Dedicated to Dr. J. Nicholas Newman in honor of his Seventieth Anniversary

A nonlinear theory for unsteady flexible wing

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Abstract. This paper extends the previous studies by Wu (2001-2006)[1]-[3] to continue developing a fully nonlinear theory for evaluation of unsteady flow generated by a two-dimensional flexible lifting surface moving in arbitrary manner through an incompressible and inviscid fluid for modeling bird/insect flight and fish swimming. The original physical concept founded by Theodore von Kármán and William R. Sears (1938)[4] in describing the complete vortex system of a wing and its wake in non-uniform motion for their linear theory is adapted and applied to a fully nonlinear consideration. The new theory employs a joint Eulerian and Lagrangian description of the lifting-surface movement to accomplish the formulation and analysis. The present investigation presents further development for addressing arbitrary variations in wing shape and trajectory to achieve a fully nonlinear integral equation generalizing Herbert Wagner's (1925)[5] linear version for enhancing determination of exact solutions in general.

Key words: nonlinear unsteady flexible wing theory, unsteady camber function, arbitrary trajectory.

1. Introduction

In the world of self-locomotion of aquatic and aerial animals by using lifting surfaces such as wings and appended fins, there are several salient features of significance. First, the wings are in general large in aspect-ratio, a feature that would suit for an unsteady lifting-line approach. Secondly, the periodic flapping of the wing generally involves changes in surface profile shape (or *shape function*), e.g. from a stretched-straight pronation in downward stroke to a form with an arched camber and spanwise bending in upward supination stroke. Further, in swift maneuvering, the wings may bend and twist asymmetrically to change and turn in orientation and trajectory, e.g. in the beautiful performance of a humming bird using a figure-eight wing flapping in keeping its body fixed in front of a flower, and then suddenly fleeting off in a flash. All these features are so strongly nonlinear and time-dependent that a comprehensively valid theory would have to take all these factors fully into account.

Recently, a nonlinear unsteady wing theory has been introduced by Wu [1]-[3] along this approach to provide an optimally unified analytical and numerical method for computation of solutions on specific premises. This nonlinear theory has been applied by Stredie (2004)[6] and Hou et al. (2006)[7,8] to perform computations of a number of unsteady motions of bodies shedding vortex sheet(s), attaining results of high accuracy (as measured versus relative errors and experiments available) in all the cases pursued. The present work is a continuation to this series of studies, here addressing further on the general issue of arbitrary changes in wing shape and trajectory along the line discussed by Wu[2,3] with intent to optimize the analytical and computational efforts required for attaining exact solutions efficiently.

2. Wing movement with arbitrary changes in shape and trajectory

We first recapitulate the nonlinear theory[1-3] of a two-dimensional arbitrary flexible lifting surface for modeling aquatic and aerial animal locomotion at high Reynolds number. We opt two-dimensional theory for its simplicity to provide a foundation for further development of unsteady wing theory and for general applications.

In this respect, we find that of the existing linear theories, the simple and clear physical concept crystallized by von Kármán and Sears[4] in providing such a general view on an ingenious restructuring of the vorticity distribution over the wing and its trailing wake is readily found to afford powerful generalizations. So it has been extended by Wu (2001, Sect. 6)[1] to account fully for all possible nonlinear effects in theory, and bring Herbert Wagner's pioneering work (1925)[5] to more general applications. The principal step is to employ a joint Eulerian and Lagrangian description of the lifting-surface movement for the formulation and analysis which we will delineate synoptically next. This useful description of unsteady bodily movement has also been applied by Lighthill [9] to develop a large-amplitude elongated-body theory.

[Figure 1 here]

Thus, we consider the irrotational flow of an incompressible and inviscid fluid generated by a two-dimensional flexible lifting surface $S_b(t)$ of negligible thickness, moving with time t through the fluid in arbitrary manner.

