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# On Polynomial Lieb–Robinson Bounds for the XY Chain in a Decaying Random Field

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**Abstract** We consider the isotropic XY quantum spin chain in a random external field in the $z$ direction, with single site distributions given by i.i.d. random variables times the critical decaying envelope $j^{-1/2}$. Our motivation is the study of many-body localization. We investigate transport properties in terms of polynomial Lieb–Robinson (PLR) bounds. We prove a zero-velocity PLR bound for large disorder strength $\lambda$ and for small $\lambda$ we show a partial converse, which suggests the existence of a transition to non-trivial transport in the model.

## 1 Introduction

It is well known that a single quantum particle in one dimension which is subjected to an arbitrarily weak random potential exhibits exponential Anderson localization [3,22]. In the presence of interactions, one enters the subject of many-body localization (MBL) which has been a hot topic of condensed-matter physics in recent years, see e.g. [4,5,11,13,17,25] and references therein. On a heuristic level, MBL is described as *absence of thermalization*. Proposed criteria for this include the validity of an area law for the entanglement entropy and absence of information propagation (e.g. a zero-velocity Lieb–Robinson bound and logarithmic in time growth of the entanglement entropy). For an extensive list of possible criteria, see the review [14]. The very special MBL phase is expected to break down for sufficiently weak randomness, in what is called the *MBL transition* [28,32].

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A possible starting point for understanding MBL is the XY quantum spin chain in an i.i.d. random field. This is an *integrable toy model* which can be mapped to non-interacting fermions in a random environment. Since the fermions are then localized in the usual Anderson sense, it can be shown rigorously that this model enjoys an area law for the entanglement entropy for large classes of states [1,2,29] and a zero-velocity Lieb–Robinson bound [6,16]. A continuum analogue of this toy model, the disordered Tonks–Girardeau gas, was recently shown to display features of MBL for bosons, such as the absence of BEC and superfluidity [30], even at zero temperature.

However, a shortcoming of the toy model (apart from integrability) is that it will never display a transition to a non-MBL phase because the fermions are localized at arbitrarily small disorder strength (which is equivalent to arbitrarily large interaction strength).

In this paper, we propose a variation of the XY chain with disorder which *rigorously displays features suggesting that such a phase transition might occur* as the disorder strength is varied. The model is the isotropic XY chain on the half line with a random and *decaying* external field in the $z$ direction. The Hamiltonian reads

$$H_n^{XY}(\omega) := -\sum_{j=1}^{n-1}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y\right) + \lambda\sum_{j=1}^{n}\frac{V_j(\omega)}{j^{1/2}}\sigma_j^z$$

where the $V_j$ are i.i.d. random variables satisfying $\mathbb{E}[V_j] = 0$ and $\mathbb{E}[V_j^2] = 1$. Moreover, $\lambda > 0$ is a parameter describing the disorder strength. Note the decaying envelope $j^{-1/2}$ for the random field. It is "critical" in the sense that the potential is just barely not in $\ell^2(\mathbb{N})$. For other decay rates, the random field is either too weak or too strong to observe a qualitative transition from MBL to non-MBL features (such as transport) when $\lambda$ is varied.

We now explain in which sense our system exhibits features suggesting a phase transition from transport to localization as the disorder strength $\lambda > 0$ is increased. While our results will be more general and include bounds on the particle number transport as well, the key notion for quantifying many-body transport for this model are new *anomalous polynomial Lieb–Robinson (PLR) bounds*. The traditional Lieb–Robinson (LR) bounds [23,27] apply to general local Hamiltonians defined on a lattice and establish the existence of a certain "light cone" in spacetime outside of which correlations are exponentially small.

We say PLR$(a, b)$ holds for parameters $0 \le a \le 1$ and $b > 0$, if there exists a universal constant $C > 0$ such that for any observables $A$ supported at site 1 and $B$ supported at site $k > 1$, we have the bound

$$\|[\tau_t^n(A), B]\| \le C\|A\|\|B\|\left(\frac{t^a}{k}\right)^b. \tag{1}$$

Here $\tau_t^N$ is the Heisenberg time evolution generated by the Hamiltonian $H_n^{XY}$, see (3), and $\|\cdot\|$ is the standard operator norm. Intuitively, PLR$(a, b)$ says that in time $t$, information (as measured by the commutator of the initially localized observables) propagates at most a distance of order $t^a$, up to errors decaying like $x^{-b}$ away from the bent "light cone" $t^a = k$ in spacetime. The case $a = 1$ corresponds to ballistic transport.

We now discuss our results in words; the precise statements are given later. For simplicity, in this discussion $A$ is supported at site 1 and $B$ is supported at site $k > 1$.

