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D. M. Kochmann and W. J. Drugan

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An infinitely-stiff elastic system via a tuned negative-stiffness component stabilized by rotation-produced gyroscopic forces

D. M. Kochmann[1,a)] and W. J. Drugan[2,b)]
[1]Graduate Aerospace Laboratories, California Institute of Technology, Pasadena, California 91125, USA
[2]Department of Engineering Physics, University of Wisconsin–Madison, 1500 Engineering Drive, Madison, Wisconsin 53706, USA

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An elastic system containing a negative-stiffness element tuned to produce positive-infinite system stiffness, although statically unstable as is any such elastic system if unconstrained, is proved to be stabilized by rotation-produced gyroscopic forces at sufficiently high rotation rates. This is accomplished in possibly the simplest model of a composite structure (or solid) containing a negative-stiffness component that exhibits all these features, facilitating a conceptually and mathematically transparent, completely closed-form analysis. *Published by AIP Publishing.*
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Elastic composite materials and structures containing a tuned negative-stiffness component have the fascinating capability to exhibit positive-infinite overall stiffness,[1,2] but can they be stable? A sufficiently stiff component can stabilize a negative-stiffness component,[3,4] but only for a limited range of negative stiffness that is not sufficient to make the overall composite infinitely stiff.[5–7] We showed that dynamic excitation can provide the needed additional stabilization to permit stable infinite system stiffness, for the case of a long negative-stiffness cylinder with a positive stiffness coating that is dynamically excited by spinning about its axis.[8] The mathematical complexity inherent in that problem's analysis obscures the concepts at work, prevents closed-form results, and treats only that special solid composite.

Here, we introduce and analyze possibly the simplest model system, composed of springs and masses, that very clearly exhibits: how tuning a negative-stiffness component can produce an infinitely stiff system; that this system is statically unstable under force boundary conditions; and that rotation at a sufficient, physically achievable rate will stabilize it. The system's simplicity permits a conceptually and mathematically transparent, completely closed-form analysis of all these phenomena. Further, the system is a direct model of a physically realizable structure exhibiting stable infinite stiffness. The results derived open the way and provide a blueprint for the creation of structures (and solids) with stable ultrahigh stiffness.

First, we demonstrate that rotation can stabilize a negative-stiffness component, via analysis of an extremely simple two-spring system (studied by Ziegler[9] for positive-stiffness springs). Showing this simple analysis first greatly aids understanding of the ensuing analysis of the primary model, which proceeds in a conceptually identical manner. The model system is shown in Fig. 1(a). A point mass m is attached to two linear elastic springs (stiffnesses $c_1, c_2 \in \mathbb{R}$) whose ends are attached to but may slide freely along the axes. For the static system, stability requires $c_i > 0$. We next demonstrate the expanded regime of stability under system rotation.

The system is assumed to rotate with angular frequency $\Omega(t) = \dot{\varphi}(t)$ about the z-axis (dots denote material time derivative, and $\varphi(t)$ is the in-plane angle), and the point mass' time-dependent position is given by $\boldsymbol{x}(t) = \boldsymbol{R}(t)[\boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t)]$. $\boldsymbol{X}$ is the initial position in the co-rotating frame of reference ($\dot{\boldsymbol{X}} = 0$), $\boldsymbol{R} \in SO(2)$ is a rotation in the x-y-plane of angle $\varphi(t)$, and $\boldsymbol{u}(\boldsymbol{X}, t)$ is the in-plane displacement field with respect to the co-rotating frame. Consequently, the governing equations of motion written in the co-rotating reference frame with total force vector $\boldsymbol{F}$ and rotation vector $\boldsymbol{\Omega}(t) = \Omega(t)\,\boldsymbol{e}_z$ become

$$\boldsymbol{F}/m = \ddot{\boldsymbol{u}} - \Omega^2\boldsymbol{u} + \dot{\boldsymbol{\Omega}} \times \boldsymbol{u} + 2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}} + \dot{\boldsymbol{\Omega}} \times \boldsymbol{X} - \Omega^2\boldsymbol{X}. \quad (1)$$

