On Sufficient Conditions for the Existence of Stable Matchings with Contracts

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Abstract

We introduce two new sufficient conditions for the existence of stable outcomes in many-to-one matching with contracts. The conditions subsume the *observable substitutability* of Hatfield et al. (2015) and the *substitutable completability* of Hatfield and Kominers (2016) as special cases. We also prove that *unilaterally substitutability* and *irrelevance of rejected contracts* imply *substitutable completability*.

Keywords: Many-to-one Matching, Stability, Substitutability, Irrelevance of Rejected Contracts

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1 Introduction

Recently there are two advances in expanding the frontier of sufficient conditions for the existence of stable outcomes in the many-to-one matching with contracts model. One is achieved by Hatfield and Kominers (2016) who propose a condition called *substitutable completability*. Simply speaking, a choice function is substitutably completable if it has a completion in the many-to-many framework that satisfies substitutability (Hatfield and Milgrom, 2005). Hatfield and Kominers prove that if the choice functions of all firms have substitutable completions that satisfy *irrelevance of rejected contracts* (IRC; Aygün and Sönmez, 2013), then the worker-proposing cumulative offer algorithm can find stable outcomes.

The other advance is achieved by Hatfield et al. (2015). They propose the notion of *observable offer processes*, which, simply speaking, are the sequences of contracts that can appear in the worker-proposing cumulative offer algorithm. They call a firm's choice function *observably substitutable* if it satisfies substitutability on observable offer processes and IRC. They prove that if all firms have observably substitutable choice functions, the worker-proposing cumulative offer algorithm can find stable outcomes.

Before this note it is not clear how these two advances are related. In particular, even though a choice function has a substitutable completion that satisfies IRC, the choice function itself may not satisfy IRC. So it is not observably substitutable. On the other hand, Hatfield et al. show that an observably substitutable choice function may not have a substitutable completion. So the two advances seem independent of each other.

In this note we prove that if a choice function has a substitutable completion that satisfies IRC, then it must satisfy both substitutability and IRC on its observable offer processes. We call choice functions having this property *weakly observably substitutable*. We prove that weakly observable substitutability is sufficient for stable outcomes to exist. Since this condition also subsumes observable substitutability, the above two advances are unified in this note.

Inspired by the substitutable completion idea we conjecture that if all firms' choice functions have *weakly observably substitutable* completions, then stable outcomes may also exist. Interestingly, we find that a choice function has a completion that is weakly observably substitutable if and only if the choice function is weakly observable substitutable by itself. So the conjecture is indeed correct. This also implies that we cannot extend the substitutable completion idea beyond weakly observable substitutability.

We are also interested in the relation between *unilateral substitutability* introduced by Hatfield and Kojima (2010) and *substitutable completability*. Hatfield et al. show that unilateral substitutability is a special case of observable substitutability. In this note we constructively prove that if a choice function satisfies unilateral substitutability and IRC, it must have a substitutable completion. This result is more general than that of Kadam (2014) since he only proves the result for choice functions derived from preferences.

In the literature there are multiple sufficient conditions for the existence of stable outcomes in the many-to-one matching with contracts model. In particular, Hatfield et al propose another sufficient condition called *observable substitutability across workers*, which is weaker than observable substitutability. In this note we similarly define *weakly observable substitutability across workers* by requiring IRC hold only on observable offer processes. We prove that it is also a sufficient condition for stable outcomes to exist. In the following graph we summarize the relations between multiple sufficient conditions. It also clarifies our contribution in this note. In the graph "condition a → condition b" means that condition a implies condition b.

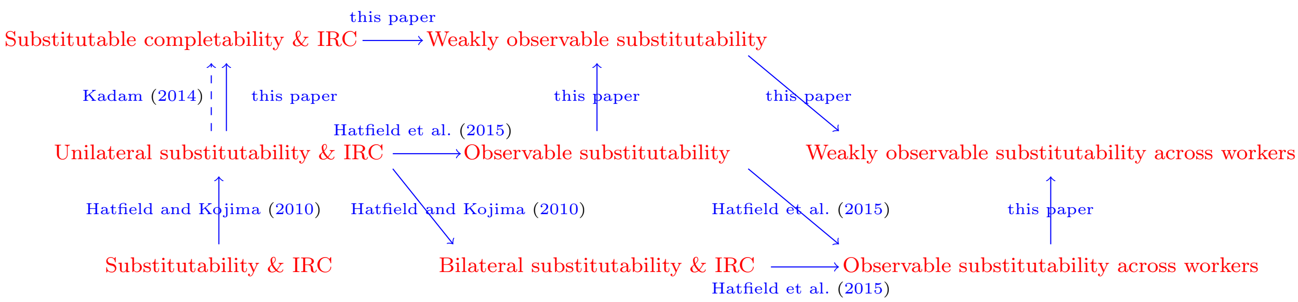

Figure 1: Relations between multiple sufficient conditions.