- When $\lambda$ is *large enough*, the system is "polynomially localized" in the sense that

$$\mathbb{E}\Big[\sup_{t\in\mathbb{R}}\|[\tau_t^n(A), B]\|\Big] \le C\|A\|\|B\|\left(\frac{1}{k}\right)^{\kappa\lambda^2-5/4} \tag{2}$$

for a coefficient $0 < \kappa \leq \frac{5}{16}$ (Theorem 3.2). This is a disorder-averaged version of PLR$(0, \kappa\lambda^2 - 5/4)$ and may be understood as a *zero-velocity PLR bound*. It is of course only effective when $\kappa\lambda^2 - 5/4 > 0$.

- When $\lambda$ is *small enough*, PLR$(a, b)$ cannot hold if $a$ is too small or $b$ is too large (Corollary 3.9). In other words, there exist observables $A$, $B$ for which the bound (1) fails and in this sense transport is at least of order $t^a$. Concretely, in Corollary 3.9 we show that for $\lambda < 2$, (1) fails with probability one if $0 \leq a \leq 1$ and $b > 1/2$ satisfy

$$a\left(1 + \frac{1}{2b-1}\right) < 1.$$

In particular, for any $0 \leq a < 1$, there exists $b > 1/2$ large enough such that (1) fails with probability one.

*Remark 1.1* (i) It follows from [8, Thm. 2.6] and Proposition 3.8 that if only exponentially small errors are tolerated in an LR bound, then our model will exhibit ballistic transport for all $\lambda > 0$. This fits with the localization being only polynomial in type, even for large $\lambda$.

(ii) We emphasize that our results do *not* exclude that for small $\lambda$, an analogue of (2) holds with the exponent $\kappa\lambda^2 - 5/4$ replaced by a number $b \leq 1/2$. If this were true, it would be misleading to speak of a true transition from non-trivial transport to localization and it is for this reason that we do not claim to prove such a transition.

(iii) For the PLR$(a, b)$ bounds defined by (1) and (2), we only consider observables $A$ supported at site 1. If $A$ is supported at a site $j > 1$, the decaying factor is *not* replaced by the distance of the supports $|j - k|$ (as would be the case in a direct polynomial generalization of the LR bound, compare [7,8]), but instead by $\min\{j, k\}/\max\{j, k\}$. The precise statement is in Theorem 3.2. The reason why one cannot expect the distance $|j - k|$ is that the system is far from being translation-invariant.

To prove the results, we use the standard method of expressing the $XY$ chain in terms of free fermions via the Jordan–Wigner transformation [24]. The basic idea is to take bounds for the corresponding one-body system [10,15,19,20] and to pull them through the (non-local) Jordan–Wigner transformation by using ideas of Hamza, Sims and Stolz [16].

[16] considered a non-decaying random external field which yields an exponentially localized system, see also [21,31]. Here we apply the method of [16] to a situation in which errors decay only polynomially. Related papers which study the dependence of parameters in generalized LR bounds on the external field are [7–9,18]. The idea of studying polynomial LR bounds was conceived in [7,8], but there it was only shown that the idea does *not* apply to the random dimer model (a model with anomalous one-body transport).

For large $\lambda$, we use the fact that the Kunz–Souillard method utilized in [10] actually yields a polynomial bound on the eigenfunction correlator (16). We are grateful to David Damanik for pointing this out to us.

As mentioned before, we also show similar results for particle number transport. For this we adapt the techniques from [1], where such bounds were studied for non-decaying i.i.d. randomness, to our situation with polynomial decay. Similar bounds on particle number transport were also proved in the recent paper [30] on the disordered Tonks-Girardeau gas, a continuum analogue of the disordered XY chain.

Overall, our results follow rather directly by combining the above mentioned methods. Nonetheless, we believe that this alternative toy model provides an opportunity to study a phase transition, in terms of transport properties, from a mathematical and physics perspective and can stimulate further research. In particular, we have also attempted without success to prove analogous results for the entanglement entropy of eigenstates in the spirit of the recent works [1,2,12,29]. However we ran into difficulty bounding the entanglement entropy of eigenstates in the "localization regime" of large $\lambda$ because of the growth in $j$ of the bound (16). We believe that this question constitutes an interesting open problem.