For simplicity, we assume a constant frequency Ω. The last term is then constant so it does not affect system stability and will thus be ignored in the stability analysis. (It will be needed when deriving the effective stiffness defined with respect to the rotating system.) In the absence of external loading, the equations of motion for in-plane displacements $\boldsymbol{u} = (u_x, u_y)^{\mathrm{T}}$ then reduce to

$$m\,\ddot{u}_x - 2\,m\,\Omega\,\dot{u}_y + (c_1 - m\,\Omega^2)u_x = 0, \quad (2a)$$

$$m\,\ddot{u}_y + 2\,m\,\Omega\,\dot{u}_x + (c_2 - m\,\Omega^2)u_y = 0. \quad (2b)$$

Symmetry of displacements is not assumed. Assuming the separable form for the displacement field $\boldsymbol{u}(t) = \boldsymbol{U}\,e^{i\,\omega\,t}$ with constant amplitude $\boldsymbol{U}$ results in

$$\begin{pmatrix} c_1 - m\,(\omega^2 + \Omega^2) & -2\,i\,m\,\Omega\,\omega \\ 2\,i\,m\,\Omega\,\omega & c_2 - m\,(\omega^2 + \Omega^2) \end{pmatrix} \cdot \boldsymbol{U} = \boldsymbol{0}. \quad (3)$$

A non-trivial solution requires the coefficient matrix to be singular, which yields the eigenfrequencies

$$\omega^2 = \frac{c_1 + c_2}{2\,m} + \Omega^2 \pm \sqrt{\left(\frac{c_2 - c_1}{2\,m}\right)^2 + 2\,\frac{c_1 + c_2}{m}\Omega^2}. \quad (4)$$

a)kochmann@caltech.edu
b)drugan@engr.wisc.edu

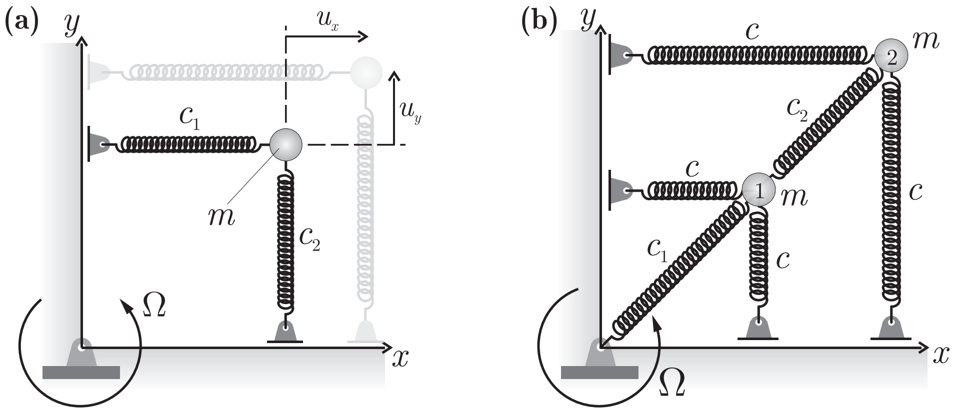

FIG. 1. (a) System of two elastic springs and one point mass; (b) system of two point masses attached to elastic springs having three different stiffnesses. Both systems rotate about the z-axis with constant angular frequency Ω, and all spring attachments slide freely parallel to their attachment axes, as illustrated in (a).

For vanishing rotation ($\Omega = 0$), these reduce to the correct static system eigenfrequencies: $\omega_1^2 = c_1/m$, $\omega_2^2 = c_2/m$. Stability requires that displacements not become large with time, which requires $\mathrm{Im}(\omega_i) \geq 0$ for all (generally complex) eigenfrequencies. Applied to (4), this means all eigenfrequencies must be pure real for stability, requiring

$$\left(\frac{c_2 - c_1}{2\,m}\right)^2 + 2\,\frac{c_1 + c_2}{m}\Omega^2 \geq 0 \tag{5}$$

and

$$\frac{c_1 + c_2}{2\,m} + \Omega^2 - \sqrt{\left(\frac{c_2 - c_1}{2\,m}\right)^2 + 2\,\frac{c_1 + c_2}{m}\Omega^2} \geq 0. \tag{6}$$