In Section 2 we define the standard many-to-one matching with contracts model. In Section 3 we define *weakly observable substitutability* and *weakly observable substitutability across workers* and prove that they are sufficient conditions. In Section 4 we prove that *substitutable completability* and *IRC* imply *weakly observable substitutability*. In Section 5 we prove that *unilateral substitutability* and *IRC* imply *substitutable completability*. In Section 5 we have some discussions.

2 Model

We introduce the standard many-to-one matching with contracts model. There are a finite set F of firms and a finite set W of workers, with $I \equiv F \cup W$ being the set of all agents. Firms sign contracts with workers, but each worker can at most one contract. Let X be the finite set of all contracts. For each contract $x \in X$, w_x and f_x are respectively the worker and the firm involved in x. For any subset of contracts $Z \subseteq X$, Z_i is the set of contracts in Z that involve any agent $i \in I$, F_Z is the set of firms involved in Z, W_Z is the set of workers involved in Z, and $I_Z \equiv F_Z \cup W_Z$ is the set of all agents involved in Z. A subset of contracts $A \subseteq X$ is called an *outcome*, and it is *feasible* if $|A_w| \leq 1$ for all $w \in W$.

Each worker w has a strict preference relation $\succ_w$ over $X_w \cup \{\emptyset\}$, the set of all contracts involving w and being unemployed. Every preference relation $\succ_w$ induces a choice function C_w such that $C_w(Z) \equiv \arg\max_{x \in Z} \succ_w$ for all $Z \subseteq X_w \cup \{\emptyset\}$. Each firm $f \in F$ has a choice function C_f such that for any $Z \subseteq X$, $C_f(Z) \subseteq Z_f$ contains at most one contract with each worker. In the terminology of Kominers (2012) we call such choice functions *unitary*. Let $C_W \equiv \{C_w\}_{w \in W}$ and $C_F \equiv \{C_f\}_{f \in F}$. The rejection function of each agent $i \in I$ is denoted by R_i such that $R_i(Z) \equiv Z_i \backslash C_i(Z)$ for all $Z \subseteq X$. For all $Z \subseteq X$ and all $f \in F$, we use $C_f^W(Z)$ to denote the set of workers involved in $C_f(Z)$.

An outcome $A \subseteq X$ is *individually rational* if for all $i \in I$, $C_i(A) = A_i$. So an individually rational outcome must be feasible. An outcome A is *blocked* by a set of contracts $Z \subseteq X$ if $Z \cap A = \emptyset$ and for all $i \in I_Z$, $Z_i \subseteq C_i(A \cup Z)$. An outcome A is *stable* if it is individually rational and unblocked.

There are two conditions of choice functions that are important in matching with contracts: substitutability and irrevelence of rejected contracts (Hatfield and Milgrom, 2005; Aygün and Sönmez, 2013). If all firms' choice functions satisfy both conditions, then stable outcomes exist.

Definition 1. *A firm's choice function C_f is* ***substitutable*** *if for all $Z' \subseteq Z \subseteq X$, $R_f(Z') \backslash R_f(Z) = \emptyset$.*

Definition 2. *A firm's choice function C_f satisfies* ***irrelevance of rejected contracts*** *if for all $x \in Z \subseteq X$, if $x \notin C_f(Z)$, then $C_f(Z) = C_f(Z \backslash \{x\})$.*

In next section we will propose two new conditions that are weaker than substitutability and IRC but are still sufficient for stable outcomes to exist. We will use the worker-proposing cumulative offer (COM) algorithm to prove the existence of stable outcomes. The formal definition of COM is below.

The worker-proposing cumulative offer algorithm:

- **Step 0**: Choose an arbitrary ordering $\rhd_W$ of all workers. Initialize the set of cumulative offers that firms have received by step 0 as $A^0 \equiv \emptyset$.

