Supplemental Information for Design Principles of Length Control of Cytoskeletal Structures

1. Unregulated filament

1.1 Time- dependent mean filament length in the regime $r > \gamma$

The master equation for an unregulated filament is given by Equation 3 in Section 1.3 in main text, namely

$$\frac{dP(l,t)}{dt} = r\,P(l-1) - rP(l) + \gamma\,P(l+1) - \gamma\,P(l). \qquad (1)$$

The equation for the average length is obtained by multiplying both sides of Equation 1 by l and summing over all possible lengths, namely,

$$\frac{d\langle l \rangle}{dt} = \sum_{l=0}^{\infty} \frac{d}{dt} l\,P(l,t)$$

$$= r\sum_{l=0}^{\infty} l\,P(l-1) - r\sum_{l=0}^{\infty} l\,P(l) + \gamma\sum_{l=0}^{\infty} l\,P(l+1) - \gamma\sum_{l=0}^{\infty} l\,P(l). \quad (2)$$

The terms on the right hand side of this equation can all be rewritten as moments of the $P(l,t)$ disribution. Using the definition of moments, $\langle l^n \rangle = \sum_{l=0}^{\infty} l^n\,P(l,t)$, the second and fourth term are simply $-r\langle l \rangle$ and $-\gamma\langle l \rangle$ respectively. The first and third terms require a bit more attention.

We begin with the first term $r \sum_{l=0}^{\infty} l\, P(l-1)$ on the RHS of equation 2. Since $P(-1) = 0$,

we can write this terms as $\sum_{l=1}^{\infty} l\, P(l-1)$. Changing the variables using $m = l - 1$, we

obtain

$$r \sum_{l=1}^{\infty} l\, P(l-1) = r \sum_{m=0}^{\infty} (m+1)P(m) . \quad (3)$$

Next we consider the third term $\gamma \sum_{l=0}^{\infty} l\, P(l+1)$ in the RHS of equation 2, where, by

adding and subtracting $P(0)$, we can write this term as $\gamma\left(\sum_{l=-1}^{\infty} l\, P(l+1) + P(0)\right)$.

Changing the variables $m = l + 1$, we obtain

$$\gamma \sum_{l=-1}^{\infty} l\, P(l+1) = \gamma \left(\sum_{m=0}^{\infty} (m-1)P(m) + P(0) \right). \quad (4)$$

Substituting Equation 3 and 4 in Equation 2 and using the definition of moments

$\langle l^n \rangle = \sum_{l=0}^{\infty} l^n\, P(l,t)$, we obtain

$$\frac{d}{dt}\langle l(t) \rangle = r - \gamma + \gamma\, P(0). \quad (5)$$

When $r > \gamma$, the filament will grow more than shrink and hence at later times, $P(0) \sim 0$.

Hence, Equation 5 can thus be solved to obtain $\langle l(t) \rangle = (r - \gamma)t$, which implies that the

filament will grow linearly with time and there is no steady state. Note that here we are

considering an infinite pool of monomers. We consider the effect of a finite pool in Section

2.2.1.

1.2 Solving master equations for $r < \gamma$

Consider a filament which adds and subtracts subunits at rates r and γ respectively. We are interested in computing the steady state probability distribution of lengths i.e $P(l)$, where l is the length of the filament in the regime where $r < \gamma$. We start by writing down the master equation for the states $P(l, t)$, namely

$$\frac{dP(l,t)}{dt} = r\, P(l-1) - rP(l) + \gamma\, P(l+1) - \gamma\, P(l). \qquad (6)$$