## 2 The Model

### 2.1 The XY Chain in a Random Decaying External Field

For every $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, we consider the Hilbert space

$$\mathcal{H}_n = \bigotimes_{j=1}^{n} \mathbb{C}^2.$$

On $\mathcal{H}_n$, the Hamiltonian of the isotropic XY chain with a random decaying external field is given by

$$H_n^{XY}(\omega) := -\sum_{j=1}^{n-1} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y\right) + \lambda \sum_{j=1}^{n} \frac{V_j(\omega)}{j^{1/2}} \sigma_j^z,$$

where $\lambda > 0$ is a coupling constant. The sequence $\left(V_j(\omega)\right)_{j \in \mathbb{N}}$ is a family of iid random variables on a probability space $(\Omega, \Sigma, \mathbb{P})$. We assume that its single-site distribution has zero mean and is absolutely continuous with a bounded density of compact support and $\mathbb{E}[V_j^2] = 1$. In the above,

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the *Pauli matrices* and $\sigma_j^{x,y,z}$ is short-handed for

$$\mathbb{1}_1 \otimes \ldots \mathbb{1}_{j-1} \otimes \sigma^{x,y,z} \otimes \mathbb{1}_{j+1} \ldots \otimes \mathbb{1}_n$$

for $1 \leq j \leq n$. In the following we omit the $\omega$-dependence for brevity. For a finite set $J \subset \mathbb{N}$, we define the algebra of observables supported on $J$ by

$$\mathcal{A}_J = \bigotimes_{j \in J} \mathcal{B}(\mathbb{C}^2),$$

where $\mathcal{B}(\mathbb{C}^2)$ is the set of all complex $2 \times 2$ matrices. We will often make use of the fact that for $J \subset J'$, there is a natural embedding of $\mathcal{A}_J$ into $\mathcal{A}_{J'}$ by tensoring with the identity on $J' \setminus J$. Also, we set $\mathcal{A}_j \equiv \mathcal{A}_{\{j\}}$.

Finally, the *Heisenberg dynamics* of an observable $A \in \mathcal{A}_J$ under the Hamiltonian $H_n^{XY}$ is defined by

$$\tau_t^n(A) := e^{itH_n^{XY}} A e^{-itH_n^{XY}}. \tag{3}$$

### 2.2 The Jordan–Wigner Transformation

We use the standard procedure, going back to [24], of mapping the XY chain to free fermions via the Jordan–Wigner transformation.

For the details of the diagonalization procedure, we refer to Section 3.1 in [16]. Here we only recall what we need to establish notation. The first step is to introduce the lowering operator

$$a_j = \frac{1}{2}\left(\sigma_j^x - i\sigma_j^y\right) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j \tag{4}$$

and its adjoint the raising operator $a_j^*$ for all $1 \le j \le n$. The Jordan–Wigner transformation maps these to the fermion operators

$$c_1 = a_1, \quad c_j = \sigma_1^z \ldots \sigma_{j-1}^z a_j \quad \text{for } 2 \le j \le n. \tag{5}$$

The $\{c_j\}$ then satisfy the canonical anticommutation relations (CAR). We have the identity

$$a_j^* a_j = c_j^* c_j. \tag{6}$$

In terms of the fermion operators, the Hamiltonian reads,

$$H_n^{XY} = 2\mathcal{C}^* H_n \mathcal{C} - \sum_{j=1}^{n} \tilde{V}_j \tag{7}$$

where $\mathcal{C} := (c_1, ..., c_n)^T$ and $\widetilde{V}_j := \frac{\lambda}{j^{1/2}} V_j$. The $n \times n$ matrix $H_n$ is given by

$$H_n = \begin{pmatrix} \widetilde{V}_1 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & \widetilde{V}_n \end{pmatrix}, \tag{8}$$

Note that $H_n$ can be identified with a discrete Schrödinger operator on the half line, i.e. on $\ell^2(\mathbb{N})$, with the random decaying potential $\{\tilde{V}_j\}$ and zero boundary conditions at site $n+1$. The constant $\sum_{j=1}^{n} \tilde{V}_j$ in (7) does not change the Heisenberg dynamics (3) and can thus be ignored in the following.

We will often use that the Heisenberg dynamics of the $c_j$ operators is given in the following simple fashion.

