pg. 142-147; June 1977.

(4) D. B. Edwards, "Time Domain Analysis of Switching Regulators," Doctoral Thesis, Calif. Inst. of Tech., March, 1977.

(5) Wolfgang Hahn, "Theory and Application of Liapunov's Direct Method," Section 39, Prentice-Hall, Englewood Cliffs, New Jersey, 1963.

(6) D. W. Bushaw, "Differential Equations with a Discontinuous Forcing Term," Report No. 469, Stevens Institute of Technology, January, 1953.