The master equation for $P(0, t)$, which is the probability of being at zero length at time t,

$$\frac{dP(0,t)}{dt} = \gamma\, P(1) - r\, P(0). \qquad (7)$$

needs to be considered separately since the transition of $P(0)$ decaying into $P(-1)$ is not allowed. At steady state, $\frac{dP(0,t)}{dt} = 0$. We can use the steady-state equation for $P(0)$ to obtain a general expression for steady state distribution $P(l)$ by using recursion, namely

$$P(1) = \frac{r}{\gamma}\, P(0). \qquad (8)$$

Also, at steady state $\frac{dP(l,t)}{dt} = 0$ and Equation 6 becomes

$$r\, P(l-1) + \gamma\, P(l+1) = (r+\gamma)P(l). \qquad (9)$$

Substituting $l = 1$ in Equation 9, we obtain

$$r\, P(0) + \gamma\, P(2) = (r+\gamma)P(1). \qquad (10)$$

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Now substituting Equation 8, we get a simple relationship between $P(2)$ and $P(0)$

namely, $P(2) = \left(\frac{r}{\gamma}\right)^2 P(0)$. By using this recursion scheme of recursion, we obtain

$$P(l) = \left(\frac{r}{\gamma}\right)^l P(0).$$

2. Finite monomer pool

2.1 Solving the master equation

We want to solve the master equation for an individual filament in a finite pool of subunits, namely

$$\frac{dP(l,t)}{dt} = r(l-1)P(l-1) - r(l)P(l) + \gamma\, P(l+1) - \gamma\, P(l). \quad (11)$$

Again, at steady state, we have $\frac{dP(l,t)}{dt} = 0$. Now let us consider the equation for $P(0)$, the

probability of having zero subunits. Since $P(-1) = 0$ and the $P(0)$ state cannot decay, the

equation for $P(0)$ is given by

$$\frac{dP(0,t)}{dt} = \gamma\, P(1) - r(0)P(0)\,. \quad (12)$$

In other words, the only allowed transitions for $P(0)$ are the decay from the state P(1) and

growth to the state $P(1)$. The decay from the state $P(1)$ where a filament with 1 subunit,

loses the subunit at a rate γ and has no subunits. This term adds to $P(0,t)$. The growth to

the state $P(1)$, where a filament with no subunits, adds a subunit at a rate $r(0)$ and now

has 1 subunit. This term reduces $P(0,t)$.

Once again at steady state, $\frac{dP(0,t)}{dt} = 0$. Substituting in Equation 12, we obtain $P(1) =$

$\frac{r(0)}{\gamma} P(0)$ which is $\frac{r' \times (N_f)}{\gamma} P(0)$. For $l = 1$, from Equation 12, $r(0) P(0) + \gamma P(2) =$

$(r(1) + \gamma)P(1)$. We can substitute $P(1)$ in terms of $P(0)$ to obtain an expression for $P(2)$,

namely

$$(r' \times N_t)P(0) + \gamma P(2) = (r' \times (N_t - 1)) + \gamma) \frac{r' \times N_t}{\gamma} P(0). \quad (13)$$

Rearranging the terms in equation 3, we can obtain P(2) as a function of P(0) as

$$P(2) = \frac{r' \times N_t}{\gamma} \left(\frac{r' \times (N_t - 1)}{\gamma} \right) P(0). \quad (14)$$

Using $\frac{N_t!}{(N_t-2)!} = N_t \times (N_t - 1)$, we can rewrite

$$P(2) = \left(\frac{r'}{\gamma} \right)^2 \frac{N_t!}{N_t - 2)!} P(0). \quad (15)$$

Repeating this recursively for $l = 3$ and so on, we can get a general expression for $P(l)$ in

terms of $P(0)$ as

$$P(l) = \left(\frac{r'}{\gamma} \right)^l \frac{N_t!}{(N_t - l)!} P(0), \quad (16)$$

2.2 Equation for the mean

We compute the mean of this distribution by multiplying the master equation (Equation

11) by l and summing over all possible lengths, namely

$$\frac{d\langle l \rangle}{dt} = \sum_{l=0}^{N_t} \frac{d}{dt} l\, P(l,t) = \sum_{l=0}^{N_t} r(l-1)\, l\, P(l-1) - \sum_{l=0}^{N_t} r(l)\, l\, P(l) + \gamma \sum_{l=0}^{N_t} l\, P(l+1) - \gamma \sum_{l=0}^{N_t} l\, P(l) . \quad (17)$$

