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Social Learning in a Changing World

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Abstract

We study a model of learning on social networks in dynamic environments, describing a group of agents who are each trying to estimate an underlying state that varies over time, given access to weak signals and the estimates of their social network neighbors.

We study three models of agent behavior. In the *fixed response* model, agents use a fixed linear combination to incorporate information from their peers into their own estimate. This can be thought of as an extension of the DeGroot model to a dynamic setting. In the *best response* model, players calculate minimum variance linear estimators of the underlying state.

We show that regardless of the initial configuration, fixed response dynamics converge to a steady state, and that the same holds for best response on the complete graph. We show that best response dynamics can, in the long term, lead to estimators with higher variance than is achievable using well chosen fixed responses.

The *penultimate prediction* model is an elaboration of the best response model. While this model only slightly complicates the computations required of the agents, we show that in some cases it greatly increases the efficiency of learning, and on complete graphs is in fact optimal, in a strong sense.

1 Introduction

The past three decades have witnessed an immense effort by the computer science and economics communities to model and understand people's behavior on social networks [17]. A particular goal has been the study of how people share information and learn from each other; learning from peers has been repeatedly shown to be a driving force of many economic and social processes (cf. [10, 8, 20, 9]).

1.1 Classical approaches and results

Early work by DeGroot [11] considered a set of agents, connected by a social network, that each have a prior belief: a distribution over the possible values of an *underlying state of the world* - say the market value of some company. The agents iteratively observe their neighbors' beliefs and update their own by averaging the distributions of their neighbors. Since DeGroot, a plethora of models for social learning have been proposed and studied.

DeGroot's simple averaging of neighbors' beliefs may seem naive and arbitrary; economists often opt for *rational* models instead. In rational models the agents update their belief not by a fixed rule, but in an attempt to maximize a utility function. It is often assumed that agents are *Bayesian*: they assume some prior distribution on the underlying state and on other agents' behavior, have access to some observations, and maximize the expected value of their utility, using Bayes' Law. Bayesian social learning has a wide literature, with noted work by Aumann [4] and the related *common knowledge* work (cf. [14]), as well as McKelvey and Page [21], Parikh and Krasucki [24], Bala and Goyal [6], Gale and Kariv [13], and many others.

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Aumann [4] and Geanakoplos [15] show that a group of Bayesian agents, who each have an initial estimate of an *underlying state*, and repeatedly announce their estimate (in particular, expected value) of this state, will eventually converge to the same estimate. McKelvey and Page [21] extend this result to processes in which "survey results", rather than all the estimates, are repeatedly shared. The social network in these models is the *complete network*; indeed, it seems that non-trivial dynamics and results are achieved already for this (seemingly simple) topology. Aaronson [1] studies the complexity of the computations required of the agents, again with highly non-trivial results.

1.2 Rationality and bounded rationality

The term *rational* in economic theory refers to any behavior that maximizes (or even attempts to maximize) some utility function. This is in contrast to, for example, behavior that is heuristic or fixed. Bayesian rationality optimizes in a probabilistic framework that includes a prior and observations, and is, as mentioned above, a commonly used paradigm.

The disadvantage of fully rational, Bayesian models is that the calculations required of the agents can very quickly become intractable, making their applicability to real-world settings questionable; this tension between rationality and tractability is an old recurring theme in behavioral economics models (cf. [25]).

A solution often advocated is *bounded rationality*. Agents still act optimally in bounded rationality models, but only optimize with respect to a restricted set of choices. This usually simplifies the optimization problem that needs to be solved. For example, agents may be required to disregard some of their available information or be restricted in the manner that they calculate their strategy. In addition to serving the goal of more realistically modeling agents, a usual added benefit of bounded rationality is that the analysis of the model becomes easier. We too follow this course.

A standard assumption in this literature is that "actions speak louder than words" (cf. Smith and Sørensen [26]); agents do not participate in a communication protocol intended to optimize the exchange of information, but rather make inferences about each others' private information by observing actions. For example, by observing the price at which a person bids for a stock one may learn her estimate for the future price, but yet not learn all of the information which she used to arrive at this estimate.

1.3 Informal statement of the model

We consider a model where the underlying state S is not a constant number - as it is in all of the above mentioned models - but changes with time, as prices and other economic quantities tend to do. In particular we assume that the state $S = S(t)$ performs a random walk; $S(0)$ is picked from some distribution, and at each iteration an i.i.d. random variable is added to it.

The process commences with each agent having some estimator of $S(0)$. We make only very weak assumptions about the joint distribution of these estimators. Then, at each discrete time period t, each agent receives an independent (and identical over time) measurement of $S(t)$, and uses it to update its estimator. Also available to it are the previous *estimates* of its neighbors on a social network. Thus social network neighbors share their beliefs (or rather, observe each others' actions), and information propagates through the network.

While conceivably agents could optimally use all the information available to them to estimate the underlying state, it appears that such calculations are extremely complex. Instead, we explore *bounded rationality* dynamics, assuming that agents are restricted to calculating linear combinations of their observations. We note that if the random walk and the measurements are taken to be Gaussian, then the minimum variance unbiased linear estimator (MVULE) is also the maximum likelihood estimator. A Gaussian random walk is a good first-order approximation for many Economic processes (cf. classical work by Bachelier [5]).

For the first part of the paper, we also require that these linear combinations only involve the agents' neighbors' estimates from the previous time period (and not earlier), as well as their new measurement. In the last model we slightly relax this requirement.

We consider three models. In the *fixed response* model each agent, at each time period, estimates the underlying state by a *fixed* linear combination of its new measurement and the estimates of its neighbors in the previous period. This is a straightforward extension of the DeGroot model to our setting.

In the *best response model*, at each iteration, each agent calculates the MVULE of the underlying state, based on its peers' estimate from the previous round, together with its new measurement. We assume here that at each iteration the agents know the covariance matrix of their estimators. While this may seem like a strong assumption, we note that, under some elaboration of our model, this covariance matrix may be estimated by observing the process for some number of rounds before updating one's estimator. Furthermore, it seems that assumptions in this spirit - and often much stronger assumptions - are necessary in order for agents to perform any kind of optimization. For example, it is not rare in the literature of social Bayesian learning to assume that the agents know the structure of the entire social network graph (e.g., [13, 24, 2]).

Finally, we introduce the *penultimate prediction* model, which is a simple extension of the best response model, additionally allowing the agents to remember exactly one value from one round to the next. While only slightly increasing the computational requirements on the agents, this model exhibits a sharp increase in learning efficiency.

1.4 Informal statement of results

While our long term goal is to understand this process on general social network graphs, we focus in this paper on the *complete network*, which already exhibits mathematical richness.

We consider the system to be in a *steady state* when the covariance matrix of the agents' estimators is constant. On general graphs we show that fixed response dynamics converge to a steady state. On the complete graph we show that best response dynamics also converge to a steady state. Both of these results hold regardless of the initial conditions (i.e., the agents' estimators at time $t = 0$).

We show that the steady state of best response dynamics is not necessarily optimal; there exist fixed response dynamics in which the agents converge to estimators which all have lower variance then the estimators of the steady state of best response dynamics. This shows that every agent can do better than the result of best response by following a socially optimal rule; thus a certain *price of anarchy* is to be paid when agents choose the action that maximizes their short term gain.

Finally, we show that in the penultimate prediction model, for the complete graph, the agents learn estimators which are the optimal (in the minimum variance sense) amongst all linear estimators, and thus outperform those of fixed and best response dynamics.

We define a notion of "socially asymptotic learning": A model has this property when the variance of the agents' steady-state estimators tends towards the information-theoretical optimum with the number of agents. We show that the penultimate prediction model exhibits socially asymptotic learning on the complete graph, while best response and fixed response dynamics fail to do the same.

2 Previous work

Our model is an elaboration of models studied by DeMarzo, Vayanos and Zwiebel [12], as well as Mossel and Tamuz [23, 22]. There, the state S is a fixed number picked at time $t = 0$, and each agent receives a single measurement of it. The process thereafter is deterministic, with each agent, at each iteration, recalculating its estimate of S based on its observation of its neighbors' estimates.

In [22] it is shown that if the agents calculate the minimum variance unbiased linear estimator (MVULE) at every turn (remembering all of their observations) then all the agents converge to the optimal estimator of S, i.e. the average of the original measurements. Furthermore, this happens in time that is at most $n \cdot d$, where d is the diameter of the graph.

When agents calculate estimates that are only based on their observations from the previous round, then they do not necessarily converge to the optimal estimator [23]. In fact, it is not known whether they converge at all.

A similar model is studied by Jadbabaie, Sandroni and Tahbaz-Salehi [18]. They explore a bounded rationality setting in which agents receive new signals at each iteration. An agent's private signals may be informative only when combined with those of other agents, and yet their model achieves efficient learning.

