Extremal Black Holes in Dynamical Chern-Simons Gravity

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Rapidly rotating black hole solutions in theories beyond general relativity play a key role in experimental gravity, as they allow us to compute observables in extreme spacetimes that deviate from the predictions of general relativity (GR). Such solutions are often difficult to find in beyond-GR theories due to the inclusion of additional fields that couple to the metric non-linearly and non-minimally. In this paper, we consider rotating black hole solutions in one such theory, dynamical Chern-Simons gravity, where the Einstein-Hilbert action is modified by the introduction of a dynamical scalar field that couples to the metric through the Pontryagin density. We treat dynamical Chern-Simons gravity as an effective field theory and thus work in the decoupling limit, where corrections are treated as small perturbations from general relativity. We perturb about the maximally-rotating Kerr solution, the so-called extremal limit, and develop mathematical insight into the analysis techniques needed to construct solutions for generic spin. First we find closed-form, analytic expressions for the extremal scalar field, and then determine the trace of the metric perturbation, giving both in terms of Legendre decompositions. Retaining only the first three and four modes in the Legendre representation of the scalar field and the trace respectively suffices to ensure a fidelity of over 99% relative to full numerical solutions. The leading-order mode in the Legendre expansion of the trace of the metric perturbation contains a logarithmic divergence at the extremal Kerr horizon, which is likely to be unimportant as it occurs inside the perturbed dynamical Chern-Simons horizon. The techniques employed here should enable the construction of analytic, closed-form expressions for the scalar field and metric perturbations on a background with arbitrary rotation.

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I. INTRODUCTION

Einstein's theory of general relativity has passed a plethora of Solar System and binary pulsar tests [1], but it has not been tested in depth in the *extreme gravity* regime [2] where the gravitational interaction is non-linear and dynamical. A number of new observations will allow us to test this regime of Einstein's theory: in the gravitational wave spectrum through advanced LIGO and its partners, when compact objects collide; in the radio spectrum with the Event Horizon Telescope, when an accretion disk illuminates its host black hole and creates a 'shadow'; and in the X-ray spectrum with the Chandra Telescope, when gas heats up and glows as it accretes in the black hole spacetime. Such future observations will either confirm Einstein's theory at unprecedented levels or reveal new phenomena in the extreme gravity regime.

Solutions that represent rotating black holes (BHs) in theories of gravity beyond general relativity (GR) are an essential ingredient of tests in the extreme gravity regime. Constraining these theories requires a metric with which to calculate observables. Once a metric is available, one can investigate the linear and non-linear stability of the solution through a mode analysis, calculate the gravitational waves emitted as two BHs inspiral, compute the 'shadow' cast by a BH when illuminated by an accretion disk, and determine the energy spectrum of the radiation emitted by gas accreting into the BH.

One beyond-GR gravity theory in which generic rotating BH solutions have not yet been found is dynamical Chern-Simons (dCS) gravity [3]. This theory modifies the Einstein-Hilbert action by introducing a dynamical (pseudo) scalar field that couples non-minimally to the metric through the Pontryagin density. This interaction leads to a scalar field evolution equation that is sourced by the Pontryagin density, and modified metric field equations with third derivatives. The latter have cast doubt on whether full dCS is well-posed as an initial value problem [4], and also on whether stable BH solutions exist. We take the point of view that dCS must be treated as an effective field theory, since it is motivated from the low-energy limit of compactified heterotic string theory [5, 6] (for a review see [3]), from effective field theories of inflation [7], and from loop quantum gravity [8]. Thus the theory is treated in the *decoupling limit*, where deformations from GR are treated perturbatively, order-reducing the field equations.

When treated as an effective theory, BH solutions in dCS have been found in certain limits. Jackiw and Pi [9] showed that the Schwarzschild metric is also a solution in dCS that represents a non-rotating BH. Linear stability of high-frequency waves about the Schwarzschild background was suggested by [10]. The axionic hair on a slowly-rotating BH was first found in [11]. The metric solution to linear order in spin was found contemporaneously and independently by Yunes and Pretorius [12], and Konno, et al. [13]. This was expanded to quadratic order by Yagi, Yunes and Tanaka [14]. The first rapidly-rotating dCS BH studies were carried out by Konno and Takahashi [15] and Stein [16] who investigated the behavior of the dynamical scalar field about a rapidly rotating Kerr background. Stein also investigated the trace of the metric perturbation, and found that the extremal limit may be singular, which partly motivated the present work. At present, nobody has succeeded in constructing the full metric of generic rotating BHs in dCS gravity, despite over two decades of work in that direction [9, 11–19].

As a first step toward finding full rotating BH solutions in effective dCS gravity, we study the extremal limit, i.e. the limit in which the BH spin is close to the maximal Kerr value. The extremal limit is of interest not only because the mathematics simplify significantly, but also because of the Kerr/CFT conjecture that posits a dual holographic description in terms of a two-dimensional conformal field theory [20–25]. Working in the extremal limit, we have found a general, closed-form expression for the Legendre mode-decomposed scalar field. The radial structure of the scalar field is more complicated than that of the slowly-rotating case. Whereas the slow-rotation case only requires a finite polynomial expansion, the rapid-rotation case is characterized by natural logarithms and arctangents. The angular structure of the scalar field is still predominantly dipolar. We find that retaining only the first 3 non-vanishing modes of the scalar field suffices to achieve a fidelity above 99% in the entire domain relative to the full numerical solution.

With analytic, closed-form expressions for the extremal scalar field in hand, we then focus on the BH metric, treating the dCS correction as a small perturbation of the Kerr background. Using a convenient (harmonic) gauge, we show that the modified field equations for the trace of the metric perturbation can be solved in quadrature in terms of another Legendre mode decomposition. As in the scalar field case, the radial structure of the metric perturbation is quite different from that of slowly-rotating BHs, with a logarithmic divergence at the extremal Kerr horizon for the dominant monopole mode. This divergence confirms the one which was conjectured in [16] by one of the present authors. This divergence, however, may be unphysical since the Kerr horizon is likely "inside" the perturbed dCS horizon. The angular structure of the trace of the metric perturbation is still predominantly monopolar. We find that retaining only the first 4 non-vanishing modes of the trace of the metric perturbation suffices to achieve a fidelity above 99% in the entire domain relative to the full numerical solution.

The results we have obtained have various consequences for the study of BHs in beyond-GR gravity theories. The logarithmic divergence of the trace of the metric suggests a conjecture: that the dCS-corrected horizon is "outside" of the Kerr horizon for all possible values of angular momentum, protecting against a naked singularity. Further, our results suggest that generic rotating BH solutions in beyond-GR theories will not have the simple rational-polynomial form that the Kerr metric enjoys, and may require more complicated functional forms. The mode-decomposition technique we employed in the extremal limit is also applicable to sub-extremal BHs [16], and if the tensor equations admit decoupling and separation of variables, could be used to find the full metric deformation.

The remainder of this paper presents details of the techniques developed, the solutions obtained and their properties. Henceforth, we use the following conventions. Latin letters $(a, b, c, \ldots)$ in index lists stand for abstract indices. Parentheses and square brackets in index lists stand for symmetrization and anti-symmetrization respectively. The metric signature will be $(-,+,+,+)$ and we choose units in which $c = 1$. However, we do not set G or h to unity. All other conventions follow the standard treatment of [26, 27].

II. THE ABC OF DCS

Dynamical Chern-Simons gravity [3, 9] is a four-dimensional theory defined by the action

$$I = I_{\rm EH} + I_{\rm CS} + I_{\vartheta} + I_{\rm Mat}\,. \tag{1}$$

The first term is the Einstein-Hilbert action

$$I_{\rm EH} = \int d^4x \, \sqrt{-g} \, \left(\frac{1}{2\kappa^2}\, R\right) , \tag{2}$$

where $\kappa^2 = 8\pi G$, R is the Ricci scalar associated with the metric tensor $g_{\mu\nu}$, and g is the metric determinant. The last term in Eq. (1) is the action for all matter degrees of freedom, which couple minimally to the metric tensor and do not couple to ϑ.

The Chern-Simons correction is mediated by a canonically-normalized scalar field ϑ, whose kinetic term in the action is

$$I_{\vartheta} = \int d^4x \sqrt{-g} \left(-\frac{1}{2}\,(\partial_a \vartheta)\,(\partial^a \vartheta)\right) . \tag{3}$$

This scalar field couples non-minimally to the metric through the potential term in the action

$$I_{\rm CS} = \int d^4x \sqrt{-g} \, \left(-\frac{1}{4}\frac{\alpha}{\kappa}\, \vartheta \, {}^*RR\right) , \tag{4}$$

where the Pontryagin density is defined via

$${}^*RR := {}^*R^{abcd} R_{abcd} = \frac{1}{2}\epsilon^{abef} R_{ef}{}^{cd} R_{abcd}\,, \tag{5}$$

and ϵ^{abcd} is the Levi-Civita tensor. Notice that the definition of the Pontryagin density here differs from that of [3] by a minus sign, which is compensated by an additional minus sign in $I_{\rm CS}$.

Variation of the action with respect to the metric yields the field equations

$$G_{ab} + 2\,\alpha\kappa\, C_{ab} = \kappa^2 T_{ab}\,, \tag{6}$$

where G_{ab} is the Einstein tensor, and the traceless 'C-tensor' is defined as

$$C^{ab} = (\nabla_c \vartheta)\ \epsilon^{cde(a}\nabla_e R^{b)}{}_d + (\nabla_c \nabla_d \vartheta)\ {}^*R^{d(ab)c} . \quad (7)$$

The stress-energy tensor decomposes linearly into a term that depends only on the matter degrees of freedom and a term that depends only on the scalar field, i.e. $T_{ab} = T_{ab}^{\rm Mat} + T_{ab}^{\vartheta}$, where the latter is

$$T_{ab}^{\vartheta} := (\nabla_a \vartheta)(\nabla_b \vartheta) - \frac{1}{2} g_{ab} (\nabla_c \vartheta)(\nabla^c \vartheta) . \quad (8)$$

Variation of the action with respect to the scalar field yields its evolution equation

$$\square \vartheta = \frac{\alpha}{4\kappa} {}^*RR , \quad (9)$$

where $\square$ stands for the d'Alembertian operator. Notice that there is no potential associated with the scalar field, which implies it is a long-ranged field. This vanishing (or flat) potential means that ϑ retains a global shift symmetry, $\vartheta \to \vartheta$+const., because ${}^*RR$ is related to a topological invariant [28]. Retaining this shift symmetry may be important to protect against certain quantum corrections.

The theoretical motivation to study dCS is varied. From a string-theory standpoint, non-minimal scalar couplings of the form of Eq. (4) arise in the low-energy limit of heterotic string theory upon four-dimensional compactification [5, 6] (for a review see [3]). From a loop quantum gravity standpoint, dCS arises when the Barbero-Immirzi parameter is promoted to a scalar field in the presence of fermions [8, 29]. From a cosmology standpoint, the interaction in Eq. (4) arises as one of three terms that remain in an effective field theory treatment of single-field inflation [7].

The choice of conventions made here differs from that of [3]. The mapping between the two sets is $\kappa_{\rm AY} = 1/(2\kappa^2)$, $\beta_{\rm AY} = 1$ and $\alpha_{\rm AY} = \alpha/\kappa$. Moreover, we retain all factors of G, or equivalently of κ, since we do not set G to unity. Without requiring the action to have any specific sets of units, demanding consistency between $I_{\rm EH}$, $I_{\rm CS}$ and I_{ϑ} implies $[\vartheta] = [\kappa]^{-1}$ and $[\alpha] = L^2$, where L stands for units of length. Given some GR solution with characteristic length scale $\mathcal{L}$, corrections are then controlled by the dimensionless parameter $\zeta := \alpha^2/\mathcal{L}^4$. One can see this by noting that $|\partial_{ab}\vartheta| \propto (\alpha/\kappa)\mathcal{L}^{-4}$ from Eq. (9), which implies that $|C_{ab}| \propto (\alpha/\kappa)\mathcal{L}^{-6}$ from Eq. (7). Then the fractional corrections to GR are proportional to $(\alpha\kappa|C_{ab}|/|G_{ab}|) \propto \alpha^2\mathcal{L}^{-4} = \zeta$.

Current constraints on dCS are rather weak because dCS corrections are relevant only in scenarios where the spacetime curvature is large. One can see this by noting that dCS corrections to the gravitational field are sourced by the scalar field, which in turn is only sourced by the spacetime curvature. In fact, one can easily show through the argument given in the previous paragraph that constraints on the α parameter of dCS will be roughly proportional to a power of $\mathcal{L}$. Let us assume that some observation places the constraints $|\zeta| < \delta$, where δ is related to the observation and its uncertainties. This constraint can then be mapped to a constraint on α to find $\sqrt{|\alpha|} < \delta^{1/4}\mathcal{L}$. Currently, the best constraint on the dCS coupling parameter is $\sqrt{|\alpha|} \lesssim 10^8$ km and it comes from observations of Lense-Thirring precession from satellites in orbit around Earth [30]. Such a weak constraint makes sense when one realizes that for these kind of experiments the characteristic length scale $\mathcal{L} = [R_\oplus^3/(GM_\oplus)]^{1/2} \approx 2 \times 10^8$ km.