The length of each filament can theoretically vary from 0 to N_t, where all the monomers are incorporated in the filament itself. we consider each of the sums on the right hand side separately. We begin with the first term $\sum_{l=0}^{N_t} r(l-1)\, l\, P(l-1)$ on the RHS of equation 17. Since $P(-1) = 0$, we can write this terms as $\sum_{l=1}^{N_t} r(l-1)\, l\, P(l-1)$. Changing the variables using $m = l - 1$, we obtain

$$\sum_{l=1}^{N_t} r(l-1)\, l\, P(l-1) = \sum_{m=0}^{N_t} r(m)(m+1)P(m) . \quad (18)$$

Note that in Equation18, we are assuming that N_t is large enough that $N_t - 1 \sim N_t$. Next we consider the third term $\gamma \sum_{l=0}^{N_t} l\, P(l+1)$ in the RHS of equation 17. By adding and subtracting $P(0)$, we can write this term as $\left(\sum_{l=-1}^{N_t} l\, P(l+1) + P(0) \right)$. Changing the variables $m = l + 1$, we obtain

$$\gamma \sum_{l=-1}^{N_t} l\, P(l+1) = \gamma \sum_{m=0}^{N_t} (m-1)P(m) . \quad (19)$$

Again, note that in Equation 14, we are assuming that N_t is large enough that $N_t + 1 \sim N_t$. $P(0)$ is 1 at early times but at later times, for $r' > \gamma$, $P(0) \sim 0$. Substituting Equation 18 and 19 in Equation 17 and using the definition of moments $\langle l^n \rangle = \sum_{l=0}^{N_t} l^n\, P(l,t)$, we obtain

$$\frac{d}{dt} \langle l \rangle = r'\, N_t \langle l \rangle - r'\langle l^2 \rangle + r'\, N_t - r'\langle l \rangle - r'\, N_t\langle l \rangle + r'\langle l^2 \rangle + \gamma\langle l^2 \rangle - \gamma - \gamma\langle l^2 \rangle, \quad (20)$$

which further simplifies to $\frac{d}{dt}\langle l \rangle = r' \, N_t - r'\langle l \rangle - \gamma$. At steady state, $\frac{d}{dt}\langle l \rangle = 0$ and we

obtain $\langle l \rangle = N_t - \frac{\gamma}{r'}$.

2.3 Equation for the variance

For calculating variance, we multiply the master equation in Equation 11 with l^2 and sum

over all possible lengths, namely

$$\sum_{l=0}^{N_t} \frac{d}{dt} l^2 \, P(l,t) = \sum_{l=0}^{N_t} r(l-1)l^2 \, P(l-1) - \sum_{l=0}^{N_t} r(l)l^2 \, P(l) + \gamma \sum_{l=0}^{N_t} l^2 \, P(l+1) - \gamma \sum_{l=0}^{N_f} l^2 \, P(l). \quad (21)$$

Let us consider the first term $\sum_{l=0}^{N_t} r(l-1)l^2 \, P(l-1)$ in the RHS of equation 21. Since

$P(-1) = 0$, we can write this terms as $\sum_{l=1}^{N_t} r(l-1)l^2 \, P(l-1)$. Changing the variables

$m = l - 1$, we obtain

$$\sum_{l=1}^{N_t} r(l-1)l^2 \, P(l-1) = \sum_{m=0}^{N_t} r(m)(m+1)^2 P(m). \quad (22)$$

