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Entanglement entropy of composite Fermi liquid states on the lattice: In support of the Widom formula

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Quantum phases characterized by surfaces of gapless excitations are known to violate the otherwise ubiquitous boundary law of entanglement entropy in the form of a multiplicative log correction: $S \sim L^{d-1} \log L$. Using variational Monte Carlo, we calculate the second Rényi entropy for a model wave function of the $\nu = 1/2$ composite Fermi liquid (CFL) state defined on the two-dimensional triangular lattice. By carefully studying the scaling of the total Rényi entropy and, crucially, its contributions from the modulus and sign of the wave function on various finite-size geometries, we argue that the prefactor of the leading $L \log L$ term is equivalent to that in the analogous free fermion wave function. In contrast to the recent results of Shao *et al.* [Phys. Rev. Lett. **114**, 206402 (2015)], we thus conclude that the "Widom formula" holds even in this non-Fermi liquid CFL state. More generally, our results further elucidate—and place on a more quantitative footing—the relationship between nontrivial wave function sign structure and $S \sim L \log L$ entanglement scaling in such highly entangled gapless phases.

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In recent years, bipartite entanglement entropy has emerged as an indispensable tool in the study of quantum many-body states [1,2]. It can reveal highly universal, even nonlocal, information about a quantum phase given a ground state wave function. While entanglement entropy has had remarkable success for gapped phases exhibiting topological order [3–6] and gapless Luttinger liquids [7], an interesting question concerns its ability to characterize two-dimensional (2D) *highly entangled* systems containing a surface of gapless excitations in momentum space. These states are known to exhibit a multiplicative log violation of the boundary law [8]:

$$S = \kappa L_A \log L_A, \tag{1}$$

where S is the entanglement entropy between a large real-space subregion of characteristic length L_A and its complement (see Fig. 1).

The free Fermi gas with a sharp Fermi surface is the simplest example of such a system [see Fig. 1(a)]. In real space, however, the free fermion wave function is highly nontrivial, exhibiting complicated *sign structure* [9] which is believed to be closely related to the anomalously large entanglement present in Eq. (1). For free fermions, the coefficient κ depends only on the *shapes* of the subregion and Fermi surface and is given by an elegant geometric integral expression commonly referred to as the "Widom formula" [10–13].

In fact, κ is expected to be surprisingly universal and given by the Widom result κ_W even for an interacting Fermi liquid [14–16], as well as for more exotic states with emergent surfaces of gapless excitations [17–19] which have the same Fermi surface content as the corresponding free Fermi gas. Loosely speaking, κ can thus generally be interpreted as measuring the "gaplessness" of the quantum state as contributed by the critical surface(s), emergent or otherwise [12,15,16,18–21].

At present, several interesting open questions remain that we set out to address in this Rapid Communication. Which types of wave functions may violate the Widom formula? More precisely, can a (possibly nonperturbatively strongly interacting) wave function with identical critical surfaces as the free Fermi gas have an entanglement scaling with $\kappa \neq \kappa_W$? Since $\kappa = \kappa_W$ is expected to hold for interacting Fermi liquids [14–16], can measuring κ in a numerical simulation thus serve as a long-sought-after positive indicator of non-Fermi liquid behavior [22]? Finally, in practice, what is the best way to

