CONVERGENCE, UNANIMITY AND DISAGREEMENT IN MAJORITY DYNAMICS ON UNIMODULAR GRAPHS AND RANDOM GRAPHS

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ABSTRACT. In majority dynamics, agents located at the vertices of an undirected simple graph update their binary opinions synchronously by adopting those of the majority of their neighbors.

On infinite unimodular transitive graphs (e.g., Cayley graphs), when initial opinions are chosen from a distribution that is invariant with respect to the graph automorphism group, we show that the opinion of each agent almost surely either converges, or else eventually oscillates with period two; this is known to hold for finite graphs, but not for all infinite graphs.

On Erdős-Rényi random graphs with degrees $\Omega(\sqrt{n})$, we show that when initial opinions are chosen i.i.d. then agents all converge to the initial majority opinion, with constant probability. Conversely, on random 4-regular finite graphs, we show that with high probability different agents converge to different opinions.

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1. Introduction

Let $G = (V, E)$ be a finite or countably infinite, locally finite, undirected simple graph. Consider time periods $t \in \{0, 1, 2, \ldots\}$ and, for each time t and $i \in V$, let $X_t(i) \in \{-1, +1\}$ be the opinion of vertex i at time t.

We define *majority dynamics* by

$$(1.1) \qquad X_{t+1}(i) = \operatorname{sgn} \sum_{j \in \partial(i)} X_t(j),$$

where $\partial(i)$ is the set of neighbors of i in G. To resolve (or avoid) ties, we either add or remove i from $\partial(i)$ so that $|\partial(i)|$ is odd. This ensures that the sum in the r.h.s. of (1.1) is never zero. Equivalently, we let ties be broken by reverting to the agent's existing opinion.

A well known result is the *period two property* of finite graphs, due to Goles and Olivos [8].

Theorem 1.1 (Goles and Olivos). *For every finite graph $G = (V, E)$, initial opinions $\{X_0(i)\}_{i \in V}$ and vertex i it holds that $X_{t+2}(i) = X_t(i)$ for all sufficiently large t.*

That is, every agent's opinion eventually converges, or else enters a cycle of length two.

This theorem also holds for some infinite graphs [7, 11]; in particular for those of bounded degree and subexponential growth, or slow enough exponential growth. In [15] it is furthermore shown that on graphs of maximum degree d the number of times t for which $X_{t+2}(i) \neq X_t(i)$ is at most

$$\frac{d+1}{d-1} \cdot d \cdot \sum_{r=0}^{\infty} \left(\frac{d+1}{d-1} \right)^{-r} n_r(G, i),$$

where $n_r(G, i)$ is the number of vertices at graph distance r from i in G.

However, on some infinite graphs there exist initial configurations of the opinions such that no agent's opinion converges to any period; this is easy to construct on regular trees. A natural question is whether such configurations are "rare", in the sense that they appear with probability zero for some natural probability distribution on the initial configurations. In [9] it was shown that on a regular trees, when initial opinions are chosen i.i.d. with sufficient bias towards $+1$, then all opinions converge to $+1$ with probability one. It was shown also that this is not the case in some odd degree regular trees, when the bias is sufficiently small. However, the question of whether opinions converge at all when the bias is small was not addressed.

We show that indeed opinions almost surely converge (or enter a cycle with period two) on regular trees, whenever the initial configuration is chosen i.i.d. In fact, we prove a much more general result.

A *graph isomorphism* between graphs $G = (V, E)$ and $G' = (V', E')$ is a bijection $h : V \to V'$ such that $(i, j) \in E$ iff $(h(i), h(j)) \in E'$. Intuitively, two graphs are isomorphic if they are equal, up to a renaming of the vertices.

The *automorphism group* $\mathrm{Aut}(G)$ is the set of isomorphisms from G to G, equipped with the operation of composition. G is said to be *transitive* if $\mathrm{Aut}(G)$ acts transitively on V. That is, if there is a single orbit V/G, or, equivalently, if for every $i, j \in V$ there exists an $h \in \mathrm{Aut}(G)$ such that $h(i) = j$. G is said to be *unimodular* if $\mathrm{Aut}(G)$ is unimodular (see, e.g., Aldous and Lyons [1])[1]. G is unimodular if and only i the following "mass transport principle" holds: informally, in every flow on the graph that is invariant to $\mathrm{Aut}(G)$, the sum of what flows into a node is equal to the sum of what flows out. Formally, for every $F \colon V \times V \to \mathbb{R}^+$ that is invariant with respect to the diagonal action of $\mathrm{Aut}(G)$ it holds that

$$\sum_{j \in \partial(i)} f(i, j) = \sum_{j \in \partial(i)} f(j, i),$$

where $i \in V$ is arbitrary.

Many natural infinite trasitive graphs are unimodular. These include all Cayley graphs, all transitive amenable graphs, and, for example, transitive planar graphs with one end [10].

Our first result is the following.

Theorem 1 (The almost sure period two property for unimodular transitive graphs)**.** *Let G be a unimodular transitive graph, and let the agents' initial opinions $\{X_0(i)\}_{i \in V}$ be chosen from a distribution that is $\mathrm{Aut}(G)$-invariant. Then, under majority dynamics,*

$$(1.2) \qquad \mathbb{P}\left[\lim_t X_{t+2}(i) - X_t(i) = 0\right] = 1,$$

and furthermore

$$\mathbb{E}\left[\#\{t \,:\, X_{t+2}(i) \neq X_t(i)\}\right] \leq 2d,$$

where d is the degree of G.

That is, each node's opinion almost surely converges to a cycle of period at most two.

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[1]See [1] for an example (the "grandfather graph") of a transitive graph that is not unimodular.