**Proposition 2.1** ([16, Sec. 3]) *For all $1 \le j, k \le n$, the identity*

$$\tau_t^n(c_j) = \sum_{m=1}^{n} \langle \delta_j, e^{-2itH_n} \delta_m \rangle c_m \tag{9}$$

*holds and consequently*

$$\|[\tau_t^n(a_j), B]\| \le 2 \sum_{l=1}^{j} \sum_{m=1}^{n} |\langle \delta_l, e^{-2itH_n} \delta_m \rangle| \left( \|[c_m, B]\| + \|[c_m^*, B]\| \right). \tag{10}$$

*Proof* The first equality follows from diagonalizing the one-particle operator $H_n$. For details see [16, Eq. (3.15)]. Taking adjoints, the same is also true for $c_k^*$. Using the Leibniz rule for commutators, i.e.

$$[AB, C] = A[B, C] + [A, C]B \tag{11}$$

we obtain the estimate

$$\|[\tau_t^n(c_j^*c_j), B]\| \le \sum_{m=1}^{n} \langle \delta_j, e^{-2itH_n}\delta_m\rangle \left(\|[c_m, B]\| + \|[c_m^*, B]\|\right). \tag{12}$$

The latter inequality also holds for the adjoint $c_j c_j^*$.

To see inequality (10), we note that $(\sigma_j^z)^{-1} = \sigma_j^z$ for all $1 \le j \le n$ gives

$$a_j = \sigma_{j-1}^z \dots \sigma_1^z c_j. \tag{13}$$

Thus, an iteration of the Leibniz rule (11) implies

$$\begin{aligned} \|[\tau_t^n(a_j), B]\| &= \|[\tau_t^n(\sigma_{j-1}^z \dots \sigma_1^z c_j), B]\| \\ &\le \|[\tau_t^n(c_j), B]\| + \sum_{l=1}^{j-1} \|[\tau_t^n(\sigma_l^z), B]\|. \end{aligned} \tag{14}$$

Since $\sigma_l^z = 2c_l^* c_l - \mathrm{id}_{\mathbb{C}^2}$, the identity (9) and the bound (12) imply

$$\begin{aligned} (14) \le &\sum_{m=1}^{n} |\langle \delta_j, e^{-2itH_n}\delta_m\rangle| \|[c_m, B]\| \\ &+2\sum_{l=1}^{j-1}\sum_{m=1}^{n} |\langle \delta_l, e^{-2itH_n}\delta_m\rangle \left(\|[c_m, B]\| + \|[c_m^*, B]\|\right). \end{aligned} \tag{15}$$

□

## 3 Polynomial Lieb–Robinson Bounds

### 3.1 Localization for Large Enough λ

We start with recalling an old result by [10] which provides bounds on the eigenfunction correlator of the Anderson model with a random decaying potential.

**Lemma 3.1** *Let $H_n$ be the operator given in (8). Then there exist constants $C, \kappa > 0$ such that for all $n \in \mathbb{N}$ and all $1 \le j \le k \le n$, we have*

$$\mathbb{E}\Big[\sup_{|g|\le 1} |\langle \delta_j, g(H_n)\delta_k\rangle|\Big] \le \frac{C}{\lambda}(jk)^{1/4}\left(\frac{j}{k}\right)^{\kappa\lambda^2}. \tag{16}$$

In particular, one can choose $g(x) = e^{-itx}$ in the above. The exponent $\kappa$ will feature in all of the following bounds and we show later that it satisfies $\kappa \le \frac{5}{16}$, see Corollary 3.11.

*Proof* We estimate

$$\mathbb{E}\Big[\sup_{|g|\le 1} |\langle \delta_j, g(H_n)\delta_k\rangle|\Big] \le \mathbb{E}\Big[\sum_{E\in\sigma(H_n)} |\psi_E^n(j)||\psi_E^n(k)|\Big] =: \overline{\rho}^n(j,k,\mathbb{R}) \tag{17}$$

where the sequence $\left(\psi_E^n\right)_{E\in\sigma(H_n)}$ denotes the normalized eigenvectors of $H_n$ counted with multiplicity. An adaption of [10, Prop. III.1] implies

$$\overline{\rho}^n(j,k,\mathbb{R}) \le \frac{C}{\lambda^2}(jk)^{1/4}\left(\frac{j}{k}\right)^{\kappa\lambda^2}. \tag{18}$$

The latter follows from inequality [10, Eq. III.16] using the bounds [10, Eq. III.14 and Eq. III.15] and we remark that in the result [10, Eq. III.4] the 1/2-exponent should be replaced by a 1/4-exponent. □

As a consequence, we obtain a disorder-averaged polynomial Lieb–Robinson bound with $a = 0$ for the spin chain $H_n^{XY}$.

**Theorem 3.2** *Let $\kappa$ be as in Lemma 3.1 above. Suppose that $\kappa\lambda^2 > \frac{5}{4}$. Then there exists a constant $C > 0$ such that for all choices of $1 \le j \le k \le n$,*

$$\mathbb{E}\Big[\sup_{t\in\mathbb{R}} \|[\tau_t^n(A), B]\|\Big] \le C\|A\|\|B\|(jk)^{5/4}\left(\frac{j}{k}\right)^{\kappa\lambda^2} \tag{19}$$

*holds for all observables $A \in \mathcal{A}_j$ and $B \in \mathcal{A}_{k,\dots,n}$.*

We emphasize that the constant $C$ is uniform in $n$.