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Its motion can be described parametrically by using a Lagrangian coordinate system (ξ, η) to identify a point $X(\xi,t), Y(\xi,t)$ on the boundary surface $S(t) = S_b(t) + S_w(t)$ comprising the body surface S_b and a wake surface S_w, with $S(t)$ lying at time $t = 0$ over a stretch of the ξ-axis (at $\eta = 0$) and moving with time $t(\geq 0)$ as can be prescribed by $z = x + iy = Z(\xi, t)$ (see Fig. 1),

$$Z(\xi,t) = X(\xi,t) + iY(\xi,t) \quad \text{on} \quad S_b(t) + S_w(t), \tag{1}$$

where $S_b(t) : (-1 < \xi < 1)$ and $S_w(t) : (1 < \xi < \xi_m)$, parametrically in ξ, with $\xi = -1$ marking the leading edge and $\xi = 1$ the trailing edge of the wing, from the latter of which a vortex sheet is assumed being shed smoothly (i.e. under the Kutta condition) to form a prolonging wake S_w, and ξ_m identifies the path $Z(\xi_m, t)$ of the starting vortex shed at $t = 0$ to reach $\xi_m = \xi_m(t)$ at time t. A simple choice for $(\xi + i\eta)$ is the initial material position of $S_b(t = 0)$, taken to be in its stretched-straight shape such that $Z(\xi, 0) = \xi \ \ (-1 < \xi < 1, \ \eta = 0)$, lying in an unbounded fluid initially at rest in an inertial frame of reference (see Figure 1). The flexible $S_b(t)$ is assumed to be inextensible ($|Z_\xi| \equiv |\partial Z/\partial \xi| = 1, \ |\xi| < 1$) and the point ξ on $S_b(t)$ moves with a *prescribed* (complex) velocity $W(\xi,t) = U - iV$,

$$W(\xi,t) = U - iV = \partial \overline{Z}/\partial t = X_t - iY_t \qquad (|\xi| < 1, \ t \geq 0; \ \overline{Z} = X - iY), \tag{2}$$

which has a tangential component, $U_s(\xi, t)$, and a normal component, $U_n(\xi, t)$, given by

$$W \partial Z/\partial \xi = (X_\xi X_t + Y_\xi Y_t) - i(X_\xi Y_t - Y_\xi X_t) = U_s - iU_n, \tag{3}$$

and with the same expression for the wake surface $S_w(t)$ for $(1 < \xi < \xi_m)$.

In the spirit of von Kármán and Sears, we adopt for $t > 0$ the following vorticity distribution:

on $S_b(t)$: $\quad \gamma(\xi,t) = \gamma_0(\xi,t) + \gamma_1(\xi,t) \qquad (-1 < \xi < 1)$,
on $S_w(t)$: $\quad \gamma(\xi,t) = \gamma_w(\xi,t) \qquad (1 < \xi < \xi_m)$,

where $\gamma_0(\xi,t)$ is the bound vortex distributed over S_b representing the "quasi-steady" flow past S_b such that the time t in the original prescribed $W(\xi,t)$ is frozen to serve merely as a parameter in evaluating the quasi-steady γ_0 (by steady airfoil theory), and $\gamma_1(\xi,t)$ is the additional bound vortex induced on S_b by the trailing wake vortices $\gamma_w(\xi,t)$ such that γ_1 and γ_w jointly will bear no change to U_n (but not so to U_s) over S_b so as to reinstate the original time-varying normal velocity $U_n(\xi,t)$ on $S_b(t)$.

Thus, we represent the velocity field by a vorticity distribution, $\gamma(\xi,t)$, per unit length spanwise over the body and wake surfaces to give at time t the complex velocity $w(z,t) = u - iv$ of the fluid at a field point z as

$$w(z,t) = \frac{1}{2\pi i} \int_{-1}^{\xi_m} \frac{\gamma(\xi,t)}{Z(\xi,t) - z} \, d\xi \qquad (z = x + iy \notin S, \ t \geq 0). \tag{4}$$