In fact, this result is a special case of our Theorem 4 below, which applies to *unimodular random networks*. These include many natural random graphs such as invariant percolation clusters, uniform infinite planar triangulations [4] and any limit of finite graphs, in the sense of [5]; see Section 2 for a formal definition. In fact, this is such a large family that one may guess that any graph has what we call the *almost sure period two property*: if initial opinions are chosen i.i.d. from the uniform distribution over $\{-1, -1\}$, then each node's opinion almost surely converges to a cycle of period at most two. This, however, is not true, as we show in the next example.

Example 1.2. *There exists an infinite graph G that does not have the almost sure period two property.*

As a reading of the details of this example will reveal, this graph is not of bounded degree. We conjecture that

Conjecture 1.3. *Every bounded degree graph has the almost sure period two property.*

We next consider the process of majority dynamics on a random finite graph, where initial opinions $\{X_0(i)\}_{i \in V}$ are chosen i.i.d. from the uniform distribution over $\{-1, +1\}$. Here convergence to period two is guaranteed by the Goles-Olivos Theorem. The question we tackle is whether agents all converge to the *same* opinion.

The Erdős-Rényi graph $G(n, p)$ is the distribution over graphs with n vertices in which each edge exists independently with probability p. A random regular graph $R(n, d)$ is the uniform distribution over all d-regular connected graphs with n vertices.

We first study $G(n, p_n)$, where $p_n = \Omega(\sqrt{n})$. Following the usual convention, we say that an event happens *with high probability* when it happens with probability that tends to one as n tends to infinity. Let $\mu_0 = \text{avg}_{i \in V}\{X_0(i)\}$.

Theorem 2 (Unanimity on high degree Erdős-Rényi graphs). *Assume $n \geq n_0$ and $p \geq cn^{-1/2}$, where $n_0, c > 0$ are sufficiently large universal constants. Then with probability at least .4 over the choice of $G \sim G(n, p)$ and the initial opinions, the vertices unanimously hold opinion $\text{sgn}(\mu_0)$ at time 4.*

Next, we consider $R(n, d)$, with $d = 4$. In this setting we prove the following result. We say that *unanimity* is reached at time t when $X_t(i) = X_t(j)$ for all $i, j \in V$.

Theorem 3 (Disagreement on random regular low degree graphs). *Let G_n be drawn from $R(n, 4)$, or be any sequence of 4-regular expanders*

with growing girth. Choose the initial opinions independently with probability $1/3 < p < 2/3$. Then, with high probability, unanimity is not reached at any time.

The following result on finite graphs is an immediate corollary of Theorem 4, which is a statement on infinite graphs.

Corollary 1.4. *Let G be drawn from $R(n, d)$ with $d \geq 3$, or from $G(n, d/n)$ with $d > 1$.*

Then for every $\varepsilon > 0$ there exists a time t such that, with high probability, $X_{t+2}(i) = X_t(i)$ for all $i \in V$ except a set of size $\varepsilon \cdot |V|$. Furthermore, at this time t, the fraction of nodes for which $X_t(i) = 1$ is, with high probability, in $[1/2 - \varepsilon, 1/2 + \varepsilon]$.

Hence at some time t almost all nodes will have already reached period at most two (at least temporarily), and without having reached agreement. This, together with the results above, motivates the following conjecture.

Conjecture 1.5. *Let G be drawn from $G(n, d_n/n)$.*

- *When d_n is a bounded, then for every $\varepsilon > 0$, with high probability, the fraction of nodes for which $\lim_t X_{2t}(i) = +1$ will be in $[1/2 - \varepsilon, 1/2 + \varepsilon]$.*
- *When $d_n \to \infty$, then for every $\varepsilon > 0$, with high probability, the fraction of nodes for which $\lim_t X_{2t}(i) = +1$ will be in $[0, \varepsilon] \cup [1 - \varepsilon, 1]$.*

That is, stark disagreement is reached for constant degrees, and unanimity is reached for super-constant degrees. An alternative, equally reasonable conjecture stipulates that this phase transition occurs, in fact, when degrees become high enough so that locally the graph ceases to resemble a tree.

Given a vertex i in a large finite transitive graph and random uniform initial opinions, consider the Boolean function which is the eventual opinion of the majority dynamics at i, say at even times. An interesting question is whether this function is local; that is, is it determined with high probability by the initial opinions in a bounded neighbourhood of i? If it is non-local, can it be *noise-sensitive* [14] or it is correlated with the majority of the initial opinions? Our results so far heuristically suggest that in the bounded degree regime, majority dynamics is local, while when the degrees are growing fast enough the majority of the initial opinions determines the final outcome. In this respect we still did not find (or even conjecturally suggest) a family of graphs in which more interesting global behaviour occurs, such as in noise-sensitive Boolean functions. Indeed, we are curious to know if such a family exists.

2. The almost sure period two property

In this section we shall consider *generalized majority dynamics*, or *weighted* majority dynamics. In this case we fix a function $w : E \to \mathbb{R}^+$ and let

$$(2.1) \qquad X_{t+1}(i) = \operatorname{sgn} \sum_{j \in \partial(i)} X_t(j) \cdot w(i, j).$$

Note that $w(i, j) = w(j, i)$, since w is a function of the (undirected) edges. Note also that $w(i, i)$ is possibly positive. We here too assume that w is chosen so that the sum in the r.h.s. can never be zero.

A *network* is a triplet $N = (G, w, X)$, where $G = (V, E)$ is a graph as above, $X : V \to \{-1, +1\}$ is a labeling of the nodes, and $w : E \to \mathbb{R}^+$ is a weighting of the edges.

In the context of networks, we think of the process of generalized majority dynamics as a sequence of networks $\{N_t\}$, which all share the same graph $G_t = G = (V, E)$ and edge weights $w_t = w$, and where the node labels X_t are updated by (2.1).