*Proof* Note that $\mathcal{A}_j$ is spanned by the matrices $\{a_j, a_j^*, a_j a_j^*, a_j^* a_j\}$. According to Proposition 2.1, we can estimate

$$\|[\tau_t^n(a_j), B]\| \le 2\sum_{l=1}^{j}\sum_{m=1}^{n} |\langle\delta_l, e^{-2itH_n}\delta_m\rangle| \left(\|[c_m, B]\| + |\|[c_m^*, B]\|\right) \tag{20}$$

We note that $[c_m, B] = 0$ for all $m < k$. Hence, Lemma 3.1 implies

$$\begin{aligned}\mathbb{E}[\|[\tau_t^n(a_j)B]\|] &\le \tfrac{4C}{\lambda^2}\|B\|\sum_{l=1}^{j}\sum_{m=k}^{n}(lm)^{1/4}\left(\tfrac{l}{m}\right)^{\kappa\lambda^2}\\ &\le \tfrac{4C}{\lambda^2}\|B\|\sum_{l=1}^{j}\sum_{m=k}^{\infty}(lm)^{1/4}\left(\tfrac{l}{m}\right)^{\kappa\lambda^2}\\ &\le \tfrac{C}{\lambda^2}\|B\|(jk)^{5/4}\left(\tfrac{j}{k}\right)^{\kappa\lambda^2}\end{aligned} \tag{21}$$

for some constant $C > 0$ which is finite for $\lambda > \sqrt{\frac{5}{4\kappa}}$. Taking adjoints the same estimate is true for $a_j^*$. For the products $a_j^* a_j$ and $a_j a_j^*$, we use the Leibniz rule (11). □

*Remark 3.3* Instead of the distance $|j - k|$ of the supports of the observables, which would appear in a straightforward polynomial generalization of the traditional LR bound as was proposed in [7,8], the right hand side depends on the quotient $j/k$. Note that the distance $|j - k|$ is not so natural for our model, because it is far from being translation-invariant.

However, if we consider observables $A$ supported at a *fixed* site, say the site 1, the bound (19) reduces to a polynomial Lieb–Robinson bound involving the distance of the supports. Let $A \in \mathcal{A}_1$. Then the bound

$$\mathbb{E}\Big[\sup_{t\in\mathbb{R}} \|[\tau_t^n(A), B]\|\Big] \le C\|A\|\|B\|\left(\frac{1}{k}\right)^{\kappa\lambda^2-5/4} \tag{22}$$

holds uniformly in $n \in \mathbb{N}$ and $B \in \mathcal{A}_{k,\dots,n}$ for any $1 < k \le n$.

For small $t$ the above is not satisfactory. One can improve the result:

**Proposition 3.4** *Let $\kappa$ be as in Lemma 3.1. There exists a constant $C$ such that for all choices of $1 \le j \le k \le n$,*

$$\mathbb{E}\left[\|[\tau_t^n(A), B]\|\right] \le C\|A\|\|B\||t|\left(\frac{1}{k}\right)^{\kappa\lambda^2-5/4} \tag{23}$$

*holds for all observables* $A \in \mathcal{A}_1$, $B \in \mathcal{A}_{k,\dots,n}$.

*Proof* We follow the proof of [16, Cor. 3.4]. Define

$$f(t) := [\tau_t(A), B]. \tag{24}$$

Then, $f(t)$ solves the ODE

$$f'(t) = i[f(t), \tau_t^n(H_1)] - i[[B, \tau_t^n(H_1)], \tau_t^n(A)]. \tag{25}$$

where $H_1 := \sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y + V_1\sigma_1^z$. Following [26, App.A] we obtain

$$\|f(t)\| \le \int_0^{|t|} \mathrm{d}s\, \|[\tau_s^n(H_1), B]\|. \tag{26}$$

Since $H_1$ is supported on $\mathcal{A}_1 \otimes \mathcal{A}_2$ we use Theorem 3.2 to obtain a time independent bound on the integrand which yields the theorem. □

### 3.2 Lower Bounds on Transport for Small Enough λ

In this section we restrict ourselves to pairs of observables for which one of the observables is supported at the site 1.