Applying Plemelj's formula to (4) yields for $w^\pm = \lim \ w(z(\xi + i\eta), t)$ as $\eta \to \pm 0$ on the two sides of S as,

$$u_s^\pm - iu_n^\pm = w^\pm(\xi,t)\frac{dZ}{d\xi} = \pm\frac{1}{2}\gamma(\xi,t) + \frac{1}{2\pi i}\frac{dZ}{d\xi}\int_S \frac{\gamma(\xi',t)}{Z' - Z} \, d\xi', \tag{5}$$

with $Z = Z(\xi,t), Z' = Z(\xi',t)$ both on $S = S_b + S_w$. From (5) we have $\gamma(\xi,t) = (u_s^+ - u_s^-)$, and

$$u_n^+(\xi,t) = u_n^-(\xi,t) \quad = \quad Re\left\{\frac{1}{2\pi}\frac{dZ}{d\xi}\int_S \frac{\gamma(\xi',t)}{Z' - Z} \, d\xi'\right\}, \tag{6}$$

$$u_{sm} \equiv \frac{1}{2}(u_s^+ + u_s^-) \quad = \quad Im\left\{\frac{1}{2\pi}\frac{dZ}{d\xi}\int_S \frac{\gamma(\xi',t)}{Z' - Z} \, d\xi'\right\}. \tag{7}$$

Here, (6) shows the continuity of normal velocity $u_n^+ = u_n^- = u_n$ across S and (7) gives the algebraic mean of tangential velocity u_s on S. From (6)-(7) we deduce the contributions separately made by γ_0, γ_1, and γ_w as:

$$U_n(\xi,t) \quad = \quad Re\left\{\frac{1}{2\pi}\frac{dZ}{d\xi}\int_{-1}^{1} \frac{\gamma_0(\xi',t)}{Z' - Z} \, d\xi'\right\} \qquad (Z = Z(\xi,t) \in S_b), \tag{8}$$

$$U_{1n}(\xi,t) \quad = \quad Re\left\{\frac{1}{2\pi}\frac{dZ}{d\xi}\int_{1}^{\xi_m} \frac{\gamma_w(\xi',t)}{Z' - Z} \, d\xi'\right\} \qquad (Z = Z(\xi,t) \in S_b), \tag{9}$$

$$-U_{1n}(\xi,t) = Re\left\{\frac{1}{2\pi}\frac{dZ}{d\xi}\int_{-1}^{1}\frac{\gamma_1(\xi',t)}{Z'-Z}\,d\xi'\right\} \qquad (Z=Z(\xi,t)\in S_b), \tag{10}$$

$$W_w(\xi,t) = \frac{1}{2\pi i}\frac{dZ}{d\xi}\int_{S_b+S_w}\frac{\gamma(\xi',t)}{Z'-Z}\,d\xi' \qquad (Z=Z(\xi,t)\in S_w), \tag{11}$$

where $W_w(\xi,t)=U_{ws}-iU_{wn}$ is the (complex) flow velocity on the wake.

The problem can now be recast to delineate the course for solution as follows. Equation (8) results from invoking condition that $u_n(\xi,t)=U_n(\xi,t)$, which is given at S_b, to give an integral equation for γ_0 which is to be solved, with time t frozen and without any unsteady wake, by applying steady airfoil theory. The velocity induced on S_b by wake vorticity γ_w (while being transported with velocity W_w of the fluid particles on the wake) has the normal component U_{1n} given by (9), which is canceled out as is required of γ_1 on S_b according to (10) so that the sum (9)+(10) gives an integral equation for γ_1 in terms of γ_w. This solution for γ_1, which is to be determined under the Kutta condition (on the continuity of vorticity at the trailing edge) may be expressed, in principle, *symbolically* with a kernel $K(\xi';\xi,t)$ in the form

$$\gamma_1(\xi,t)=\int_1^{\xi_m}K(\xi';\xi,t)\gamma_w(\xi',t)\,d\xi' \qquad (|\xi|\le 1). \tag{12}$$

Finally, we apply Kelvin's theorem that the total circulation around S_b+S_w must vanish $\forall t\ge 0$, i.e. $\Gamma_0+\Gamma_1+\Gamma_w=\int_{S_b}(\gamma_0+\gamma_1)\,d\xi+\int_{S_w}\gamma_w\,d\xi=0$ (if it is zero initially), or, symbolically,

$$\Gamma_0+\int_1^{\xi_m}\left\{1+\int_{-1}^{1}K(\xi';\xi,t)\,d\xi\right\}\gamma_w(\xi',t)\,d\xi'=0. \tag{13}$$

This is in essence the desired form of "generalized Wagner's integral equation" for wake vorticity γ_w. Its original linear version has been attained by Wagner[3] and shown by him and by von Kármán and Sears[2] to play a key role in providing accurate solutions for the entire vorticity distributions and hence for the final solution to the linearized problem.