A *rooted network* is a pair (N, i) with N a network and $i \in V$. An isomorphism between two rooted networks (N, i) and (N', i') is a graph isomorphism h between G and G' such that $h(i) = i'$, $X = X' \circ h$ and $w = w' \circ h$, where we here extend h to a bijection from E to E'. A *directed edge rooted network* is a triplet (N, i, j) with $(i, j) \in E$. Isomorphisms of directed edge rooted networks are defined similarly to those of rooted networks.

A rooted network isomorphism class $[N, i]$ is the set of rooted graphs isomorphic to (N, i). The set of connected, rooted network isomorphism classes, which we shall denote by $\mathcal{G}_\bullet$, is equipped with the natural topology of convergence of finite balls around the root (see [2, 5]). This topology provides a Borel structure for probability measures on this space.

A *random network*, or, more precisely, a *random rooted network isomorphism class* (we shall use the former term), is a rooted-network-isomorphism-class-valued random variable $[N, I]$; its distribution is a measure on $\mathcal{G}_\bullet$. Denote by $\mathcal{G}_{\bullet\bullet}$ the space of isomorphism classes of directed edge rooted networks $[N, i, j]$. $[N, I]$ is a *unimodular random*

network if, for every Borel $f : \mathcal{G}_{\bullet\bullet} \to [0, \infty]$, it holds that

$$(2.2) \qquad \mathbb{E}\left[\sum_{j \in \partial(I)} f(N, I, j)\right] = \mathbb{E}\left[\sum_{j \in \partial(I)} f(N, j, I)\right].$$

We direct the reader to Aldous and Lyons [1] for an excellent discussion of this definition.

Let $\{[N_t, I]\}_{t \in \mathbb{N}}$ be a sequence of random networks defined as follows. Fix some random network $[N_0, I] = [G, w, X_0, I]$. For $t > 0$, let $[N_t, I] = [G, w, X_t, I]$, where

$$(2.3) \qquad X_t(i) = \operatorname{sgn} \sum_{j \in \partial(i)} X_{t-1}(j) \cdot w(i, j).$$

This sequence of random networks is coupled to share the same (random) graph, weights and root; only the labeling of the nodes X_t changes with time. We say that such a sequence is *related by generalized majority dynamics*. We impose the condition that w is such that almost surely no ties occur (i.e., the sum in (2.3) is nonzero).

Claim 2.1. *If $[N_0, I]$ is a unimodular random network then so is $[N_t, I]$, for all $t \in \mathbb{N}$.*

This follows immediately from the fact that the majority dynamics map $(G, w, X_{t-1}) \mapsto X_t(i)$ given by (2.3) is indeed a function of the rooted network isomorphism class $[N_{t-1}, i] \in \mathcal{G}_{\bullet}$.

For $W, \varepsilon > 0$ we say that (the weights w of) a random network $[N, I]$ is (ε, W)-*regular* if the following two conditions hold. First, we require that

$$\mathbb{E}\left[\sum_{j \in \partial(I)} w(I, j)\right] \leq W.$$

Note that in the case that w is the constant function one, this is equivalent to having finite expected degree. Next, we require that

$$\min_{x \in \{-1, +1\}^{\partial(I)}} \left|\sum_{j \in \partial(I)} X_t(j) w(i, j)\right| \geq \varepsilon$$

almost surely. This is an "ellipticity" condition that translates to requiring that one is always ε-far from a tie. In the case that w is the constant function one and degrees are odd, this holds with $\varepsilon = 1$.

We are now ready to state our main result of this section, which is a generalization of Theorem 1 from a fixed unimodular graph setting to a unimodular random network setting.

Theorem 4. *Let* $\{[N_t, I]\}$ *be a sequence of* (ε, W)-*regular, unimodular random networks related by majority dynamics. Then*

(2.4) $$\mathbb{P}\left[\lim_t X_{t+2}(I) - X_t(I) = 0\right] = 1,$$

and furthermore

$$\mathbb{E}\left[\#\{t \,:\, X_{t+2}(I) \neq X_t(I)\}\right] \leq \frac{2W}{\varepsilon}.$$

Before proving this theorem, we show that it implies Theorem 1. When the underlying graph of a random network is a fixed transitive unimodular graph, and when the distribution of the labels $X_t(i)$ is invariant to the automorphism group of this graph, then this random network is a unimodular random network [1]. Furthermore, since majority dynamics is generalized majority dynamics with weights 1, this random network is $(1, d)$-regular, where d is the degree. Hence Theorem 1 follows.

2.1. **Proof of Theorem 4.** In this section we prove Theorem 4. Our proof follows the idea of the proof of the period two property for finite graphs by Goles and Olivos [8].

Let $\{[N_t, I]\}$ be a sequence of finite expected weighted degree, unimodular random networks related by majority dynamics.

Define the function $f : \mathcal{G}_{\bullet\bullet} \to [0, \infty]$ by

$$f(N, i, j) = w(i, j)\left(1 + X_j \operatorname{sgn} \sum_{k \in \partial(i)} w(i, k)X_k\right),$$

where $N = (G, w, X)$ is a network and (i, j) is an edge in G. If $[N, I]$ is unimodular then

(2.5) $$\mathbb{E}\left[\sum_{j \in \partial(I)} f(N, I, j)\right] = \mathbb{E}\left[\sum_{j \in \partial(I)} f(N, j, I)\right].$$

Note that $X_{t+1}(i) = \operatorname{sgn} \sum_{k \in \partial(i)} w(i, k)X_k$, and so

$$f(N_t, I, j) = w(I, j)\left(1 + X_{t+1}(I)X_t(j)\right)$$

and

$$f(N_t, j, I) = w(I, j)\left(1 + X_{t+1}(j)X_t(I)\right).$$

Hence we can write 2.5 for N_t as

(2.6)
$$\mathbb{E}\left[\sum_{j\in\partial(I)} w(I,j)X_{t+1}(I)X_t(j)\right] = \mathbb{E}\left[\sum_{j\in\partial(I)} w(I,j)X_{t+1}(j)X_t(I)\right].$$

Next, we define a "potential"

$$\ell_t = \frac{1}{4}\,\mathbb{E}\left[\sum_{j\in\partial(I)} w(I,j)\left(X_{t+1}(I) - X_t(j)\right)^2\right].$$

Note that ℓ_t is positive for all t, and also that it is finite for all t, since it is bounded from above by W, as a consequence of the (ε, W)-regularity of w.