**Definition 3.5** Let $0 \le a \le 1$ and $b \ge 0$. We say that $H_n^{XY}$ exhibits the polynomial Lieb–Robinson bound PLR$(a, b)$, if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$\|[\tau_t^n(A), B]\| \le C\|A\|\|B\| \left(\frac{t^a}{k}\right)^b \tag{27}$$

holds for all $A \in \mathcal{A}_1$, $B \in \mathcal{A}_{k,\dots,n}$.

Let $H$ be the discrete Schrödinger operator on $\ell^2(\mathbb{N})$ which arises as the inductive limit of the family $(H_n)_{n\in\mathbb{N}}$.

**Definition 3.6** We define the $p$-th moment of the position operator

$$|X|^p(t) := \sum_{k\in\mathbb{N}} k^p \left|\left\langle e^{-itH}\delta_j, \delta_k\right\rangle\right|^2 \tag{28}$$

and its *time-average*

$$\langle|X|^p\rangle(T) := \frac{2}{T}\int_0^\infty \mathrm{d}t\, e^{-2t/T}|X|^p(t) \tag{29}$$

for all $T > 0$. The upper and lower transport exponents are defined by

$$\beta^-(p) := \liminf_{t\to\infty} \frac{\ln|X|^p(t)}{p\ln t} \quad \text{and} \quad \beta^+(p) := \limsup_{t\to\infty} \frac{\ln|X|^p(t)}{p\ln t} \tag{30}$$

and their time averaged versions are defined by

$$\langle\beta^-(p)\rangle := \liminf_{T\to\infty} \frac{\ln\langle|X|^p\rangle(T)}{p\ln T} \quad \text{and} \quad \langle\beta^+(p)\rangle := \limsup_{T\to\infty} \frac{\ln\langle|X|^p\rangle(T)}{p\ln T}. \tag{31}$$

**Theorem 3.7** *Assume* PLR$(a, b)$ *holds for some* $0 \le a \le 1$ *and* $b > 1/2$. *Then,*

$$\limsup_{\epsilon\to 0} \beta^+(2b - 1 - \epsilon) \le a\left(1 + \frac{1}{2b-1}\right). \tag{32}$$

*Proof* The strong resolvent-convergence of $H_n$ to $H$ (this follows e.g. from the geometric resolvent identity) implies the convergence

$$\lim_{n\to\infty}\langle e^{itH_n}\delta_1,\delta_k\rangle = \langle e^{itH}\delta_1,\delta_k\rangle, \tag{33}$$

for any $1 \le k \le n$. Hence, Fatou's lemma implies the inequality

$$\sum_{k\in\mathbb{N}} k^{2b-1-\epsilon}|\langle e^{-itH}\delta_1,\delta_k\rangle|^2 \le \liminf_{n\to\infty}\sum_{k\in\mathbb{N}} k^{2b-1-\epsilon}|\langle e^{-itH_n}\delta_1,\delta_k\rangle|^2, \tag{34}$$

where $\epsilon > 0$ is arbitary.

Now, we bound the one-body propagation in terms of the many-body propagation using [8, Lm. 4.1]. It implies that for any $1 \le k \le n$

$$|\langle e^{-itH_n}\delta_1,\delta_k\rangle| \le \|[\tau_t^n(c_1), a_k^*]\|. \tag{35}$$

Using this and the assumption that PLR$(a, b)$ holds, we bound

$$(34) \le t^{2ab}\sum_{k\in\mathbb{N}} k^{-1-\epsilon}. \tag{36}$$

Since the latter is summable for any $\epsilon > 0$, this implies

$$\beta^+(2b-1-\epsilon) \le \frac{2ab}{2b-1-\epsilon} \tag{37}$$

and therefore (32) follows. □

**Proposition 3.8** *Let $p > 0$. If $\lambda < 4p$, the lower bound*

$$\beta^+(p) \ge 1 - \frac{\lambda}{4p} \tag{38}$$

*holds $\mathbb{P}$-almost surely. If $\lambda < 2$ one even has*

$$\beta^+(p) = 1 \tag{39}$$

*$\mathbb{P}$-almost surely.*

Before we give the proof, which is based on results in [15,19,20], we discuss the consequences of combining Theorem 3.7 and Proposition 3.8. What we obtain can be interpreted as lower bounds on transport, as we explained in the introduction, however see also the caveat in Remark 1.1(iii).

**Corollary 3.9** *Let $(a, b)$ be a pair of $0 \le a \le 1$ and $b > 1/2$. If either of the following two conditions applies, then, with probability one,* PLR$(a, b)$ *cannot hold.*

- $\lambda < 2$ *and* $a\left(1 + \frac{1}{2b-1}\right) < 1$
- $\lambda < 4(2b-1)$ *and* $a\left(1 + \frac{1}{2b-1}\right) < 1 - \frac{\lambda}{4(2b-1)}$.