The aforementioned theoretical motivations suggest that one treat dCS as an *effective theory* valid up to some cut-off scale, i.e., the scale above which higher-order curvature terms in the action cannot be neglected [2]. We will here restrict attention to physical scenarios in which the effective theory is valid, and since we are interested only in black holes, this means we restrict attention to those with masses $GM \gg \sqrt{\alpha}$. When this is the case, we can work in the decoupling limit of the theory, i.e. we perform a perturbative expansion of the field equations and their solutions in powers of ζ. Henceforth, dCS is exclusively treated in the decoupling limit.

The decoupling limit can be implemented in practice by expanding the metric tensor and the scalar field in powers of ζ. In this paper, we will expand the metric and the scalar field as follows:

$$g_{ab} = g_{ab}^{(0)} + \zeta^{1/2}\, g_{ab}^{(1/2)} + \zeta\, g_{ab}^{(1)} + \mathcal{O}(\zeta^{3/2}) , \quad (10)$$

$$\vartheta = \frac{1}{\kappa}\tilde{\vartheta}^{(0)} + \frac{1}{\kappa}\zeta^{1/2}\, \tilde{\vartheta}^{(1/2)} + \frac{1}{\kappa}\zeta\, \tilde{\vartheta}^{(1)} + \mathcal{O}(\zeta^{3/2}) , \quad (11)$$

where the superscript denotes the order in ζ of each term. Notice that a factor of κ^{-1} in the expansion for the scalar field ensures that $\tilde{\vartheta}^{(n)}$ is dimensionless. As we are perturbing about $\zeta = 0$, our background solution $(g^{(0)}, \tilde{\vartheta}^{(0)})$ must solve the field equations for GR and a free massless scalar field. Choosing trivial initial data for $\tilde{\vartheta}^{(0)}$ gives $\tilde{\vartheta}^{(0)} = 0$ at all times, so we find $(g_{ab}^{(0)}, \tilde{\vartheta}^{(0)}) = (g_{ab}^{\rm GR}, 0)$ at zeroth order, where $g_{ab}^{\rm GR}$ is some known GR solution. If we next examine the system at order $\zeta^{1/2}$, we find that $g_{ab}^{(1/2)}$ satisfies a homogeneous linear equation due to the vanishing of $\tilde{\vartheta}^{(0)}$. Therefore, again, trivial initial data gives $g_{ab}^{(1/2)} = 0$ at all times.

Thus to the order we are working, our expansion is

$$g_{ab} = g_{ab}^{\rm GR} + \zeta\, g_{ab}^{(1)} + \mathcal{O}(\zeta^{3/2}) , \quad (12)$$

$$\vartheta = 0 + \frac{1}{\kappa}\zeta^{1/2}\, \tilde{\vartheta}^{(1/2)} + \mathcal{O}(\zeta^{1}) . \quad (13)$$

Henceforth, we will focus on BH solutions, with the $\mathcal{O}(\zeta^0)$ term in the metric, $g_{ab}^{\rm GR}$, being simply the Kerr metric. The $\mathcal{O}(\zeta^{1/2})$ term in the scalar field, $\vartheta^{(1/2)}$, is sourced by the Kerr metric and, in turn, this sources the $\mathcal{O}(\zeta)$ correction to the metric, $g_{ab}^{(1)}$. To be within the regime of validity of the perturbative expansion, we require $\zeta \ll 1$, and since for the Kerr black hole the typical curvature length scale is $\mathcal{L} = GM$, we take

$$\zeta = \frac{\alpha^2}{(GM)^4} \ll 1 . \quad (14)$$

Notice that this definition differs from others in the literature [3] in that we do not include a factor of $1/\kappa^2$ in ζ, but rather we factor it out in the scalar field directly.

In this paper we are concerned with solutions that represent rotating BHs spinning near extremality, so in addition to the decoupling expansion we will also perform a *near-extremal expansion*. Letting the BH spin angular momentum be $|\vec{J}|$, we can define the BH dimensionless spin parameter $\chi := |\vec{J}|/(GM)^2$. We can then expand all fields in the problem in a bivariate expansion, i.e. a simultaneous expansion in both $\zeta \ll 1$ and $\chi \sim 1$, namely

$$g_{ab}^{(n)} = g_{ab}^{(n,0)} + \varepsilon\, g_{ab}^{(n,1)} + \varepsilon^2\, g_{ab}^{(n,2)} + \mathcal{O}(\varepsilon^3)\,, \tag{15}$$

$$\tilde{\vartheta}^{(n)} = \tilde{\vartheta}^{(n,0)} + \varepsilon\, \tilde{\vartheta}^{(n,1)} + \varepsilon^2\, \tilde{\vartheta}^{(n,2)} + \mathcal{O}(\varepsilon^3)\,, \tag{16}$$

where $\varepsilon := (1-\chi^2)^{1/2}$ is a near-extremality parameter, i.e. $\varepsilon \ll 1$ for near-extremal BHs.

III. SCALAR FIELD: SOLUTION

We wish to solve the evolution equation for the scalar field [Eq. (9)] to leading order in ζ. To this order, the Pontryagin density on the right-hand side of Eq. (9) is evaluated on the unmodified Kerr spacetime. The wave operator on the left-hand side can also be evaluated on the Kerr spacetime, since corrections will be of $\mathcal{O}(\zeta)$. In polynomial Boyer-Lindquist coordinates, the scalar field evolution equation is evaluated on the line element [31]

$$\begin{aligned} g_{ab}^{(0)} dx^a dx^b = & -\frac{\Delta}{\Sigma}\left[dt - a\,\Gamma\, d\phi\right]^2 + \frac{\Sigma}{\Delta}dr^2 \\ & + \frac{\Sigma}{\Gamma}d\psi^2 + \frac{\Gamma}{\Sigma}\left((r^2+a^2)d\phi - a\,dt\right)^2, \end{aligned} \tag{17}$$

where the usual polar angle θ has been replaced with a coordinate $\psi = \cos\theta$, and $\Gamma := 1-\psi^2 = \sin^2\theta$. The mass of the black hole is M and it rotates with angular momentum per unit mass $a = |\vec{J}|/(GM)$, where $-GM \le a \le GM$. The functions Σ and Δ are

$$\Sigma = r^2 + a^2\psi^2 \tag{18}$$

$$\Delta = r^2 - 2GMr + a^2\ , \tag{19}$$

so that the background horizons, where $\Delta = (r-r_+)(r-r_-) = 0$, are located at $r_\pm = GM \pm \sqrt{(GM)^2 - a^2}$.

It will be convenient to replace all quantities with dimensionless variables by scaling out factors of GM: $\tilde{r} = r/(GM)$ and $\chi = a/(GM)$, so that the rescaled functions $\tilde{\Delta} = \Delta/(GM)^2 = (\tilde{r}-1)^2 - (1-\chi^2)$ and $\tilde{\Sigma} = \Sigma/(GM)^2 = \tilde{r}^2 + \chi^2\psi^2$. Assuming a stationary and axisymmetric solution for the scalar field, the $\mathcal{O}(\alpha)$ term in Eq. (9) then takes the form

$$\partial_{\tilde{r}}\left(\tilde{\Delta}\partial_{\tilde{r}}\tilde{\vartheta}^{(1/2)}\right) + \partial_\psi\left(\Gamma\partial_\psi\tilde{\vartheta}^{(1/2)}\right) = s^{(1/2)}(\tilde{r},\psi) \tag{20}$$

where factors of (α/κ) and (GM) have canceled from both sides of the equation. The source $s^{(1/2)}(\tilde{r},\psi)$ is proportional to $\Sigma\ ({}^*RR^{(0)})$ and given explicitly by

$$s^{(1/2)}(\tilde{r},\psi) = 24\,\frac{\chi\tilde{r}\psi(3\tilde{r}^2 - \chi^2\psi^2)(\tilde{r}^2 - 3\chi^2\psi^2)}{\tilde{\Sigma}^5}\ . \tag{21}$$

Equation (20) admits a solution via separation of variables, by expanding the solution

$$\tilde{\vartheta}^{(1/2)} = \sum_{\ell=0}^{\infty} \tilde{\vartheta}_\ell^{(1/2)}(\tilde{r}) P_\ell(\psi)\ , \tag{22}$$

where $P_\ell(\cdot)$ are Legendre functions of the first kind. The radial modes $\tilde{\vartheta}_\ell^{(1/2)}(\tilde{r})$ then satisfy the equation

$$\partial_{\tilde{r}}\left(\tilde{\Delta}\partial_{\tilde{r}}\tilde{\vartheta}_\ell^{(1/2)}\right) - \ell(\ell+1)\tilde{\vartheta}_\ell^{(1/2)} = s_\ell^{(1/2)}(\tilde{r})\ , \tag{23}$$

with source functions $s_\ell^{(1/2)}(\tilde{r})$ given by the modes in the Legendre decomposition of Eq. (21)

$$s_\ell^{(1/2)}(\tilde{r}) = \frac{2\ell+1}{2}\int_{-1}^{1} d\psi\, P_\ell(\psi) s^{(1/2)}(\tilde{r},\psi)\ . \tag{24}$$

Note that the source function in Eq. (21) is odd in the variable ψ, so its Legendre expansion (as well as that of the scalar field) will only contain odd modes: $\ell = 2n+1$ for all $n \in \mathbb{N}$.

The integral in Eq. (24) can be evaluated in closed form in terms of known functions:

$$\begin{aligned} s_\ell^{(1/2)}(\tilde{r}) = & (-1)^{\frac{\ell+1}{2}}\,\frac{\Gamma(\frac{1}{2})\Gamma(\ell+4)}{2^\ell\Gamma(\ell+\frac{1}{2})}\,\frac{\chi^\ell}{\tilde{r}^{\ell+4}} \\ & \times\Big[3\ {}_2F_1\left(\tfrac{\ell+4}{2}, \tfrac{\ell+5}{2}; \ell+\tfrac{3}{2}; -\tfrac{\chi^2}{\tilde{r}^2}\right) \\ & -(\ell+5)\ {}_2F_1\left(\tfrac{\ell+4}{2}, \tfrac{\ell+7}{2}; \ell+\tfrac{3}{2}; -\tfrac{\chi^2}{\tilde{r}^2}\right)\Big]\ , \end{aligned} \tag{25}$$

where ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ is the ordinary hypergeometric function and ℓ is odd. One can show, via identities for hypergeometric functions, that this expression is equivalent to one given previously in [32].

The solution of Eq. (23) can be obtained through the method of variation of parameters (see Appendix A). Defining a new variable $\eta = (\tilde{r}-1)/\sqrt{1-\chi^2}$, the solution of Eq. (23) [see also Eq. (A3)] for the mode function $\tilde{\vartheta}_\ell^{(1/2)}$ is

$$\begin{aligned} \tilde{\vartheta}_\ell^{(1/2)}(\tilde{r}) = & \ P_\ell(\eta)\int_\infty^\eta d\eta'\, s_\ell^{(1/2)}(1+\eta'\sqrt{1-\chi^2})\, Q_\ell(\eta') \\ & - Q_\ell(\eta)\int_1^\eta d\eta'\, s_\ell^{(1/2)}(1+\eta'\sqrt{1-\chi^2})\, P_\ell(\eta')\,, \end{aligned} \tag{26}$$

where $Q_\ell(\cdot)$ are Legendre functions of the second kind. This solution is regular at $\tilde{r}_+$, and approaches zero as $\tilde{r} \to \infty$.

Our eventual goal is to evaluate Eq. (26) in closed form for the full range of the rotation parameter, $-1 \le \chi \le 1$. The slow rotation limit of the field, i.e. the solution in a $|\chi| \ll 1$ expansion, is already well-understood; it was first derived in [12], verified in [13], and extended to second order in rotation in [14]. Similarly, it is also possible to systematically solve the scalar field equation of motion in the near-extremal expansion introduced in Sec. II. Expanding the source functions of Eq. (25) for $\varepsilon \ll 1$, we find

$$s_\ell^{(1/2)}(\tilde{r}) = s_\ell^{(1/2,\,0)}(\tilde{r}) + \varepsilon^2 s_\ell^{(1/2,\,2)}(\tilde{r}) + \varepsilon^4 s_\ell^{(1/2,\,4)}(\tilde{r}) + \mathcal{O}(\varepsilon^6)\,. \tag{27}$$

Recall that the second superscript in each of these terms represents the order in ε at which it enters the near-extremal expansion. Because the $\chi \to 1$ limit is regular for $s_\ell^{(1/2)}(\tilde{r})$, $s_\ell^{(1/2,\,0)}(\tilde{r})$ is simply $s_\ell^{(1/2)}(\tilde{r})$ evaluated at $\chi = 1$.