Now let us consider the third term $\gamma \sum_{l=0}^{N_t} l^2 \, P(l+1)$ in the RHS of equation 21. Adding

and subtracting $P(0)$, we can write this terms as $\gamma \left(\sum_{l=-1}^{N_t} l^2 \, P(l+1) - P(0) \right)$. Changing

the variables $m = l + 1$, we obtain

$$\gamma \sum_{l=-1}^{N_t} l^2 \, P(l+1) = \gamma \sum_{m=0}^{N_t} (m-1)^2 P(m). \quad (23)$$

Once again, Note that in Equation 22 and 23, we are assuming that N_t is large enough that

$N_t \pm 1 \sim N_t$ and $P(0) \sim 0$. Substituting Equation 22 and 23 in Equation 21, we get

$$\frac{d}{dt}\sum_{l=0}^{N_t} l^2\, P(l,t) = \sum_{m=0}^{N_t} r' \times (N_t - m)(m+1)^2 P(m) - \sum_{l=0}^{N_t} r' \times N_t l^2\, P(l) + \gamma \sum_{m=0}^{N_t}(m-1)^2 P(m) - \gamma \sum_{l=0}^{N_t} l^2\, P(l). \quad (24)$$

Once again using the definition of moments $\langle l^n \rangle = \sum_{l=0}^{N_f} l^n\, P(l,t)$, we obtain

$$\frac{d}{dt}\langle l^2 \rangle = r'\, N_t \langle l^2 \rangle - r'\, \langle l^3 \rangle + r'\, N_t - r'\langle l \rangle + 2\, r''\, N_t \langle l \rangle - 2\, r'\langle l^2 \rangle - r'\, N_t \langle l^2 \rangle + r'\, \langle l^3 \rangle$$

$$+ \gamma \langle l^2 \rangle - \gamma - 2\gamma \langle l \rangle - \gamma \langle l^2 \rangle. \quad (25)$$

which further simplifies to

$$\frac{d}{dt}\langle l^2 \rangle = r'\, N_t - r'\langle l \rangle + 2r''\, N_t \langle l \rangle - 2r''\langle l^2 \rangle + \gamma - 2\gamma \langle l \rangle. \quad (26)$$

At steady state, $\frac{d}{dt}\langle l^2 \rangle = 0$, Hence $2r'\, \langle l^2 \rangle = r'\, N_t - r'\langle l \rangle + 2r''\, N_t \langle l \rangle + \gamma -$

$2\gamma \langle l \rangle$. Substituting $\langle l \rangle = N_t - \frac{\gamma}{r'}$, we obtain

$$2r'\, \langle l^2 \rangle = r'\left(\langle l \rangle + \frac{\gamma}{r'}\right) - r'\langle l \rangle + 2r''\left(\langle l \rangle + \frac{\gamma}{r'}\right)\langle l \rangle + \gamma - 2\gamma \langle l \rangle, \quad (27)$$

which further simplifies to $2r'\, \langle l^2 \rangle - 2r'\langle l \rangle^2 = 2\gamma$, yielding an expression for the variance

of the distribution as

$$\langle l^2 \rangle - \langle l \rangle^2 = \frac{\gamma}{r'}. \quad (28)$$

3. Directed dampers

3.1 Solving the master equation

The average time that the filament spends in the ON state, when the elongator is active and

the filament is growing at rate r, is $\frac{1}{k_{on}(l)}$, while the average time the filament spends in

the OFF state is $1/k_{off}$. Since we assume that the rate of growth in the OFF state is zero, the

average rate of polymerization is $\bar{r}(l) = r\left(\frac{k_{off}}{k_{off}+k_{on}(l)}\right)$, where the factor appearing in

parenthesis is the fraction of time that the filament spends in the ON state. In the fast

switching limit, the filament can be assumed to have an instantaneous polymerization rate,

$\bar{r}$, which is length dependent since $k_{on}(l) = wl$, and a depolymerisation rate γ. Using the

detailed balance condition $P(l)\,\bar{r}(l) = \gamma P(l + 1)$ we obtain

$$P(l)\, r\frac{k_{off}}{k_{off} + wl} = \gamma\, P(l+1). \qquad (29)$$

Writing equation 29 for $l = 0$, we get an equation for $P(1)$ in terms of $P(0)$, the probability

of zero subunits present at the elongator, namely $P(1) = \frac{r}{\gamma}\, P(0)$. Writing equation 29 for