We take $c_1 < 0$, $c_2 > 0$ to explore the possibility of stabilizing a negative stiffness c_1. The analogous analysis ($c_1 > 0$, $c_2 < 0$) is unnecessary since (4) is symmetric in c_1 and c_2. If $\Omega^2 > \omega_2^2 = c_2/m$, then (5) together with the fact that (6) clearly requires $(c_1 + c_2)/(2\,m) + \Omega^2 \geq 0$ give the lower limit on spring stiffness c_1 for stability:

$$\frac{c_1}{c_2} \geq 1 - 4\left(\frac{\Omega}{\omega_2}\right)^2 + 4\,\frac{\Omega}{\omega_2^2}\sqrt{\Omega^2 - \omega_2^2}. \tag{7}$$

If $\Omega^2 < c_2/m$, (5) is always satisfied and hence gives no restriction on c_1. The special case $\Omega = \sqrt{c_2/m}$ corresponds to resonance, for which the particular solution may always yield an unbounded response. Thus, we treat resonant cases as unstable. With these restrictions in mind, we can explore admissible (negative) values of the spring stiffness c_1 that ensure stability, i.e., which in addition to the aforementioned conditions satisfy (6). When (5) is satisfied, (6) is equivalent to requiring

$$\left(\frac{c_1}{m} - \Omega^2\right)\left(\frac{c_2}{m} - \Omega^2\right) \geq 0. \tag{8}$$

There are three cases to consider. First, if $\Omega > \sqrt{c_2/m}$, (8) requires $c_1 \leq \Omega^2 m$. Second, if $\Omega < \sqrt{c_2/m}$, (8) requires $c_1 \geq \Omega^2 m$ (this implies positive c_1). Third, if $\Omega = \sqrt{c_2/m}$, (8) does not restrict c_1, but this is resonance, treated as unstable as explained. Thus, only the first case permits negative c_1. In summary, we have derived the sufficient conditions of stability permitting $c_1 < 0$ when $c_2 > 0$: $\Omega > \omega_2 = \sqrt{c_2/m}$ and

$$1 - 4\left(\frac{\Omega}{\omega_2}\right)^2 + 4\,\frac{\Omega}{\omega_2^2}\sqrt{\Omega^2 - \omega_2^2} < \frac{c_1}{c_2} < \left(\frac{\Omega}{\omega_2}\right)^2. \tag{9}$$

The lower bound in (9) is always negative; for $\Omega \to \infty$, it reduces to -1.

We have shown that rotation of the simple two-elastic-spring system can stabilize a significant range of negative stiffness of one spring. Importantly, negative stiffness can only be stabilized if the rotational frequency is above resonance. Fig. 2 illustrates the stable regime for spring stiffness c_1 (normalized by c_2) versus rotational frequency Ω (normalized by $\omega_2 = \sqrt{c_2/m}$). Below resonance ($\Omega/\omega_2 < 1$), stiffness c_1 must satisfy $c_1/c_2 > (\Omega/\omega_2)^2$, meaning that rotation below resonance destabilizes a range of positive c_1 that increases with Ω. Above resonance ($\Omega/\omega_2 > 1$), stiffness c_1 must lie within the range (9). The resulting landscape of stable and unstable regimes (in light and dark gray, respectively) is shown in Fig. 2. Recall: (4) is symmetric in c_1 and c_2.