In this paper we will only consider the extremal limit, $\varepsilon \to 0$, which is the leading term in the near-extremal expansion. This corresponds to the limit $\chi \to \pm 1$ of the dimensionless spin parameter χ. The homogeneous solutions are regular in this limit [see Eqs. (A8)-(A9)], and the solution for the scalar field [see Eq. (A3)] at $\varepsilon = 0$ is

$$\tilde{\vartheta}_\ell^{(1/2,\,0)}(\tilde{r}) = \frac{1}{2\ell+1}\left[(\tilde{r}-1)^\ell \int_\infty^{\tilde{r}} d\tilde{r}'\, \frac{s_\ell^{(1/2,\,0)}(\tilde{r}')}{(\tilde{r}'-1)^{\ell+1}} - \frac{1}{(\tilde{r}-1)^{\ell+1}} \int_1^{\tilde{r}} d\tilde{r}'\,(\tilde{r}'-1)^\ell s_\ell^{(1/2,\,0)}(\tilde{r}')\right]\,. \tag{28}$$

The source functions at leading-order in ε, $s_\ell^{(1/2,\,0)}(\tilde{r})$, are given by Eq. (24) or Eq. (25) evaluated at $|\chi| = 1$. The boundary conditions are the same as before: each mode $\tilde{\vartheta}_\ell^{(1/2,\,0)}$ is regular at $\tilde{r}_+ = 1$, and goes to zero as $\tilde{r} \to \infty$.

The integrals in Eq. (28) can be readily evaluated for specific values of ℓ. For example, the $\ell = 1$ radial mode is given by

$$\tilde{\vartheta}_1^{(1/2,\,0)}(\tilde{r}) = 3(\tilde{r}-1)\log\left(\frac{\tilde{r}-1}{\sqrt{\tilde{r}^2+1}}\right) + 3(\tilde{r}-1)\,\mathrm{arccot}\,\tilde{r} + \frac{3(\tilde{r}-1)(2\tilde{r}^2+\tilde{r}+3)}{2(\tilde{r}^2+1)^2}. \tag{29}$$

With more work, we can also give an expression for general values of ℓ in terms of finite-order rational polynomials, $\mathrm{arccot}(\tilde{r})$, and the log which appears above. But before we can give the general form, we first have to establish two results for the behavior of the modes at large $\tilde{r}$ and at $\tilde{r} = 1$. The far-field behavior of the modes is dominated by the second integral in Eq. (28), since the first one decays with a higher power of r. This second integral converges in this limit, and thus, $\tilde{\vartheta}_\ell^{(1/2,\,0)} \sim \tilde{r}^{-(\ell+1)}$ as $\tilde{r} \gg 1$. The near-horizon behavior of the modes is dominated by the first integral in Eq. (28), but its asymptotic behavior as $\tilde{r} \sim 1$ cannot be easily discerned from that equation. Instead, it is easier to return to Eq. (26) and set $\chi = 1$, remembering that the horizon limit $\tilde{r} \to 1$ is equivalent to $\eta \to 1$ (this is the case for all values of χ). In this limit, only the first line of Eq. (26) contributes, leading to

$$\tilde{\vartheta}_\ell^{(1/2,\,0)}(1) = -\frac{1}{\ell(\ell+1)}\, s_\ell^{(1/2,\,0)}(1)\,. \tag{30}$$

The overall ℓ-dependent factor comes from the definite integration of $Q_\ell(\eta')$ in the range $\eta' \in (1,\infty)$, while $s_\ell^{(1/2,\,0)}(1)$ is Eq. (25) evaluated at $\chi = 1$ and $\tilde{r} = 1$.

With these results, we can now give a general expression for the radial modes of the scalar field. They take the form

$$\tilde{\vartheta}_\ell^{(1/2,\,0)}(\tilde{r}) = A_\ell(\tilde{r}) + B_\ell(\tilde{r})\,\mathrm{arccot}(\tilde{r}) + C_\ell(\tilde{r})\,\log\left(\frac{\tilde{r}-1}{\sqrt{\tilde{r}^2+1}}\right)\,, \tag{31}$$

where the functions $A_\ell(\tilde{r})$, $B_\ell(\tilde{r})$, and $C_\ell(\tilde{r})$ are

$$A_\ell(\tilde{r}) = (-1)^{\frac{\ell-1}{2}}\,\frac{\ell(\ell-1)}{(\tilde{r}-1)^{\ell+1}}\sum_{k=0}^{\ell}\gamma_k\,(\tilde{r}-1)^k - \frac{2\ell+1}{(\tilde{r}^2+1)^2} + \frac{(2\ell+1)(4\tilde{r}-\ell(\ell+1))}{4\,(\tilde{r}^2+1)} + \sum_{k=0}^{\ell-1}\alpha_{\ell,k}\,\tilde{r}^k \tag{32}$$

$$B_\ell(\tilde{r}) = (-1)^{\frac{\ell+1}{2}}\,\frac{\ell(\ell-1)}{(\tilde{r}-1)^{\ell+1}} + \sum_{k=0}^{\ell}\beta_{\ell,k}\,\tilde{r}^k \tag{33}$$

$$C_\ell(\tilde{r}) = (-1)^{\frac{\ell+1}{2}}\,\frac{(\ell+1)(\ell+2)}{2}\,(\tilde{r}-1)^\ell. \tag{34}$$

The constants γ_k appearing in $A_\ell(\tilde{r})$ are the first $\ell+1$ terms in the Taylor expansion of $\mathrm{arccot}(\tilde{r})$ around $\tilde{r} = 1$

$$\gamma_k = \frac{1}{k!}\left(\frac{\partial}{\partial x^k}\,\mathrm{arccot}(x)\right)\Bigg|_{x=1}\,. \tag{35}$$

The remaining $2\ell+1$ coefficients $\alpha_{\ell,k}$ and $\beta_{\ell,k}$ are fixed by imposing the boundary conditions: each mode falls off as $\tilde{r}^{-(\ell+1)}$ at large $\tilde{r}$, and takes the value Eq. (30) at $\tilde{r} = 1$. Alternately, the condition at $\tilde{r} = 1$ can be replaced with the requirement that the leading asymptotic behavior of the mode is given by Eq. (C25). The coefficients $\alpha_{\ell,k}$ and $\beta_{\ell,k}$ for the first several modes are given explicitly in Appendix B.

IV. SCALAR FIELD: PROPERTIES

Let us now discuss some properties of the scalar field solution obtained in the previous section. We begin by plotting the first five (odd) modes in Fig. 1. Observe that the integrated norm of $\tilde{\vartheta}_\ell^{(1/2,\,0)}$ decays exponentially with ℓ. This is because this function is a spectral solution to a differential equation with a C^∞ source, so it must converge exponentially with mode number. Observe also from Fig. 1 that the $\ell = 1$ mode of the field vanishes at the horizon. Modes with $\ell > 1$ are finite but non-zero at the horizon, with values that scale like $\ell^{5/2}(1+\sqrt{2})^{-(\ell+1)}$ for $\ell \gg 1$.

By including contributions from a sufficient number of modes, we can construct an arbitrarily accurate approximation of the full, extremal scalar field. An approximation

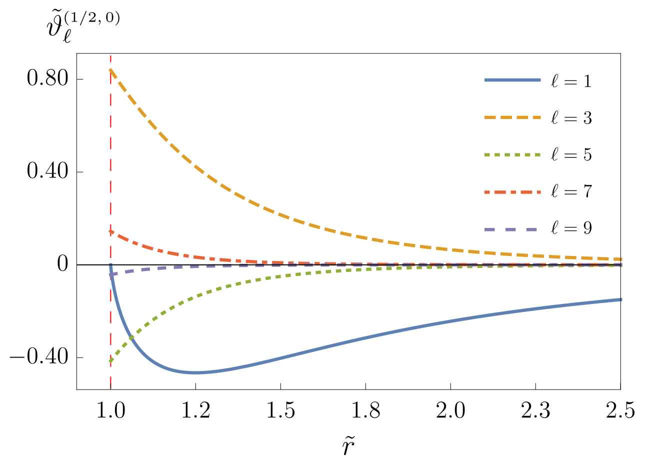

FIG. 1. The first five radial mode functions of the scalar field. The vertical dashed line indicates the location of the event horizon of an extremal black hole. The $\ell = 1$ mode vanishes at the horizon, while modes with $\ell > 1$ are all non-zero at $\tilde{r} = 1$.

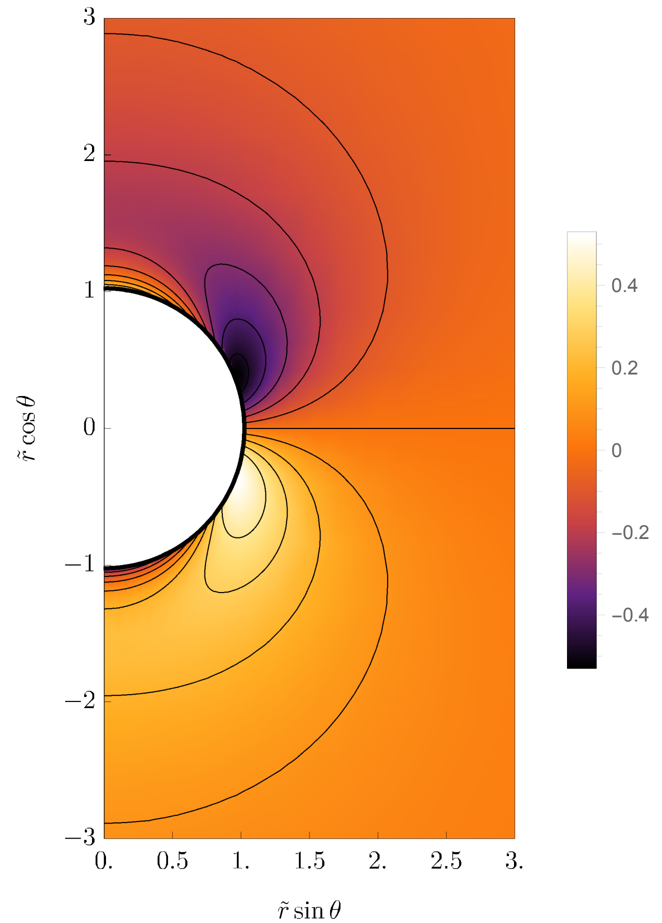

FIG. 2. The behavior of the scalar field on the extremal background, approximated by its first five Legendre modes. The coordinates $\tilde{r}\psi$ and $\tilde{r}\sqrt{1-\psi^2}$ correspond to $\tilde{r}\cos\theta$ and $\tilde{r}\sin\theta$, respectively, in conventional Boyer-Lindquist coordinates.

using the first five modes is shown in Fig. 2, as a function of both radius and polar angle. The accuracy of such an approximation can be characterized using a slicing-independent measure of the scalar field energy through the ADM energy. Let u^a be a timelike unit vector normal to a hypersurface $\mathcal{S}$, with γ_{ab} the induced metric on $\mathcal{S}$. Then the scalar field's contribution to the energy is

$$E = \int_{\mathcal{S}} d^3x\, \sqrt{\gamma}\, u^a T^{\vartheta}_{ab} t^b \tag{36}$$

where t^b is the Killing vector $\partial/\partial t$. This energy can be perturbatively expanded in powers of ζ,

$$E = \zeta\, E^{(1)} + \zeta^2\, E^{(2)} + \dots \tag{37}$$

The scalar field's ADM energy at leading order can further be computed via the spectral decomposition,

$$E^{(1)} = M \sum_{k=1}^{\infty} \tilde{E}_k^{(1)}, \tag{38}$$

with the dimensionless $\tilde{E}_k^{(1)}$ functions given by

$$\tilde{E}_k^{(1)} = \frac{1}{4}\frac{1}{2k+1}\int_{\tilde{r}_+}^{\infty} d\tilde{r}\left[\tilde{\Delta}(\partial_{\tilde{r}}\tilde{\vartheta}_k^{(1/2)})^2 + k(k+1)(\tilde{\vartheta}_k^{(1/2)})^2\right] . \tag{39}$$

The fractional difference between the total energy in the scalar field and the energy in the first N modes is then

$$\delta_N = 1 - \frac{M}{E^{(1)}}\sum_{k=1}^{N}\tilde{E}_k^{(1)} . \tag{40}$$

Figure 3 shows the fractional difference δ_N for the first seven nonvanishing modes. The contribution from the first five – up to $\ell = 9$ – differs from the total energy by less than one part in 10^4. Observe that the accuracy increases exponentially with N. Observe also that if we wish to capture 99% of the energy in the field, it suffices to keep only up to the first three odd modes, i.e. $N = 5$. Finally, note that the energy in the scalar field is dominated by the behavior of the scalar field close to the horizon. If one is interested in regimes of spacetime outside some two-sphere with radius $r \gg M$, then the full scalar field can be accurately modeled using just the dipole ($\ell = 1$) and octupole ($\ell = 3$) modes.