Our model is a special case of a model studied by Acemoglu, Nedic, and Ozdaglar [3]. They extend these models by allowing the state to change from period to period. They don't require the change in the state to be i.i.d, but only to have zero mean and be independent in time. Their agents also receive a new, independent measurement of the state at every period, which again need not be identically distributed. They focus on a different regime than the one we study; their main result is a proof of convergence in the case that the variations in the state diminish with time, with variance tending to zero.

In our model the change in the underlying state has constant variance, as does the agents' measurement noise. This allows us to explore steady states, in which the covariance matrix of the agents' estimators does not change from iteration to iteration.

Our model is non-trivial already for a single agent, although here a complete solution is simple, and can be calculated using tools developed for the analysis of *Kalman filters* [19].

3 Notation, formal models, and results

Let $[n] = \{1, 2, \dots, n\}$ be a set of agents. Let $G = ([n], E)$ be a directed graph representing the agents' social network. We denote by $\partial i = \{j | (i, j) \in E\}$ the neighbors of i, and assume that always $i \in \partial i$.

We consider discrete time periods $t \in \{0, 1, \dots\}$. The *underlying state of the world* at time t, $S(t)$, is defined as follows. $S(0)$ is a real random variable with arbitrary distribution, and for $t > 0$

$$S(t) = S(t-1) + X(t-1), \tag{1}$$

where $\mathbb{E}[X(t)] = 0$, $\text{Var}[X(t)] = \sigma^2$, and σ is a parameter of the model. The random variables $X(0), X(1), \dots$ are independent. Hence the underlying state $S(t)$ performs a random walk with zero mean and standard deviation σ.

At time $t = 0$ each agent i receives $Y_i(0)$, an estimator of $S(0)$. The only assumptions we make on their joint distribution is that $\mathbb{E}[Y_i(0)|S(0)] = S(0)$, i.e. the estimators are unbiased, and that $\text{Var}[Y_i(0) - S(0)]$ is finite for all i.

At each subsequent period $t > 0$, each agent i receives $M_i(t)$, an independent measurement of $S(t)$, defined by

$$M_i(t) = S(t) + D_i(t), \tag{2}$$

where $\mathbb{E}[D_i(t)] = 0$, $\text{Var}[D_i(t)] = \tau_i^2$, and the τ_i's are parameters of the model. Hence $D_i(t)$ is the measurement error of agent i at time t. Again, the random variables $D_i(t)$ are independent.

At each period $t > 0$, each agent i calculates $Y_i(t)$, agent i's estimate of $S(t)$, using the information available to it. Precisely what information is available varies by the model (and is defined below), but in all cases $Y_i(t)$ is a (deterministic) convex linear combination of agent i's measurements up to and including time t, $\{M_i(t')|t' \le t\}$, as well as the previous *estimates* of its social network neighbors, $\{Y_j(t')|t' < t, j \in \partial i\}$. Additionally, in the penultimate prediction model, at each round t each agent computes a value $R_i(t)$, and at round $t+1$ uses this value to compute $R_i(t+1)$ and $Y_i(t+1)$. Like $Y_i(t)$, $R_i(t)$ is also a convex linear combination of the same random variables.

In general, we shall assume that the agents are interested in minimizing the expected squared error of their estimators, $\mathbb{E}\left[(Y_i(t) - S(t))^2\right]$; assuming $Y_i(t)$ is unbiased (i.e., $\mathbb{E}[Y_i(t)|S(t)] = S(t)$) this is equivalent to minimizing $\text{Var}[Y_i(t) - S(t)]$, which we refer to as the *"variance of the estimator $Y_i(t)$."* We shall assume throughout that the estimators $Y_i(t)$ are indeed unbiased; we elaborate on this in the definitions of the models below.

We shall (mostly) restrict ourselves to the case where the agents use only their neighbors' estimates from the previous iteration, and not from the ones before it. In these cases we write

$$Y_i(t) = A_i(t)M_i(t) + \sum_j P_{ij}(t)Y_j(t-1). \tag{3}$$

for some $A_i(t)$ and $P_{ij}(t)$ such that $P_{ij} = 0$ whenever $j \notin \partial i$.

We will find it convenient to express such quantities in matrix form. To that end we let $\mathbf{m}(t), \mathbf{y}(t), \mathbf{d}(t) \in \mathbb{R}^n$ be column vectors with entries $M_i(t), Y_i(t), D_i(t)$, and let $\mathbf{A}(t), \mathbf{P}(t), \mathbf{T} \in \mathbb{R}^{n\times n}$ be the weight matrices, with $\mathbf{A}(t) = \text{Diag}(A_1(t), \dots, A_n(t))$, $\mathbf{P} = (P_{ij})_{ij}$ and $\mathbf{T} = \text{Var}[\mathbf{d}(t)] = \text{Diag}(\tau_1^2, \dots, \tau_n^2)$. Using this notation Eq. (3) becomes

$$\mathbf{y}(t) = \mathbf{A}(t)\mathbf{m}(t) + \mathbf{P}(t)\mathbf{y}(t-1). \tag{4}$$

We will also make use of the *covariance matrix*

$$\mathbf{C}(t) = \mathrm{Var}\left[\mathbf{y}(t) - \mathbf{1}S(t)\right], \tag{5}$$

where $\mathbf{1} \in \mathbb{R}^n$ denotes the column vector of all ones. Hence $C_{ij}(t) = \mathrm{Cov}\left[Y_i(t) - S(t), Y_j(t) - S(t)\right]$, which we refer to as the *"covariance of the estimators $Y_i(t)$ and $Y_j(t)$."*

3.1 Dynamics models

3.1.1 Best response

The main model we study is the best response dynamics. Here we assume that at round t, each agent i has access to $M_i(t)$, $\mathbf{y}(t-1)$ and the covariance matrix for these values. At each iteration t, agent i picks $A_i(t)$ and $\{P_{ij}(t)\}_j$ that minimize $C_{ii}(t) = \mathrm{Var}\left[Y_i(t) - S(t)\right]$, under the constraints that (a) $P_{ij}(t)$ may be non-zero only if $j \in \partial i$, and (b) $\mathbb{E}\left[Y_i(t)|S(t)\right] = S(t)$, i.e. $Y_i(t)$ is an unbiased estimator of $S(t)$. In Section 6.1 we show that these minimizing coefficients are a deterministic function of $\mathbf{C}(t-1)$, σ and $\{\tau_i\}$. Hence we assume here that the agents know these values. By this definition $Y_i(t)$ is the *minimum variance unbiased linear estimator* (MVULE) of $S(t)$, given $M_i(t)$ and $\mathbf{y}(t-1)$.

Note that it follows from our definitions that if the estimators $\{Y_i(t-1)\}$ at time $t-1$ are unbiased then, in order for the estimators at time t to be unbiased, it must be the case that

$$A_i(t) + \sum_j P_{ij}(t) = 1. \tag{6}$$

Since at time zero the estimators are unbiased then it follows by induction that Eq. (6) hold for all $t > 0$.

3.1.2 Fixed response

We shall also consider the case of estimators which are *fixed* linear combinations of the agent's new measurement $M_i(t)$ and its neighbors' estimators at time $t-1$. These we call fixed response estimators. In this case we would have, using our matrix notation:

$$\mathbf{y}(t) = \mathbf{A}\mathbf{m}(t) + \mathbf{P}\mathbf{y}(t-1). \tag{7}$$

The matrices $\mathbf{A}$ and $\mathbf{P}$ are arbitrary matrices that satisfy the following conditions: (a) P_{ij} is positive and non-zero only if $j \in \partial i$, and (b) $\mathbf{y}_i(t)$ is a convex linear combination of $M_i(t)$ and $\{Y_j(t-1)\}_j$. Equivalently, $\mathbf{A}_i + \sum_j P_{ij} = 1$, which is the same condition described in Equation (6).

3.1.3 Penultimate prediction

Finally, we consider the penultimate prediction model where each agent i can remember one value, which we denote $R_i(t)$, from one round t to the next round $t+1$. We assume that at round t, each agent i has access to $M_i(t)$, $\mathbf{y}(t-1)$, $R_i(t-1)$ and the covariance matrix for these values. We denote $\mathbf{r}(t) = (R_1(t), \ldots, R_n(t))$.

We fix $R_i(0) = 0$, and let $R_i(t)$ be agent i's MVULE of $S(t-1)$, given $R_i(t-1)$ and $\mathbf{y}(t-1)$ (note that this is in general *not* equal to $Y_i(t-1)$). $Y_i(t)$ now becomes the MVULE of $S(t)$ given $R_i(t)$ and $M_i(t)$.