For the present nonlinear theory, it is of interest to derive the kernel $K(\xi,\xi',t)$ in closed form for efficient applications to wing movement in arbitrary manner. Such a desired integral equation has been first explicitly given by Wu[1], however in a rather lengthy series form by perturbation expansion. Another attempt has been made by Wu[3] to obtain an integral equation for γ_w which can be resolved efficiently by iteration, however with the body shape function still imbedded with a linear approximation. The present work attempts to achieve the theory fully generalized by including all possible nonlinear effects exactly.

3. A unified method of solution

Here, the method proposed by Wu[1-3] based on a unified analytical-and-numerical scheme is further pursued to completion. Thus, following Wu[3], we first rewrite (8) as

$$U_n(\xi,t) = \frac{1}{2\pi}\int_{-1}^{1}\{1+g(\xi',\xi,t)\}\frac{\gamma_0(\xi',t)}{\xi'-\xi}\,d\xi',$$

$$g(\xi',\xi,t) = Re\left\{\frac{dZ}{d\xi}\frac{\xi'-\xi}{Z'-Z}\right\}-1 \qquad (\forall(\xi,\xi')\in S_b). \tag{14}$$

As has been noted, if S_b is a flat wing, held at an arbitrary angle θ with the x-axis, we have

$$Z(\xi)-Z(\xi')=e^{i\theta(t)}(\xi-\xi') \qquad \longrightarrow \qquad g(\xi',\xi,t)=0, \tag{15}$$

valid for arbitrary movement of the flat wing. For wing with small camber, as is usually seen, $g(\xi',\xi,t)$ is found to be regular and quadratic in the camber (see (19)). We can therefore call $g(\xi',\xi,t)$ the *residual kernel*, and its integral, the *residual integral*, which is of a form apt for iteration with rapid convergence. This is the principle we shall follow in this unified approach.

[Figure 2 here]

For the body shape, S_b can always assume a *shape function* $Z(\xi,t)$ and a general *camber function* $\hat{Z}(\xi,t) \in C^1\ \forall\ \xi[-1,1]$ pertaining to the 'frame of reference' of the deformable wing so that $Z(\xi,t)$ can be prescribed as

$$\begin{aligned} Z(\xi,t) &= Z_0(t) + e^{i\theta}\ \hat{Z}(\xi,t) \qquad\qquad (-1 \le \xi \le 1), \\ \hat{Z}(\xi,t) &= \hat{X}(\xi,t) + i\hat{Y}(\xi,t) = \hat{X}(\xi,t) + iF(\hat{X}(\xi,t),t), \end{aligned} \tag{16}$$

(see Fig. 2). Here, $Z_0(t)$ is a reference point to be chosen as the origin of $\hat{Z}(\xi,t)$, $\theta(t)$ is the slope angle measured from the x-axis to the wing chord which passes through the leading edge at $Z(-1,t)$ and the trailing edge at $Z(+1,t)$ of the wing, and $\hat{Y}(\xi,t) = F(\hat{X}(\xi,t),t)$ stands for the real camber function, assumed regular. For convenience, we choose $Z_0(t)$ to be

$$Z_0(t) = Z(0,t) - ie^{i\theta}\ \hat{Y}(0,t), \tag{17}$$

which is the projection of the wing center point $Z(0,t)$ onto the chord, which is at the mid-chord if the wing has the fore-and-aft symmetry ($\hat{X}(\xi,t)$ being odd, $\hat{Y}(\xi,t)$ even in ξ). In the wing frame, the leading edge is at $\hat{Z}(-1,t) = -a(t)$, and the trailing edge at $\hat{Z}(+1,t) = b(t)$, ordinarily with $0 < a, b \le 1$, and $a = b$ for symmetric wings.