*In particular, if $\lambda < 2$, then for any fixed $0 \le a < 1$ there exists $b > 1/2$ large enough such that PLR$(a, b)$ cannot hold.*

*Remark 3.10* A shortcoming of our results is that we need to assume $b > 1/2$, see Remark 1.1(iii). This is ultimately a consequence of summing up one-body transport bounds when inverting the Jordan–Wigner transformation (compare Proposition 2.1) and is therefore intimately connected to the core of the method.

We also get a bound on the maximal power of the polynomial decay coefficient $\kappa$ which was introduced in the previous section.

**Corollary 3.11** *The constant $\kappa$ from Proposition 3.1 satisfies $\kappa \leq \frac{5}{16}$.*

*Proof* Note that $\kappa$ is independent of $\lambda$. Fix $\lambda < 2$ and $p > 0$. By Proposition 3.8, $\sup_{t>0} |X|^p(t) = \infty$. Recalling the definition (28) of $|X|^p(t)$ and using the estimate in Lemma 3.1 then gives $p + 1/4 - \kappa\lambda^2 \geq -1$. Sending $\lambda \to 2$ and $p \to 0$ yields $\kappa \leq \frac{5}{16}$. □

It remains to give the

*Proof of Prop. 3.8* For equation (38), we apply the lower bound [15, Thm. 5.1, Eq. (5.3)] to the function $f \in C_c^\infty(\mathbb{R})$ with $f \equiv 1$ on $\sigma(H)$. This provides for any $\epsilon > 0$ the bound

$$\langle |X| \rangle_j^p(T) \geq C_\omega(p, \epsilon) T^{p-2\gamma-\epsilon}, \tag{40}$$

$\mathbb{P}$-almost surely, where $\gamma := \inf_{E\in(-2,2)} \frac{\lambda}{8-2E^2}$. This implies

$$\langle \beta^-(p) \rangle \geq 1 - \frac{\lambda}{4p}. \tag{41}$$

The chain of inequalities $\langle \beta^-(p) \rangle \leq \langle \beta^+(p) \rangle \leq \beta^+(p)$ gives the result. To see the last inequality, note that $\beta := \beta^+(p) > 0$ implies for any $\epsilon > 0$, $|X|_1^p(t) \leq Ct^{p\beta+\epsilon}$. This readily gives

$$\langle |X|_1^p \rangle(T) = \frac{2}{T} \int_0^\infty \mathrm{d}t\, e^{-2t/T} |X|_1^p(t) \leq CT^{p\beta+\epsilon} \tag{42}$$

and the inequality $\langle \beta^+(p) \rangle \leq \beta$.

For equation (39), we use [19, Thm. 5.1] with $m = p$, where we have to prove its assumption, which is $P_c\delta_1 \neq 0$. Here, $P_c$ is the orthogonal projection onto continuous part of the spectrum. Since $|\lambda| < 2$, the operator $H$ exhibits singular continuous spectrum [20], thus $P_c \neq 0$. Now, $P_c\delta_1 \neq 0$ follows from cyclicity of $\delta_1$, which can be proven by induction because the Hamiltonian acts on the half space $\ell^2(\mathbb{N})$ only. □

## 4 Propagation Bounds for the Number Operator

In this section, we derive bounds on the propagation of the number operator by combining ideas from [1] with the bounds on the one-body dynamics discussed before. We recall that [1] derived such bounds for the case of non-decaying randomness (see also [30] for a continuum analogue).

We define the number operator and the local number operator by

$$\mathcal{N} := \sum_{j=1}^n a_j^* a_j \quad \text{and} \quad \mathcal{N}_S := \sum_{j\in S} a_j^* a_j, \tag{43}$$

where $a_j$ is given in (4) and $S \subset \{1, ..., n\}$. This measures the number of up-spins in $S$. Let

$$\rho = \bigotimes_{j=1}^n \rho_j, \qquad \rho_j := \begin{pmatrix} \eta_j & 0 \\ 0 & 1-\eta_j \end{pmatrix} \tag{44}$$

and $0 \leq \eta_j \leq 1$. We denote by $\rho_t := e^{-itH_n} \rho e^{itH_n}$ the time evolution of the state $\rho$ and by $\langle A \rangle_\rho := \mathrm{tr}\,[A\rho]$ the expectation of an observable $A$ with respect to the state $\rho$.