3.2 Steady states and efficient learning

We say that the system converges to a *steady state* $\mathbf{C}$ when

$$\lim_{t\to\infty} \mathbf{C}(t) = \mathbf{C}.$$

Assuming that agents are constrained to calculating linear combinations of their measurements and neighbors' estimators, the variance of the estimators $Y_i(t)$ of $S(t)$ at time t can be bounded from below by the variance of $Z_i(t)$, where we define $Z_i(t)$ to be the MVULE of $S(t)$ given the initial estimators $\mathbf{y}(0)$, all measurements up to time $t-1$ $\{M_j(s)|j \in [n], s < t\}$ and $M_i(t)$. We therefore define that a process achieves *perfect learning* when $\mathrm{Var}[Y_i(t) - S(t)] = \mathrm{Var}[Z_i(t) - S(t)]$. Note that this definition is a natural one for the complete graph and should be altered for general networks, where a tighter lower bound exists.

If an agent were to know $S(t-1)$ exactly at time t, then, together with $M_i(t)$, its minimum variance unbiased linear estimator for $S(t)$ would be a linear combination of just $S(t-1)$ and $M_i(t)$, because of the Markov property of $S(t)$. In this case it is easy to show (see Proposition 2) that $C_{ii}(t) = \mathrm{Var}[Y_i(t) - S(t)]$ would equal $\sigma^2\tau_i^2/(\sigma^2+\tau_i^2)$. We say that a model achieves *socially asymptotic learning* if for n sufficiently large, as the number of agents tends to infinity, the steady state $\mathbf{C}$ exists and C_{ii} tends to $\sigma^2\tau_i^2/(\sigma^2+\tau_i^2)$ for all i. We stress that this definition only makes sense in models where the number of agents n grows to infinity and therefore is incomparable to perfect learning, which is defined for a particular graph.

4 Statement of the main results

The following are our main results. Let $\beta(t) = 1/(\mathbf{1}^\top \mathbf{C}(t)^{-1}\mathbf{1})$.

Theorem 8. *When G is a complete graph, best-response dynamics converge to a unique steady-state, for all starting estimators $\mathbf{y}(0)$ and all choices of parameters $\{\tau_i\}$ and σ. Moreover, the convergence is fast, in the sense that $-\log|\beta(t) - \beta^*| = O(t)$, where $\beta^* = \lim_{t\to\infty}\beta(t)$.*

Theorem 11. *In fixed response dynamics, if $A_i > 0$ for all $i \in [n]$ then system converges to a steady state $\mathbf{C} = \lim_{t\to\infty}\mathbf{C}(t)$ such that*

$$\mathbf{C} = \mathbf{A}^2\mathbf{T} + \sigma^2\mathbf{P}\mathbf{1}\mathbf{1}^\top\mathbf{P}^\top + \mathbf{P}\mathbf{C}\mathbf{P}^\top. \tag{8}$$

In particular, $\mathbf{C}$ is independent of the starting estimators $\mathbf{y}(0)$.

Theorem 13. *Let G be a graph with $[n]$ vertices. Fix σ and $\{\tau_i\}_{i\in[n]}$.*

Consider best response *dynamics for n agents on G with σ and $\{\tau_i\}_{i\in[n]}$. Let $\mathbf{C}^{br}$ denote the steady state the system converges to.*

Consider fixed response *dynamics with some $\mathbf{P}$ and $\mathbf{A}$ for n agents on G with σ and $\{\tau_i\}_{i\in[n]}$. Let $\mathbf{C}^{fr}$ denote the steady state the system converges to.*

Then there exists a choice of n, G, σ, $\{\tau_i\}$, $\mathbf{A}$ and $\mathbf{P}$ such that $C_{ii}^{br} > C_{ii}^{fr}$ for all $i \in [n]$.

Theorem 16. *If $\sigma, \tau > 0$, no fixed response dynamics can achieve socially asymptotic learning.*

Theorem 17. *Penultimate prediction on the complete graph achieves perfect learning.*

5 Background results

5.1 Time evolution of the covariance matrix

We commence by proving a preliminary proposition on the relation between the coefficients matrices $\mathbf{P}(t)$ and $\mathbf{A}(t)$, and the covariance matrix $\mathbf{C}(t)$ in the best response and fixed response models. This result does not depend on how $\mathbf{P}(t)$ and $\mathbf{A}(t)$ are calculated, and therefore applies to both models.

First, let us calculate the covariance matrix directly. By the definition of $\mathbf{C}(t)$ and by Eq. (4) we have that

$$\mathbf{C}(t) = \mathrm{Var}[\mathbf{y}(t) - \mathbf{1}S(t)] = \mathrm{Var}[\mathbf{A}(t)\mathbf{m}(t) + \mathbf{P}(t)\mathbf{y}(t-1) - \mathbf{1}S(t)].$$

Since $S(t) = S(t-1) + X(t-1)$ then we can write

$$\mathbf{C}(t) = \mathrm{Var}\left[\mathbf{A}(t)\big(\mathbf{m}(t) - \mathbf{1}S(t)\big) + \mathbf{P}(t)\mathbf{y}(t-1) - (\mathbf{I} - \mathbf{A}(t))\mathbf{1}\big(S(t-1) + X(t-1)\big)\right].$$

Since the estimators $\{Y_i(t)\}$ are unbiased then $\mathbf{P}(t)\mathbf{1} = (\mathbf{I} - \mathbf{A}(t))\mathbf{1}$; see the definitions of the models in Section 3.1, and in particular Eq. (6). Hence

$$\mathbf{C}(t) = \mathbf{A}(t)\mathbf{T}\,\mathbf{A}(t)^\top + \mathrm{Var}[\mathbf{P}(t)(\mathbf{y}(t-1) - \mathbf{1}S(t-1))] + \mathrm{Var}[\mathbf{P}(t)\mathbf{1}X(t-1)],$$

since $\mathrm{Var}[\mathbf{m}(t) - \mathbf{1}S(t)] = \mathrm{Var}[\mathbf{d}(t)] = \mathbf{T}$. Finally, since $\mathrm{Var}[\mathbf{y}(t-1)] = \mathbf{C}(t-1)$ we can write

$$\mathbf{C}(t) = \mathbf{A}(t)^2\mathbf{T} + \mathbf{P}(t)\mathbf{C}(t-1)\mathbf{P}(t)^\top + \sigma^2\mathbf{P}(t)\mathbf{1}\mathbf{1}^\top\mathbf{P}(t)^\top. \tag{9}$$

Proposition 1. *Let* $\mathbf{Q}(r,t) = \prod_{s=r+1}^{t} \mathbf{P}(s)$ *and* $\mathbf{W}(t) = \mathbf{A}(t)^2\mathbf{T} + \sigma^2\mathbf{P}(t)\mathbf{1}\mathbf{1}^\top\mathbf{P}(t)^\top$ *with* $\mathbf{W}(0) = \mathbf{C}(0)$. *Then for all* $t \geq 1$,

$$\mathbf{C}(t) = \sum_{r=0}^{t} \mathbf{Q}(r,t)\mathbf{W}(r)\mathbf{Q}(r,t)^\top. \tag{10}$$

Proof. First note that equation (9) becomes simply $\mathbf{C}(t) = \mathbf{W}(t) + \mathbf{P}(t)\mathbf{C}(t-1)\mathbf{P}(t)^\top$. The base of the induction $t = 1$ is now immediate, since $\mathbf{W}(0) = \mathbf{C}(t-1)$. Now assume that it holds to time t. Then

$$\begin{aligned}
\mathbf{C}(t+1) &= \mathbf{W}(t+1) + \mathbf{P}(t+1)\mathbf{C}(t)\mathbf{P}(t+1)^\top \\
&= \mathbf{W}(t+1) + \mathbf{P}(t+1)\left[\sum_{r=0}^{t} \mathbf{Q}(r,t)\mathbf{W}(r)\mathbf{Q}(r,t)^\top\right]\mathbf{P}(t+1)^\top \\
&= \mathbf{W}(t+1) + \sum_{r=0}^{t} \mathbf{Q}(r,t+1)\mathbf{W}(r)\mathbf{Q}(r,t+1)^\top \\
&= \sum_{r=0}^{t+1} \mathbf{Q}(r,t)\mathbf{W}(r)\mathbf{Q}(r,t)^\top.
\end{aligned}$$

□

5.2 Minimum variance unbiased linear estimator

We show in this subsection how in general a minimum variance unbiased linear estimator is calculated, given a collection of estimators with a known covariance matrix.

Let X be a random variable and let $(Z_1, \ldots, Z_n)$ be random variables such that $\mathbb{E}[Z_i|X] = X$ for all $i \in [n]$. Let $C_{ij} = \mathrm{Cov}[Z_i - X, Z_j - X]$, with $\mathbf{C}$ being the matrix with entries C_{ij}.

Let $M = \sum_i b_i Z_i$ be the minimum variance unbiased linear estimator of X, i.e., let $(b_1, \ldots, b_n)$ minimize $\mathrm{Var}[M - X]$ under the constraint that $\mathbb{E}[M|X] = X$, which is equivalent to $\sum_i b_i = 1$, since $\mathbb{E}[Z_i|X] = X$ for all i.

Denote $\mathbf{b} = (b_1, \ldots, b_n)$.