We would like to show that ℓ is non-increasing. By definition,

$$\ell_t - \ell_{t-1} = -\frac{1}{2}\,\mathbb{E}\left[\sum_{j\in\partial(I)} w(I,j)X_{t+1}(I)X_t(j)\right] + \frac{1}{2}\,\mathbb{E}\left[\sum_{j\in\partial(I)} w(I,j)X_t(I)X_{t-1}(j)\right].$$

By (2.6) we can, in the expectation on the right, switch the roles of I and j. Rearranging, we get

$$\ell_t - \ell_{t-1} = -\frac{1}{2}\,\mathbb{E}\left[\left(X_{t+1}(I) - X_{t-1}(I)\right)\sum_{j\in\partial(I)} w(I,j)X_t(j)\right].$$

Now, $X_{t+1}(I) = \operatorname{sgn}\sum_{j\in\partial(I)} w(I,j)X_t(j)$, and so

$$\left(X_{t+1}(I) - X_{t-1}(I)\right)\sum_{j\in\partial(I)} w(I,j)X_t(j)$$

$$= |X_{t+1}(I) - X_{t-1}(I)| \cdot \left|\sum_{j\in\partial(I)} w(I,j)X_t(j)\right|$$

$$= \mathbb{1}_{\{X_{t+1}(I)\neq X_{t-1}(I)\}} \cdot \left|\sum_{j\in\partial(I)} w(I,j)X_t(j)\right|.$$

Hence

$$\ell_t - \ell_{t-1} = -\frac{1}{2}\,\mathbb{E}\left[\mathbb{1}_{\{X_{t+1}(I)\neq X_{t-1}(I)\}} \cdot \left|\sum_{j\in\partial(I)} w(I,j)X_t(j)\right|\right],$$

and we have shown that ℓ_t is non-increasing.

Now, by the (ε, W)-regularity of w we have that

$$\left| \sum_{j \in \partial(I)} w(I, j) X_t(j) \right| \geq \varepsilon,$$

and so

$$\ell_t - \ell_{t-1} \leq -\frac{1}{2} \, \mathbb{P}\left[X_{t+1}(I) \neq X_{t-1}(I)\right] \cdot \varepsilon.$$

Since $\ell_1 \leq W$, and since $\ell_1 \geq \sum_{t=2}^{\infty} \ell_{t-1} - \ell_t = \ell_1 - \lim_t \ell_t$, we can conclude that

$$\sum_{t=2}^{\infty} \mathbb{P}\left[X_{t+1}(I) \neq X_{t-1}(I)\right] \leq \frac{2W}{\varepsilon}.$$

Hence

$$\mathbb{E}\left[\#\{t \,:\, X_{t+2}(I) \neq X_t(I)\}\right] < \frac{2W}{\varepsilon},$$

and by the Borel-Cantelli lemma

$$\mathbb{P}\left[\lim_t X_{t+2}(I) - X_t(I) = 0\right] = 1.$$

This completes the proof of Theorem 4.

2.2. **Example 1.2: an infinite graph without the almost sure period two property.** Consider an infinite, locally finite graph defined as follows. Divide the set of nodes into "levels" $L_1, L_2, \ldots$, where level L_n has $2^n - 1$ vertices. Connect each node in L_n with each of the nodes in L_{n-1}, L_n and L_{n-1}, except for the nodes in L_0, which are connected only to L_1. It follows that

- Every pair of nodes in the same level have the same set of neighbours.
- The majority of the neighbors of $i \in L_n$ are in L_{n+1}.

Therefore, for all n and for all $i, j \in L_n$, it holds that $X_1(i) = X_1(j)$. By induction, it follows that $X_t(i) = X_t(j)$ for all $t \geq 1$, and we accordingly denote $X_t(L_n) = X_t(i)$ for some $i \in L_n$. Furthermore, $X_t(L_n) = X_{t-1}(L_{n+1})$ for $t \geq 2$, and so $X_t(L_0) = X_1(L_{t+1})$ for $t \geq 2$. Finally, $\{X_1(L_{3n})\}_{n \in \mathbb{N}}$ are independent random variables, each uniformly distributed over $\{-1, +1\}$. Hence so are the random variables $\{X_{3t-1}(L_0)\}_{t \geq 1}$, and the single node in L_0 (and in fact all the other nodes too) does not converge to period two.

3. MAJORITY DYNAMICS ON $G(n, p)$

3.1. Heuristic analysis for the high degree case.

Herein we describe a "heuristic" analysis suggesting what should happen for majority dynamics in $G(n, d_n/n)$ when $d_n = \omega(1)$ is sufficiently large. We suggest the reader keep in mind the parameter range $d_n = n^\delta$ where $0 < \delta < 1$ is an absolute constant. Our heuristic reasoning will suggest that unanimity is reached at time roughly $1/\delta + O(1)$. Unfortunately, we will only be able to make some of this reasoning precise in the case that $\delta \geq 1/2$. That case is handled formally in Section 3.2.