- **Step** $t \geq 1$: Consider the set of workers who want to make an offer at step t:

$$W^t = \{w \in W : w \notin C_F^W(A^{t-1}), x_w^t \equiv \arg \max_{x \in X_w \cup \{\emptyset\} \setminus A^{t-1}} \succ_w \neq \emptyset\},$$

  If $W^t \neq \emptyset$, let the worker $w \in W^t$ who is ranked highest in $\rhd_W$ makes the offer x_w^t. Then let $A^t \equiv A^{t-1} \cup \{x_w^t\}$. So $C_F(A^t) \equiv \cup_{f \in F} C_f(A_f^t)$ is the set of accepted contracts.

  If $W^t = \emptyset$, then the algorithm terminates and the outcome is $C_F(A^{t-1})$.

3 Weakly Observable Substitutability (Across Workers)

In this section we propose the notions of *weakly observable substitutability* and *weakly observable substitutability across workers*. To define them we first introduce some notions.

An *offer process for firm* f is a finite sequence of distinct contracts $\mathbf{x} = (x^1, \cdots, x^M)$ such that for all $m = 1, \cdots, M$, $x^m \in X_f$. For all $m = 1, \cdots, M$, we denote $\mathbf{x}^m \equiv (x^1, \cdots, x^m)$ and call it a *subprocess* of $\mathbf{x}$. We call $\mathbf{x}$ *observable* if for all $m = 1, \cdots, M$, $w_{x^m} \notin C_f^W(\{x^1, \cdots, x^{m-1}\})$. That is, the worker involved in x^m is different from all workers involved in $C_f(\{x^1, \cdots, x^{m-1}\})$.

A firm's choice function C_f satisfies *no observable violation of substitutability* (NOVS) if for any two observable offer processes $\mathbf{x}, \mathbf{x}'$ for f such that $\mathbf{x}'$ is a subprocess of $\mathbf{x}$, $R_f(\mathbf{x}') \setminus R_f(\mathbf{x}) = \emptyset$.[1] In a weaker notion C_f satisfies *no substantial violation of substitutability* (NSVS) if for any two observable offer processes $\mathbf{x}, \mathbf{x}'$ for f such that $\mathbf{x}'$ is a

[1] If $\mathbf{x} = (x^1, \cdots, x^M)$, we use $R_f(\mathbf{x})$ to denote $R_f(\{x^1, \cdots, x^M\})$. In the original definition of Hatfield et al. (2015), their "if" condition is stated as "for any observable offer process $(x^1, \cdots, x^M)$ for f, $R_f(\{x^1, \cdots, x^{M-1}\}) \setminus R_f(\{x^1, \cdots, x^M\}) = \emptyset$." This condition is equivalent to the one in this note.

subprocess of $\mathbf{x}$, if $x \in R_f(\mathbf{x}')\backslash R_f(\mathbf{x}) \neq \emptyset$, then $w_x \in C_f^W(\mathbf{x}')$. Then C_f is *observably substitutable* if it satisfies NOVS and IRC, and C_f is *observably substitutable across workers* if it satisfies NSVS and IRC.

Now we define a weak version of IRC which only requires IRC hold on observable offer processes.

Definition 3. *A firm's choice function C_f satisfies* ***observable IRC*** *if for all observable offer process $\boldsymbol{x}$ for f and all $Z \subseteq R_f(\boldsymbol{x})$, $C_f(\boldsymbol{x}) = C_f(\boldsymbol{x}\backslash Z)$.*

A choice function must satisfy IRC if it is derived from a strict preference relation. In other words, if a choice function satisfies *observable IRC* but not *IRC*, it cannot be derived from a strict preference relation.

Then we define the weaker versions of observable substitutability and observable substitutability across workers by replacing IRC with observable IRC.

Definition 4. *A firm's choice function C_f is* ***weakly observably substitutable*** *if it satifies NOVS and observable IRC.*

Definition 5. *A firm's choice function C_f is* ***weakly observably substitutable across workers*** *if it satifies NSVS and observable IRC.*

Note that *weakly observable substitutability across workers* is weaker than *weakly observable substitutability.* In the following we prove the existence of stable outcomes when the choice functions of all firms are weakly observably substitutable across workers. The proof is almost same as that of Hatfield et al.