Proposition 2.

$$\mathbf{b} = \frac{\mathbf{1}^\top\mathbf{C}^{-1}}{\mathbf{1}^\top\mathbf{C}^{-1}\mathbf{1}}. \tag{11}$$

Proof. By definition

$$\mathrm{Var}[M - X] = \mathrm{Var}\left[\sum_i b_i Z_i - X\right] = \mathrm{Cov}\left[\sum_i b_i Z_i - X, \sum_j b_j Z_j - X\right].$$

Since $\sum_i b_i = 1$ then we can write

$$\mathrm{Var}[M - X] = \mathrm{Cov}\left[\sum_i b_i(Z_i - X), \sum_j b_j(Z_j - X)\right],$$

and then by the bilinearity of covariance we have that

$$\mathrm{Var}[M - X] = \sum_{ij} b_i b_j \mathrm{Cov}[Z_i - X, Z_j - X] = \mathbf{b}^\top\mathbf{C}\mathbf{b}.$$

Note that we again used here the fact that $\sum_i b_i = 1$.

To find $\mathbf{b}$ that minimizes this expression under the constraint that $\sum_i b_i = 1$ we use Lagrange multipliers to minimize

$$\mathbf{b}^\top \mathbf{C}\mathbf{b} + \lambda(\mathbf{1}^\top \mathbf{b} - 1),$$

which is a straightforward calculation yielding Eq. (11). □

We assumed in this proof that $\mathbf{C}$ is an invertible matrix. When $\mathbf{C}$ is not invertible then it is easy to show that the same statement holds, with $\mathbf{C}^{-1}$ being a pseudo-inverse of $\mathbf{C}$. While for different such pseudo-inverses one gets different values of $\mathbf{b}$, the variance of the different M's is identical.

The following two corollaries follow directly from Proposition 2.

Corollary 3. *If $\mathbf{C}$ is a diagonal matrix, then b_i, the weight given to each variable Z_i, in the minimum variance unbiased estimator is proportional to $Var[Z_i - X]^{-1}$ and the variance of the minimal variance unbiased estimator is $1/(\sum_i Var[Z_i - X]^{-1})$.*

Corollary 4. *If*

$$\mathbf{C} = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

then the minimum variance unbiased estimator is

$$M = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} Z_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} Z_2,$$

with

$$Var[M - X] = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}.$$

Note that in the best response model $Y_i(t)$ is the minimum variance unbiased linear estimator of $S(t)$ given $M_i(t)$ and $\{Y_j(t-1)\}$. Hence to calculate it is suffices to know the covariances of these estimators. It follows from the definitions that

$$\mathrm{Cov}\left[Y_j(t-1) - S(t), Y_k(t-1) - S(t)\right] = \mathbf{C}_{jk}(t-1) + \sigma^2,$$

$$\mathrm{Cov}\left[Y_j(t-1) - S(t), M_i(t) - S(t)\right] = \tau_j^2 + \sigma^2 + \tau_i^2,$$

and

$$\mathrm{Cov}\left[M_i(t) - S(t), M_i(t) - S(t)\right] = \tau_i^2.$$

Thus knowing $\mathbf{C}(t-1)$, σ and $\{\tau_j\}$ is sufficient to calculate the coefficients $A_i(t)$ and $\{P_{ij}(t)\}$ in the best response model.

5.3 Best response with a single agent

We provide the following proposition without proof. It is a consequence of basic Kalman filter theory [19]; it is shown there that, for $n = 1$, the MVULE of $S(t)$ given all the measurements up to time t is identical to the MVULE of $S(t)$ given the new measurement at time t and the previous estimator. Formally:

Proposition 5. *Best response achieves perfect learning when $n = 1$.*

6 Best response dynamics

Recall that in best response dynamics at time $t > 0$ agent i chooses $A_i(t)$ and $\{P_{ij}(t)\}_j$ that minimize $C_{ii}(t)$, under the constraints that $P_{ij}(t) = 0$ if $j \notin \partial i$, and $A_i(t) + \sum_j P_{ij}(t) = 1$. Thus $Y_i(t)$ is in fact the MVULE (see definition in Section 3.1.1) of $S(t)$, given $M_i(t)$ and $\{Y_j(t-1)\}_{j \in \partial i}$.

Note that to calculate $A_i(t)$ and $\{P_{ij}(t)\}_j$ it is necessary (and, in fact, sufficient, as we note in Section 5.2) to know $\mathbf{C}(t-1)$, σ and $\{\tau_j\}$, and so this model indeed assumes that the agents know the covariance matrix of their neighbors' estimators.

6.1 Understanding best-response dynamics

The condition that estimators are unbiased, or $A_i(t) + \sum_j P_{ij}(t) = 1$, means that given $\{P_{ij}(t)\}_j$ one can calculate $A_i(t)$, or alternatively given $\mathbf{P}$ one can calculate $\mathbf{A}$. Hence, fixing σ and $\{\tau_j\}$, $\mathbf{P}(t)$ is a deterministic function of $\mathbf{C}(t-1)$. Since by Eq. (9) $\mathbf{C}(t)$ is a function of $\mathbf{A}(t)$, $\mathbf{P}(t)$ and $\mathbf{C}(t-1)$, then under best response dynamics, $\mathbf{C}(t)$ is in fact a function of $\mathbf{C}(t-1)$. We will denote this function by F, so that $\mathbf{C}(t) = F(\mathbf{C}(t-1))$. Our goal is to understand this map F, and in particular to determine its limiting behavior.

We next analyze in more detail the best response calculation for agent i. This can conceptually be divided into two stages: calculating a best estimator for $S(t)$ from $\mathbf{y}(t-1)$, and then combining that with $M_i(t)$ for a new estimator of $S(t)$.

Let the vector $\mathbf{y}_{\partial i}(t-1) = \{Y_j(t-1) | j \in \partial i\}$ and let $\mathbf{C}_i(t-1) = C_{\partial i, \partial i}(t-1)$ be the covariance matrix of the estimators of the neighbors of agent i.

Denote by $\mathbf{q}_i(t)$ the vector of coefficients for $\mathbf{y}_{\partial i}(t-1)$ that make $Z_i = \mathbf{q}_i(t)^\top \mathbf{y}_{\partial i}(t-1)$ a minimum variance unbiased linear estimator for $S(t)$; note that this is also the estimator for $S(t-1)$. Then by Proposition 2 we have that

$$\mathbf{q}_i(t) = \beta_i(t-1)\mathbf{1}^\top \mathbf{C}_i(t-1)^{-1},$$

where $\beta_i(t-1) = 1/\mathbf{1}^\top \mathbf{C}_i(t-1)^{-1}\mathbf{1}$. It is easy to see that $\mathrm{Var}\left[Z_i - S(t-1)\right] = \beta_i(t-1)$ and thus $\mathrm{Var}\left[Z_i - S(t)\right] = \beta_i(t-1) + \sigma^2$.

$M_i(t)$ is an independent estimator of $S(t)$ with variance τ_i^2. To combine it optimally with Z_i we set

$$A_i(t) = \frac{\beta_i(t-1) + \sigma^2}{\tau_i^2 + \beta_i(t-1) + \sigma^2} \geq \frac{\sigma^2}{\tau_i^2 + \sigma^2}, \tag{12}$$

by Corollary 4. The optimal weight vector $\mathbf{p}_i(t)$ for agent i (i.e., $\{P_{ij}\}_{j \in \partial i}$) is therefore $\mathbf{p}_i(t) = (1 - A_i(t))\mathbf{q}_i(t)$.

6.2 Complete graph case

When G is the complete graph, the agents best-respond similarly, since they all observe the same set of estimators from the previous iteration. We now have $\mathbf{C}_i(t-1) = \mathbf{C}(t-1)$, $\mathbf{q}_i(t) = \mathbf{q}(t)$, and $\beta_i(t-1) = \beta(t-1)$, for all i. For the moment, we will suppress the t. Letting $\boldsymbol{a}$ be the vector with coefficients A_i, we then have $\mathbf{P} = (\mathbf{1} - \boldsymbol{a})\mathbf{q}^\top = \beta(\mathbf{1} - \boldsymbol{a})\mathbf{1}^\top \mathbf{C}^{-1}$. Using this form for $\mathbf{P}$, we can now see that $\mathbf{P}\mathbf{C}\mathbf{P}^\top = \beta(\mathbf{1} - \boldsymbol{a})(\mathbf{1} - \boldsymbol{a})^\top$. Putting this all together, and adding back the t, we have by Eq. (9) that

$$\mathbf{C}(t) = \mathbf{A}(t)^2\, \mathbf{T} + (\beta(t-1) + \sigma^2)(\mathbf{1} - \boldsymbol{a}(t))(\mathbf{1} - \boldsymbol{a}(t))^\top. \tag{13}$$

Since by equation (12), $A_i(t)$ depends only on $\beta(t-1)$, τ_i, and σ, we see that $\mathbf{C}(t) = F(\mathbf{C}(t-1))$ depends on $\mathbf{C}(t-1)$ only through $\beta(t-1) = 1/\mathbf{1}^\top \mathbf{C}(t-1)^{-1}\mathbf{1}$. Hence we can write $\mathbf{C}(t) = \mathbf{C}(\beta(t-1))$. We now see that we can completely describe the state of the system by a single parameter β, and our map F reduces to the map $f : \beta \mapsto 1/\mathbf{1}^\top \mathbf{C}(\beta)^{-1}\mathbf{1}$. We wish to analyze this function f as a single-parameter discrete dynamical system.