$l = 1$, we get an equation for $P(2)$ in terms of $P(0)$ and repeating the same scheme for

$l = 2, 3$ and so on, we can get an expression for $P(l)$ in terms of $P(0)$, namely

$$P(l) = \left(\frac{r}{\gamma}\right)^{l} \prod_{i=0}^{l-1}\left(\frac{k_{off}}{k_{off} + i\,w}\right) P(0). \qquad (30)$$

We use the normalization condition for $P(l)$ i.e. $\sum_{l=0}^{\infty} P(l) = 1$ to obtain $P(0)$, which then

gives us an analytic formula for the length distribution

$$P(l) = \frac{\left(\frac{rk_{off}}{\gamma w}\right)^{\frac{k_{off}}{w}+l-1} e^{-\left(\frac{k_{off}\, r}{\gamma w}\right)} \Gamma\left(\frac{k_{off}}{w} - 1\right)}{\Gamma\left(\frac{k_{off}}{w} + l\right)} \left(\Gamma\left(\frac{k_{off}}{w} - 1\right) - \Gamma\left(\frac{k_{off}}{w} - 1, \frac{k_{off}\, r}{\gamma w}\right)\right)^{-1}, \quad (31)$$

where $\Gamma(x)$ is the gamma function.

3.2 Estimation of parameters for actin cables in budding yeast

This mechanism of active transport of dampers is specified by four parameters $(r, \gamma, w$ and $k_{off})$, which can be estimated based on published experiments (1–3). The value $r = 1$ µm/s is estimated based on the observed rate of cable growth in vivo. GFP-labelled Smy1 proteins are seen to pause at the bud neck for about a second in wild type cells and so we estimate $k_{off} = 1/s$ for the rate of Smy1 falling off of the formins.

The myosin-aided delivery rate of Smy1 to the formin, leads to a length dependent on rate $k_{on}(l) = wl$. We estimate the value of the parameter w using the observed number of myosin+Smy1 complexes on the cable. If we model the actin cable as a polymer with l subunits, at every subunit we can consider all the processes by which the myosin+Smy1 complexes arrive and depart the particular subunit. In steady state, the number of complexes arriving and departing need to balance. In particular, myosin+Smy1 can either reach the xth subunit ($1 < x < l$) diffusively from the cell cytosol with a rate k_{on}^0 (which is proportional to the concentration of Smy1 proteins), or by translocating from the $x - 1$ subunit, with a rate v. We assume that the motors do not fall off the polymer and therefore the only way that they leave the xth subunit is by translocating to subunit $x + 1$. At steady state, the number of complexes arriving and departing the xth subunit are equal and therefore the steady state number is $N(x) = \frac{x \, k_{on}^0}{v}$. Using this quantity we can compute the total number of motors (myosin+Smy1 complexes) on the polymer (or cable) by summing over all subunits i.e. $N_{tot} = \sum_{x=0}^{l} N(x) = \frac{k_{on}^0}{v} \frac{l(l+1)}{2}$ (2).

The rate of delivery of Smy1 to the formin at the barbed end is equal to the number of complexes that translocate from the lth subunit to the formin, i.e. $k_{on}(l) = vN(l) = lk_{on}^0$; therefore k_{on}^0 is equal to the previously defined parameter w. We can solve for k_{on}^0, to

obtain the relation $N_{tot} = w \frac{L(L+L_0)}{2L_0 V}$, where $V = vL_0$, is the myosin velocity in units of

microns per second, and $L = l\, L_0$ is the cable length in microns; $L_0 = 2.7$ nm is the size of

an actin subunit in the cable. In the in-vivo experiments, about 5 Smy1+myosin complexes

are observed moving at a rate 3.5 μm/s towards the budneck, hence for the purpose of our

calculations, $N_{tot} = 5$, $L = 5$ μm, and $V = 3.5$ μm/s which yields $w = 0.004$ s^{-1}.