The *global mean* at time t is defined to be $\mu_t = \mathrm{avg}_{i \in V}\{X_t(i)\}$. To analyze convergence to unanimity we will track the progression of μ_t^2 over time. The quantity is nonnegative and it is easy to estimate it initially:

Proposition 3.1. $\mathbb{E}\left[\mu_0^2\right] = \frac{1}{n}$.

On the other hand, we also have $\mu_t^2 \leq 1$ with equality if and only if there is unanimity at time t.

We suggest the following heuristic:[2]

Heuristic 3.2. *In $G(n, d_n/n)$, assuming $d = d_n = \omega(1)$ is sufficiently large, we expect $\mu_{t+1}^2 \gtrsim d\mu_t^2$, provided $d\mu_t^2 \leq 1$.*

Granting this heuristic, we expect the sequence $\mu_0^2, \mu_1^2, \mu_2^2, \ldots, \mu_t^2$ to behave (up to constant factors) as $\frac{1}{n}, \frac{d}{n}, \frac{d^2}{n}, \ldots, \frac{d^t}{n}$ until $d^t \approx n$. Once d^t is within a constant factor of n we expect to reach near-unanimity in one more step, and to reach perfect unanimity after an additional step. For these reasons, we suggest that for $d_n = n^\delta$, one may expect convergence to unanimity after $\frac{1}{\delta} + O(1)$ steps. Our intuition for how well this heuristic should hold when δ is "subconstant" is not very strong, but perhaps it indeed holds so long as $d_n = \omega(1)$.

The remainder of this section is devoted to giving some justification for Heuristic 3.2. Let us suppose that we have reached time t and that $d\mu_t^2 \ll 1$. Computing just the expectation we have

$$\mathbb{E}\left[\mu_{t+1}^2\right] = \mathrm{avg}_{i,j \in V}\mathbb{E}\left[X_{t+1}(i)X_{t+1}(j)\right] \approx \mathrm{avg}_{i \neq j}\mathbb{E}\left[X_{t+1}(i)X_{t+1}(j)\right].$$

Here the approximation neglects the case $i = j$; this only affects the average by an additive quantity on the order of $\frac{1}{n}$, which is negligible even compared to $d\mu_1^2$. In a random graph drawn from $G(n, d/n)$ we

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[2]We here use the notation $A \gtrsim B$ for $A = \Omega(B)$.

expect all pairs of distinct vertices i, j to behave similarly, so we simply consider $\mathbb{E}\left[X_{t+1}(i)X_{t+1}(j)\right]$ for some *fixed* distinct $i, j \in V$.

Here we come to the weakest point in our heuristic justification; we imagine that the neighbors of i and j are "refreshed" — i.e., that we can view them as chosen anew from the $G(n, d/n)$ model. For simplicity, we also assume that i and j both have exactly d neighbors (an odd number). We might also imagine that they have roughly $\frac{d^2}{n}$ neighbors in common, though we won't use this. Under these assumptions we have

$$\mathbb{E}\left[X_{t+1}(i)X_{t+1}(j)\right] = \mathbb{E}\left[\operatorname{sgn}(R_1 + \cdots + R_d)\operatorname{sgn}(S_1 + \cdots + S_d)\right]$$

where $R_1, \ldots, R_d$ are independent $\{-1, +1\}$-valued random variables with $\mathbb{E}\left[R_i\right] = \mu_t$, the same is true of $S_1, \ldots, S_d$, and we might assume that some $\frac{d^2}{n}$ of the R_i's and S_i's are identical. In any case, by the FKG Inequality (say), we have

$$\mathbb{E}\left[\operatorname{sgn}(R_1 + \cdots + R_d)\operatorname{sgn}(S_1 + \cdots + S_d)\right]$$
$$\geq \mathbb{E}\left[\operatorname{sgn}(R_1 + \cdots + R_d)\right]\mathbb{E}\left[\operatorname{sgn}(S_1 + \cdots + S_d)\right].$$

Thus to finish our heuristic justification of $\mu_{t+1}^2 \gtrsim d\mu_t^2$ it suffices to argue that

$$(3.1) \qquad\qquad \left|\mathbb{E}\left[\operatorname{sgn}(R_1 + \cdots + R_d)\right]\right| \gtrsim \sqrt{d}|\mu_t|.$$

Without loss of generality we assume $\mu_t \geq 0$. By the Central Limit Theorem, $R_1 + \cdots + R_d$ is distributed essentially as $Z \sim N(d\mu_t, d(1 - \mu_t^2)) \approx N(d\mu_t, d)$. (We are already assuming $d\mu_t^2 \ll 1$, so $\mu_t^2 \ll 1$ as well.) By the symmetry of normal random variables around their mean we have

$$\mathbb{E}\left[\operatorname{sgn}(Z)\right] = \mathbb{P}\left[0 \leq Z \leq 2\mathbb{E}\left[Z\right]\right] = \mathbb{P}\left[-\sqrt{d}\mu_t \leq Z' \leq \sqrt{d}\mu_t\right],$$

where Z' is a standard normal random variable. This last quantity is asymptotic to $\sqrt{\frac{2}{\pi}} \cdot \sqrt{d}\mu_t$ assuming $\sqrt{d}\mu_t \ll 1 \iff d\mu_t^2 \ll 1$, "confirming" (3.1)

3.2. **Constant time to unanimity for the very high degree case.** In this section we give a precise argument supporting the heuristic analysis from Section 3.1 in the case of $G(n, p_n)$ when $p = p_n \gg 1/\sqrt{n}$. The main task is to analyze what happens at time 1; after that we can apply a result from [12], relying on the fact that a random graph is a good expander. For simplicity we will assume n is odd, so that μ_t is never 0.