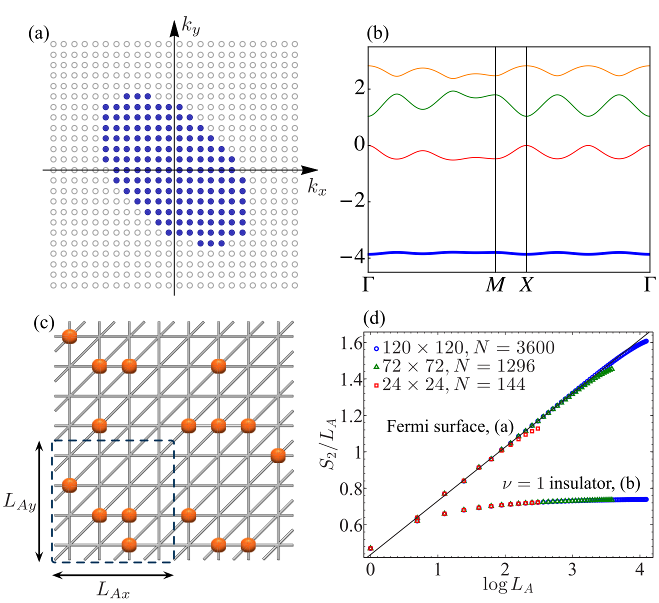

FIG. 1. (a) Filled Fermi sea with a sharp Fermi surface used to construct Ψ_f^{FS} on a 24×24 lattice with $N = 144$ electrons. (b) Band structure for $d_{1,2}$; the $\Psi_{d_{1,2}}^{(\nu=1)}$ Slater determinants are constructed by filling the lowest, nearly flat band (blue) which has Chern number $C = 1$. (c) We work on the 2D triangular lattice and consider subregions of size $L_{Ax} \times L_{Ay}$ for our calculations of S_2 [52]. (d) S_2 scaling for the free fermion states in (a) and (b) for $L_A \times L_A$ subregions embedded in various $L \times L$ systems at $\rho = 1/4$ (see legend); the black line indicates the Widom formula slope κ_W (see text).

detect Widom-formula violation in numerical studies given the well-known signal-to-noise ratio problems inherent in Monte Carlo measurements of the entanglement entropy on large systems?

We now turn to the composite Fermi liquid (CFL) phase of the half-filled Landau level ($\nu = 1/2$). The CFL still stands today as the paradigmatic example of a strongly interacting gapless non-Fermi-liquid state [23–28] (see also Refs. [29–40] for several recent exciting developments). Following Halperin, Lee, and Read (HLR) [23], a model wave function for the CFL reads [41–43]

$$\Psi_{\mathrm{HLR}}(\{\mathbf{r}_i\}) = \Psi_b^{(\nu=1/2)}(\{\mathbf{r}_i\})\Psi_f^{\mathrm{FS}}(\{\mathbf{r}_i\}), \tag{2}$$

where $\Psi_b^{(\nu=1/2)}$ is a Laughlin-type wave function for bosons at $\nu = 1/2$ [44,45], Ψ_f^{FS} is a wave function for fermions in zero field exhibiting a Fermi surface (FS), and $\{\mathbf{r}_i\}$ are the coordinates of the N electrons at which both $\Psi_b^{(\nu=1/2)}$ and Ψ_f^{FS} are to be evaluated.

Recently, Ref. [22] presented a numerical study of the second Rényi entropy S_2 [46] for a continuum wave function in the form of Eq. (2) projected into the lowest Landau level on the torus. These authors found that for square $L_A \times L_A$ subregions the prefactor κ in the leading $L_A \log L_A$ term of S_2 is approximately *twice* the corresponding Widom formula result, i.e., twice what is obtained for the zero-field free fermion wave function Ψ_f^{FS}. This is a very striking result. Since $\Psi_b^{(\nu=1/2)}$ is a fully gapped state with a clear boundary law [5] (albeit a wave function with interesting structure of zeros and complex phases) and the Guztwiller projection implicit in Eq. (2) generally only tends to (slightly) decrease entanglement [17,47,48], such a dramatic increase in κ for this wave function is very unexpected and, if correct, could point to new physics at play which is currently not understood.

Here, we study the entanglement entropy of analogous HLR-type wave functions on the lattice, which to our knowledge have not been considered before in detail in any capacity. Our wave functions are particularly easy to define and straightforward to handle using variational Monte Carlo [17,49,50], yet they should be in the same quantum phase as the state considered in Ref. [22]. We consider N spinless electrons moving on a toroidal 2D triangular lattice [see Fig. 1(c)] of dimension $L_x \times L_y$ with uniform magnetic flux penetrating the sample [51]. For concreteness, we take an electron density $\rho = N/(L_x L_y) = 1/4$ with $\pi/2$ external magnetic flux per triangle. Our model HLR wave function for this $\nu = 1/2$ system reads

$$\Psi_{\mathrm{HLR}}^{\mathrm{ferm}}(\{\mathbf{r}_i\}) = \Psi_{d_1}^{(\nu=1)}(\{\mathbf{r}_i\})\Psi_{d_2}^{(\nu=1)}(\{\mathbf{r}_i\})\Psi_f^{\mathrm{FS}}(\{\mathbf{r}_i\}) \tag{3}$$