V. TRACE OF THE METRIC PERTURBATION: SOLUTION

The leading correction to the metric in Eq. (12) is determined by the $\mathcal{O}(\zeta)$ term in the metric equation of motion [Eq. (6)]. We work in a gauge where the covariant divergence of $g_{ab}^{(1)}$ is proportional to the derivative of its

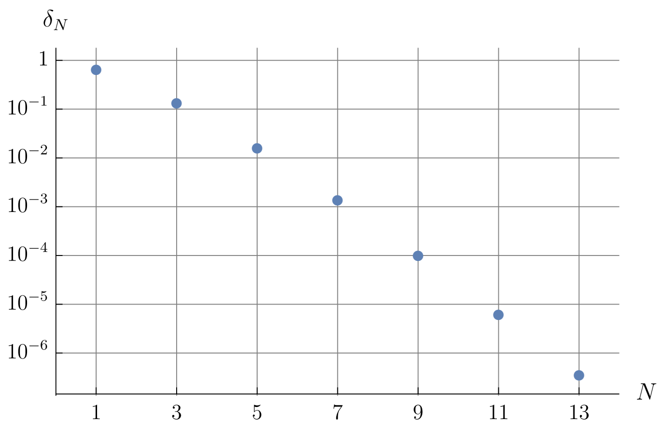

FIG. 3. The fractional difference between the scalar field's contribution to the ADM energy, and the contribution from the field's first N modes.

trace $g^{(1)} = g^{ab}_{(0)} g^{(1)}_{ab}$ with respect to the background metric:

$$\nabla^a g^{(1)}_{ab} = \frac{1}{2} \nabla_b g^{(1)} . \tag{41}$$

This gauge leads to simplifications in the $\mathcal{O}(\zeta)$ term of Eq. (6), but its structure is still too complicated to allow for a simple solution. As a first step towards determining $g^{(1)}_{ab}$, we take the trace of the $\mathcal{O}(\zeta)$ correction to Eq. (6) to find

$$\nabla^2 g^{(1)} = -2(\nabla\vartheta^{(1/2)})^2 . \tag{42}$$

Assuming a stationary and axisymmetric solution and transforming to dimensionless variables, this reduces to

$$\left[\partial_{\tilde r}\tilde\Delta\partial_{\tilde r} + \partial_\psi\Gamma\partial_\psi\right] g^{(1)} = -2\tilde\Delta(\partial_{\tilde r}\tilde\vartheta^{(1/2)})^2 - 2\Gamma(\partial_\psi\tilde\vartheta^{(1/2)})^2 . \tag{43}$$

As with the scalar field, we can express $g^{(1)}$ in a Legendre decomposition as

$$g^{(1)} = \sum_\ell g^{(1)}_\ell(\tilde r)\, P_\ell(\psi) . \tag{44}$$

Then, the equation of motion [Eq. (43)] again separates, giving the radial equation

$$\left[\partial_{\tilde r}\tilde\Delta\partial_{\tilde r} - \ell(\ell+1)\right] g^{(1)}_\ell(\tilde r) = S_\ell(\tilde r) . \tag{45}$$

The source functions $S_\ell(\tilde r)$ are the Legendre modes of the right-hand side of Eq. (42), i.e.

$$S_\ell(\tilde r) = \frac{2\ell+1}{2}\int_{-1}^{1} d\psi P_\ell(\psi) S_g(\tilde r,\psi) , \tag{46}$$

where the source function S_g is simply

$$S_g(\tilde r,\psi) := -2\sqrt{-g^{(0)}}\,(\nabla\tilde\vartheta^{(1/2)})^2 . \tag{47}$$

The solution for the mode functions $g^{(1)}_\ell(\tilde r)$ is then given by Eq. (A3), which in this case becomes

$$g^{(1)}_\ell = \frac{1}{W_\ell}\Bigg(H^+_\ell(\tilde r)\int_\infty^{\tilde r} d\tilde r' H^-_\ell(\tilde r') S_\ell(\tilde r') - H^-_\ell(\tilde r)\int_{\tilde r_+}^{\tilde r} d\tilde r' H^+_\ell(\tilde r') S_\ell(\tilde r')\Bigg) . \tag{48}$$

Note that the source is quadratic in the scalar field, which has odd Legendre modes. Thus, both the trace of the metric perturbation and its source function have even Legendre modes: $\ell = 2n$ for all $n \in \mathbb{N}$.

One approach to evaluating the integrals in Eq. (48) is to express the Legendre modes of the source function in terms of the scalar field modes and their radial derivatives. The resulting integrals are significantly more complicated than the ones we encountered in Sec. III, so we will opt for a different approach. One can express Eq. (48) in terms of a simpler set of integrals through multiple integrations-by-parts (noting that the source S_ℓ depends on S_g, which in turn is proportional to the squared derivative of the scalar field) and application of the scalar field evolution equation [Eq. (20)]. Doing so, the modes of the trace of the metric perturbation are given by

$$\begin{aligned} g^{(1)}_\ell(\tilde r) = \frac{2\ell+1}{W_\ell}\Bigg(& H^+_\ell(\tilde r)\int_\infty^{\tilde r} d\tilde r' \int_{-1}^{1} d\psi\; H^-_\ell(\tilde r') P_\ell(\psi)\tilde\vartheta^{(1/2)}(\tilde r',\psi) s(\tilde r',\psi) \\ & - H^-_\ell(\tilde r)\int_{\tilde r_+}^{\tilde r} d\tilde r' \int_{-1}^{1} d\psi\; H^+_\ell(\tilde r') P_\ell(\psi)\tilde\vartheta^{(1/2)}(\tilde r',\psi) s(\tilde r',\psi) \\ & + H^+_\ell(\tilde r)\int_\infty^{\tilde r} d\tilde r' \int_{-1}^{1} d\psi\; \partial_\mu V^\mu_-(\tilde r',\psi) - H^-_\ell(\tilde r)\int_{\tilde r_+}^{\tilde r} d\tilde r' \int_{-1}^{1} d\psi\; \partial_\mu V^\mu_+(\tilde r',\psi)\Bigg) , \end{aligned} \tag{49}$$

where $s(\tilde r',\psi)$ is the scalar field source given in Eq. (21), and we have defined

$$\begin{aligned} V^\mu_\pm := & \frac{1}{2}(\tilde\vartheta^{(1/2)})^2\,\sqrt{-g^{(0)}}\; g^{\mu\nu}_{(0)}\partial_\nu\left[H^\pm_\ell P_\ell\right] \\ & - \frac{1}{2} H^\pm_\ell P_\ell\,\sqrt{-g^{(0)}}\; g^{\mu\nu}_{(0)}\partial_\nu(\tilde\vartheta^{(1/2)})^2 . \end{aligned} \tag{50}$$

The integrals of total derivatives in Eq. (49) can be simplified by noting that (i) $\sqrt{-g^{(0)}}g^{\psi\psi}_{(0)} = \Gamma$, which vanishes when evaluated at the limits of integration $\psi = \pm 1$, and (ii) contributions at spatial infinity and at the horizon vanish due to the behavior of the scalar field modes $\tilde{\vartheta}^{(1/2)}$, the homogeneous solutions $H^{\pm}_{\ell}$, and $\sqrt{-g^{(0)}}g^{rr}_{(0)} = \tilde{\Delta}$. The modes of the trace of the metric perturbation are then

$$g^{(1)}_{\ell}(\tilde{r}) = \frac{2\ell+1}{W_\ell}\left(H^{+}_{\ell}(\tilde{r})\int_{\infty}^{\tilde{r}}d\tilde{r}'\int_{-1}^{1}d\psi\; H^{-}_{\ell}(\tilde{r}')P_\ell(\psi)\tilde{\vartheta}^{(1/2)}s^{(1/2)} - H^{-}_{\ell}(\tilde{r})\int_{\tilde{r}_+}^{\tilde{r}}d\tilde{r}'\int_{-1}^{1}d\psi\; H^{+}_{\ell}(\tilde{r}')P_\ell(\psi)\tilde{\vartheta}^{(1/2)}s^{(1/2)}\right) - \frac{2\ell+1}{2}\int_{-1}^{1}d\psi\; P_\ell(\psi)\;\left(\tilde{\vartheta}^{(1/2)}(\tilde{r},\psi)\right)^2, \qquad (51)$$

where in the first and second lines $\tilde{\vartheta}^{(1/2)}$ and $s^{(1/2)}$ are both functions of $\tilde{r}'$ and ψ. We have simplified the last line by extracting a factor of W_ℓ, defined in Eq. (A4), which is a constant.

Let us now focus on the extremal limit. With the normalizations defined in Appendix A, the factor $W_\ell = 2\ell+1$ and the homogeneous solutions are given by Eqs. (A8)-(A9). We can then write

$$g^{(1,0)}_{\ell}(\tilde{r}) = (\tilde{r}-1)^\ell\int_{\infty}^{\tilde{r}}d\tilde{r}'\int_{-1}^{1}d\psi\;\frac{P_\ell(\psi)\tilde{\vartheta}^{(1/2,0)}s^{(1/2,0)}}{(\tilde{r}-1)^{\ell+1}} - \frac{1}{(\tilde{r}-1)^{\ell+1}}\int_{1}^{\tilde{r}}d\tilde{r}'\int_{-1}^{1}d\psi\;(\tilde{r}-1)^\ell P_\ell(\psi)\tilde{\vartheta}^{(1/2,0)}s^{(1/2,0)} - \frac{(2\ell+1)}{2}\int_{-1}^{1}d\psi P_\ell(\psi)\;(\tilde{\vartheta}^{(1/2,0)})^2\,. \qquad (52)$$

This completes the formal solution for the modes of the trace of the metric perturbation in the extremal limit in integral form.

The angular integrals in Eq. (52) can be evaluated in closed form using the Legendre decomposition of the scalar field and the source function. From Eq. (22) and Eq. (24) we have

$$\tilde{\vartheta}^{(1/2,0)}(\tilde{r},\psi) = \sum_{k=1}^{\infty}\tilde{\vartheta}^{(1/2,0)}_{k}(\tilde{r})P_k(\psi)$$
$$s^{(1/2,0)}(\tilde{r},\psi) = \sum_{j=1}^{\infty}s^{(1/2,0)}_{j}(\tilde{r})P_j(\psi)\,,$$

where the sums are over odd integers in both cases. Using the orthonormality of Legendre functions, the $\ell=0$ mode is given by

$$g^{(1,0)}_{0}(\tilde{r}) = \sum_{k=1}^{\infty}\frac{2}{2k+1}\left[\int_{\infty}^{\tilde{r}}d\tilde{r}'\;\frac{\tilde{\vartheta}^{(1/2,0)}_{k}(\tilde{r}')s^{(1/2,0)}_{k}(\tilde{r}')}{\tilde{r}'-1} - \frac{1}{\tilde{r}-1}\int_{1}^{\tilde{r}}d\tilde{r}'\;\tilde{\vartheta}^{(1/2,0)}_{k}(\tilde{r}')s^{(1/2,0)}_{k}(\tilde{r}') - \frac{1}{2}\tilde{\vartheta}^{(1/2,0)}_{k}(\tilde{r})^2\right]. \qquad (53)$$

For general ℓ, the integration over ψ can be expressed in terms of the standard $3j$-symbols. The resulting expression is

$$g^{(1,0)}_{\ell}(\tilde{r}) = \sum_{k,j}2\begin{pmatrix}\ell & k & j\\ 0 & 0 & 0\end{pmatrix}^2\times \left[(\tilde{r}-1)^\ell\int_{\infty}^{\tilde{r}}d\tilde{r}'\,\frac{\tilde{\vartheta}^{(1/2,0)}_{k}(\tilde{r}')s^{(1/2,0)}_{j}(\tilde{r}')}{(\tilde{r}'-1)^{\ell+1}} - \frac{1}{(\tilde{r}-1)^{\ell+1}}\int_{\infty}^{\tilde{r}}d\tilde{r}'(\tilde{r}'-1)^\ell\,\tilde{\vartheta}^{(1/2,0)}_{k}(\tilde{r}')s^{(1/2,0)}_{j}(\tilde{r}') - \frac{2\ell+1}{2}\,\tilde{\vartheta}^{(1/2,0)}_{k}(\tilde{r})\,\tilde{\vartheta}^{(1/2,0)}_{j}(\tilde{r})\right] \qquad (54)$$

The radial integrals in Eqs. (53) and (54), though still complicated, are more tractable than the integrals that result from expressing the source function in terms of the modes of the scalar field in Eq. (48).