Theorem 1. *If the choice function of each firm is weakly observably substitutable across workers, then the outcome of the worker-proposing cumulative offer algorithm is stable.*

Proof. Let T be the last step of COM, then A^T is the set of cumulative offers received by all firms. Let $Y = C_F(A^T) \equiv \cup_{f\in F} C_f(A_f^T)$, then Y is the outcome of COM. So we need to prove that Y is stable.

First, we prove that Y is feasible. Suppose the contrary that there exist two contracts $x, x' \in Y$ such that $w_x = w_{x'} = w$ and without loss of generality $x \succ_w x'$. So w must propose x earlier than proposing x'. Suppose w proposes x' at step t, then it must be that $x \in R_{f_x}(A_{f_x}^{t-1})$. That is, x is not accepted at the end of step $t-1$. However, $x \in Y$ implies that $x \notin R_{f_x}(A_{f_x}^T)$. Since $A_{f_x}^{t-1}$ is a subprocess of $A_{f_x}^T$ and C_{f_x} satisfies

NSVS, $w \in C^W_{f_x}(A^{t-1}_{f_x})$. That is, there exists a contract $x'' \in X$ such that $w_{x''} = w$ and $x'' \in C_{f_x}(A^{t-1}_{f_x})$. But this implies that w should not make an offer at step t, which is a contradiction.

Second, we prove that Y is individually rational. Since all workers must propose only acceptable contracts to firms, every contract in Y is acceptable to the relevant worker. Since Y is also feasible, it must be that for every w, $C_w(Y) = Y_w$. By construction Y is individually rational for firms.

Lastly, we prove that Y is unblocked. Consider any set $Z \subseteq X$ such that $Z \cap Y = \emptyset$ and $Z_w \subseteq C_w(Y \cup Z)$ for all $w \in W_Z$. For all $w \in W_Z$, Z_w must be a single contract and $Z_w \succ_w Y_w$. So w must propose Z_w at some earlier step and is rejected before step T. Hence, $Z_w \in R_{f_{Z_w}}(A^T_{f_{Z_w}}) \subseteq A^T_{f_{Z_w}}$. Since $A^T_{f_{Z_w}}$ is an observable offer process for f_{Z_w} and $C_{f_{Z_w}}$ satisfies observable IRC, $Y_{f_{Z_w}} = C_{f_{Z_w}}(A^T_{f_{Z_w}}) = C_{f_{Z_w}}(Y_{f_{Z_w}} \cup R_{f_{Z_w}}(A^T_{f_{Z_w}})) = C_{f_{Z_w}}(Y_{f_{Y_w}} \cup Z_{f_{Z_w}})$. So Z is not a blocking set. □

4 Substitutable Completability and IRC Imply Weakly Observable Substitutability

Hatfield and Kominers (2016) have an innovative observation that although the choice functions of some firms are not substitutable, the firms may have underlying "substitutable preferences" which cannot be expressed in the many-to-one framework.

Definition 6. *$\bar{C}_f$ is a **completion** of C_f if for all $Z \subseteq X$, either $\bar{C}_f(Z) = C_f(Z)$ or there exists distinct $z, z' \in \bar{C}_f(Z)$ such that $w_z = w_{z'}$. Then C_f is **substitutably completable** if it has a substitutable completion.*

Hatfield and Kominers prove that if the choice functions of all firms have substitutable completions that satisfy IRC, then stable outcomes exist. However, although the substitutable completion of a choice function satisfies IRC, the choice function itself may not satisfy IRC. We provide a simple example to illustrate this fact.

Example 1. *A firm f has a choice function C_f shown in Table 1. $w_x = w_{\hat{x}} = w_1$ and $w_y = w_2$. $\bar{C}_f$ is a completion of C_f.*

Here $(x, \hat{x})$ and $(x, \hat{x}, y)$ are unobservable offer processes. $\bar{C}_f$ is substitutable and satisfies IRC. However, C_f does not satisfy IRC since $\hat{x} \notin C_f(\{x, \hat{x}, y\})$ but $C_f(\{x, \hat{x}, y\}) =$

offer set Z	$C_f(Z)$	$\bar{C}_f(Z)$
$\{a\} : \forall a \in \{x, \hat{x}, y\}$	$\{a\}$	$\{a\}$
$\{x, y\}$	$\{x, y\}$	$\{x, y\}$
$\{x, \hat{x}\}$	$\{x\}$	$\{x, \hat{x}\}$
$\{\hat{x}, y\}$	$\{\hat{x}\}$	$\{\hat{x}\}$
$\{x, \hat{x}, y\}$	$\{y\}$	$\{x, \hat{x}\}$