Proposition 3.3. *(Assuming n is odd,)* $\mathbb{E}\left[\operatorname{sgn}(\mu_0)\mu_1\right] \geq \frac{2}{\pi}\sqrt{p} - \frac{1}{n\sqrt{p}}.$

Proof. We have $\mathbb{E}\left[\mathrm{sgn}(\mu_0)\mu_1\right] = \mathrm{avg}_{i \in V}[\mathrm{sgn}(\mu_0)X_1(i)]$ and by symmetry the expectation is the same for all i. Let's therefore compute it for a fixed $i \in V$; say, $i = n$. Now suppose we condition on vertex n having exactly d neighbors when the graph is chosen from $G(n,p)$. The conditional expectation does not depend on the identities of these neighbors; thus we may as well assume they are vertices $1, \ldots, d$. Writing $X(j) = X_0(j)$ for brevity, we therefore obtain

$$\mathbb{E}\left[\mathrm{sgn}(\mu_0)\mu_1\right] = \sum_{d=0}^{n-1} \Pr[\mathrm{Bin}(n-1,p) = d] \times$$

$$(3.2) \qquad \mathbb{E}\left[\mathrm{Maj}_n(X(1), \ldots, X(n))\mathrm{Maj}_{d'}(X(1), \ldots, X(d), X(n))\right].$$

Here d' denotes d when d is odd and $d+1$ when d is even, and $\mathrm{Maj}_k(x_1, \ldots, x_\ell)$ denotes $\mathrm{sgn}(x_1 + \cdots + x_k)$. We can lower-bound the expectation in line (3.2) using Fourier analysis; by Parseval's identity,

$$(3.2) = \sum_{S \subseteq [n]} \widehat{\mathrm{Maj}_n}(S)\widehat{\mathrm{Maj}_{d'}}(S).$$

By symmetry, the value of $\widehat{\mathrm{Maj}_k}(S)$ only depends on $|S|$; furthermore, it's well known that the sign of $\widehat{\mathrm{Maj}_k}(S)$ depends only on $|S|$ and not on k [14]. Thus all summands above are nonnegative so we obtain

$$(3.2) \geq \sum_{|S|=1} \widehat{\mathrm{Maj}_n}(S)\widehat{\mathrm{Maj}_{d'}}(S).$$

Finally, for odd k we have the explicit formula $\widehat{\mathrm{Maj}_k}(S) = \frac{2}{2^k}\binom{k-1}{\frac{k-1}{2}} \geq \frac{\sqrt{2/\pi}}{\sqrt{k}}$ for any $|S| = 1$. Since the two majorities have exactly d' coordinates in common, we conclude

$$(3.2) \geq d'\frac{\sqrt{2/\pi}}{\sqrt{n}}\frac{\sqrt{2/\pi}}{\sqrt{d'}} = \frac{2}{\pi}\sqrt{\frac{d'}{n}} \geq \frac{2}{\pi}\sqrt{\frac{d}{n}}.$$

Putting this into the original identity we deduce

$$\mathbb{E}\left[\mathrm{sgn}(\mu_0)\mu_1\right] \geq \frac{2}{\pi}\frac{1}{\sqrt{n}}\mathbb{E}\left[\sqrt{\mathrm{Bin}(n-1,p)}\right].$$

We have the standard estimates[3]

$$\mathbb{E}\left[\sqrt{\mathrm{Bin}(n-1,p)}\right] \geq \sqrt{(n-1)p} - \frac{1}{2\sqrt{(n-1)p}} \geq \sqrt{np} - 1.5/\sqrt{np}.$$

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[3]For the first inequality see, e.g., `http://mathoverflow.net/questions/121411/expectation-of-square-`

Thus we finally obtain

$$\mathbb{E}\left[\mathrm{sgn}(\mu_0)\mu_1\right] \geq \frac{2}{\pi}\sqrt{p} - \frac{1}{n\sqrt{p}},$$

as claimed. $\qquad\square$

Proposition 3.4. *We have*

$$\mathbb{E}\left[(\mathrm{sgn}(\mu_0)\mu_1)^2\right] = \mathbb{E}\left[\mu_1^2\right] \leq p + \tfrac{3}{pn}.$$

Proof. We have $\mathbb{E}\left[\mu_1^2\right] = \frac{1}{n} + \mathrm{avg}_{i\neq j}\{\mathbb{E}\left[X_1(i)X_1(j)\right]\}$; by symmetry it therefore certainly suffices to show

(3.3) $$\mathbb{E}\left[X_1(i)X_1(j)\right] \leq p + \tfrac{2}{pn}$$

for some fixed pair of vertices $i \neq j$. Let us condition on the neighborhood structure of vertices i and j. Write $X(j) = X_0(j)$ as in the proof of Proposition 3.3, and write $N_1' = \partial(i) \setminus \{j\}$, $N_2' = \partial(j) \setminus \{i\}$. Then

$$\mathbb{E}\left[X_1(i)X_1(j)\right] = \mathbb{E}\left[\mathrm{Maj}((X(k))_{k\in N_1}) \cdot \mathrm{Maj}((X(k))_{k\in N_2})\right]$$

for some sets $N_1' \subseteq N_1 \subseteq N_1 \cup \{i,j\}$ and similarly N_2. Writing $M = N_1 \cap N_2$ and also $\mathrm{Maj}_N = \mathrm{Maj}(X(r) : k \in N)$ for brevity, the above is equal to

$$\mathbb{E}_{N_1,N_2,(X(k))_{k\in M}}\left[\mathbb{E}\left[\mathrm{Maj}_{N_1}\big|(X(k))_{k\in M}\right] \cdot \mathbb{E}\left[\mathrm{Maj}_{N_2}\big|(X(k))_{k\in M}\right]\right]$$

$$\leq \sqrt{\mathbb{E}_{N_1,N_2,(X(k))_{k\in M}}\left[\mathbb{E}\left[\mathrm{Maj}_{N_1}\big|(X(r))_{k\in M}\right]^2\right]}$$

$$\times \sqrt{\mathbb{E}_{N_1,N_2,(X(k))_{k\in M}}\left[\mathbb{E}\left[\mathrm{Maj}_{N_2}\big|(X(r))_{k\in M}\right]^2\right]}$$

(3.4)
$$= \mathbb{E}_{N_1,N_2,(X(k))_{k\in M}}\left[\mathbb{E}\left[\mathrm{Maj}_{N_2}\big|(X(r))_{k\in M}\right]^2\right],$$

where the inequality is Cauchy–Schwartz and the final equality is by symmetry of i with j.