(see Fig. 1 and [52] for details). Within a "parton" approach [53,54], Eq. (3) corresponds to decomposing the physical electron as $c = d_1 d_2 f$ subject to the constraint $d_1^\dagger d_1 = d_2^\dagger d_2 = f^\dagger f = c^\dagger c$ at each site. We will also consider a bosonic analog of the HLR state appropriate for bosons at $\nu = 1$ [55,56]. The construction parallels the fermionic state of Eq. (3) with a final wave function given by $\Psi_{\mathrm{HLR}}^{\mathrm{bos}}(\{\mathbf{r}_i\}) = \Psi_{d_1}^{(\nu=1)}(\{\mathbf{r}_i\})\Psi_f^{\mathrm{FS}}(\{\mathbf{r}_i\})$.

We begin by considering square $L_A \times L_A$ subregions embedded within total systems of size $L \times L$ at $\rho = 1/4$. The second Rényi entropy S_2 for the free fermion state Ψ_f^{FS} on systems with $L = 24,72,120$ as calculated via the correlation matrix technique [57,58] is shown in Fig. 1(d). Plotting S_2/L_A versus $\log L_A$ clearly reveals the multiplicative log violation. We fit the $L = 120$ data with L_A between 4 and 36 to obtain an accurate linear fit $S_2/L_A = \kappa \log L_A + a$ with $\kappa = \kappa_W \equiv 0.2950(6)$ and $a = 0.436(2)$. The fitted value κ_W is expected to be very close to that predicted by the Widom formula [11,22,59]. The free fermion entropy for the gapped $d_{1,2}$ partons at $\nu = 1$ is also shown in Fig. 1(d); in this case, saturation to a boundary law is evident.

We now turn to Monte Carlo measurements of S_2. As has become standard, we compute S_2 via the expectation value of the "swap" operator [17,60]: $S_2 = -\log[\mathrm{Tr}(\rho_A^2)] = -\log\langle \mathrm{SWAP}_A\rangle$. (An alternative approach in the context of fermionic determinantal quantum Monte Carlo was developed in Ref. [61]; see also Refs. [62,63].) Importantly, we employ [52] the mod/sign decomposition [17] to compute the total Rényi entropy as a sum of two terms: $S_2 = S_{2,\mathrm{total}} = S_{2,\mathrm{mod}} + S_{2,\mathrm{sign}}$ [64]. We will argue that it is $S_{2,\mathrm{sign}}$ which is responsible for Eq. (1) on long scales (cf. Ref. [17]); hence, this approach allows us to glean more valuable long-distance information about κ than what is contained in $S_{2,\mathrm{total}}$ alone.

We show in Fig. 2 calculations of $S_{2,\mathrm{total}}$ (left panel), $S_{2,\mathrm{mod}}$ (middle panel), and $S_{2,\mathrm{sign}}$ (right panel) for both the fermionic and bosonic HLR wave functions, as well as for the free fermion wave function, on a 24×24 system with $N = 144$ electrons. As is evident in the left panel of Fig. 2, the total entropy for the HLR wave functions indeed appears to have a slope κ significantly enhanced over the free fermion/Widom value. For example, fits to the fermionic HLR data indicate a κ at least 60% larger than that obtained by similar fits to the free fermion data. We can thus corroborate the result of Ref. [22]: For square subregions with $O(100)$ electrons, the HLR wave function appears to violate the Widom formula by nearly a factor of 2.

However, a closer inspection of the contributions from the modulus and sign of the wave functions, as shown in the middle and right panels of Fig. 2, reveals that this data is likely plagued by strong finite-size effects. The dramatic increase in entanglement for the HLR wave functions is almost entirely due to contributions from $S_{2,\mathrm{mod}}$ on these sizes, while $S_{2,\mathrm{sign}}$ is remarkably nearly equal for all three wave functions. However, $S_{2,\mathrm{mod}}$ displays eventual boundary law behavior (with quite large boundary law coefficients for the HLR wave functions). On the other hand, it is clearly $S_{2,\mathrm{sign}}$ which is ultimately responsible for the long-distance $L_A \log L_A$ scaling behavior. Hence, in order to make conclusions about κ by analyzing only $S_{2,\mathrm{total}}$, one should be deep in a regime of L_A where $S_{2,\mathrm{mod}}$ has saturated to a boundary law.