Next, for the inextensibility condition on the wing arc, we invoke $|\partial Z/\partial\xi| = |\partial\hat{Z}/\partial\xi| = 1$, giving

$$\left\{1 + \left(\frac{\partial F}{\partial \hat{X}}\right)^2\right\}^{1/2} \left|\frac{\partial \hat{X}}{\partial \xi}\right| = 1.$$

For $\partial\hat{X}/\partial\xi > 0$, which is ordinarily the case, we have

$$\xi = \int_0^{\hat{X}} \left\{1 + \left(\frac{\partial F}{\partial \hat{X}}\right)^2\right\}^{1/2} d\hat{X} \qquad (-1 < \xi < 1), \tag{18}$$

from which follows $\hat{X}(\xi,t)$ by quadrature and inversion, and $\hat{Y}(\xi,t) = F(\hat{X}(\xi,t),t)$. This relation of providing $\hat{X}(\xi,t), \hat{Y}(\xi,t)$ is of importance to applying the boundary conditions exactly. In some particular cases with certain symmetry, e.g. a circular arc wing, suitable parametric representations of the camber function may prevail for possible simplification of the analysis and computations involved.

With $\hat{X}(\xi,t), \hat{Y}(\xi,t)$ so determined, the residual kernel becomes

$$g(\xi,\xi',t) = \frac{\delta\hat{X}\Delta\hat{X} + \delta\hat{Y}\Delta\hat{Y}}{(\Delta\hat{X})^2 + (\Delta\hat{X})^2} - 1 \qquad (\delta\hat{X} \equiv \frac{\partial\hat{X}}{\partial\xi},\ \ \Delta\hat{X} \equiv \frac{\hat{X}(\xi,t) - \hat{X}(\xi',t)}{\xi - \xi'}), \tag{19}$$

and similarly for $\delta\hat{Y}$ and $\Delta\hat{Y}$, $(-1 \le \xi, \xi' \le 1)$. For a flat wing, $\hat{X}(\xi,t) = \xi,\ \hat{Y}(\xi,t) = 0$, hence $g(\xi,\xi',t) = 0$. For wings of small camber, (19) shows g being quadratic in the camber.

For given $Z(\xi,t)$ of $S_b(t)$, its surface (complex) velocity is, with using (16), given by

$$W_b(\xi,t) = \frac{\partial}{\partial t}\overline{Z}(\xi,t) = \frac{\partial}{\partial t}(X - iY) = [(U_0 - iV_0) + i\Omega(\hat{X} - i\hat{Y}) + (\hat{X}_t - i\hat{Y}_t)]e^{-i\theta}. \tag{20}$$

This W_b prescribes the wing movement consisting in general of a translation with velocity $dZ_0/dt = (U_0 + iV_0)\exp i\theta$, a rotation of the wing chord about Z_0 with clockwise angular velocity $\Omega = -d\theta/dt$ (+ive for nose-up by convention), and a camber variation at the rate $(\hat{X}_t + i\hat{Y}_t)]\exp(i\theta)$. This wing surface velocity has its tangential component $U_s(\xi,t)$ and normal component $U_n(\xi,t)$ given by $W_b\partial\overline{Z}/\partial\xi = U_s - iU_n$ (see (3)), where

$$U_s(\xi,t) = (U_0 + \Omega\hat{Y} + \hat{X}_t)\hat{X}_\xi + (V_0 - \Omega\hat{X} + \hat{Y}_t)\hat{Y}_\xi, \tag{21}$$

$$U_n(\xi,t) = (V_0 - \Omega\hat{X} + \hat{Y}_t)\hat{X}_\xi + (U_0 + \Omega\hat{Y} + \hat{X}_t)\hat{Y}_\xi. \tag{22}$$

The normal component U_n will provide the kinematic flow condition (8) at S_b, and the tangential component U_s may be used to verify the wing being inextensible, if needed. Of course, with body motion (16) given, the surface velocity is completely determined.