**Theorem 4.1** *Let $\kappa > 0$ be as in Lemma 3.1. There exists a constant $C > 0$ such that for every $n \geq 1$ and $S \subset \{1, \ldots, n\}$,*

$$\mathbb{E}\Big[\sup_{t\geq 0}\langle \mathcal{N}_S\rangle_{\rho_t}\Big] \leq \frac{C}{\lambda}\sum_{j\in S}\sum_{k=1}^{n}\eta_k (jk)^{1/4}\left(\frac{\min\{j,k\}}{\max\{j,k\}}\right)^{\kappa\lambda^2}. \tag{45}$$

This follows directly by combining results of [1] with Lemma 3.1.

*Remark 4.2* To illustrate the above we split $\{1, ..., n\} = I \cup J$ with $I := \{1, ..., m\}$ and $J := \{m+1, ..., n\}$ for $n > m \in \mathbb{N}$. We set $\eta_j = 0$ on $I$ and $\eta_j = 1$ on the complement $J$. In other words $\rho = |\varphi\rangle\langle\varphi|$ with the vector

$$|\varphi\rangle = |\downarrow\rangle^{\otimes m} \otimes |\uparrow\rangle^{\otimes(n-m+1)} \tag{46}$$

in standard notation. Let $m > l \in \mathbb{N}$ and $S = \{1, ..., l\}$. For $\kappa\lambda^2 > 5/4$, the above theorem implies the bound

$$\mathbb{E}\Big[\sup_{t\geq 0}\langle \mathcal{N}_S\rangle_{\rho_t}\Big] \leq C\left(\frac{l}{m}\right)^{\kappa\lambda^2}(lm)^{5/4} \tag{47}$$

for a constant $C > 0$ uniform in $l, m, n$. This is a time-independent bound on the number of up-spins which propagate from $J$ into $S$ and it decays as the distance $m \to \infty$ (when $\lambda$ is large enough to guarantee $\kappa\lambda^2 > 5/4$).

*Proof* The same computation that gives [1, Eq. (41)] shows

$$\langle \mathcal{N}_S\rangle_{\rho_t} = \sum_{j\in S}\sum_{k=1}^{n}|\langle \delta_j, e^{2itH_n}\delta_k\rangle|^2\eta_k. \tag{48}$$

Using this, Lemma 3.1 implies

$$\mathbb{E}\Big[\sup_{t\geq 0}\langle \mathcal{N}_S\rangle_{\rho_t}\Big] \leq \sum_{j\in S}\sum_{k=1}^{n}\eta_k\mathbb{E}\Big[\sup_{t\geq 0}|\langle \delta_j, e^{2itH_n}\delta_k\rangle|^2\Big] \tag{49}$$

The assertion now follow from $|\langle \delta_j, e^{2itH_n}\delta_k\rangle|^2 \leq |\langle \delta_j, e^{2itH_n}\delta_k\rangle|$ and Lemma 3.1. □

**Theorem 4.3** *If for some $0 \leq a \leq 1 < b$ and all $k, n \in \mathbb{N}$ with $k \leq n$*

$$\langle \mathcal{N}_1\rangle_{\rho_t} \leq \left(\frac{t^a}{k}\right)^b \tag{50}$$

*holds for all $\rho$ of the form (44) and $\eta_j = 0$ for $j < k$. Then, the upper transport exponent satisfies the bound*

$$\limsup_{\epsilon\to 0}\beta^+(b-1-\epsilon) \leq \frac{ab}{b-1}. \tag{51}$$

Again, Proposition 3.8 then gives restrictions on the possible values of $0 \leq a \leq 1 < b$ for which (50) can hold. Therefore Theorem 4.3 may be interpreted as a lower bound on the transport of particles (from sites $k$ and larger to the site 1) if at most error of order $x^{-b}$ with $b > 1$ can ignored, compare Remark 1.1(iii).

*Proof* Let $\rho_k$ be given as in (44) with $\eta_j = \delta_{j,k}$. By (48)

$$\langle \mathcal{N}_1 \rangle_{\rho_t^k} = |\langle \delta_1, e^{-itH_n} \delta_k \rangle|^2. \tag{52}$$

Hence, the computation in (34) and assumption (50) imply that for any $p > 0$

$$\begin{aligned} |X|^p(t) &\le \liminf_{n\to\infty} \sum_{k\in N} k^p |\langle e^{-itH_n} \delta_1, \delta_k \rangle|^2 \\ &\le \sum_{k\in\mathbb{N}} k^p \left(\frac{t^a}{k}\right)^b = t^{ab} \sum_{k\in\mathbb{N}} k^{p-b}. \end{aligned} \tag{53}$$

Taking $p = b - 1 - \epsilon$ for an $\epsilon > 0$, the last sum is finite and this gives the assertion. □

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