While for the HLR states we are not yet in such a regime on the 24×24, $N = 144$ system [65], there are already telling indications in the $S_{2,\mathrm{sign}}$ data that these wave functions indeed do obey the Widom formula. In the right panel of Fig. 2, we show a line with slope κ_W (intercept is arbitrary here and in Fig. 3). For L_A beyond just a couple of lattice spacings, we see that $S_{2,\mathrm{sign}}$ very nearly obeys the Widom formula for all three wave functions, perhaps most accurately for the fermionic HLR state itself. Finally, in Fig. 3 we show an alternative view of the fermionic HLR (left panel) and free fermion (right panel) data from Fig. 2, where we also include data from

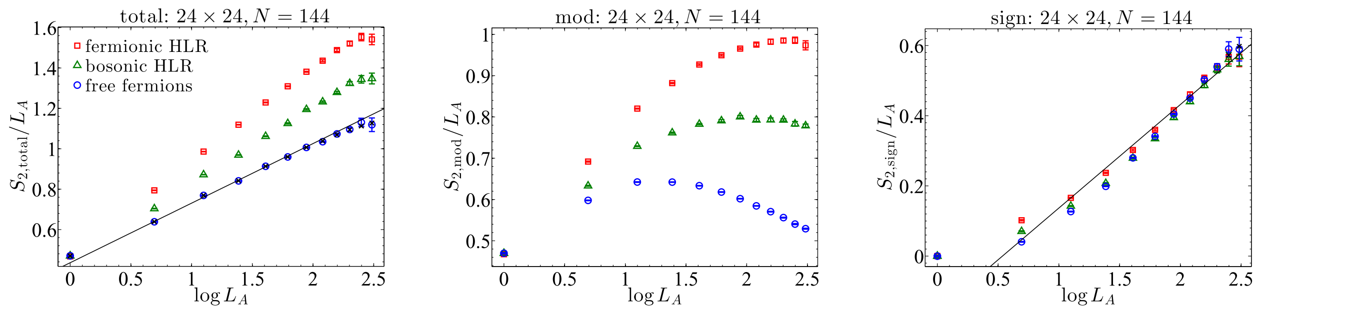

FIG. 2. Monte Carlo calculations of the total Rényi entropy (left panel) and the modulus (middle panel) and sign (right panel) components for the fermionic HLR, bosonic HLR, and free fermion wave functions on the $24{\times}24$, $N = 144$ system with $L_A \times L_A$ subregions. Here, and in Figs. 1(d) and 3, L_A ranges from 1 to $L/2$. The black "×" symbols indicate the numerically exact S_2 values for free fermions [57,58] (also in Figs. 3 and 4), and the black lines indicate the Widom formula slope κ_W from Fig. 1(d).

a smaller system: $16{\times}16$, $N = 64$. As in the right panel of Fig. 2, the black lines near the sign data indicate the Widom slope κ_W. The following three points are now clear: (i) $S_{2,\mathrm{mod}}$ for $\Psi_{\mathrm{HLR}}^{\mathrm{ferm}}$ indeed saturates to a boundary law; (ii) $S_{2,\mathrm{sign}}$ for $\Psi_{\mathrm{HLR}}^{\mathrm{ferm}}$ is well described by the Widom formula [66]; and (iii) the apparent Widom formula violation in $S_{2,\mathrm{total}}$ for $\Psi_{\mathrm{HLR}}^{\mathrm{ferm}}$ is mainly due to significant short-distance entanglement increase in the modulus of the wave function which results from strong correlations contained in the Jastrow-like factor $|\Psi_b^{(\nu=1/2)}|$ [67]. Collectively, these three points suggest that the Widom formula will eventually be satisfied in the thermodynamic limit.

We now further bolster our arguments that the fermionic HLR state obeys the Widom formula by considering S_2 scaling on *strip* geometries. That is, we take $X \times L_y$ subregions embedded within $L_x \times L_y$ systems and vary X. In this case, for free fermions the Widom formula essentially reduces to the familiar quasi-one-dimensional (quasi-1D) form:

$$S_2(X,L_x) = \frac{c}{4}\log\left[\frac{L_x}{\pi}\sin\left(\frac{\pi X}{L_x}\right)\right] + A, \qquad (4)$$

where $c = N_{\mathrm{slices}}$ is simply the number of "slices" through which the quantized k_y momenta pierce the Fermi surface, and we have used the familiar chord length ℓ inside the log [68] (appropriate for X comparable to L_x). More generally, at least in the quasi-1D limit ($L_x \gg L_y$), c is the *central charge* [7], i.e., the number of (nonchiral) gapless modes present in the realized multimode Luttinger liquid [47,48,69–71].