Table 1: Example 1

$\{y\} \neq \{x, y\} = C_f(\{x, \hat{x}, y\} \backslash \{\hat{x}\})$. *So* C_f *is not observably substitutable.*

We will prove that if a choice function has a substitutable completion that satisfies IRC, it must satisfy observable IRC and NOVS. So it is weakly observably substitutable. Actually we will show that it is a corollary of a more general result. Since weakly observable substitutability (across workers) is sufficient for stable outcomes to exist, inspired by the substitutable completion idea we conjecture that if each firm's choice function C_f has a completion $\bar{C}_f$ which is *weakly observably substitutable (across workers)*, then stable outcomes may also exist. Because $\bar{C}_f$ may not be unitary, we cannot apply Theorem 1 to prove this conjecture. However, we will prove that $\bar{C}_f$ and C_f have the same set of observable offer processes, and for every observable offer process $\mathbf{x}$, $\bar{C}_f(\mathbf{x}) = C_f(\mathbf{x})$. Then if $\bar{C}_f$ is *weakly observably substitutable (across workers)*, C_f must be also *weakly observably substitutable (across workers)*. Then Theorem 1 implies that the conjecture is correct.

To prove the above result we first prove two lemmas.

Lemma 1. *Let* $\bar{C}_f$ *be a completion of a firm's choice function* C_f. *If* $\bar{C}_f$ *satisfies NSVS, then for all observable offer process* $\boldsymbol{x}$ *for* $\bar{C}_f$, $\bar{C}_f(\boldsymbol{x})$ *is feasible.*

Proof. We prove it by contradiction. Suppose for some observable offer process $\mathbf{x} = (x^1, \cdots, x^M)$ for $\bar{C}_f$, $\bar{C}_f(\mathbf{x})$ is not feasible. Then there exist two distinct contracts $x^i, x^j \in \bar{C}_f(\mathbf{x})$ with $i < j$ such that $w_{x^i} = w_{x^j}$. Since $\mathbf{x}$ is observable, $x_i \in \bar{R}_f(\{x^1, \cdots, x^{j-1}\})$. But $x_i \notin \bar{R}_f(\mathbf{x})$, so $x_i \in \bar{R}_f(\{x^1, \cdots, x^{j-1}\}) \backslash \bar{R}_f(\mathbf{x})$. Then since $\bar{C}_f$ satisfies NSVS, there exists $x_k \in \bar{C}_f(\{x^1, \cdots, x^{j-1}\})$ such that $w_{x^k} = w_{x^i} = w_{x^j}$, which contradicts the observability of $(x^1, \cdots, x^j)$. So $\bar{C}_f(\mathbf{x})$ is feasible. □

Lemma 2. *Let $\bar{C}_f$ be a completion of a firm's choice function C_f. If $\bar{C}_f$ satisfies NSVS, then $\bar{C}_f$ and C_f have the same set of observable offer processes and for every observable offer process $\boldsymbol{x}$, $\bar{C}_f(\boldsymbol{x}) = C_f(\boldsymbol{x})$.*

Proof. First, for any observable offer process $\mathbf{x}$ for $\bar{C}_f$, by Lemma 1, $\bar{C}_f(\mathbf{x})$ is feasible. So $C_f(\mathbf{x}) = \bar{C}_f(\mathbf{x})$. This implies that $\mathbf{x}$ is also observable for C_f.