We have not yet obtained a closed-form expression for the trace of the metric perturbation on the extremal background. The main difficulty, apparent in Eqs. (52)-(54), is that the source term depends on the full tower of Legendre modes of the scalar field. Using our expressions for the modes of the scalar field and its source, Eq. (31) and Eq. (25), it is possible to evaluate individual terms in these sums. However, we have not been able to perform the sums themselves. Indeed, the analytic results for the individual terms are sufficiently complicated that we turn to approximations and numerical analysis, which we discuss in the next section.

VI. TRACE OF THE METRIC PERTURBATION: PROPERTIES

The results of Sec. IV suggest that the first three or four modes of the scalar field capture most of its physics, and should be sufficient for analyzing the behavior of the trace of the metric perturbation. But first, let us consider a few important properties of the modes $g^{(1,0)}_{\ell}$ that can be extracted from the integral form of the solution.

At large radius, $\tilde{r} \gg 1$, the second line of Eq. (52) dominates and the leading behavior of the mode is $g^{(1,0)}_{\ell} \sim \tilde{r}^{-(\ell+1)}$. This is because the first line of Eq. (52) decays with a higher power of $\tilde{r}$, while the third line is proportional to $(\tilde{\vartheta}^{(1/2,0)})^2$ and therefore decays as $\tilde{r}^{-2(\ell+1)}$. Near the horizon the first and third line of Eq. (52) dominate. The asymptotic behavior as $\tilde{r}\to 1$ is most easily extracted by first evaluating Eq. (48) at $\tilde{r} = \tilde{r}_+ = 1+\varepsilon$, changing the integration variable to $\eta = (\tilde{r}'-1)/\varepsilon$, and then taking the $\varepsilon \to 0$ limit, which gives

$$g^{(1,0)}_{\ell}(1) = S^{(1,0)}_{\ell}(1)\int_{\infty}^{1}d\eta\,Q_\ell(\eta)\;. \qquad (55)$$

The overall factor of $S^{(1,0)}_{\ell}(1)$, the source function evaluated at the extremal Kerr horizon, can be expressed in

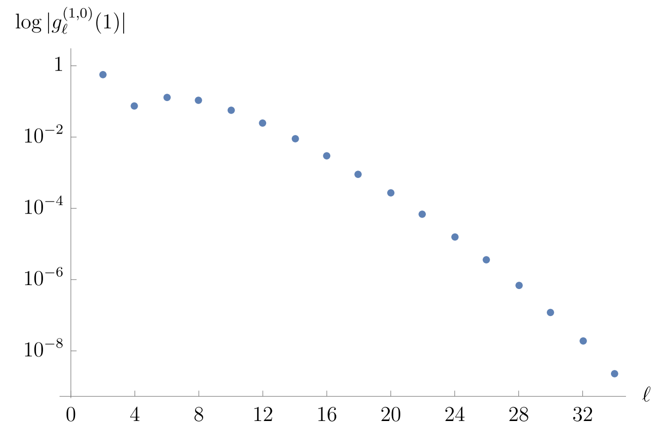

FIG. 4. The (absolute value of) Legendre modes of the trace of the metric perturbation evaluated at the horizon of the extremal background, on a logarithmic scale, as a function of harmonic number ℓ.

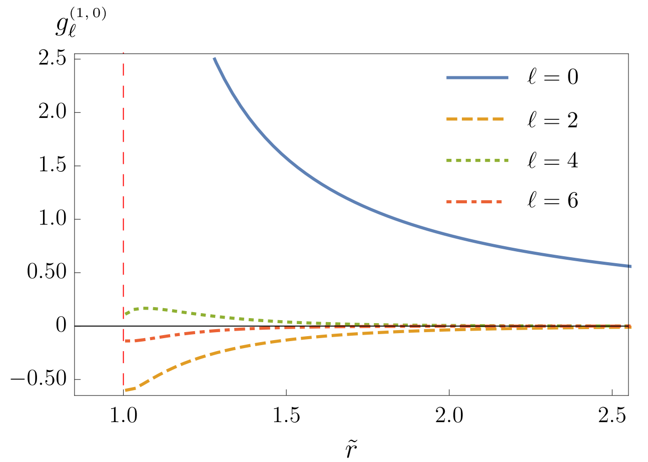

FIG. 5. The first four radial modes of the trace of the metric perturbation as a function of $\tilde r$. The dashed vertical line indicates the horizon of the extremal background. The $\ell = 0$ mode exhibits a logarithmic divergence as $\tilde r \to 1$.

terms of the source functions for the scalar field. Evaluating Eq. (46) at $\tilde r = 1$, using $\lim_{\tilde r\to 1}\tilde\Delta(\partial_{\tilde r}\tilde\vartheta)^2 = 0$, and performing the angular integral yields

$$S_\ell^{(1,0)}(1) = -\sum_{k,j=1}^{\infty}\frac{2(2\ell+1)}{\sqrt{j(j+1)k(k+1)}}\begin{pmatrix}\ell & j & k\\ 0 & 0 & 0\end{pmatrix} \times\begin{pmatrix}\ell & j & k\\ 0 & 1 & -1\end{pmatrix} s_j^{(1/2,0)}(1)s_k^{(1/2,0)}(1)\ , \quad (56)$$

where again the result is expressed in terms of $3j$-symbols.

For $\ell \geq 2$ (recall that ℓ is even) the integral in Eq. (55) converges to

$$g_\ell^{(1,0)}(1) = -\frac{1}{\ell(\ell+1)}\,S_\ell^{(1,0)}(1)\ . \quad (57)$$

In this case the mode is finite at the horizon of the extremal background, just like the modes of the scalar field. The values $g_\ell^{(1,0)}(1)$ are plotted against ℓ in Fig. 4, where observe that rather than falling off monotonically with ℓ, the $\ell = 4$ mode is suppressed relative to the $\ell = 6$ and $\ell = 8$ modes.

For the $\ell = 0$ mode the integral in Eq. (55) does not converge, and the mode has a logarithmic divergence as $\tilde r \to 1$:

$$\lim_{\tilde r\to 1} g_0^{(1,0)}(\tilde r) \sim S_0^{(1,0)}(1)\log(\tilde r - 1)\ . \quad (58)$$

The meaning of this divergence will be discussed in Sec. VII; suffice it to say here that this logarithmic divergence does *not* imply the existence of a naked singularity at the perturbed horizon. For $\ell = 0$, Eq. (56) reduces to a single sum which can be evaluated numerically

$$S_0^{(1,0)}(1) = -\sum_{k=1}^{\infty}\frac{2}{k(k+1)(2k+1)}\left(s_k^{(1/2,0)}(1)\right)^2 \simeq -3.52572\ . \quad (59)$$

This precisely matches the log behavior of the solution near $\tilde r = 1$, which can be extracted from both analytical and numerical results for $g_0^{(1,0)}(\tilde r)$.

As explained at the end of the previous section, analytical results for the $\tilde r$-dependence of the modes $g_\ell^{(1,0)}$ are sufficiently complicated that a numerical analysis is called for. The first four modes of the trace of the metric perturbation are shown in Fig. 5. These are numerical solutions, obtained with closed form expressions for the source that include contributions from modes of the scalar field with $\ell \leq 21$. It is immediately apparent that the $\ell = 0$ mode dominates the trace of the metric perturbation, even away from the log-divergent behavior near $\tilde r = 1$.

We expect, based on the behavior shown in Fig. 5 that $g^{(1,0)}$ is well-approximated by its first few Legendre modes. In the case of the scalar field, a similar conclusion was justified by examining mode-by-mode contributions to the ADM energy. There is not an obvious analog for the trace of the metric perturbation, so instead we consider the fractional difference, as a function of $\tilde r$, between $g^{(1,0)}$ and its approximation by the first N modes

$$\delta_N(\tilde r) = 1 - \frac{1}{g^{(1,0)}(\tilde r)}\left(\sum_{\ell=0}^{N} g_\ell^{(1,0)}(\tilde r)\right)\ . \quad (60)$$

The fractional difference for $N = 0, 2, 4, 6$ is shown in Fig. 6. As expected, the log-divergence of the $\ell = 0$ modes means that the fractional difference $\delta_0(\tilde r) \to 0$ as $\tilde r \to 1$. Figure 6 shows that if one wishes an accuracy of no more than about 10%, then retaining only the $\ell = 0$ mode suffices. To obtain a higher accuracy, more modes are needed. In particular, since the $\ell = 4$ mode is suppressed at $\tilde r = 1$ relative to the $\ell = 6$ and 8 modes, one must include modes up to $\ell = 6$ to obtain uniform accuracy of at least one percent. Observe, however, that if one is interested in the trace of the metric perturbation

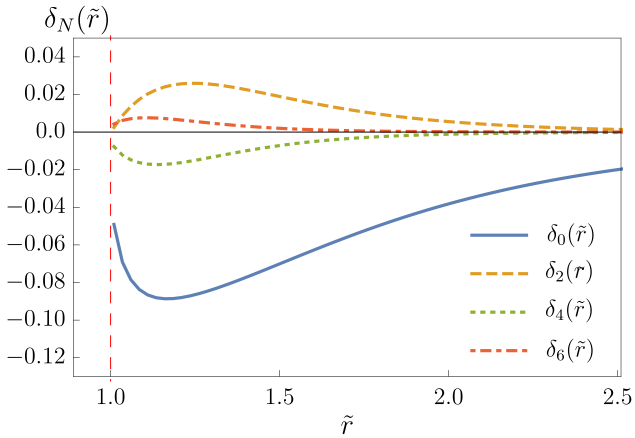

FIG. 6. The fractional difference between the trace of the metric perturbation at $\theta = 0$, and its approximation including only modes with $\ell \leq N$.

outside a larger radius, such as for $\tilde{r} \gtrsim 2$, then $g^{(1,0)}$ is approximated at better than percent precision with only the first two or three modes.

We conclude that Legendre modes $g_\ell^{(1,0)}$ with $\ell \leq 4$ capture almost all of the physics of $g^{(1,0)}$, except near $\tilde{r} = 1$ where the $\ell = 6$ and $\ell = 8$ modes may be required. Note that the fractional error defined in Eq. (60) puts a bound on the fidelity of our approximation of $g^{(1,0)}$ at $\psi = 1$ ($\theta = 0$), where $P_\ell(\psi) = 1$ for all ℓ. However, since the $\ell = 0$ mode dominates, and $-1 \leq P_\ell(\psi) \leq 1$ for $\ell \geq 2$, the fractional error $\delta_N(\tilde{r})$ gives an upper bound on the fidelity of the approximation in the full $(\tilde{r}, \psi)$ plane. An approximation of $g^{(1,0)}$ by its first four Legendre modes is shown in Fig. 7.

The analysis above uses modes of the scalar field with $\ell \leq 21$ to approximate the source for the trace of the metric perturbation. However, as we saw in Sec. IV, the first three modes of the scalar field account for most of its contribution to the ADM energy. Indeed, the behavior of the first few modes of $g^{(1,0)}$ is largely unchanged if we include fewer modes of the scalar field. In particular, we achieve comparable results for the mode $g_\ell^{(1,0)}$ by approximating the scalar field by its first $N = \ell + 1$ modes.