The narrowest nontrivial strip that we can consider has $L_y = 4$ [52] and $N_{\mathrm{slices}} = 3$. For free fermions, we thus expect an effective central charge $c = N_{\mathrm{slices}} = 3$. For the fermionic HLR state, on the other hand, we expect the Gutzwiller projection in Eq. (2) to *remove* one gapless mode [33,47,69–71] giving $c = N_{\mathrm{slices}} - 1 = 2$ (since $\Psi_b^{(\nu=1/2)}$ is fully gapped). Indeed, we can unambiguously confirm this prediction on a $48{\times}4$, $N = 48$ system (see the Supplemental Material).

We have performed measurements on increasingly wide strips to approach the 2D limit. By performing fits to the data using Eq. (4), we can extract the central charge associated with the total entropy, denoted c_{total}, as well as contributions to the central charge from the mod and sign individually, denoted c_{mod} and c_{sign} (with $c_{\mathrm{total}} = c_{\mathrm{mod}} + c_{\mathrm{sign}}$). Figure 4 shows an example of such data and the associated fits for $48{\times}12$, $N = 144$. This system has $N_{\mathrm{slices}} = 7$, and indeed we find $c_{\mathrm{total}} \approx 7$ for free fermions. For the HLR state, c_{total} is reduced compared to free fermions and roughly consistent with $c \approx N_{\mathrm{slices}} - 1$.

The middle and right panels of Fig. 4 again demonstrate that it is $S_{2,\mathrm{sign}}$ which is mainly responsible for the boundary law violation in these systems. Remarkably, the fermionic HLR and free fermion $S_{2,\mathrm{sign}}$ results continue to track each other, both accurately following the scaling form Eq. (4). On the other hand, $S_{2,\mathrm{mod}}$ grows relatively weakly with $\log \ell$ for both wave functions. In fact, the main qualitative difference between the two states is simply a larger intercept A in Eq. (4) for the HLR state, which is coming entirely from the modulus of the wave function (consistent with Fig. 2) and due to the presence of the $\Psi_{d_{1,2}}^{(\nu=1)}$. However, such physics is clearly distinct from that giving rise to the multiplicative log boundary law violation.

In the Supplemental Material, we present the entirety of our strip geometry study showing (in addition to Fig. 4) simulations for $L_y = 4$, 8, 16, and 20 with $L_x = 48$, 48, 36, and 24, respectively, all at $\rho = 1/4$. As L_y (and thus N_{slices}) is increased, the *scaling* of the entropy becomes concentrated in c_{sign} for both states (cf. Fig. 2) while c_{mod} remains of order one. This itself constitutes a very interesting result—even for the free Fermi gas—which nicely elucidates the intimate relationship between sign structure and entanglement for these wave functions in the 2D limit.

All in all, we find no evidence that the HLR state violates the Widom formula in our strip geometry study, even in the total S_2 entropy itself. That is, for the total entropy we have found $c_{\mathrm{HLR}} \approx c_{\mathrm{FF}}$ in all cases. These results also put on firm footing the expression $c = N_{\mathrm{slices}} - 1$ for the CFL used in the recent

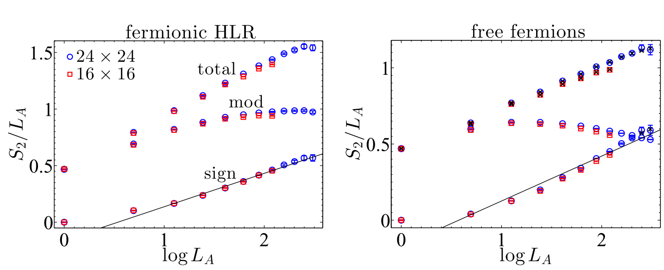

FIG. 3. Fermionic HLR (left panel) and free fermion (right panel) data for $24{\times}24$, $N = 144$ and $16{\times}16$, $N = 64$ showing $S_{2,\mathrm{total}}$, $S_{2,\mathrm{mod}}$, and $S_{2,\mathrm{sign}}$ on the same axes.