We now construct possibly the simplest system capable of exhibiting rotation-stabilized infinite stiffness. Referring to Fig. 1(b), the most primitive system capable of exhibiting infinite stiffness is the two masses with the two radial

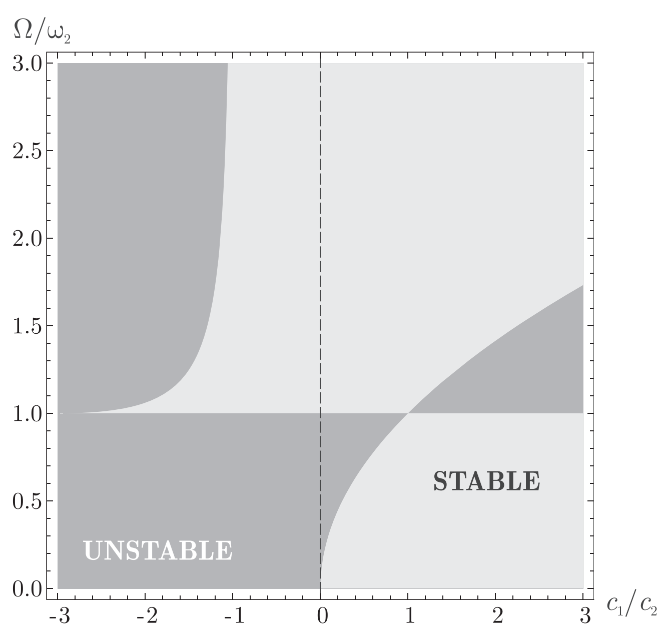

FIG. 2. Stable (light) and unstable (dark) regimes of spring stiffness c_1 vs. rotational frequency Ω (both normalized), for the system of Fig. 1(a).

springs, confined to purely radial displacement (e.g., by a rigid tube): with $c_2 > 0$ and c_1 tuned from below to $-c_2$, overall radial system stiffness becomes positive-infinite. But this system is unstable at that c_1 value. Rotation of this radial system (in a rigid tube) will not stabilize it. The key to rotational stabilization is the Coriolis acceleration. This must modify the radial equations of motion, which is effected by permitting circumferential displacement of the masses (no rigid tube), and having this coupled to their radial displacement. This is accomplished by adding two springs to the radial two mass/two spring system that are neither purely radial nor purely circumferential, like the two vertical springs shown in Fig. 1(b). We have also added the two horizontal springs shown so that the resulting symmetric system has a unique definition of effective radial static stiffness. The system has stiffness values $c_1 < 0$, $c_2 > 0$, $c > 0$.

The effective radial static stiffness c_{eff} of this system is determined by applying an infinitesimal symmetric load $(F_{2,x}, F_{2,y}) = (F, F)$ to mass 2 (the outer mass) and computing the resulting symmetric displacements $(u_{2,x}, u_{2,y}) = (u_2, u_2)$ so that $c_{\text{eff}} = F/u_2$. This gives

$$c_{\text{eff}} = \frac{c(c+c_1) + (2c+c_1)c_2}{c+c_1+c_2}. \tag{10}$$

Note that when the system rotates, centrifugal forces will displace both masses before application of the external load, but since the system is linear, the resulting displacements superimpose. Therefore, the computed effective stiffness c_{eff} defines the true stiffness against a symmetric external force applied to the rotating system. For the same reason, this is also the effective stiffness in the non-rotating static case ($\Omega = 0$), which can easily be verified since (10) can be rewritten as

$$c_{\text{eff}} = c + \left(\frac{1}{c_1+c} + \frac{1}{c_2}\right)^{-1}. \tag{11}$$

These show that unbounded effective stiffness $c_{\text{eff}} \to +\infty$ will occur when $c_1 \uparrow -(c+c_2)$. We show below that stability in the static case requires $c_1 \geq -c\,(c+2c_2)/(c+c_2)$, meaning that unbounded stiffness of the spring system under static conditions cannot be stable. This is exactly analogous to the solid composite material case.[5–7]

Now we analyze rotation-produced system stabilization. As before, the system is rotated about the z-axis at constant frequency Ω. Because displacements due to centripetal