Second, for any observable offer process $\mathbf{x} = \{x^1, \cdots, x^M\}$ for C_f, we prove by induction that it is also observable for $\bar{C}_f$. It is obvious that $C_f(\{x^1\}) = barC_f(\{x^1\})$ since $\bar{C}_f(\{x^1\})$ must be feasible. Suppose $C_f(\{x^1, \cdots, x^i\}) = \tilde{C}_f(\{x^1, \cdots, x^i\})$ for all $i \leq m < M$. Since $(x^1, \cdots, x^{m+1})$ is observable for C_f, $w_{x^{m+1}} \notin C^W_f(\{x^1, \cdots, x^m\}) = \bar{C}^W_f(\{x^1, \cdots, x^m\})$. So $(x^1, \cdots, x^{m+1})$ is also observable for $\bar{C}_f$. By Lemma 1 $\tilde{C}_f(\{x^1, \cdots, x^{m+1}\})$ is feasible. So $C_f(\{x^1, \cdots, x^{m+1}\}) = \bar{C}_f(\{x^1, \cdots, x^{m+1}\})$. By induction $\mathbf{x}$ is observable for $\bar{C}_f$, and $C_f(\mathbf{x}) = \tilde{C}_f(\mathbf{x})$. □

Now we are ready to prove the main result in this section.

Proposition 1. *A firm's choice function C_f has a weakly observably substitutable (across workers) completion if and only if C_f is weakly observably substitutable (across workers).*

Proof. Let $\bar{C}_f$ be a completion of C_f that is weakly observably substitutable (across workers). Then by Lemma 2, for every observable offer process $\mathbf{x}$ for C_f, $\bar{C}_f(\mathbf{x}) = C_f(\mathbf{x})$. Then if $\bar{C}_f$ satisfies NOVS, for any two observable offer processes $\mathbf{x}, \mathbf{x}'$ for f such that $\mathbf{x}'$ is a subprocess of $\mathbf{x}$, $\bar{R}_f(\mathbf{x}')\backslash\bar{R}_f(\mathbf{x}) = \emptyset$. This implies that $R_f(\mathbf{x}')\backslash R_f(\mathbf{x}) = \emptyset$. So C_f also satisfies NOVS.

If $\bar{C}_f$ satisfies NSVS, for any two observable offer processes $\mathbf{x}, \mathbf{x}'$ for C_f such that $\mathbf{x}'$ is a subprocess of $\mathbf{x}$, if $\bar{R}_f(\mathbf{x}')\backslash\bar{R}_f(\mathbf{x}) \neq \emptyset$, for every $x \in \bar{R}_f(\mathbf{x}')\backslash\bar{R}_f(\mathbf{x})$ there exists $x' \in \bar{C}_f(\mathbf{x}')$ such that $w_{x'} = w_x$. Since $\bar{C}_f(\mathbf{x}) = C_f(\mathbf{x})$ and $\bar{C}_f(\mathbf{x}') = C_f(\mathbf{x}')$, C_f must have the same property. So C_f also satisfies NSVS.

For any observable offer process $\mathbf{x}$ for C_f and any $Z \subseteq R_f(\mathbf{x})$, since $C_f(\mathbf{x}) = \bar{C}_f(\mathbf{x})$, $\bar{C}_f(\mathbf{x})$ is feasible and $Z \subseteq \bar{R}_f(\mathbf{x})$. Since $\mathbf{x}$ is also observable for $\bar{C}_f$ and $\bar{C}_f$ satisfies observable IRC, $\bar{C}_f(\mathbf{x}) = \bar{C}_f(\mathbf{x}\backslash Z)$. So $\bar{C}_f(\mathbf{x}\backslash Z)$ is also feasible, which implies that $\bar{C}_f(\mathbf{x}\backslash Z) = C_f(\mathbf{x}\backslash Z)$. Hence $C_f(\mathbf{x}) = C_f(\mathbf{x}\backslash Z)$, which implies that C_f satisfies observable IRC. So if $\bar{C}_f$ is weakly observably substitutable (across workers), C_f must be also weakly observably substitutable (across workers).

Of course if C_f is weakly observably substitutable (across workers), it can be a completion of itself. □

Since a substitutable completion that satisfies IRC is weakly observably substitutable, the following corollary is immediate.

Corollary 1. *If a firm's choice function C_f has a substitutable completion that satisfies IRC, C_f is weakly observably substitutable.*

5 Unilateral Substitutability and IRC Imply Substitutable Completability

In this section we show that if a choice function satisfies unilateral substitutability and IRC, it must has a substitutable completion.