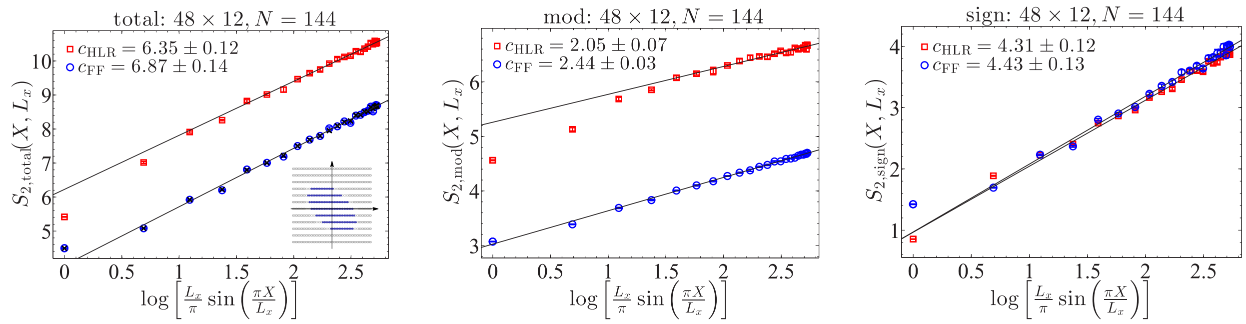

FIG. 4. From left to right, we show $S_{2,\mathrm{total}}$, $S_{2,\mathrm{mod}}$, and $S_{2,\mathrm{sign}}$ versus $\log \ell$ on a 48×12, $N = 144$ system. The free fermion (FF) state has $N_{\mathrm{slices}} = 7$ [see inset in the left panel; cf. Fig. 1(a)]. The lines correspond to fits to Eq. (4) with obtained values of c given in the legends.

DMRG study of Ref. [33]. It would be interesting to perform a similar analysis as we have in this work—for both types of subregion geometries—on the precise HLR wave function considered in Ref. [22], and also on the interacting Fermi liquid wave functions considered in Ref. [72] which were claimed to weakly violate the Widom formula.

While we have argued that our lattice HLR states have the same leading entanglement scaling as free fermions, it is interesting to think about which types of wave functions may actually violate the Widom formula [73]. On this note, we have also considered a wave function in the form of Eq. (2) but with $\Psi_b^{(\nu=1/2)} \rightarrow \Psi_b^{(\nu=1/2)}/|\Psi_b^{(\nu=1/2)}|$, i.e., a wave function with sign structure given by $\Psi_{\mathrm{HLR}}^{\mathrm{ferm}}$ but amplitudes given by Ψ_f^{FS}. Such wave functions basically model attachment of flux at the mean-field level—as opposed to attachment of vortices in Eq. (2)—and are known to have various deficiencies [74,75]. Interestingly, we find that $S_{2,\mathrm{sign}}$ for this wave function, grows extremely quickly with L_A, and the full wave function may possibly have a scaling different from the Widom formula. We leave further investigation of this result for future work. Finally, Gutzwiller projection—as employed here and, for example, in the spin liquid states in Ref. [17]—is known to only capture gauge fluctuations in a partial way [76]. Remedying this problem and subsequently studying the long-distance entanglement properties of such wave functions constitutes an exciting and challenging future direction.

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[64] $S_{2,\mathrm{mod}}$ is the entropy of the modulus of the wave function in the coordinate basis, while $S_{2,\mathrm{sign}}$ is the component of the entropy due to nontrivial signs (phases). See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.94.081110 for more details and discussion.
[65] Even doubling L (i.e., 48×48, $N=576$), which may or may not be sufficient, is already well out of current computational abilities (cf. Refs. [72,77,78]).
[66] Comparing the 16×16 and 24×24 data, $S_{2,\mathrm{sign}}/L_A$ appears to be well converged in system size for $L_A = 1$–8.
[67] Even though the pseudopotential corresponding to the Jastrow factor $|\Psi_b^{(\nu=1/2)}|$ is very long range ($\sim -\log r$), the entropy cannot increase stronger than boundary law since $S_{2,\mathrm{mod}} = S_{2,\mathrm{total}} - S_{2,\mathrm{sign}} \leqslant S_{2,\mathrm{total}}$ and $S_{2,\mathrm{total}}$ obeys a boundary law for the gapped state $\Psi_b^{(\nu=1/2)}$.
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