VII. DISCUSSION

This paper explored rotating black holes in dCS. Using an effective field theory treatment of dCS, we worked in the decoupling limit where dCS corrections are small perturbations from GR solutions. We have further focused on BHs that spin at the maximal Kerr rate, the so-called extremal limit. With these assumptions in hand, we then solved for the dynamical scalar field in closed analytic form, through a Legendre decomposition that we found was dominated by the dipole term. The radial structure of this decomposition includes natural logarithms and arctangents, unlike the simple polynomial results obtained

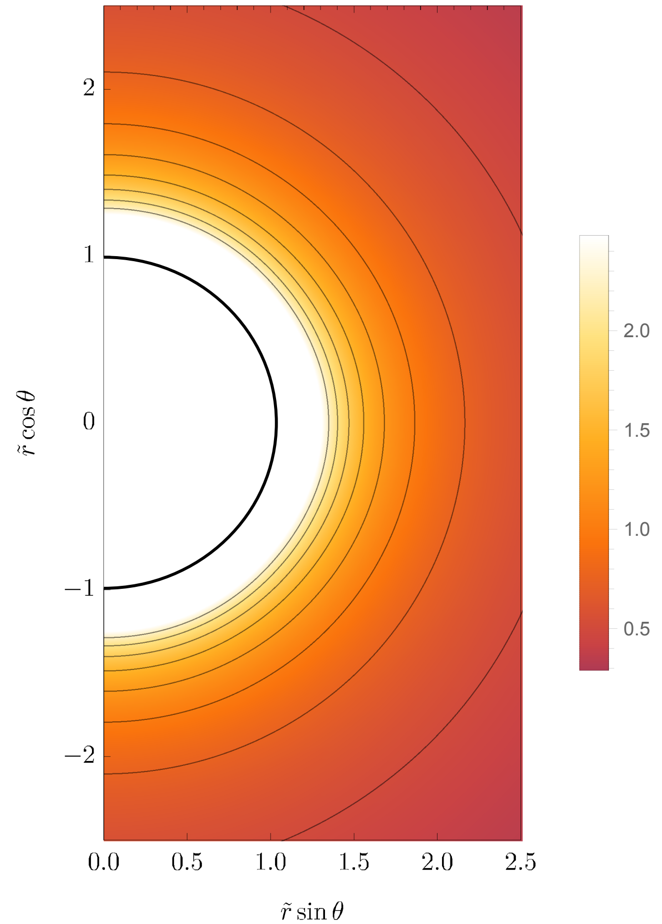

FIG. 7. The trace of the metric perturbation on the extremal background, approximated by its first four Legendre modes. Near the extremal horizon $\tilde{r} = 1$ (the solid line) the logarithmic divergence of the monopole term dominates.

in the slow-rotation limit. We then solved for the Legendre decomposition of the trace of the metric perturbation. We discovered that retaining 3 (4) terms in the Legendre expansion suffices to ensure a fidelity of at least 99% in the scalar field (trace of the metric perturbation) relative to numerical solutions.

The trace of the metric perturbation in harmonic gauge exhibits a logarithmic divergence, but this is probably not a problem. The divergence occurs at the location of the extremal Kerr horizon, which need not coincide with the location of the dCS corrected horizon. Indeed, in the slow-rotation expansion, the horizon was seen to be shifted outward to $r_{\rm hor} = r_{\rm hor,Kerr} + (915/28672)\zeta M\chi^2 + \mathcal{O}(\chi^4)$ [14]. Thus, one may expect that in the extremal limit the Kerr and the dCS horizons do not coincide either. It may also be the case that the extremality condition in dCS is shifted away from $J = M$.

The techniques found above rely heavily on Legendre expansions, but our work suggests that these can be truncated at a finite mode number without losing much of the overall behavior of the function. In particular, if one wishes to carry out astrophysical tests of GR, certain observables may be sensitive to only certain regions of spacetime that need not include the horizon. For example, BH shadow observations are most sensitive to the location of the light-ring, while astrophysical observations of the energy spectrum of radiation emitted by accretion disks are most sensitive to the location of the innermost stable circular orbit. For such observations, it may suffice to keep only the first few modes in a Legendre expansion provided the BH is not rotating maximally. The reason here is two-fold. First, the light-ring and ISCO are both pushed away from the horizon as the spin decreases, and the approximation by a finite number of Legendre modes improves away from the immediate vicinity of the horizon. Second, our studies indicate that, in general, the fidelity of an approximation at a fixed number of modes improves away from extremality, as shown in [16].

The results obtained here open the door for new investigations of rotating BHs in dCS gravity. For example, the methods we employed could be extended to the next-order term in a near-extremal expansion, or better yet, for BHs that rotate with arbitrary spins. We have already obtained partial results for the latter, with closed-form results for the first two modes of the scalar field (and numerical results for all other modes). Ultimately, of course, one would like to solve for the full metric perturbation of dCS BHs, and not just the trace.

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Appendix A: Solution of the Scalar Equation of Motion

The equations of motion for the scalar field and the trace of the metric perturbation have the same general form, so let us briefly establish some conventions for the solutions of such equations. First, consider an equation of the form

$$\partial_{\tilde{r}}(\tilde{\Delta}\partial_{\tilde{r}} I_\ell) - \ell(\ell+1)\, I_\ell = K_\ell \tag{A1}$$

with some source K_ℓ. Denote by H_ℓ^+ and H_ℓ^- the solutions of the homogeneous equation

$$\partial_{\tilde{r}}(\tilde{\Delta}\partial_{\tilde{r}} H_\ell^\pm) - \ell(\ell+1)\, H_\ell^\pm = 0\ , \tag{A2}$$

with H_ℓ^+ regular at $\tilde{r}_+$ and $H_\ell^- \to 0$ at $\tilde{r}\to\infty$. We use the method of variation of parameters to find the general solution to the inhomogeneous equation. The solution of (A1) that is both regular at $\tilde{r}_+$ and goes to zero as $\tilde{r}\to\infty$ can be expressed in terms of the homogeneous solutions and the source as

$$I_\ell = \frac{1}{W_\ell} \times \left(H_\ell^+(\tilde{r}) \int_\infty^{\tilde{r}} d\tilde{r}'\, H_\ell^-(\tilde{r}') K_\ell(\tilde{r}') \right. \\ \left. - H_\ell^-(\tilde{r}) \int_{\tilde{r}_+}^{\tilde{r}} d\tilde{r}'\, H_\ell^+(\tilde{r}') K_\ell(\tilde{r}') \right)\ . \tag{A3}$$

Here a factor of $\tilde{\Delta}$ has canceled inside each integral, allowing us to pull out a constant W_ℓ; this constant depends on the Wronskian of the homogeneous solutions,

$$\begin{aligned} W_\ell &\equiv \tilde{\Delta} \times W[H_\ell^-, H_\ell^+] \\ &= \tilde{\Delta} \times \left(H_\ell^- \partial_{\tilde{r}} H_\ell^+ - H_\ell^+ \partial_{\tilde{r}} H_\ell^- \right)\ . \end{aligned} \tag{A4}$$

It is straightforward to verify that this is constant using Eq. (A2).

For the Kerr background, the homogeneous solutions can be written as

$$H_\ell^+(\tilde{r}) = c_\ell^+\, (1-\chi^2)^{\frac{\ell}{2}}\, P_\ell\left(\frac{\tilde{r}-1}{\sqrt{1-\chi^2}} \right) \tag{A5}$$

$$H_\ell^-(\tilde{r}) = c_\ell^-\, (1-\chi^2)^{-\frac{\ell+1}{2}}\, Q_\ell\left(\frac{\tilde{r}-1}{\sqrt{1-\chi^2}} \right)\ . \tag{A6}$$

where $P_\ell(\cdot)$ and $Q_\ell(\cdot)$ are Legendre functions of the first and second kind, respectively. A standard identity for Legendre functions then gives the factor $W_\ell = c_\ell^+ c_\ell^-$.

The factors of $\sqrt{1-\chi^2}$ in Eq. (A5)-(A6) have been chosen so that the extremal limit, $\chi\to\pm1$, is regular. Otherwise, the overall normalization factors $c_\ell^\pm$ are arbitrary. A convenient choice is to set

$$c_\ell^+ = \frac{\ell!}{(2\ell-1)!!}\,, \qquad c_\ell^- = \frac{(2\ell+1)!!}{\ell!}\,. \tag{A7}$$

Then $W_\ell = 2\ell + 1$, and in the extremal limit the homogeneous solutions are simply

$$\lim_{|\chi|\to 1} H_\ell^+ = (\tilde{r}-1)^\ell \tag{A8}$$

$$\lim_{|\chi|\to 1} H_\ell^- = \frac{1}{(\tilde{r}-1)^{\ell+1}} \,. \tag{A9}$$

We adopt this normalization throughout Secs. III and V.

Appendix B: Expressions for Radial Modes

The radial mode function for general ℓ is given in Eq. (31). In this form, each mode depends on $2\ell+1$ coefficients $\alpha_{\ell,k}$ and $\beta_{\ell,k}$. The coefficients for the modes up to $\ell = 9$ are given below.

k	$\alpha_{1,k}$	$\beta_{1,k}$	$\alpha_{3,k}$	$\beta_{3,k}$	$\alpha_{5,k}$	$\beta_{5,k}$	$\alpha_{7,k}$	$\beta_{7,k}$	$\alpha_{9,k}$	$\beta_{9,k}$
0	0	-3	$\frac{70}{3}$	$\frac{15}{2}$	$-\frac{68}{5}$	$-\frac{85}{8}$	$\frac{31719}{70}$	$\frac{189}{16}$	$\frac{49019}{1260}$	$-\frac{1377}{128}$
1	-	3	-25	-75	$\frac{1015}{8}$	$\frac{2955}{8}$	$-\frac{22317}{80}$	$-\frac{4347}{4}$	$\frac{351417}{896}$	$\frac{315405}{128}$
2	-	-	15	75	$-\frac{8645}{8}$	$-\frac{735}{2}$	$\frac{90819}{20}$	$\frac{8127}{8}$	$-\frac{13832005}{896}$	$-\frac{34155}{16}$
3	-	-	-	25	$\frac{4263}{8}$	$\frac{4935}{4}$	$-\frac{44079}{16}$	$-\frac{33705}{4}$	$\frac{3351161}{384}$	$\frac{1065405}{32}$
4	-	-	-	-	$-\frac{819}{8}$	$-\frac{4935}{8}$	$\frac{21009}{2}$	$\frac{73395}{16}$	$-\frac{9069467}{128}$	$-\frac{1183545}{64}$
5	-	-	-	-	-	$\frac{987}{8}$	$-\frac{14067}{4}$	$-\frac{44793}{4}$	$\frac{3456585}{128}$	$\frac{6416487}{64}$
6	-	-	-	-	-	-	$\frac{1989}{4}$	$\frac{14931}{4}$	$-\frac{30089455}{384}$	$-\frac{292215}{8}$
7	-	-	-	-	-	-	-	$-\frac{2133}{4}$	$\frac{2523125}{128}$	$\frac{2579445}{32}$
8	-	-	-	-	-	-	-	-	$-\frac{279565}{128}$	$-\frac{2579445}{128}$
9	-	-	-	-	-	-	-	-	-	$\frac{286605}{128}$

Appendix C: Representations of the Scalar Field

Equation (31) provides one representation of the solution to Eq. (23) for arbitrary harmonic number ℓ after a Legendre decomposition and in an expansion to leading order in ζ (i.e. in the GR deformation) and in ε (i.e., in the extremal limit). This form of the solution depends on $2\ell+1$ coefficients ($\alpha_{\ell,k}$ and $\beta_{\ell,k}$) that are fixed by imposing appropriate boundary conditions on the mode.

In this appendix, we present two additional representations of the solution to the scalar field evolution equation that may be preferable in some applications. As in the case of Eq. (31), these representations will have both advantages and disadvantages that we will describe in detail. The solutions start by representing the source function in the extremal limit in terms of a series:

$$s_\ell^{(1/2,\,0)}(\tilde{r}) = \sum_{n=0}^{\infty} \alpha_{\ell,n} \frac{2\ell+1}{\tilde{r}^{\ell+4+2n}} \,, \tag{C1}$$

where we have introduced the constants

$$\alpha_{\ell,n} = (-1)^{\frac{\ell-1}{2}} (-1)^n \frac{(\ell+2n+2)\,\Gamma(\ell+4+2n)\,\Gamma(\frac{1}{2})}{2^{\ell+1+2n}\,\Gamma(n+1)\,\Gamma(\ell+n+\frac{3}{2})} \,, \tag{C2}$$

in terms of the Gamma function $\Gamma(\cdot)$. The factor of $2\ell+1$ in Eq. (C1) has been introduced to simplify some expressions, by canceling a similar factor in the denominator of Eq. (28). From here on, different representations take different routes to arrive at a solution to Eq. (23) in the extremal limit, so we tackle each of them separately below.

1. Incomplete Beta Function Representation

Introducing expansion Eq. (C1) for the scalar source into the solution Eq. (28) for the scalar field:

$$\tilde{\vartheta}_\ell^{(1/2,\,0)}(\tilde{r}) = \sum_{n=0}^{\infty} \alpha_{\ell,n} \Bigg[(\tilde{r}-1)^\ell \int_\infty^{\tilde{r}} \frac{d\tilde{r}'}{(\tilde{r}'-1)^{\ell+1}\,\tilde{r}'^{\ell+4+2n}} - \frac{1}{(\tilde{r}-1)^{\ell+1}} \int_1^{\tilde{r}} d\tilde{r}' (\tilde{r}'-1)^\ell \frac{1}{\tilde{r}'^{\ell+4+2n}} \Bigg] \,, \tag{C3}$$

where we have already imposed appropriate boundary conditions. The integrals can be evaluated in closed-form to obtain

$$\tilde{\vartheta}_\ell^{(1/2,\,0)}(\tilde{r}) = \beta_1(\tilde{r}) + \beta_2(\tilde{r}) + \beta_3(\tilde{r}) \tag{C4}$$

where we have defined

$$\beta_1(\tilde{r}) = -\sum_{n=0}^{\infty} \alpha_{\ell,n} (\tilde{r}-1)^\ell B_{1/\tilde{r}}\Big(2\ell+4+2n, -\ell\Big) \,, \tag{C5}$$

$$\beta_2(\tilde r) = \sum_{n=0}^{\infty} \frac{\alpha_{\ell,n}}{(\tilde r-1)^{\ell+1}} B_{1/\tilde r}\Big(2n+3,\ell+1\Big)\,, \tag{C6}$$

$$\beta_3(\tilde r) = -\sum_{n=0}^{\infty} \frac{\alpha_{\ell,n}}{(\tilde r-1)^{\ell+1}} \frac{\Gamma(\ell+1)\,\Gamma(2n+3)}{\Gamma(\ell+4+2n)}\,, \tag{C7}$$

in terms of the incomplete Beta function $B_x(a,b)$ (see e.g. Sec. 8.17 of [37]),

$$B_x(a,b) \equiv \int_0^x t^{a-1}(1-t)^{b-1}dt\,. \tag{C8}$$

Evaluating at $x=1$ gives the ordinary Beta function, $B(a,b)=B_1(a,b)$.