To simplify our formula for f, we will make use of a matrix identity attributed to Woodbury and others (cf. [16]):

Theorem 6 (Sherman-Morrison-Woodbury formula)**.** *For any $U \in \mathbb{R}^{n \times k}$, $V \in \mathbb{R}^{k \times n}$, and nonsingular $X \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{k \times k}$ such that $Y^{-1} + VX^{-1}U$ is nonsingular,*

$$(X + UYV)^{-1} = X^{-1} - X^{-1}U\left(Y^{-1} + VX^{-1}U\right)^{-1} VX^{-1}. \tag{14}$$

Using the formula in Eq. (14), we can expresses f in terms of β:

Lemma 7. *Let $y = \sum_i \tau_i^2$ and $z = (\sum_i \tau_i^2)(\sum_i \tau_i^{-2})$. Then $f(\beta)$ has the following form:*

$$f(\beta) = \frac{y\left(\beta + \sigma^2\right)\left(y + \beta + \sigma^2\right)}{y\left(y - (n-2)n\left(\beta + \sigma^2\right)\right) + z\left(\beta + \sigma^2\right)\left(y + \beta + \sigma^2\right)} \tag{15}$$

Proof. Let $x = (\beta + \sigma^2)$ for brevity. We will compute $\mathbf{1}^\top \mathbf{C}(\beta)^{-1}\mathbf{1}$ by applying the matrix identity (14) to equation (13), with $k = 1$, $X = \mathbf{A}^2\,\mathbf{T}$, $Y = I_{1\times 1}$, and $U^\top = V = \sqrt{x}(\mathbf{1} - \boldsymbol{a})$. This gives us:

$$\begin{aligned}\mathbf{1}^\top C(\beta)^{-1}\mathbf{1} &= \mathbf{1}^\top\left(\mathbf{A}^{-2}\,\mathbf{T}^{-1} - x\frac{\mathbf{A}^{-2}\,\mathbf{T}^{-1}(\mathbf{1}-\boldsymbol{a})(\mathbf{1}-\boldsymbol{a})^\top\mathbf{A}^{-2}\,\mathbf{T}^{-1}}{1 + x(\mathbf{1}-\boldsymbol{a})^\top\mathbf{A}^{-2}\,\mathbf{T}^{-1}(\mathbf{1}-\boldsymbol{a})}\right)\mathbf{1}\\ &= \mathbf{1}^\top\mathbf{A}^{-2}\,\mathbf{T}^{-1}\mathbf{1} - x\frac{\left(\mathbf{1}^\top\mathbf{A}^{-2}\,\mathbf{T}^{-1}(\mathbf{1}-\boldsymbol{a})\right)^2}{1 + x(\mathbf{1}-\boldsymbol{a})^\top\mathbf{A}^{-2}\,\mathbf{T}^{-1}(\mathbf{1}-\boldsymbol{a})}\mathbf{1}. \qquad (16)\end{aligned}$$

We have the identities

$$\begin{aligned}\mathbf{1}^\top\mathbf{A}^{-2}\,\mathbf{T}^{-1}(\mathbf{1}-\boldsymbol{a}) &= nx + yx^2,\\ \mathbf{1}^\top\mathbf{A}^{-2}\,\mathbf{T}^{-1}\mathbf{1} &= yx^2 + 2nx + z/y,\\ (\mathbf{1}-\boldsymbol{a})^\top\mathbf{A}^{-2}\,\mathbf{T}^{-1}(\mathbf{1}-\boldsymbol{a}) &= yx^2, \qquad (17)\end{aligned}$$

from the expression (12) for $\boldsymbol{a}$. These identities allow us to simplify equation (16):

$$\mathbf{1}^\top C(\beta)^{-1}\mathbf{1} = yx^2 + 2nx + \frac{z}{y} - \frac{(n + yx)^2}{x^{-1} + y}.$$

Finally, setting $f(\beta) = 1/\mathbf{1}^\top C(\beta)^{-1}\mathbf{1}$ and simplifying gives us the result. ☐

We are now ready to prove the main theorem of this section.

Theorem 8. *When G is a complete graph, best-response dynamics converge to a unique steady-state, for all starting estimators $\mathbf{y}(0)$ and all choices of parameters $\{\tau_i\}$ and σ. Moreover, the convergence is fast, in the sense that $-\log|\beta(t) - \beta^*| = O(t)$, where $\beta^* = \lim_{t\to\infty}\beta(t)$.*

Proof. We will make use of the Banach fixed point theorem [7] which states that if there exists some $k < 1$ such that $|f'(\beta)| < k$ for all β, then there is a unique fixed point β^* of f, and iterates of f satisfy $|f^t(\beta) - \beta^*| < \frac{k^t}{1-k}|\beta - f(\beta)|$ for all starting points β. Thus, given this theorem, we need only show $|f'(\beta)| < k$ for some $k < 1$.

First note that the $n = 1$ case reduces to a Kalman filter, which we review briefly in Section 5.3.

We will find it convenient to think of horizontal shift $g(x) = f(x - \sigma^2)$, where $x = \beta + \sigma^2$, but allow any $x > 0$. That is, g can be thought of as taking the variance x of the estimate of the process *this* round using only estimates from last round. We first compute g and its first two derivatives; letting $D = y(y - (n-2)nx) + xz(x + y)$, we have

$$g(x) = xy(x + y)/D \qquad (18)$$

$$g'(x) = y^2(y - (n-2)x)(nx + y)/D^2 \qquad (19)$$

$$g''(x) = 2y^2\left((n-2)nx^3z + y^3\left((n-1)^2 - z\right) - 3x^2yz - 3xy^2z\right)/D^3 \qquad (20)$$

Before proving bounds on f', we prove a few useful observations:

A. $z \geq n^2$. This follows from the Cauchy-Schwarz inequality, since $\sum_i \tau_i\tau_i^{-1} = n$.

B. $D > 0$. Expanding D, we have three strictly positive terms $2nxy + y^2 + x^2z$ plus $xyz - n^2xy$, which is non-negative by observation A.

C. $g'(x) < 0 \iff y < (n-2)x$. Since $D > 0$ by observation B, this follows from (19).

Upper bound: We will show that there exists some k_U such that $f'(\beta) < k_U < 1$ for all $\beta > 0$ by showing $g'(x) < k_U$ for all $x > \sigma^2$. First, note that $g'(0) = 1$. Next, by observation C and some simple algebra, one can show that $g''(x) < 0$ as long as $g'(x) > 0$; that is, g' is strictly decreasing while positive, until $x = (n-2)/y$. Hence, letting $k_U := f'(0)$, we have $k_U = g'(\sigma^2) < g'(0) = 1$. Finally, we have $f'(\beta) < f'(0) = k_U$ for all $\beta > 0$, so k_U is our upper bound.

Lower bound: To show the lower bound, we minimize g' with respect to x as well as all the parameters. Let us start with z. By observation C, we know that the minimum value of $g'(x)$ must occur when $x >$

$(n-2)/y$, the region where $g'(x) < 0$ (note that if $n = 2$ we get a trivial lower bound of 0, so henceforth we will assume $n > 2$). In this region, it is clear from observations A and B that the minimum of g' with respect to z occurs when $z = n^2$. Substituting and simplifying, we have

$$g'(x) \geq \frac{y^2(y-(n-2)x)}{(nx+y)^3} := h(n,x,y).$$

We next minimize over x: solving $\frac{d}{dx}h(n,x,y) = 0$ for x yields $x = y\frac{2n-1}{n(n-2)}$, and one can check that indeed $\frac{d^2}{dx^2}h(n,x,y) > 0$ for this x. Substituting again, we are now left with

$$g'(x) \geq -\frac{(n-2)^3}{27n(n-1)^2},$$

which is now only a function of n. One can now easily see that $g' > -\frac{1}{27}$ for all n, x, y, z.

We have therefore shown that $|f'(\beta)| < k := \max(f'(0), 1/27) < 1$ for all β and all parameter values. □

As a concluding comment we analyze the steady-state β^*. From the form of f, one can show that β^* satisfies:

$$\begin{aligned} 0 = {} & y\sigma^2\left(y+\sigma^2\right) - \sigma^2\left(y(-(n-2)n+z-2)+z\sigma^2\right)\beta \\ & -\left((n-1)^2y - z\left(y+2\sigma^2\right)\right)\beta^2 - z\beta^3 \end{aligned} \tag{21}$$

As a corollary of Theorem 8, this cubic polynomial has a unique positive root.