We use these three parameters and the expression for mean cable length in Equation 2 to

obtain a value of the fourth parameter, the disassembly rate γ. By equating the mean cable

length to 5 microns (i.e. the diameter of the yeast cell), and using the parameter values

listed above, we estimate $\gamma = 0.12$ μm/s or 45 subunits/s.

4. Active transport of monomers to the site of assembly

4.1 Solving the master equations

The master equation governing the growth of a filament in this mechanism is given by

$$\frac{dP(l,t)}{dt} = \frac{r'}{l-1}P(l-1) + \gamma P(l+1) - \frac{r'}{l}P(l) - \gamma P(l) . \quad (32)$$

We will use the scheme of recursion to solve this equation. Note that the master equation

blows up at $l = 0$, hence we impose $P(0) = 0$, to avoid this issue. At steady state,

substituting $l = 1$ in Equation 32, we obtain, $P(2) = \frac{r'''}{\gamma}P(1)$. Similarly, for $l = 2$, we get

$P(3) = \frac{1}{2!}\left(\frac{r'}{\gamma}\right)^2 P(1)$, and for $l = 3$, we get $P(4) = \frac{1}{3}\left(\frac{r'}{\gamma}\right)P(3) = \frac{1}{3!}\left(\frac{r'}{\gamma}\right)^3 P(1)$. Proceeding in a

similar fashion, we obtain $P(l) = \frac{1}{l!}\left(\frac{r\prime}{\gamma}\right)^l P(1)$, and we can use normalization i.e. .

$\sum_{l=0}^{\infty} P(l) = 1$ to obtain $P(1) = e^{-\frac{r\prime}{\gamma}}$.

5. Depolymerizers

5.1 Flux of the motors at the end of the filament

In Figure 9A in the main text, we define $N(x)$ as the number of motors on the xth subunit. Balancing the flux of motors at the xth cell we obtain,

$$vN(x-1) + k_{on} = vN(x). \qquad (33)$$

In Equation 33, the flux at the xth subunit is equal to the incoming flux of motors on the $(x-1)^{th}$ subunit and the motors arriving on the polymer from bulk diffusion. At the first subunit, there is no incoming flux; the only motors on the subunit are there through diffusion. Hence we obtain, $N(1) = \frac{k_{on}}{v}$ as $N(0) = 0$. In Equation 33, for $x = 2$, we find $vN(1) + k_{on} = vN(2)$. Substituting $N(1)$ in Equation 33, we get $N(2) = \frac{2\, k_{on}}{v}$. Similarly, for x=3, we obtain, $N(3) = \frac{3\, k_{on}}{v}$ and so on. Thus solving Equation 33 recursively, we obtain $N(x) = \frac{x\, k_{on}}{v}$. In other words, the number of motors at a particular subunit depends on the distance from the negative end of the microtubule.

5.2 Solving the master equation

The master equation for depolymerizers is given by

$$\frac{dP(l,t)}{dt} = r\, P(l-1) - rP(l) + \gamma' \times (l+1)P(l+1) - \gamma' \times l\, P(l). \qquad (34)$$

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As is shown in Equation 34, in order to have a filament of length l, the filament can either grow from $l - 1$ by adding a subunit with a rate r or depolymerize from $l + 1$ with a length dependent rate $\gamma' \times (l + 1)$. Note that these terms add to the probability $P(l, t)$, probability that the filament has length l at time t. Alternatively the filament of length l add a subunit and become $l + 1$ with a rate r or lose a subunit with a length dependent depolymerization rate γl. These terms reduce the probability $P(l, t)$.