The sums over n can be evaluated in closed form for β_2 and β_3. The latter can be summed into

$$\beta_3(\tilde r) = \frac{(-1)^{\frac{\ell+1}{2}}\,\ell\,\Gamma(\ell+1)\Gamma(\frac12)}{2^{\ell+2}\Gamma(\ell+\frac32)}\frac{1}{(\tilde r-1)^{\ell+1}}\left[\ell\,(2\ell+1)-2(\ell-1)(\ell+1){}_2F_1\big(-\tfrac12,1;\ell+\tfrac32;-1\big)\right]\,, \tag{C9}$$

which gives the leading behavior of $\tilde\vartheta_\ell^{(1,0)}$ at large $\tilde r$. The sum over n for β_2 can be evaluated using the series representation of the incomplete Beta function appropriate for Eq. (C6),

$$B_x(m,n) = \sum_{j=0}^{n-1}\frac{(-1)^j}{m+j}\frac{\Gamma(n)}{\Gamma(j+1)\,\Gamma(n-j)}x^{m+j}\,. \tag{C10}$$

Permuting the order of the sums over j and n yields

$$\begin{aligned}\beta_2(\tilde r) = &-\frac{(-1)^{(\ell+1)/2}}{(\tilde r-1)^{\ell+1}}\frac{\Gamma(\ell+1)\Gamma(1/2)\Gamma(4+\ell)}{2^{2+\ell}\Gamma(5/2+\ell)}\sum_{j=0}^{\ell}(-1)^j\frac{1}{(3+j)(5+j)}\frac{1}{\Gamma(1+j)\Gamma(1+\ell-j)}\frac{1}{\tilde r^{5+j}}\\ &\times\Bigg[(5+j)(2+\ell)(3+2\ell)\,\tilde r^2\,{}_3F_2\left(\frac{3+j}{2},\frac{4+\ell}{2},\frac{5+\ell}{2};\frac{5+j}{2},\frac{3+2\ell}{2};-\frac{1}{\tilde r^2}\right)\\ &-(3+j)(4+\ell)(5+\ell)\,{}_3F_2\left(\frac{5+j}{2},\frac{6+\ell}{2},\frac{7+\ell}{2};\frac{7+j}{2},\frac{5+2\ell}{2};-\frac{1}{\tilde r^2}\right)\Bigg]\,,\end{aligned} \tag{C11}$$

where ${}_PF_Q(\cdot;\cdot;\cdot)$ is the generalized hypergeometric function. We have not succeeded in finding a closed-form expression for the above sum over j, but the sum can be performed explicitly given any value of ℓ.

One is then only left with the sum over n for β_1. To obtain an expression for this sum, we start with the following representation of the incomplete Beta function relevant for Eq. (C5):

$$\begin{aligned}B_{1/\tilde r}\,(2\ell+4+2n,-\ell) = &\frac{(-1)^{\ell+1}(\ell+4+2n)\Gamma(2\ell+4+2n)}{\Gamma(\ell+1)\Gamma(\ell+5+2n)}\left[\ln\left(\frac{\tilde r-1}{\tilde r}\right)+\frac{1}{\tilde r-1}\left(1-\sum_{k=1}^{\ell+3+2n}\frac{1}{k(k+1)}\frac{1}{\tilde r^k}\right)\right]\\ &-\frac{\Gamma(2\ell+4+2n)}{\Gamma(\ell+1)}\frac{1}{\tilde r^{\ell+3+2n}}\sum_{k=0}^{\ell-2}(\tilde r-1)^{k-\ell}\frac{(-1)^{k+1}\Gamma(\ell-k)}{\Gamma(2\ell+4+2n-k)}\,.\end{aligned} \tag{C12}$$

This allows us to write $\beta_1(\tilde r) := \beta_4(\tilde r)+\beta_5(\tilde r)$, where we have defined

$$\beta_4(\tilde r) = -(\tilde r-1)^\ell\sum_{n=0}^{\infty}\alpha_{\ell,n}\frac{(-1)^{\ell+1}(\ell+4+2n)\Gamma(2\ell+4+2n)}{\Gamma(\ell+1)\Gamma(\ell+5+2n)}\left[\ln\left(\frac{\tilde r-1}{\tilde r}\right)+\frac{1}{\tilde r-1}\left(1-\sum_{k=1}^{\ell+3+2n}\frac{1}{k(k+1)}\frac{1}{\tilde r^k}\right)\right] \tag{C13}$$

$$\beta_5(\tilde r) = -\frac{1}{\Gamma(\ell+1)}\frac{1}{\tilde r^{\ell+3}}\sum_{n=0}^{\infty}\alpha_{\ell,n}\Gamma(2\ell+4+2n)\frac{1}{\tilde r^{2n}}\sum_{k=0}^{\ell-2}\frac{(-1)^k\,\Gamma(\ell-k)}{\Gamma(2\ell+4+2n-k)}(\tilde r-1)^k\,. \tag{C14}$$

The function $\beta_5(\tilde r)$ can be simplified further by performing the sums to obtain

$$\begin{aligned}\beta_5(\tilde r) = &(-1)^{(\ell+1)/2}2^{1+\ell}(1+\ell)(2+\ell)\Gamma(4+\ell)\frac{1}{\tilde r^{5+\ell}}\sum_{k=0}^{\ell-2}(-1)^k\frac{\Gamma(\ell-k)}{\Gamma(6-k+2\ell)}(\tilde r-1)^k\\ &\times\Bigg[(k-2\ell-5)\,(3+2\ell)\,(k-4-2\ell)\;\tilde r^2\;{}_4F_3\left(\frac{4+\ell}{2},\frac{5+\ell}{2},2+\ell,\frac{5+2\ell}{2};\frac{3+2\ell}{2},\frac{4-k+2\ell}{2},\frac{5-k+2\ell}{2};-\frac{1}{\tilde r^2}\right)\\ &-2\,(4+\ell)\,(5+\ell)\,(5+2\ell)\;{}_4F_3\left(\frac{6+\ell}{2},\frac{7+\ell}{2},3+\ell,\frac{7+2\ell}{2};\frac{5+2\ell}{2},\frac{6-k+2\ell}{2},\frac{7-k+2\ell}{2};-\frac{1}{\tilde r^2}\right)\Bigg]\,.\end{aligned} \tag{C15}$$

The function $\beta_4(\tilde{r})$ can also be simplified by performing some of the sums in closed form to obtain

$$\beta_4(\tilde{r}) = \frac{(-1)^\ell}{\Gamma(\ell+1)}(\tilde{r}-1)^\ell \sum_{n=0}^{\infty} \alpha_{\ell,n} \frac{(\ell+4+2n)\Gamma(2\ell+4+2n)}{\Gamma(\ell+5+2n)} \left[\ln\left(\frac{\tilde{r}-1}{\tilde{r}}\right) + \frac{1}{\tilde{r}-1}\left(1 - \sum_{k=1}^{\ell+3+2n} \frac{1}{k(k+1)} \frac{1}{\tilde{r}^k}\right)\right]$$
$$= \frac{(-1)^{(3\ell+1)/2}}{2} \frac{\Gamma(\ell+3)}{\Gamma(\ell+1)} (\tilde{r}-1)^{\ell-1} \left[1 + (\tilde{r}-1)\ln\left(\frac{\tilde{r}-1}{\tilde{r}}\right)\right] + \frac{(-1)^{\ell+1}}{\Gamma(\ell+1)}(\tilde{r}-1)^{\ell-1} \sum_{n=0}^{\infty} \sum_{k=1}^{\ell+3+2n} \gamma_{\ell,n} \frac{1}{k(k+1)} \frac{1}{\tilde{r}^k}\,, \tag{C16}$$

where we have defined

$$\gamma_{\ell,n} := \alpha_{\ell,n} \frac{(\ell+4+2n)\Gamma(2\ell+4+2n)}{\Gamma(\ell+5+2n)}\,. \tag{C17}$$

The last term in Eq. (C16) can also be represented as follows:

$$\sum_{n=0}^{\infty} \sum_{k=1}^{\ell+3+2n} \gamma_{\ell,n} \frac{1}{k(k+1)} \frac{1}{\tilde{r}^k} = \frac{(-1)^{(\ell+1)/2}}{2} \Gamma(\ell+3)\, G\left(\frac{1}{\tilde{r}}, \ell+3\right) \sum_{j=0}^{\infty} \left(\sum_{n=j+1}^{\infty} \gamma_{\ell,n}\right) \left[\frac{1}{(\ell+4+2j)(\ell+5+2j)} \frac{1}{\tilde{r}^{\ell+4+2j}} + \right.$$
$$\left. + \frac{1}{(\ell+5+2j)(\ell+6+2j)} \frac{1}{\tilde{r}^{\ell+5+2j}}\right]\,, \tag{C18}$$

where we have defined the new function

$$G(x, N) := \sum_{k=1}^{N} \frac{x^k}{k(k+1)}\,, \tag{C19}$$

for some $x \in \Re$ and $N \in \mathbb{N}$. This function is the first N terms of the Taylor series for $1 - \log(1-x) + x^{-1}\log(1-x)$ about $x = 0$. Notice that the sum in this new function is finite, and thus $G(1/\tilde{r}, N)$ is simply a polynomial in $1/\tilde{r}$. Given a particular value of ℓ, the remaining sum over j can be performed explicitly.

2. Radial Series Representation

Instead of using variation of parameters to solve Eq. (23), we will search for a series solution. We thus insert the ansatz

$$\tilde{\vartheta}_\ell^{(1,0)}(\tilde{r}) = \sigma_1(\tilde{r}) + \sigma_2(\tilde{r})\,, \tag{C20}$$

with

$$\sigma_1(\tilde{r}) := \sum_{n=0}^{\infty} a_{\ell,n} \frac{1}{\tilde{r}^{\ell+1+n}}\,, \tag{C21}$$

$$\sigma_2(\tilde{r}) := \sum_{n=0}^{\infty} b_{\ell,n} \frac{1}{\tilde{r}^{\ell+4+2n}}\,, \tag{C22}$$

into Eq. (23) and find recursion relations for the $a_{\ell,n}$ and $b_{\ell,n}$ coefficients.

The recursion relations for the $a_{\ell,n}$ can be solved to obtain

$$a_{\ell,n} = \frac{(\ell+n)!}{\ell!\, n!}\, a_{\ell,0}\,, \tag{C23}$$

which then leads to

$$\sigma_1(\tilde{r}) = \frac{a_{\ell,0}}{(\tilde{r}-1)^{\ell+1}}\,. \tag{C24}$$

Since this is the leading behavior of the scalar field at large $\tilde{r}$, we can determine the coefficient $a_{\ell,0}$ by comparing it with the incomplete Beta function representation of the previous subsection:

$$a_{\ell,0} = -\sum_{n=0}^{\infty} \alpha_{\ell,n}\, B(2n+3, \ell+1) = \frac{(-1)^{\frac{\ell+1}{2}} \sqrt{\pi}\, \ell!}{2^\ell} \left[\frac{(\ell+2)}{\Gamma(\ell+\frac{3}{2})}\, {}_2F_1\left(\tfrac{3}{2}, 2; \ell+\tfrac{3}{2}; -1\right) - \frac{6}{\Gamma(\ell+\frac{5}{2})}\, {}_2F_1\left(\tfrac{5}{2}, 3; \ell+\tfrac{5}{2}; -1\right)\right]. \tag{C25}$$

Resumming the coefficients $b_{\ell,n}$ is more complicated. We can solve the recursion relations to express the $b_{\ell,n}$ coefficient as finite sums that depend on the coefficients $\alpha_{\ell,n}$ in the series expansion of the source:

$$b_{\ell,n} = \frac{\Gamma(\ell+4+n)}{\Gamma(4+n)} \sum_{j=0}^{j_{max}} \frac{\Gamma(3+2j)}{\Gamma(\ell+4+2j)} \alpha_{\ell,j} - \frac{\Gamma(\ell+4+n)}{\Gamma(2\ell+5+n)} \sum_{j=0}^{j_{max}} \frac{\Gamma(2\ell+4+2j)}{\Gamma(\ell+4+2j)} \alpha_{\ell,j}\,, \tag{C26}$$

where $j_{max} = n/2$ if n is even, and $j_{max} = (n+1)/2$ if n is odd.