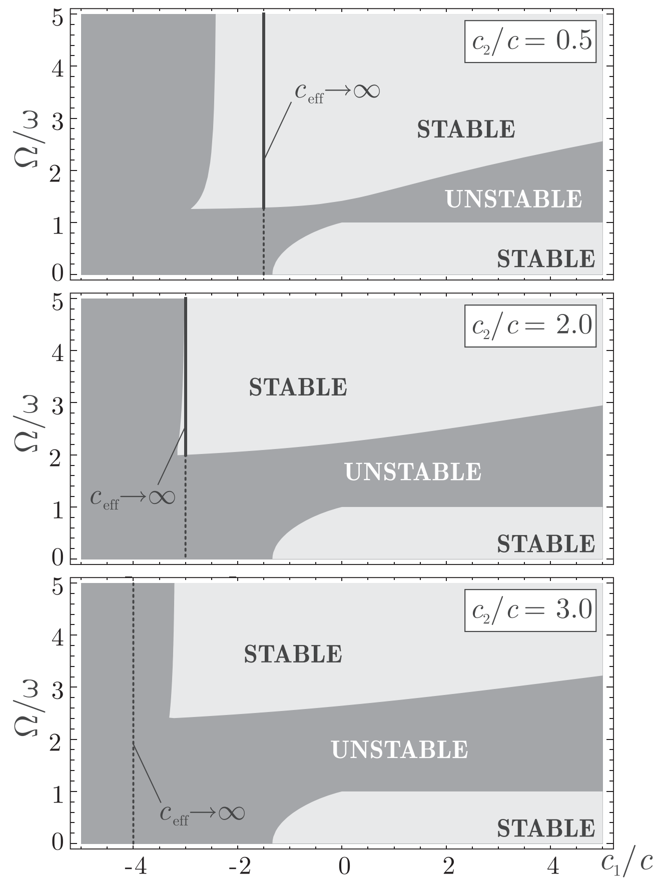

FIG. 3. Stable (light) and unstable (dark) regimes of normalized spring stiffness c_1/c vs. normalized rotational frequency $\Gamma = \Omega/\omega$ ($\omega = \sqrt{c/m}$), for three c_2/c values. The bold lines show all combinations for which $c_{\text{eff}} \to +\infty$; their solid portions are stable, dotted portions unstable.

effects are centric, and because $|\boldsymbol{U}|$ in the solution form below is assumed small, spring orientations remain constant. In the absence of external forces the equations of motion govern the motion of the two point masses, summarized in the displacement vector $\boldsymbol{u} = (u_{1,x}, u_{1,y}, u_{2,x}, u_{2,y})^{\mathrm{T}}$, where subscripts 1 and 2 refer to the inner and outer mass, respectively, see Fig. 1(b). Again writing the solution $\boldsymbol{u}(t)$ in separable form $\boldsymbol{u}(t) = \boldsymbol{U}\, e^{i\omega t}$ with constant amplitude vector $\boldsymbol{U}$, the equations of motion become

$$\begin{pmatrix} c + \frac{c_1+c_2}{2} - m(\omega^2+\Omega^2) & \frac{c_1+c_2}{2} - 2im\omega\Omega & -\frac{c_2}{2} & -\frac{c_2}{2} \\ \frac{c_1+c_2}{2} + 2im\omega\Omega & c + \frac{c_1+c_2}{2} - m(\omega^2+\Omega^2) & -\frac{c_2}{2} & -\frac{c_2}{2} \\ -\frac{c_2}{2} & -\frac{c_2}{2} & c + \frac{c_2}{2} - m(\omega^2+\Omega^2) & \frac{c_2}{2} - 2im\omega\Omega \\ -\frac{c_2}{2} & -\frac{c_2}{2} & \frac{c_2}{2} + 2im\omega\Omega & c + \frac{c_2}{2} - m(\omega^2+\Omega^2) \end{pmatrix} \begin{pmatrix} U_{1,x} \\ U_{1,y} \\ U_{2,x} \\ U_{2,y} \end{pmatrix} = \mathbf{0}. \tag{12}$$

This admits nontrivial solutions for $\boldsymbol{U}$ only if the matrix determinant vanishes. The resulting characteristic equation is a quartic for the squared eigenfrequencies ω^2, having the four solutions

$$\omega^2 = \frac{1}{4m}\left[\hat{c} + 4m\Omega^2 \pm\sqrt{2}\sqrt{(\hat{c}-4c)(c_1+2c_2) - 2c_1c_2 + 8\hat{c}m\Omega^2}\right], \quad (13)$$

having defined $\hat{c} = 4c + c_1 + 2c_2 \pm \sqrt{c_1^2 + 4c_2^2}$, whose square root must have the same sign in all locations in (13).