Definition 7. *A firm's choice function C_f is* ***unilaterally substitutable*** *if for all $Z \subseteq X$ and all $x, z \in X \backslash Z$, if $z \in R_f(Z \cup \{z\})$ and $z \notin R_f(Z \cup \{x, z\})$, then there exists $z' \in Z$ such that $w_z = w_{z'}$.*

Proposition 2. *If a firm's choice function C_f satisfies unilateral substitutability and IRC, then it is substitutably completable.*

To prove the proposition we first prove a lemma

Lemma 3. *If a firm's choice function C_f satisfies unilateral substitutability and IRC, then for all $Z \subseteq X$ and all $x, z \in X \backslash Z$, if $z \in R_f(Z \cup \{z\})$ and $z \notin R_f(Z \cup \{x, z\})$, then there exists $\hat{z} \in C_f(Z \cup \{z\})$ such that $w_{\hat{z}} = w_z$.*

Proof. We prove by contradiction. Let $w = w_z$. Suppose there does not exist $\hat{z} \in C_f(Z \cup \{z\})$ such that $w_{\hat{z}} = w$, then it means that $Z_w \subseteq R_f(Z \cup \{z\})$. Since $z \in C_f(Z \cup \{x, z\})$ and C_f is unitary, we also have $Z_w \subseteq R_f(Z \cup \{x, z\})$. Since C_f satisfies IRC, $z \in R_f(Z \cup \{z\})$ and $z \notin R_f(Z \cup \{x, z\})$ imply that $z \in R_f(Z' \cup \{z\})$ and $z \notin R_f(Z' \cup \{x, z\})$ where $Z' = Z \backslash Z_w$. However, by the unilateral substitutability of C_f these imply that there exists $z' \in Z'$ such that $w_{z'} = w$, which is a contradiction. □

Proof of Proposition 2:

We construct a substitutable completion $\bar{C}_f$ of C_f. At step 0 below we let $\bar{C}_f(Z) = C_f(Z)$ for all $Z \subseteq X_f$. Then there may exist multiple violations of substitutability in $\bar{C}_f$. At each following step we identify some violations of substitutability and correct them by updating $\bar{C}_f$. At the end of the procedure we obtain the choice function $\bar{C}_f$ which is substitutable.

- Step 0: Define $\bar{C}^0_f(Z) \equiv C_f(Z)$ for all $Z \subseteq X_f$. It is obvious that $\bar{C}^0_f$ satisfies unilateral substitutability and IRC.

- Step 1: We consider all $Z \subsetneq X_f$ such that $\bar{R}^0_f(Z)\backslash\bar{R}^0_f(X_f) \neq \emptyset$. Lemma 3 says that for any $z \in \bar{R}^0_f(Z)\backslash\bar{R}^0_f(X_f)$ there exists $z' \in \bar{C}^0_f(Z)$ such that $w_{z'} = w_z$. So $\bar{C}^0_f(Z) \cup (\bar{R}^0_f(Z)\backslash\bar{R}^0_f(X_f))$ is not feasible. Then we define $\bar{C}^1_f(Z) \equiv \bar{C}^0_f(Z) \cup (\bar{R}^0_f(Z)\backslash\bar{R}^0_f(X_f))$. For all other Z, define $\bar{C}^1_f(Z) \equiv \bar{C}^0_f(Z)$.

  By this definition there is no violation of substitutability associated with X_f in $\bar{C}^1_f$.

- Step $k \geq 2$: We consider all $Y \subsetneq X_f$ such that $|Y| = |X_f| - k + 1$ and all $Z \subsetneq Y$ such that $\bar{R}^{k-1}_f(Z)\backslash\bar{R}^{k-1}_f(Y) \neq \emptyset$. By our construction $\bar{C}^{k-1}_f$ always chooses weakly more contracts than $\bar{C}^{k'-1}_f$ from the same set of contracts if $k > k'$. So for any $z \in \bar{R}^{k-1}_f(Z)\backslash\bar{R}^{k-1}_f(Y)$, $z \in \bar{R}^{k-1}_f(Z)$ implies that $z \in \bar{R}^0_f(Z) = R_f(Z)$. Now we prove that $z \notin R_f(Y)$.