After substituting the exact expression (22) for $U_n(\xi,t)$ in integral equation (14) for $\gamma_0(\xi,t)$, the leading term with the Cauchy kernel can be inverted by steady airfoil theory[10], i.e.

$$U_n(\xi,t) = \frac{1}{2\pi}\int_{-1}^{1}\frac{\gamma_0(\xi',t)}{\xi'-\xi}\,d\xi' \equiv G_0\gamma_0, \quad \longrightarrow \quad \gamma_0(\xi,t) = G_0^{-1}U_n \quad (|\xi|<1), \tag{23}$$

$$\gamma_0(\xi,t) = -\frac{2}{\pi}\sqrt{\frac{1-\xi}{1+\xi}}\int_{-1}^{1}\sqrt{\frac{1+\xi'}{1-\xi'}}\frac{U_n(\xi',t)}{\xi'-\xi}\,d\xi' \equiv G_0^{-1}U_n, \tag{24}$$

where G_0 denotes the integral operator and G_0^{-1} its inverse (i.e. $G_0^{-1}G_0 = 1$) as designated. Applying this inversion to (14) in its entirety yields the following reduced integral equation for γ_0 as

$$\gamma_0(\xi,t) = \gamma_{00}(\xi,t) + H\gamma_0, \qquad \gamma_{00}(\xi,t) = G_0^{-1}U_n,$$

$$H\gamma_0 \equiv \int_{-1}^{1}\gamma_0(\xi',t)h(\xi',\xi,t)d\xi', \qquad h(\xi',\xi,t) = \frac{1}{\pi^2}\sqrt{\frac{1-\xi}{1+\xi}}\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\frac{g(\xi',\zeta,t)\,d\zeta}{(\xi'-\zeta)(\zeta-\xi)}. \tag{25}$$

The above expression for $h(\xi',\xi,t)$ arrived at with interchanging the order of integration is justified by the Poincare-Bertrand formula[10]. For a flat wing, the solution for γ_0 terminates with $H\gamma_0 = 0$, due to $g = 0$ by (15). For cambered wings, γ_0 can be readily solved by iteration, either numerically or analytically using $\gamma_0^{(k)}(\xi,t) = \gamma_{00}(\xi,t) + H\gamma_0^{(k-1)}$, $(k = 1,2,\cdots)$ under the integral operator H, with $\gamma_0^{(0)} = 0$. In fact, this iterative scheme by analysis is easily seen, by successive substitutions, to yield

$$\gamma_0(\xi,t) = (1 + H + H^2 + \cdots)\gamma_{00}(\xi,t) = (\textstyle\sum_{m=0}^{\infty} H^m)G_0^{-1}U_n,$$

which we write, for convenience, as

$$\gamma_0(\xi,t) = G_0^{-1}(1+N_0)U_n(\xi,t) \qquad (N_0 = G_0(\textstyle\sum_{m=1}^{\infty} H^m)G_0^{-1}) \tag{26}$$

in which N_0 is the nonlinear integral operator as has been determined exactly by iteration to represent the nonlinear effects due to the camber distribution which is instantaneously frozen at each time step. Being quadratic in the wing camber, N_0 vanishes for flat wing and its series expansion for cambered wings is expected to converge reasonably rapidly in general. This solution for γ_0 contributes a circulation $\Gamma_0(t)$ around the wing as

$$\Gamma_0(t) = \int_{-1}^{1}\gamma_0(\xi,t)d\xi = -2\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}(1+N_0)U_n(\xi,t)d\xi, \tag{27}$$

in which the multi-integrals have all been reduced in number by one (with G_0^{-1} integrated out), leaving the term with N_0 to give the nonlinear camber effects on $\Gamma_0(t)$.

For the wake-induced bound vortex γ_1, the complete analogy between (10) and (8) can be used to imply for γ_1 the solution which can first be written *formally* by analogy with (26) as

$$\gamma_1(\xi,t) = -G_0^{-1}(1+N_0)U_{1n}(\xi,t),$$

followed by having the unknown $U_{1n}(\xi,t)$ eliminated by applying (9) which we rewrite, like (14) for (8), as

$$U_{1n}(\xi,t) = \frac{1}{2\pi}\int_{1}^{\xi_m}\{1+g_1(\xi',\xi,t)\}\,\frac{\gamma_w(\xi',t)}{\xi'-\xi}\,d\xi', \tag{28}$$

where $g_1(\xi',\xi,t)$ has the same expression as $g(\xi',\xi,t)$ of (14) but differs from it in range by having $\xi \in S_b$ but $\xi' \in S_w$. As a result, unlike $g(\xi',\xi,t)$ being always small for S_b with a small camber, as shown by (19), $g_1(\xi',\xi,t)$ in general can be finite in magnitude, especially when S_b displaces itself by finite amount at fast rate from a straight trajectory in the space. It is in such general cases that the wake vortices can give rise to finite nonlinear effects on the flow field in addition to the local nonlinear effects due to changes in body shape according to (14).