The requirement for system stability is that all eigenfrequencies satisfy $\mathrm{Im}(\omega_i) \geq 0$. Applying this to (13), the procedure is very similar to that followed for the two-spring system: we must require the square root argument, and the entire right side, of (13) to be nonnegative always. The former requirement gives the left side of (15), and the latter the right sides of (14) and (15). Defining $\Gamma = \Omega/\sqrt{c/m}$, we find two stable domains: the system is stable if

$$\text{(i)} \quad \Gamma \leq 1 \quad \text{and} \quad \frac{c_1}{c} \geq -\frac{(1-\Gamma^2)(1+2c_2/c-\Gamma^2)}{1+c_2/c-\Gamma^2}, \quad (14)$$

or

$$\text{(ii)} \quad \Gamma \geq \Gamma_0 \quad \text{and} \quad 8\Gamma\frac{\left[(c_2/c)^2 + 4(2+c_2/c)\Gamma^2\right]\left(\sqrt{\Gamma^2-1}-\Gamma\right) - 2\Gamma c_2/c}{(c_2/c)^2 + 8(2+c_2/c)\,\Gamma^2} \leq \frac{c_1}{c} \leq \frac{(\Gamma^2-1)(\Gamma^2-1-2c_2/c)}{\Gamma^2-1-c_2/c}, \quad (15)$$

where the value of Γ_0 is obtained as the intersection of upper and lower bounds in (15), and $\Gamma_0 > 1$. Condition (14) contains the static stability limit: inserting $\Gamma = 0$ yields $c_1 \geq -c\,(c+2c_2)/(c+c_2)$. This result illustrates that when $c > 0$, there is a significant range of negative c_1 for which the static system is stable, but not enough to stabilize infinite system stiffness as noted above.

Example results of system effective stiffness and system stability for various spring stiffness ratios and rotation frequencies are illustrated in Fig. 3. From these results, it is clear that if c_2/c is sufficiently small and the normalized rotational frequency Ω/ω is sufficiently high—always above resonance—infinite effective stiffness can be stable. Comparing stability limit (15) to effective stiffness (11) shows that unbounded effective stiffness is stable if

$$0 \leq \frac{c_2}{c} \leq \frac{3(5+8\sqrt{10})}{41} \approx 2.220 \quad (16)$$

for all rotation frequencies satisfying (with $\bar{c}_2 = c_2/c$)

$$\sqrt{\frac{1+\bar{c}_2+\sqrt{1+2\bar{c}_2+5\bar{c}_2^2}}{2}} \leq \frac{\Omega}{\sqrt{c/m}} \leq \begin{cases} \sqrt{\dfrac{2+\bar{c}_2+6\bar{c}_2^2-\bar{c}_2^3+(2-\bar{c}_2+\bar{c}_2^2)\sqrt{1+2\bar{c}_2+5\bar{c}_2^2}}{16(\bar{c}_2-2)(\bar{c}_2+1)}}, & \text{if } 2 < \bar{c}_2 \leq 2.220, \\ +\infty, & \text{if } 0 \leq \bar{c}_2 \leq 2. \end{cases} \quad (17)$$

In summary, we have demonstrated that a discrete spring-mass composite system can, in principle, exhibit positive-infinite effective stiffness while being stable overall, if the system is spinning at an appropriate frequency. The concepts modeled here admit practical implementation, in composite structures and composite solids; in the former, the negative-stiffness spring is realized, e.g., by pre-stressed spring or buckled-column elements that exhibit negative stiffness if their snap-through behavior is held in limbo by a sufficiently stiff surrounding structure.[2] For appropriate mass-spring tuning, frequencies can lie well within technologically relevant regimes.

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