From Equation 34 for $l = 0$, we get $\frac{dP(0,t)}{dt} = r\,P(-1) - rP(0) + \gamma'P(1)$. Since l is length of the polymer, it cannot be negative, hence $P(-1) = 0$. Thus $\frac{dP(0,t)}{dt} = \gamma'\,P(1) - rP(0)$. At steady state, $\frac{dP(0,t)}{dt} = 0$. Thus we obtain $P(1) = \frac{r}{\gamma'}\,P(0)$.

In Equation 34, for $l = 1$, $rP(0) + 2\gamma'\,P(2) = (r + \gamma')P(1)$. Substituting the expression for $P(0)$ we obtain an expression for $P(2)$ in terms of $P(0)$, namely $r\,P(0) + 2\gamma'P(2) = (r + \gamma')\frac{r}{\gamma'}\,P(0)$, which simplifies to $P(2) = \frac{1}{2}\left(\frac{r}{\gamma'}\right)^2 P(0)$. Similarly we can obtain an expression for $P(3)$ by substituting $l = 2$ in the Equation 34, i.e. $P(3) = \frac{1}{3}\frac{1}{2}\left(\frac{r}{\gamma'}\right)^3 P(0)$.

Thus Equation 34 can be recursively solved to obtain $P(l) = \left(\frac{r}{\gamma'}\right)^l \frac{1}{l!}\,P(0)$.

5.3 Equation for the mean

We multiply the master equation in Equation 29 by l and sum over all possible lengths, namely

$$\sum_{l=0}^{\infty} \frac{d}{dt} l\,P(l,t) = \sum_{l=0}^{\infty} r\,l\,P(l-1) - \sum_{l=0}^{\infty} r\,l\,P(l) + \sum_{l=0}^{\infty} \gamma'\,l\,(l+1)P(l+1) - \sum_{l=0}^{\infty} \gamma'l^2\,P(l). \quad (35)$$

Let's first consider the first term $\sum_{l=0}^{\infty} r\, l\, P(l-1)$. As defined before, $P(-1) = 0$. Then the

first term can be written as $\sum_{l=1}^{\infty} r\, l\, P(l-1)$. Now making a change of variable where

$m = l - 1$, we get

$$\sum_{l=1}^{\infty} r\, l\, P(l-1) = \sum_{m=0}^{\infty} r(m+1)P(m). \quad (36)$$

Now let's consider the third term $\sum_{l=0}^{\infty} \gamma'\, l\, (l+1)P(l+1)$. As defined before, $P(-1) = 0$.

Then this term can be written as $\sum_{l=-1}^{\infty} \gamma'\, ''l\, (l+1)\, P(l+1)$. Now making a change of

variable where $m = l + 1$, we get

$$\sum_{l=-1}^{\infty} \gamma'\, l\, (l+1)\, P(l+1) = \sum_{m=0}^{\infty} \gamma'(m-1)m\, P(m). \quad (37)$$

Also, $\sum_{l=0}^{\infty} \frac{d}{dt} l\, P(l,t) = \frac{d}{dt}\sum_{l=0}^{\infty} l\, P(l,t)$. Substituting Equation 35 and 36 in Equation 34,

we get

$$\frac{d}{dt}\sum_{l=0}^{\infty} l\, P(l,t) = \sum_{m=0}^{\infty} r(m+1)P(m) - \sum_{l=0}^{\infty} r\, l\, P(l) + \sum_{m=0}^{\infty} \gamma'(m-1)m\, P(m)$$

$$- \sum_{l=0}^{\infty} \gamma' l^2\, P(l). \quad (38)$$

Equation 37 simplifies to

$$\frac{d}{dt}\sum_{l=0}^{\infty} l\, P(l,t) = r\sum_{m=0}^{\infty} mP(m) + r\sum_{m=0}^{\infty} P(m) - r\sum_{l=0}^{\infty} l\, P(l) + \gamma\sum_{m=0}^{\infty} m^2\, P(m)$$

$$- \gamma'\sum_{m=0}^{\infty} m\, P(m) - \sum_{l=0}^{\infty} \gamma' l^2\, P(l). \quad (39)$$