7 Fixed response dynamics

Recall that in fixed response dynamics each agent i, at each round t, has access to its neighbors' estimators from the previous round, $\{Y_j(t-1)|j \in \partial i\}$, as well as its current measurement $M_i(t)$. The new estimates are $\mathbf{y}(t) = \mathbf{A}\cdot\mathbf{m}(t) + \mathbf{P}\cdot\mathbf{y}(t-1)$, i.e., fixed convex linear combinations of these values.

We first show the system converges to a steady state: as t tends to infinity the covariance matrix $\mathbf{C}(t) = \mathrm{Var}\left[\mathbf{y}(t) - \mathbf{1}S(t)\right]$ tends to some matrix $\mathbf{C}$.

Note that as in Section 6.2 above, a result of the convexity condition is that the choice of $\mathbf{P}$ uniquely determines $\mathbf{A}$.

7.1 Convergence of fixed response dynamics

To prove our theorem we shall need the following lemma, as well as an easy corollary of Proposition 1.

Lemma 9. *Let $P_i \in \mathbb{R}_+^{n\times n}$ satisfy $\|P_i\|_\infty \leq \gamma$ for all $1 \leq i \leq t$. Then for any $Q \in \mathbb{R}_+^{n\times n}$*

$$\left\|\left(\prod_{i=1}^{t} P_i\right) Q \left(\prod_{i=t}^{1} P_i^\top\right)\right\|_\infty \leq n\gamma^{2t}\|Q\|_\infty$$

Proof. This follows from two facts about the infinity norm. First, $\|\cdot\|_\infty$ is submultiplicative, meaning for all $A, B \in \mathbb{R}^{n\times n}$, $\|AB\|_\infty \leq \|A\|_\infty\|B\|_\infty$. Second, for all A we have $\|A\|_\infty \leq n\|A^\top\|_\infty$. These two combined give us

$$\left\|\prod_{i=1}^{t} P_i\right\|_\infty \leq \gamma^t \quad \text{and} \quad \left\|\prod_{i=t}^{i} P_i^\top\right\|_\infty \leq n\left\|\prod_{i=1}^{t} P_i\right\|_\infty \leq n\gamma^t,$$

and the result then follows from submultiplicativity. □

Corollary 10.

$$\mathbf{C}(t) = \mathbf{P}^t\mathbf{C}(0){\mathbf{P}^\top}^t + \sum_{r=0}^{t-1}\mathbf{P}^r(\mathbf{A}^2 + \sigma^2 P\mathbf{1}\mathbf{1}^\top\mathbf{P}^\top){\mathbf{P}^\top}^r. \tag{22}$$

The main theorem of this subsection is the following.

Theorem 11. *In fixed response dynamics, if $A_i > 0$ for all $i \in [n]$ then system converges to a steady state $\mathbf{C} = \lim_{t\to\infty}\mathbf{C}(t)$ such that*

$$\mathbf{C} = \mathbf{A}^2\mathbf{T} + \sigma^2\mathbf{P}\mathbf{1}\mathbf{1}^\top\mathbf{P}^\top + \mathbf{P}\mathbf{C}\mathbf{P}^\top. \tag{23}$$

In particular, $\mathbf{C}$ is independent of the starting estimators $\mathbf{y}(0)$.

Proof. Let $\gamma = 1 - \max_i A_i < 1$. Then since the entries of $\mathbf{P}$ are non-negative, the absolute row sums of $\mathbf{P}$ are less than γ, so we have $\|\mathbf{P}\|_\infty \le \gamma$. Letting $\mathbf{Z} = \mathbf{A}^2 + \sigma^2\mathbf{P}\mathbf{1}\mathbf{1}^\top\mathbf{P}^\top$, we have by Lemma 9 that

$$\begin{aligned}\|\mathbf{C}(t+1) - \mathbf{C}(t)\|_\infty &= \left\|\sum_{r=0}^{t}\mathbf{P}^r X{\mathbf{P}^\top}^r + \mathbf{P}^{t+1}\mathbf{C}(0){\mathbf{P}^\top}^{t+1} - \sum_{r=0}^{t-1}\mathbf{P}^r\mathbf{Z}{\mathbf{P}^\top}^r - \mathbf{P}^t\mathbf{C}(0){\mathbf{P}^\top}^t\right\|_\infty \\ &= \left\|\mathbf{P}^t(\mathbf{Z} + \mathbf{P}\mathbf{C}(0)\mathbf{P}^\top - \mathbf{C}(0)){\mathbf{P}^t}^\top\right\|_\infty \\ &\le n\gamma^{2t}\|\mathbf{Z} + \mathbf{P}\mathbf{C}(0)\mathbf{P}^\top - \mathbf{C}(0)\|_\infty.\end{aligned}$$

Thus since $\gamma < 1$, we have $\lim_{t\to\infty}\|\mathbf{C}(t+1) - \mathbf{C}(t)\|_\infty = 0$. Moreover, for all t we have

$$\begin{aligned}\|\mathbf{C}(t)\|_\infty &= \left\|\sum_{r=0}^{t-1}\mathbf{P}^r\mathbf{Z}{\mathbf{P}^\top}^r + \mathbf{P}^t\mathbf{C}(0){\mathbf{P}^\top}^t\right\|_\infty \\ &\le (\|\mathbf{Z}\|_\infty + \|\mathbf{C}(0)\|_\infty)\sum_{r=0}^{t}n\gamma^{2r} \\ &\le n\frac{\|\mathbf{Z}\|_\infty + \|\mathbf{C}(0)\|_\infty}{1-\gamma^2} < \infty,\end{aligned}$$

so clearly $\lim_{t\to\infty}\|\mathbf{C}(t)\|_\infty < \infty$. Thus, $\lim_{t\to\infty}\mathbf{C}(t)$ exists, and (23) follows from the recurrence in equation (9).

To see that the choice of $\mathbf{C}(0)$ is immaterial, consider the alternate sequence $\tilde{\mathbf{C}}(t)$ resulting from another choice $\tilde{\mathbf{C}}(0)$ (but the same $\mathbf{P}$). By definition,

$$\tilde{\mathbf{C}}(t) = \sum_{r=0}^{t-1}\mathbf{P}^r\mathbf{Z}{\mathbf{P}^\top}^r + \mathbf{P}^t\tilde{\mathbf{C}}(0){\mathbf{P}^\top}^t.$$

By Lemma 9 we have that

$$\|\mathbf{C}(t) - \tilde{\mathbf{C}}(t)\|_\infty = \|\mathbf{P}^t(\mathbf{C}(0) - \tilde{\mathbf{C}}(0)){\mathbf{P}^\top}^t\|_\infty \le n\gamma^{2t}\|\mathbf{C}(0) - \tilde{\mathbf{C}}(0)\|_\infty,$$

so we clearly have $\lim_{t\to\infty}\|\mathbf{C}(t) - \tilde{\mathbf{C}}(t)\|_\infty = 0$. Thus, $\lim_{t\to\infty}\tilde{\mathbf{C}}(t) = \lim_{t\to\infty}\mathbf{C}(t) = \mathbf{C}$. □

7.2 Non-optimality of best response steady-state

By Theorem 8 best response dynamics converges to a unique steady state. The next result shows that although, in best responding, agents minimize the variance of their estimators, in some cases they can converge to a steady state with lower variances by an appropriate choice of a fixed response. I.e., by cooperating the agents can achieve better results than by greedily choosing the short-term minimum.

We consider the case of the complete graph over n players where the agents measurement errors τ_i, are the same and equal $\tau = 1$, and also the standard deviation of the state's random walk $\sigma = 1$.

Before proving the main theorem of this subsection we establish the following lemma.