  Suppose the contrary that $z \in R_f(Y)$. Then $z \notin \bar{R}^{k-1}_f(Y)$ implies that at some earlier step k' we have $z \in \bar{R}^{k'-1}_f(Y)$ but $z \notin \bar{R}^{k'-1}_f(\hat{Y})$ for some $Y \subsetneq \hat{Y} \subseteq X_f$ so that we define $\bar{C}^{k'}_f(Y) \equiv \bar{C}^{k'-1}_f(Y) \cup (\bar{R}^{k'-1}_f(Y)\backslash\bar{R}^{k'-1}_f(\hat{Y}))$. However, at step k' we should also have $z \in \bar{R}^{k'-1}_f(Z)$ since $z \in \bar{R}^{k-1}_f(Z)$ and $k' < k$. Then $Z \subsetneq \hat{Y}$ and $\bar{R}^{k'-1}_f(Z)\backslash\bar{R}^{k'-1}_f(\hat{Y}) \neq \emptyset$ imply that at step k' we must also define $\bar{C}^{k'}_f(Z) \equiv \bar{C}^{k'-1}_f(Z) \cup (\bar{R}^{k'-1}_f(Z)\backslash\bar{R}^{k'-1}_f(\hat{Y}))$. This means that $z \in \bar{C}^{k'}_f(Z)$, which contradicts $z \in \bar{R}^{k-1}_f(Z)$. So $z \notin R_f(Y)$.

  Hence $\bar{R}^{k-1}_f(Z)\backslash\bar{R}^{k-1}_f(Y) \subseteq R_f(Z)\backslash R_f(Y) \neq \emptyset$. Then Lemma 3 says that for any $z \in \bar{R}^{k-1}_f(Z)\backslash\bar{R}^{k-1}_f(Y) \subseteq R_f(Z)\backslash R_f(Y)$, there exists $z' \in C_f(Z)$ such that $w_{z'} = w_z$. So $C_f(Z) \cup (\bar{R}^{k-1}_f(Z)\backslash\bar{R}^{k-1}_f(Y))$ is not feasible. If $\bar{C}^{k-1}_f(Z) = C_f(Z)$, then $\bar{C}^{k-1}_f(Z) \cup (\bar{R}^{k-1}_f(Z)\backslash\bar{R}^{k-1}_f(Y))$ is not feasible. If $\bar{C}^{k-1}_f(Z) \neq C_f(Z)$, by our construction $\bar{C}^{k-1}_f(Z)$ is already not feasible. So in any case $\bar{C}^{k-1}_f(Z) \cup (\bar{R}^{k-1}_f(Z)\backslash\bar{R}^{k-1}_f(Y))$ is not feasible. Define $\bar{C}^k_f(Z) \equiv \bar{C}^{k-1}_f(Z) \cup (\bar{R}^{k-1}_f(Z)\backslash\bar{R}^{k-1}_f(Y))$. For all other Z, define $\bar{C}^k_f(Z) \equiv \bar{C}^{k-1}_f(Z)$.

By this definition there is no violation of substitutability associated with any $Y \subsetneq X_f$ such that $|Y| \geq |X_f| - k + 1$ in $\bar{C}_f^k$.

When $k = |X_f|$, we stop the procedure. Then $\bar{C}_f^{|X_f|}$ is the choice function we want. Let $\bar{C}_f \equiv \bar{C}_f^{|X_f|}$. $\bar{C}_f$ is a completion of C_f since from the above procedure we know that $\bar{C}_f(Z) \neq C_f(Z)$ only when $\bar{C}_f(Z)$ is not feasible. $\bar{C}_f$ is substitutable because we eliminate all violations of substitutability in the above procedure. □

The choice function $\bar{C}_f$ we construct in the above proof may not satisfy IRC. However, since it is substitutable, by Lemma 2 we know that the choice function C_f must satisfy NOVS. Since C_f further satisfies IRC, C_f is weakly observably substitutable.

6 Discussion

Besides the existence of stable outcomes, Hatfield et al. (2015) also prove that *observable substitutability* and other two conditions are sufficient for COM to be strategy-proof. In this note we relax *IRC* to *observable IRC*. It is an open question that whether the strategy-proofness of COM still holds if *IRC* is replaced by *observable IRC* in their proof.

In our definition of COM we arbitrarily choose an ordering of all workers according to which workers make offers. However, the outcome of COM is actually independent of this ordering if all firms' choice functions are weakly observably substitutable across workers. This fact can be proved in the entirely same way as the similar result of Hatfield et al.

Lastly, in Section 5 we construct a substitutable completion of a unilaterally substitutable choice function that satisfies IRC. However, the completion may not satisfy IRC. It is also an open question that whether there exists a substitutable completion that satisfies IRC, and if yes, how to construct it.

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