In general, substituting (28) for U_{1n} in the former equation, we can derive for the total circulation around the wing due to γ_1, $\Gamma_1 = \int_{S_b}\gamma_1(\xi,t)d\xi$, to obtain the following result

$$\Gamma_1(t) = \int_{1}^{\xi_m}\left\{\sqrt{\frac{\xi+1}{\xi-1}} - 1 + N_w(\xi,t) + N_b(\xi,t)\right\}\gamma_w(\xi,t)\,d\xi, \tag{29}$$

$$N_w(\xi,t) = \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+\xi'}{1-\xi'}}\,\frac{g_1(\xi,\xi',t)}{\xi-\xi'}\,d\xi', \tag{30}$$

$$N_b(\xi,t) = \frac{1}{2\pi}\int_{-1}^{1}\sqrt{\frac{1+\xi'}{1-\xi'}}\,N_0(\xi',t)\,\frac{1+g_1(\xi,\xi',t)}{\xi-\xi'}\,d\xi'. \tag{31}$$

Finally, we apply Kelvin's theorem as we have expounded for (13) to obtain for γ_w the equation

$$\Gamma_0(t) + \int_1^{\xi_m}\left\{\sqrt{\frac{\xi+1}{\xi-1}} + N_w(\xi,t) + N_b(\xi,t)\right\}\gamma_w(\xi)\;d\xi = 0. \tag{32}$$

This is the *nonlinear wake-vorticity integral equation* for wake vorticity γ_w in closed form. It generalizes Wagner's integral equation for linear case to fully account for a flexible wing in arbitrary movement. Of the different terms in this equation, $\Gamma_0(t)$ has a component in its integral with kernel $N_0(\xi,t)$ representing a local nonlinear effect on the flow due to changes in body shape. In the wake integral, the term with $N_w(\xi,t)$ represents the nonlinear wake effects primarily due to non-uniformity of the wake vorticity resulting from finite changes in orientation and velocity of body movement. The other term with $N_b(\xi,t)$ represents the nonlinear effects due jointly to changes in body shape and their wake effects, since it vanishes completely for flat wing (by virtue of (15)). In the linear limit, N_w and N_b both vanish, reducing (32) to Wagner's integral equation. The foregoing is a simple unified derivation of Wu's results[1-3], here brought to completion with the new comprehensive exact representations provided for arbitrary wing movement.

In computation, it is convenient to start with the motion of S_b prescribed for $t \geq 0$. In a small time interval δt_k at $t = t_k > 0$ $(k = 1, 2, \cdots)$, a new segment of S_w is created (due to body moving forward) in the wake just beyond the trailing edge (at $\xi = 1$), namely $\delta z(1,t) = \overline{W}(1,t)\delta t_k$. The wake vorticity shed into this small segment of S_w can be obtained, by analysis and numerics, *accurately* by applying (32). Once the local γ_w of that fluid particle (leaving the trailing edge at $t = t_k$) is determined, its value will remain invariant with the particle, by Helmholtz's theorem, and move on with wake fluid at velocity $W_w(\xi,t)$ of (11) for $t > t_k$ $(k = 1, 2, \cdots)$. Therefore, the key step in using the nonlinear wake-vorticity integral equation (32) is at the initial time step in which the starting vortex is shed from the trailing edge and at each successive time steps when a new vortex element is shed continuously into the wake.