To analyze (3.4), suppose we condition on N_1 and N_2 (hence also M). By symmetry, the conditional expectation depends only on $|N_1| = n_1$ and $|M| = m$; by elementary Fourier analysis [14] it equals

$$\sum_{S\subseteq[m]} \widehat{\mathrm{Maj}_{n_1}}(S)^2 = \sum_{S\subseteq[n_1]} \frac{\binom{m}{|S|}}{\binom{n_1}{|S|}}\widehat{\mathrm{Maj}_{n_1}}(S)^2$$

$$\leq \sum_{S\subseteq[n_1]} \left(\frac{m}{n_1}\right)^{|S|} \widehat{\mathrm{Maj}_{n_1}}(S)^2 = \mathrm{Stab}_{\frac{m}{n_1}}[\mathrm{Maj}_{n_1}].$$

Finally, we have the bounds [13]

$$\mathrm{Stab}_{\frac{m}{n_1}}[\mathrm{Maj}_{n_1}] \leq \frac{m}{n_1},$$

(3.5) $$\mathrm{Stab}_{\frac{m}{n_1}}[\mathrm{Maj}_{n_1}] \leq \frac{2}{\pi}\arcsin\frac{m}{n_1} + O\left(\frac{1}{\sqrt{1-(m/n_1)^2}\sqrt{n}}\right).$$

Although the second bound here would save us a factor of roughly $\frac{2}{\pi}$, for simplicity we'll only use the first bound. It yields

$$\mathbb{E}\left[X_1(i)X_1(j)\right] \leq \mathbb{E}\left[\frac{|M|}{|N_1|}\right].$$

Each vertex in $N_1 \setminus \{j\}$ has an (independent) probability p of being in M; as for j, we'll overestimate by assuming that if $j \in N_1$ then it is always in M as well. This leads to

$$\mathbb{E}\left[\frac{|M|}{|N_1|}\right] \leq p + \mathbb{E}\left[\frac{1}{|N_1|}\right].$$

Finally, recall that $|N_1|$ is distributed as $\mathrm{Bin}(n-1, p)$ rounded up to the nearest even integer. Thus (see, e.g., [6])

$$\mathbb{E}\left[\frac{1}{|N_1|}\right] \leq \mathbb{E}\left[\frac{1}{(\mathrm{Bin}(n-1,p)+1)/2}\right] = \frac{2}{pn}(1-(1-p)^n) \leq \frac{2}{pn}.$$

This completes the proof.   □

Proposition 3.5. *Assume $n \geq n_0$ and $p \geq \frac{c}{\sqrt{n}}$, where $n_0, c > 0$ are sufficiently large universal constants. Then $\mathbb{P}\left[\mathrm{sgn}(\mu_0)\mu_1 \geq .006\sqrt{p}\right] \geq .4004$.*

Proof. Write $W = \mathrm{sgn}(\mu_0)\mu_1$. The "one-sided Chebyshev inequality" implies that

$$\mathbb{P}\left[W \geq .01\mathbb{E}\left[W\right]\right] \geq \frac{.99^2}{\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]^2} + .99^2 - 1}.$$

Combining Propositions 3.3, 3.4, we have

$$\frac{\mathbb{E}\left[W^2\right]}{\mathbb{E}\left[W\right]^2} \leq \frac{p + 3/(pn)}{(\frac{2}{\pi}\sqrt{p} - 1/(n\sqrt{p}))^2} \leq \frac{\pi^2}{4} + O\left(\frac{1}{p^2 n}\right).$$

As $\frac{.02}{\pi} > .006$ and $\frac{.99^2}{\pi^2/4 + .99^2 - 1} > .4004$, the claim follows.   □

For good expander graphs of degree d, the results of [12] show that unanimity will be reached quickly if the global mean ever significantly exceeds $1/\sqrt{d}$ (in magnitude). In our situation, we essentially have degree-pn graphs with a constant chance of global mean exceeding $\Omega(\sqrt{p})$

at time 1. Consequently we are able to show convergence to unanimity provided $p \gg 1/\sqrt{n}$.

We'll need the following result, which is essentially Proposition 6.2 from [12] (but slightly modified since we need not have perfectly regular graphs):

Lemma 3.6. *Assume $G = (V, E)$ satisfies the following form of the "Expander Mixing Lemma": for all $A, B \subseteq V$,*

$$\left| E(A, B) - p|A||B| \right| \leq \lambda \sqrt{|A||B|},$$

where $E(A, B)$ denotes $\#\{(u, v) \in E : u \in A, v \in B\}$. Then if majority dynamics on G ever has $\mu_t \geq |\alpha|$ then

$$\#\{i \in V : X_{t+1}(i) = -\operatorname{sgn}(\mu_t)\} \leq \frac{2\lambda^2}{\alpha^2 p^2 n}.$$

To use this, we'll also need the following claim that follows easily from a result of Vu on the spectra of $G(n, p)$ [16].

Lemma 3.7. *For $G \sim G(n, p)$ with $p \gg (\log n)^4 / n$ and n high enough, the Expander Mixing Lemma holds for G with $\lambda = 4\sqrt{np}$, except with probability at most $o(1)$.*

Proof. Let P_0 be the (random) adjacency matrix of G, and let P be given by $P = P_0 + D$, where D is a random diagonal matrix whose each diagonal entry is one with probability p and zero otherwise. P can be thought of as the adjacency matrix of a graph G' which is obtained from G by adding each self-loop with probability p.