Lemma 12. *Let* $\mathbf{C}_{(n)}$ *be the steady state of* fixed response *dynamics on the complete graph with* n *agents,* $\tau_i = \sigma = 1$*,* $A_i = \alpha$ *for all* i *and* $P_{ij} = (1-\alpha)/n$ *for all* i *and* j*. Then*

$$C_{ii}(t) = \alpha^2 + \frac{(1-\alpha)^2(1+\alpha^2/n)}{(2-\alpha)\alpha}. \tag{24}$$

Proof. Let $\beta_n^\alpha = 1/\mathbf{1}^\top \mathbf{C}_{(n)}^{-1}\mathbf{1}$, and let $Z_n(t) = \frac{1}{n}\sum_i Y_i(t)$, so that

$$\beta_n^\alpha = \lim_{t\to\infty} \mathrm{Var}\left[Z_n(t) - S(t)\right].$$

By the symmetry of the model we have that $Y_i(t+1) = (1-\alpha)Z_n(t) + \alpha M_i(t+1)$, and so

$$Z_n(t+1) = (1-\alpha)Z_n(t) + \alpha\frac{1}{n}\sum_i M_i(t+1).$$

Therefore

$$\beta_n^\alpha = \mathrm{Var}\left[Z_n(t+1) - S(t+1)\right] = \frac{\alpha^2}{n}\tau^2 + (1-\alpha)^2(\beta_n^\alpha + \sigma^2),$$

which, since $\tau = \sigma = 1$, implies by simple manipulation that

$$\beta_n^\alpha = \frac{\alpha^2/n + (1-\alpha^2)}{1-(1-\alpha^2)}.$$

Finally, because $Y_i(t) = \alpha M_i(t) + (1-\alpha)Z_n(t-1)$ then

$$\begin{aligned} C_{ii}(t) &= \lim_{t\to\infty} \mathrm{Var}\left[Y_i(t) - S(t)\right] = \alpha^2 + (1-\alpha)^2(\beta_n^\alpha + 1) \\ &= \alpha^2 + \frac{(1-\alpha)^2(1+\alpha^2/n)}{(2-\alpha)\alpha}. \end{aligned}$$

□

Theorem 13. *Let* G *be a graph with* $[n]$ *vertices. Fix* σ *and* $\{\tau_i\}_{i\in[n]}$*.*

Consider best response *dynamics for* n *agents on* G *with* σ *and* $\{\tau_i\}_{i\in[n]}$*. Let* $\mathbf{C}^{br}$ *denote the steady state the system converges to.*

Consider fixed response *dynamics with some* $\mathbf{P}$ *and* $\mathbf{A}$ *for* n *agents on* G *with* σ *and* $\{\tau_i\}_{i\in[n]}$*. Let* $\mathbf{C}^{fr}$ *denote the steady state the system converges to.*

Then there exists a choice of n*,* G*,* σ*,* $\{\tau_i\}$*,* $\mathbf{A}$ *and* $\mathbf{P}$ *such that* $C_{ii}^{br} > C_{ii}^{fr}$ *for all* $i \in [n]$*.*

Proof. Let $n = 2$, let G be the complete graph on two vertices, let $\sigma = 1$ and let $\tau_i = 1$ for all $i \in [n]$. Let $A_i = \alpha$ for all i and $P_{ij} = (1-\alpha)/n$ for all i and j. Then by Lemma 12 we have that

$$C_{ii}^{fr} = \alpha^2 + \frac{(1-\alpha)^2(1+\alpha^2/2)}{(2-\alpha)\alpha}.$$

for all i.

By Eq. (21) we have that

$$C_{ii}^{br} = 2 - \sqrt{2} \approx 0.58578,$$

for all i.

It is easy to verify that for $\alpha = 0.60352$, for example (which is in fact the minimum), it holds that $C_{ii}^{fr} \approx 0.58472$ and so $C_{ii}^{br} > C_{ii}^{fr}$. □

We note that the choice $n = 2$ was made to make the proof above simple, rather than being a pathological example; we now show that the same holds for n large enough.

Lemma 14. *Let $\mathbf{C}^{br,(n)}$ be the steady state of* best response *dynamics on the complete graph with n agents and $\tau_i = \sigma = 1$. Then*

$$\lim_{n\to\infty} C_{ii}^{br,n} = \frac{\beta+1}{\beta+2}, \qquad \text{where} \qquad \beta = \tfrac{2}{3}\sqrt{7}\cos\left(\tfrac{1}{3}\tan^{-1}(3\sqrt{3})\right) - \tfrac{4}{3}.$$

We omit the proof and mention that it follows directly by substitution into, and solution of, the cubic polynomial of Eq. (21), and Eq. (12). We likewise omit the proof of the following lemma, which is an immediate corollary of Lemma 12 above.

Lemma 15. *Let $\mathbf{C}^{fr,n,\alpha}$ be the steady state of* fixed response *dynamics on the complete graph with n agents, $\tau_i = \sigma = 1$, $A_i = \alpha$ for all i, and $P_{ij} = (1-\alpha)/n$ for all i and j. Then*

$$\lim_{n\to\infty} C_{ii}^{fr,n,\alpha} = \alpha^2 + \frac{1}{\alpha(2-\alpha)} - 1.$$

Hence setting $\alpha_\infty = 0.59075$ (again the minimum) we get

$$\lim_{n\to\infty} C_{ii}^{fr,n,\alpha_\infty} \approx 0.55017,$$

and using Lemma 14 it is easy to numerically verify that

$$\lim_{n\to\infty} C_{ii}^{br,n} \approx 0.55496.$$

Thus we have shown that for large enough n it again holds that $C_{ii}^{br} > C_{ii}^{fr}$ on the complete graph, for the appropriate choice of parameters. We conjecture that this is in fact achievable for all $n > 1$, with the correct choice of parameters. In the case of $n = 1$ best response is optimal by Proposition 5.

7.3 Socially asymptotic learning

Recall that in the complete graph setting with fixed σ and τ and $\tau_i < \tau$ for all $i \in [n]$ we say that a dynamics is socially asymptotically learning if the variance of each agent's estimator approaches $\frac{\sigma^2\tau_i^2}{\sigma^2+\tau_i^2}$ as the number of agents increases.

Surprisingly, no fixed response dynamics can achieve socially asymptotic learning unless either $\sigma = 0$ or $\tau = 0$. This, of course, includes the steady state of the best response dynamics. In the case that $\sigma = 0$ the value of $S(t)$ is constant over time, and we are in the DeGroot model which is known to converge. In the case that $\tau = 0$ each agent receives the exact value of $S(t)$ in each round and can simply set $Y_i(t) = M_i(t)$ to asymptotically learn.

Theorem 16. *If $\sigma, \tau > 0$, no fixed response dynamics can achieve socially asymptotic learning.*

Proof. For the sake of contradiction, first assume that there exists some fixed response scheme that permits socially asymptotic learning so that in the limit $C_{ii} = \frac{\sigma^2\tau_i^2}{\sigma^2+\tau_i^2}$ for all $i \in [n]$. We analyze Equation 9 which states that: $\mathbf{C}(t) = \mathbf{A}(t)^2\mathbf{T} + \sigma^2\mathbf{P}(t)\mathbf{1}\mathbf{1}^\top\mathbf{P}(t)^\top + \mathbf{P}(t)\mathbf{C}(t-1)\mathbf{P}(t)^\top$.

It is easy to see that each of the three terms on the right hand side of Equation 9 is a positive semidefinite matrix and thus will have non-negative diagonal entries. By our assumption that we have a socially asymptotic learner, we know that the sum of the i,i entries in the three matrices on the right hand side of Equation 9 equals $\frac{\sigma^2\tau_i^2}{\sigma^2+\tau_i^2}$ in the limit. Now, $(\mathbf{P}(t)\mathbf{1}\mathbf{1}^\top\mathbf{P}(t)^\top)_{i,i} = \sum_{j,k} p_{ij}p_{ik} = \sum_j p_{ij}\sum_k p_{ik} = (1-\alpha_i)^2$. By fixing ε_i so that $\alpha_i = \frac{\sigma^2+\varepsilon_i}{\sigma^2+\tau_i^2}$ we see that in the limit

$$\begin{aligned}
\frac{\sigma^2\tau_i^2}{\sigma^2+\tau_i^2} = C_{ii} &\le (\mathbf{A}(t)^2\mathbf{T})_{i,i} + (\sigma^2\mathbf{P}(t)\mathbf{1}\mathbf{1}^\top\mathbf{P}(t)^\top)_{i,i} + (\mathbf{P}(t)\mathbf{C}(t-1)\mathbf{P}(t)^\top)_{i,i} \\
&= \alpha_i^2\tau_i^2 + (1-\alpha_i)^2\sigma^2 + (\mathbf{P}(t)\mathbf{C}(t-1)\mathbf{P}(t)^\top)_{i,i} \\
&= \frac{\sigma^2\tau_i^2}{\sigma^2+\tau_i^2} + \frac{\varepsilon_i^2}{\sigma^2+\tau_i^2} + (\mathbf{P}(t)\mathbf{C}(t-1)\mathbf{P}(t)^\top)_{i,i} \qquad (25)
\end{aligned}$$

and we have that in the limit for all $i \in [n]$: ε_i goes to 0, $(\mathbf{P}(t)\mathbf{C}(t-1)\mathbf{P}(t)^\top)_{i,i}$ goes to 0, and α_i goes to $\frac{\sigma^2}{\sigma^2+\tau_i^2}$.