In conclusion, we have addressed all the issues concerning the generation of entire vortex distribution over a flexible wing moving in arbitrary manner, with all the various nonlinear effects identified for general applications to self-propulsion and related future studies. The final exact form (32) is based on series expansion of the residual term to all orders in camber, its rapid convergence is expected (primarily due to the smallness of the kernel g as stressed by Wu[3]) and can be easily assessed in practice for the contributions from consecutive orders to finding the nonlinear effects accurately in increasing orders, which should be straightforward by computation. These nonlinear effects are expected to play an active and important role in aerial and aquatic animal locomotion.

Acknowledgment. I am most appreciative for the gracious encouragement from Dr. Chinhua S. Wu and the American-Chinese Scholarship Foundation.

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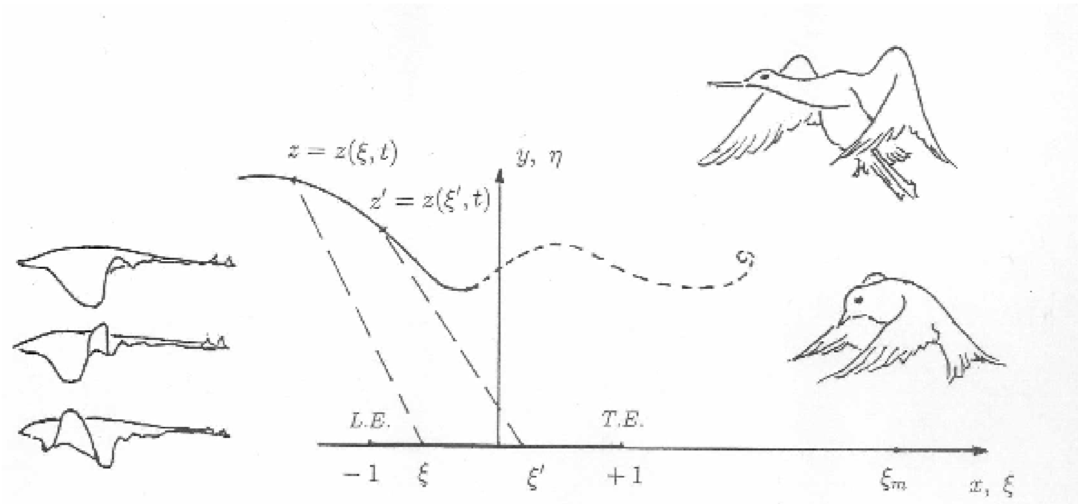

Figure 1: The Lagrangian coordinates (ξ, η) adopted to describe arbitrary motion of a two-dimensional flexible lifting surface moving along arbitrary trajectory through fluid in an inertial frame fixed with the fluid at infinity.

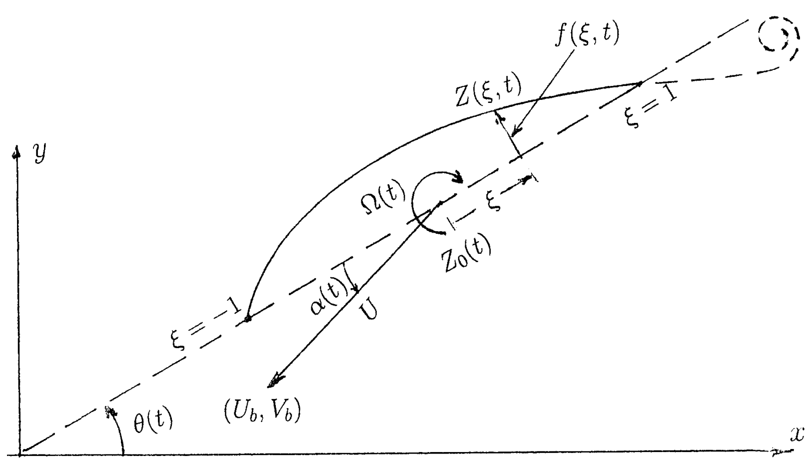

Figure 2: The wing movement consists of (i) rectilinear translation with velocity (U_b, V_b) at incidence angle $\alpha(t)$, (ii) rotation with angular velocity $\Omega(t)$, and (iii) unsteady camber function $\hat{Z}(\xi, t) = \hat{X} + i\hat{Y}, \hat{Y} = F(\hat{X}, t)$, shown as $\xi + if(\xi, t)$.