When one tries to perform the full infinite sum of the $b_{\ell,n}$ coefficients over n to find $\sigma_2(\tilde{r})$, one finds a familiar problem: the coefficients of Eq. (C26) are finite sums with an upper limit that depends on n, which must then be summed to infinity. To get around this problem, we can rewrite each finite sum as the difference of two infinite sums:

$$b_{\ell,n} = \frac{\Gamma(\ell+4+n)}{\Gamma(4+n)}\, c_{\ell,j_{max}} - \frac{\Gamma(\ell+4+n)}{\Gamma(2\ell+5+n)}\, d_{\ell,j_{max}} \ , \tag{C27}$$

where

$$\begin{aligned} c_{\ell,k} = (-1)^{\frac{\ell-1}{2}} \frac{\sqrt{\pi}}{2^{\ell+1}} \times \Bigg[& \frac{(\ell+1)(4\ell-7)}{2\,\Gamma(\ell+\frac{3}{2})} + \frac{2\,(-1)^k(2k+3)\Gamma(k+\frac{5}{2})}{\sqrt{\pi}\,\Gamma(\ell+\frac{3}{2}+k)} + \frac{(\ell^4-2\ell^2+9\ell+10)}{2\,\Gamma(\ell+\frac{5}{2})}\, {}_2F_1(-\tfrac{1}{2},1;\ell+\tfrac{5}{2};-1) \\ & - \frac{(\ell^4+5\ell^3+\ell+5)}{2\,\Gamma(\ell+\frac{5}{2})}\, {}_2F_1(\tfrac{1}{2},1;\ell+\tfrac{5}{2};-1) + \frac{2\,(-1)^k\,(\ell-1)\,\Gamma(k+\frac{5}{2})}{\sqrt{\pi}\,\Gamma(\ell+\frac{5}{2}+k)}\, {}_2F_1(1,k+\tfrac{5}{2};\ell+k+\tfrac{5}{2};-1) \\ & + \frac{2^{\ell-1}(\ell+2)}{\Gamma(\ell+\frac{3}{2})}\, {}_2F_1(\ell-\tfrac{1}{2},\ell;\ell+\tfrac{3}{2};-1) \Bigg] , \end{aligned} \tag{C28}$$

$$\begin{aligned} d_{\ell,k} = (-1)^{\frac{\ell-1}{2}} \times \Bigg[& -\frac{1}{2}\,\Gamma(\ell+3) + \frac{(-1)^k 2^{\ell+2}(k+1)(\ell+k+2)\Gamma(\ell+k+3)}{\Gamma(k+2)} \\ & + \frac{(-1)^k 2^{\ell+1}(\ell+2)\Gamma(\ell+k+3)}{\Gamma(k+2)}\, {}_2F_1(1,\ell+k+3;k+2;-1) \Bigg] . \end{aligned} \tag{C29}$$

With the $b_{\ell,n}$ coefficients expressed in this form, the second sum for the scalar field becomes

$$\sigma_2(\tilde{r}) = \sum_{n=0}^{\infty} \left[\frac{\Gamma(\ell+4+n)}{\Gamma(4+n)}\, c_{\ell,j_{max}} - \frac{\Gamma(\ell+4+n)}{\Gamma(2\ell+5+n)}\, d_{\ell,j_{max}} \right] \frac{1}{\tilde{r}^{\ell+4+2n}} \,. \tag{C30}$$

We have not succeeded in finding closed-form expressions for the sum over n given a generic ℓ value, but the sum can be performed for a given value of ℓ.

Appendix D: Series Solutions for the Trace of the Metric Perturbation

Instead of truncating the Legendre expansion of the scalar field, it is also possible to construct series approximations of the modes $g_\ell^{(1,0)}$. We first note that the source term (46) can be expanded in powers of $1/\tilde{r}$. For the $\ell = 0$ mode this series takes the form

$$S_0^{(1,0)}(\tilde{r}) = \sum_{n=0}^{\infty} e_{0,n}\, \frac{1}{\tilde{r}^{4+n}} \ , \tag{D1}$$

while for $\ell \geq 2$ it is

$$S_\ell^{(1,0)}(\tilde{r}) = \sum_{n=0}^{\infty} e_{\ell,n}\, \frac{1}{\tilde{r}^{\ell+2+n}} . \tag{D2}$$

In terms of the series coefficients for the source, the $\ell = 0$ mode is

$$\begin{aligned} g_0^{(1,0)}(\tilde{r}) = \sum_{n=0}^{\infty} e_{0,n} \times \Bigg[& \log\left(\frac{\tilde{r}-1}{\tilde{r}}\right) + \sum_{j=1}^{n+3} \frac{1}{j\,\tilde{r}^j} \\ & + \frac{1}{n+3}\, \frac{1}{\tilde{r}-1} \left(\frac{1}{\tilde{r}^{n+3}} - 1 \right) \Bigg] . \end{aligned} \tag{D3}$$

Note that the coefficient of the $\log(\tilde{r}-1)$ term is (D1) evaluated at $\tilde{r} = 1$, as in Eq. (58). The modes with $\ell \geq 2$ can be expressed as a series involving incomplete Beta functions:

$$\begin{aligned} g_\ell^{(1,0)}(\tilde{r}) = \sum_{n=0}^{\infty} e_{\ell,n} \times \Big[& -(\tilde{r}-1)^\ell\, B_{1/\tilde{r}}\,(2\ell+2+n,-\ell) \\ & - \frac{1}{(\tilde{r}-1)^{\ell+1}}\, B_{1-1/\tilde{r}}\,(\ell+1,n+1) \Big] . \end{aligned} \tag{D4}$$

Since these solutions are obtained directly from Eq. (48) they already satisfy the correct boundary conditions at $\tilde{r} \to \infty$ and $\tilde{r} = 1$.

Given the expansion of the source functions, one can obtain a series solution of the equation of motion Eq. (45) directly. For the $\ell = 0$ mode this solution takes the form

$$g_0^{(1,0)}(\tilde{r}) = \frac{f_{0,0}}{(\tilde{r}-1)} + \sum_{n=0}^{\infty} \frac{1}{\tilde{r}^{4+n}} \sum_{j=0}^{n} \frac{n+1-j}{(n+4)(j+3)}\, e_{0,j} \tag{D5}$$

while for $\ell \geq 2$ it is

$$g_\ell^{(1,0)}(\tilde r) \quad = \quad \frac{f_{\ell,0}}{(\tilde r-1)^{\ell+1}} \quad + \quad \sum_{n=0}^{\infty}\frac{(\ell+n)!}{\tilde r^{\ell+2+n}}\sum_{j=0}^{n} e_{\ell,j}\left(\frac{j!}{(n+1)!(\ell+j+1)!}-\frac{(2\ell+j+1)!}{(\ell+j+1)!(2\ell+n+2)!}\right) \quad . \tag{D6}$$

In both cases the coefficient $f_{\ell,0}$ of the leading term can be expressed in terms of one or more integrals of the source; for $\ell = 0$ it is

$$f_{0,0} = \int_\infty^1 d\tilde r\, S_0^{(1,0)}(\tilde r) \ . \tag{D7}$$

[1] C. M. Will, Living Reviews in Relativity **17** (2014), 10.1007/lrr-2014-4.
[2] N. Yunes and X. Siemens, Living Rev.Rel. **16**, 9 (2013), arXiv:1304.3473 [gr-qc].
[3] S. Alexander and N. Yunes, Phys.Rept. **480**, 1 (2009), arXiv:0907.2562 [hep-th].
[4] T. Delsate, D. Hilditch, and H. Witek, Phys. Rev. **D91**, 024027 (2015), arXiv:1407.6727 [gr-qc].
[5] M. B. Green and J. H. Schwarz, Phys. Lett. **B149**, 117 (1984).
[6] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology* (Cambridge: Cambridge University Press, 1988).
[7] S. Weinberg, Phys.Rev. **D77**, 123541 (2008), arXiv:0804.4291 [hep-th].
[8] V. Taveras and N. Yunes, Phys.Rev. **D78**, 064070 (2008), arXiv:0807.2652 [gr-qc].
[9] R. Jackiw and S. Pi, Phys.Rev. **D68**, 104012 (2003), arXiv:gr-qc/0308071 [gr-qc].
[10] D. Ayzenberg, K. Yagi, and N. Yunes, Phys. Rev. **D89**, 044023 (2014), arXiv:1310.6392 [gr-qc].
[11] B. A. Campbell, M. J. Duncan, N. Kaloper, and K. A. Olive, Phys. Lett. **B251**, 34 (1990).
[12] N. Yunes and F. Pretorius, Phys.Rev. **D79**, 084043 (2009), arXiv:0902.4669 [gr-qc].
[13] K. Konno, T. Matsuyama, and S. Tanda, Prog.Theor.Phys. **122**, 561 (2009), arXiv:0902.4767 [gr-qc].
[14] K. Yagi, N. Yunes, and T. Tanaka, Phys.Rev. **D86**, 044037 (2012), arXiv:1206.6130 [gr-qc].
[15] K. Konno and R. Takahashi, Phys. Rev. **D90**, 064011 (2014), arXiv:1406.0957 [gr-qc].
[16] L. C. Stein, Phys. Rev. **D90**, 044061 (2014), arXiv:1407.2350 [gr-qc].
[17] D. Grumiller and N. Yunes, Phys. Rev. **D77**, 044015 (2008), arXiv:0711.1868 [gr-qc].
[18] N. Yunes and C. F. Sopuerta, Phys. Rev. **D77**, 064007 (2008), arXiv:0712.1028 [gr-qc].
[19] C. F. Sopuerta and N. Yunes, Phys. Rev. **D80**, 064006 (2009), arXiv:0904.4501 [gr-qc].
[20] M. Guica, T. Hartman, W. Song, and A. Strominger, Phys. Rev. **D80**, 124008 (2009), arXiv:0809.4266 [hep-th].
[21] A. M. Ghezelbash, JHEP **08**, 045 (2009), arXiv:0901.1670 [hep-th].
[22] Y. Matsuo and T. Nishioka, JHEP **12**, 073 (2010), arXiv:1010.4549 [hep-th].
[23] S. Carlip, JHEP **04**, 076 (2011), [Erratum: JHEP01,008(2012)], arXiv:1101.5136 [gr-qc].
[24] G. Compere, Living Rev. Rel. **15**, 11 (2012), arXiv:1203.3561 [hep-th].
[25] O. J. C. Dias, J. E. Santos, and M. Stein, JHEP **10**, 182 (2012), arXiv:1208.3322 [hep-th].
[26] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).
[27] R. M. Wald, *General Relativity* (University of Chicago Press, 1984).
[28] K. Yagi, L. C. Stein, and N. Yunes, (2015), arXiv:1510.02152 [gr-qc].
[29] S. Mercuri and V. Taveras, Phys.Rev. **D80**, 104007 (2009), arXiv:0903.4407 [gr-qc].
[30] Y. Ali-Haimoud and Y. Chen, Phys.Rev. **D84**, 124033 (2011), arXiv:1110.5329 [astro-ph.HE].
[31] R. P. Kerr, Phys.Rev.Lett. **11**, 237 (1963).
[32] L. C. Stein, (2014), arXiv:1407.0744 [gr-qc].
[33] "GRTensorII," This is a package which runs within Maple but distinct from packages distributed with Maple. It is distributed freely on the World-Wide-Web from the address: `http://grtensor.org`.
[34] J. M. Martin-Garcia, R. Portugal, and L. R. U. Manssur, Comput.Phys.Commun. **177**, 640 (2007), arXiv:0704.1756 [cs.SC].
[35] J. M. Martin-Garcia, D. Yllanes, and R. Portugal, Comput.Phys.Commun. **179**, 586 (2008), arXiv:0802.1274 [cs.SC].
[36] D. Brizuela, J. M. Martin-Garcia, and G. A. Mena Marugan, Gen.Rel.Grav. **41**, 2415 (2009), arXiv:0807.0824 [gr-qc].
[37] DLMF, "NIST Digital Library of Mathematical Functions," http://dlmf.nist.gov/, Release 1.0.10 of 2015-08-07, online companion to [38].
[38] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, eds., *NIST Handbook of Mathematical Functions* (Cambridge University Press, New York, NY, 2010) print companion to [37].