We will now show that it cannot be that both α_i goes to $\frac{\sigma^2}{\sigma^2+\tau_i^2}$ and that $(\mathbf{P}(t)\mathbf{C}(t-1)\mathbf{P}(t)^\top)_{i,i}$ goes to 0. Another application of Equation 9 yields that

$$\begin{aligned}(\mathbf{P}(t)\mathbf{C}(t-1)\mathbf{P}(t)^\top)_{i,i} = (\mathbf{P}(t)\mathbf{A}(t-1)^2\mathbf{P}(t)^\top)_{i,i} + \sigma^2(\mathbf{P}(t)\mathbf{P}(t-1)\mathbf{1}\mathbf{1}^\top\mathbf{P}(t-1)^\top\mathbf{P}(t)^\top)_{i,i} \\ + (\mathbf{P}(t)\mathbf{P}(t-1)\mathbf{C}(t-2)\mathbf{P}(t-1)^\top\mathbf{P}(t)^\top)_{i,i}\end{aligned}$$

and again all the matrices on the right hand side are positive semi-definite and thus have non-negative diagonals. Thus:

$$\begin{aligned}(\mathbf{P}(t)\mathbf{C}(t-1)\mathbf{P}(t)^\top)_{i,i} &\geq \sigma^2(\mathbf{P}(t)\mathbf{P}(t-1)\mathbf{1}\mathbf{1}^\top\mathbf{P}(t-1)^\top\mathbf{P}(t)^\top)_{i,i} \\ &= \sigma^2 \sum_{j,k,l,m} p_{ij}p_{jk}p_{il}p_{lm} = \sigma^2\left(\sum_j p_{ij}\sum_k p_{jk}\right)^2 = \sigma^2\left(\sum_j p_{ij}(1-\alpha_j)\right)^2 \\ &\geq \sigma^2(1-\alpha_i)^2\min_j(1-\alpha_j)^2 \geq \frac{\sigma^2\tau^8}{(\sigma^2+\tau^2)^4} > 0,\end{aligned}$$

and this is a contradiction because we saw that $(\mathbf{P}(t)\mathbf{C}(t-1)\mathbf{P}(t)^\top)_{i,i}$ limited to 0. The penultimate inequality is because α_i limits to $\frac{\sigma^2}{\sigma^2+\tau_i^2}$ and because $\tau_i \leq \tau$ for all i. □

8 Penultimate prediction dynamics

In this section we consider players that can remember one value from the previous round. We show that, in the case of the complete graph, this allows the agents to learn substantially more efficiently than in the previous models. Also on the complete graph we show that this model features perfect learning. In general, it is not clear that the optimal strategy involving one remembered value must necessarily have this property.

We call this model the penultimate prediction model because an agent using it is effectively trying to re-estimate the value of the underlying state in the previous round, disregarding its own new measurement from the current round. This may help discount the older information that contributed to the prediction of each neighbor in the previous round.

As defined in Section 3.1, in this model each agent i does the following at time t: (a) agent i first picks A_i and $\{P_{ij}\}_j$ that minimize $\mathrm{Var}[R_i(t) - S(t-1)]$ where $\mathbf{r}(t) = \mathbf{A}\cdot\mathbf{r}(t-1) + \mathbf{P}\cdot\mathbf{y}(t-1)$, and (b) agent i then chooses $k_i(t)$ that minimizes $\mathrm{Var}[Y_i(t) - S(t)]$ where $Y_i(t) = k_i(t)M_i(t) + (1-k_i(t))R_i(t)$.

8.1 Complete graph case

We show that the penultimate prediction model achieves perfect learning on the complete graph. This means that agent i learns $S(t)$ as if she had access to every agent's measurements from all the previous rounds, rather than just her neighbors' estimators from the last round.

Theorem 17. *Penultimate prediction on the complete graph achieves perfect learning.*

Proof. Let

$$\tau_* = \left(\sum_{i\in[n]}\tau_i^{-2}\right)^{-\frac{1}{2}},$$

let $q_i = \tau_*^2/\tau_i^2$ and let $\bar{M}(t) = \sum_i q_i M_i(t)$ be the average of the measurements of the agents at time t, weighted by the inverse of their variance. Then $\bar{M}(t)$ is the MVULE of $S(t)$ given $\mathbf{m}(t)$ (see Corollary 3).

Let $E(t)$ be the MVULE of $S(t)$, given *all that is known up to time* t: $\{\mathbf{m}(s)|s \leq t\}$, together with $\mathbf{y}(0)$. Let $V(t) = \mathrm{Var}[E(t) - S(t)]$.

Basic Kalman filter theory (see, e.g. [19]) shows that $E(t)$ can be written as

$$E(t) = (1 - K(t))E(t-1) + K(t)\bar{M}(t), \tag{26}$$

with

$$V(t+1) = (V(t) + \sigma^2)(1 - K(t))$$

and $K(t) = \frac{V(t)+\sigma^2}{V(t)+\sigma^2+\tau_*^2}$. Note that $V(t)$ is deterministic.

We now prove by induction that $R_i(t) = E(t-1)$. The base case of $t = 1$ follows from definitions.

By our inductive hypothesis at step t we have that $R_i(t) = E(t-1)$ and $\text{Var}\left[R_i(t) - S(t-1)\right] = V(t-1)$. Hence $R_i(t)$ is identical for all agents and we can write $R(t) = R_i(t) = E(t-1)$. Because $R(t) - S(t-1)$ and $S(t) - S(t-1)$ are independent we have that

$$\text{Var}\left[R(t) - S(t)\right] = \text{Var}\left[R(t) - S(t-1)\right] + \sigma^2 = V(t) + \sigma^2.$$

Since $R(t) - S(t)$ and $M_i(t) - S(t)$ are independent, $\text{Var}\left[M_i(t) - S(t)\right] = \tau_i^2$, and since $Y_i(t)$ is the MVULE of $S(t)$ given $R(t)$ and $M_i(t)$, then by Corollary 4, $Y_i(t)$ will satisfy

$$Y_i(t) = k_i(t)R(t) + (1 - k_i(t))M_i(t), \tag{27}$$

where $k_i(t) = \frac{\tau_i^2}{V(t-1)+\sigma^2+\tau_i^2}$ is also deterministic. Thus $Y_i(t)$ is a deterministic function of $R(t)$ and $M_i(t)$, and more importantly $M_i(t)$ is a deterministic linear combination of $Y_i(t)$ and $R(t)$. Since $R(t+1)$ is the MVULE of $S(t)$ given $R(t)$ and $\mathbf{y}(t)$, its variance is bounded from above by MVULE of $S(t)$ given $R(t)$ and $\mathbf{m}(t)$. But by Eq. 26 we have that the optimum is achieved by $E(t)$, and so $R(t+1) = E(t)$.

We have therefore established that $Y_i(t)$ is the MVULE of $S(t)$ given $E(t-1)$ and $M_i(t)$, where $E(t-1)$ is the MVULE of $S(t-1)$ given *all the measurements* up to time $t-1$. To complete the proof we note that by the Markov property of $S(t)$ this means that $Y_i(t)$ is the MVULE of $S(t)$ given all the measurements up to time t, together with $M_i(t)$.

□

Corollary 18. *In the complete graph case, for any fixed σ and τ where $\tau_i \leq \tau$ for all $i \in [n]$, the penultimate prediction heuristic is a socially asymptotic learner.*

Proof. By Theorem 17 we need only show that the optimal learner is socially asymptotic. Fix some agent i. At round t if this agent is given $\mathbf{m}(t-1)$, then, by Corollary 3, he can predict $S(t-1)$ with $R_i(t)$ such that $\text{Var}\left[R_i(t) - S(t-1)\right] = \tau_*^2 = \left(\sum_{i\in[n]} \tau_i^{-2}\right)^{-1}$. However, note that $\tau_*^2 \leq \tau^2/n$. Agent i can then compute $Y_i(t) = \frac{\sigma^2}{\sigma^2+\tau_i^2}M_i(t) + \frac{\tau_i^2}{\sigma^2+\tau_i^2}R_i(t)$ and it is easy to see that $\text{Var}\left[Y_i(t) - S(t)\right] \leq \frac{\sigma^2\tau_i^2}{\sigma^2+\tau_i^2} + \frac{\tau_i^6}{n(\sigma^2+\tau_i^2)^2}$, which approaches $\frac{\sigma^2\tau_i^2}{\sigma^2+\tau_i^2}$ as n grows. □

9 Conclusion

This work can be seen as a study of natural extensions of the DeGroot model to the setting where the value to be learned changes over time. The most direct extension is the *fixed response* model. Here we show that while the estimate will keep moving with the true values, its variance will converge to a fixed value. However, in contrast to the DeGroot model, the agents are continually receiving new independent signals, and so have a reference point from which to evaluate the validity of their neighbors' signals. This leads us to propose the *best response* model. We show that in the case of the complete graph, best response dynamics will always converge to a particular fixed response that is (myopically) optimal. However, we also show that it is not necessarily Pareto optimal amongst all fixed responses. Finally, we show that a simple strengthening of the model to allow agents to remember one value can, in certain cases, lead to much improved performance. This can be seen not only as a critique of fixed response dynamics as being too weak to capture natural dynamics, but also as an interesting model to be studied more in its own right.

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