Let Q be the $n \times n$ matrix whose entries are all equal to p. Then, since $p \gg (\log n)^4 / n$, by [16] it holds with high probability that

$$|P - Q| \leq 3\sqrt{np},$$

where $|\cdot|$ is here the L^2 operator (spectral) norm. Equivalently, for any two vectors $v, w \in \mathbb{R}^n$,

(3.6) $$|v^\top (P - Q)w| \leq 3\sqrt{np} \cdot |v||w|.$$

Let $A, B \subset V$ be any two subsets of vertices. Then the number of edges between A and B is given by

$$E(A, B) = 1_A^\top P_0 1_B = 1_A^\top (P - D) 1_B = 1_A^\top P 1_B - 1_A^\top D 1_B.$$

Now, $1_A^\top D 1_B$ is at most $|A \cap B|$ and therefore, for n high enough it holds that $1_A^\top D 1_B \leq \sqrt{np|A||B|}$.

By (3.6),

$$|1_A^\top P 1_B - 1_A^\top Q 1_B| \leq 3\sqrt{np}|1_A||1_B| = 3\sqrt{np|A||B|}.$$

Since $1_A^\top Q 1_B = p|A||B|$, with high probability

$$\left| E(A,B) - p|A||B| \right| = |1_A^\top P 1_B - 1_A^\top D 1_B - 1_A^\top Q 1_B| \leq 4\sqrt{np|A||B|}.$$

$\square$

Combining the previous two lemmas with Proposition 3.5 we obtain:

Proposition 3.8. *Assume $n \geq n_0$ and $p \geq \frac{c}{\sqrt{n}}$, where $n_0, c > 0$ are sufficiently large universal constants. Then with probability at least .4003 we have*

$$\#\{i \in V : X_2(i) \neq \mathrm{sgn}(\mu_0)\} \leq \frac{c}{p^2}.$$

In $G(n,p)$ (with $p \gg 1/\sqrt{n}$, say), almost surely each vertex has degree at least $(p/2)n$, which in turn exceeds $2c/p^2$ if $p > (4c/n)^{1/3}$. Thus we may conclude:

Theorem 3.9. *Assume $n \geq n_0$ and $p \geq cn^{-1/3}$, where $n_0, c > 0$ are sufficiently large universal constants. Then with probability at least .4 over the choice of $G \sim G(n,p)$ and the initial opinions, the vertices unanimously hold opinion $\mathrm{sgn}(\mu_0)$ at time 3.*

In case $n^{-1/2} \ll p \lesssim n^{-1/3}$ we need an extra time period. By assuming $p \gg n^{-1/2}$, the right-hand side in Proposition 3.8 can be made smaller than any desired positive constant. Then applying the two lemmas again we obtain:

Proposition 3.10. *Assume $n \geq n_0$ and $p \geq \frac{c}{\sqrt{n}}$, where $n_0, c > 0$ are sufficiently large universal constants. Then with probability at least .4002 we have*

$$\#\{i \in V : X_3(i) \neq \mathrm{sgn}(\mu_0)\} \leq \frac{c}{p}.$$

Now we can finish as in the case of $p \gg n^{-1/3}$; we get:

Theorem (2). *Assume $n \geq n_0$ and $p \geq cn^{-1/2}$, where $n_0, c > 0$ are sufficiently large universal constants. Then with probability at least .4 over the choice of $G \sim G(n,p)$ and the initial opinions, the vertices unanimously hold opinion $\mathrm{sgn}(\mu_0)$ at time 4.*

As a final remark, when $p = o(1)$ we can (with slightly more effort) improve the probability bound of .4 to any constant smaller than $2/\pi \approx .6366$ by using (3.5).

4. Majority dynamics on $R(n, d)$

Proposition 4.1. *Let G_n be drawn from $R(n, 4)$, or be a sequence of 4-regular expanders with growing girth. Choose the initial opinions independently with probability $1/3 < p < 2/3$. Then, with high probability, unanimity is not reached at any time.*

Note that Theorem 3 is rephrasing of this proposition.

Proof. Consider a growing sequence of d-regular expanders with girth growing to infinity, denoted by G_n. By [3], p-Bernoulli percolation will contains a unique giant component of size proportional to G_n, provided $p > 1/(d-1)$. The same holds for random d-regular graphs. In particular if $d = 4$ and $1/3 < p < 2/3$ an open giant component and a closed giant component will coexist.

Since $d = 4$ is even, we take majorities over the four neighbors and the vertex itself, as we explain above. To show that with high probability unanimity is not reached on these graphs, it is enough then to show that the giant component of $1/2$-Bernoulli percolation contains cycles. The opinions on the cycles will not change in the process of majority dynamics, since each node will have three (including itself) neighbors on the cycle with which it agrees.

To see this, perform the percolation in two stages: first carry out $(p - \epsilon)$-Bernoulli percolation (such that $1/3 < p - \epsilon$), and then sprinkle on top of it an independent ϵ-Bernoulli percolation. If the first percolation already contains a cycle we are done. Otherwise the giant component is a tree.

Pick an edge of the random giant tree that splits the tree to two parts, so that each part has size at least $1/4$ of the tree. Denote these two parts by A and B. As in the uniqueness proof of [3], since G_n is an expander and A and B has size proportional to the size of G_n. there are order $\Theta(n)$ disjoint paths of length bounded by a function depending only on the expansion. Thus the ϵ-sprinkling connects A and B with order $\Theta(n)$ disjoint open paths, creating many cycles with probability tending to 1 with n, and we are done. □

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