Using the definition of moments, i.e. $\sum_{l=0}^{\infty} l^n\, P(l,t) = \langle l^n \rangle$, and the normalization condition

$\sum_{l=0}^{\infty} P(l,t) = 1$, we get $\frac{d}{dt}\langle l \rangle = r\langle l \rangle + r - r\langle l \rangle + \gamma'\langle l^2 \rangle - \gamma'\langle l \rangle - \gamma'\langle l^2 \rangle$, which simplifies to

$\frac{d}{dt}\langle l \rangle = r - \gamma'\langle l \rangle$. Note that this is the equation for time dependent average length, i.e.

$\langle l \rangle = \langle l(t) \rangle$

Integrating the equation of $\frac{d}{dt}\langle l \rangle$ from l_0 to l we get, $\langle l(t) \rangle = \frac{r}{\gamma} + e^{-\gamma't}\left(l_0 - \frac{r}{\gamma'}\right)$, where l_0 is

the starting filament length at $t = 0$. At steady state $(t \to \infty)$, the average length is $\left(\frac{r}{\gamma'}\right)$

(Figure S1).

5.4 Effect of finite number of binding sites

In our simple calculation, we assumed that the concentration of motors is small enough, so

the capture rate of motors on the microtubule is just $k_{on}^0(s)$ where $k_{on}^0(s) =$

$k_{on}'\,[motors]$. But in reality this rate will depend on the number of binding sites on the

microtubule, given by $k_{on}(s) = k_{on}^0\left(1 - \frac{N(x)}{N_{max}}\right)$, where $N(x)$ is the linear density of motors

on a microtubule lattice and N_{max} is the maximum number of motors that a microtubule

can accommodate. The change in the number of motors will be then given by

$$\frac{dN(x)}{dt} = v\, N(x-1) + k_{on} - v\, N(x). \quad (40)$$

We want to find out how does having a limited number of binding sites changes $N(x)$. As we will see, if the number of available binding sites are large, then this will lead us to the expression we obtained before, namely $N(x) = \frac{k_{on}^0 x}{v}$. Taylor expanding $N(x-1)$ leads to

$$N(x-1) = N(x) - \frac{dN}{dx} + \frac{1}{2}\left(\frac{dN}{dx}\right)^2 + \cdots \quad . \quad (41)$$

Truncating $N(x-1)$ to first order, we can substitute in Equation 35 to obtain

$$\frac{dN}{dt} = -v\frac{dN}{dx} + k_{on} \quad . \quad (42)$$

At steady state, Equation 41 becomes $v\frac{dN}{dx} = k_{on}$. Now we substitute our new expression for $k_{on}(s)$ to get

$$\frac{dN(x)}{dx} = \frac{k_{on}^0}{v}\left(1 - \frac{N(x)}{N_{max}}\right). \quad (43)$$

Now integrating both sides, we get $\int_0^{N(x)}\left(\frac{dN(x)}{\left(1-\frac{N(x)}{N_{max}}\right)}\right) = \frac{k_{on}^0}{v}\int_0^x dx$, which yields,

$\log\left(\frac{N_{max}-N(x)}{N_{max}}\right) = -\frac{k_{on}^0 x}{v\, N_{max}}$. Rearranging the terms, we get $\frac{N_{max}-N(x)}{N_{max}} = e^{-\frac{k_{on}^0 x}{v\, N_{max}}}$, which finally becomes

$$N(x) = N_{max}\left(1 - e^{-\frac{k_{on}^0 x}{v\, N_{max}}}\right). \quad (44)$$

Equation 44 reduces to the expression we got before, i.e. $N(x) = N_{max} \frac{k_{on}^0 x}{v\, N_{max}} = \frac{k_{on}^0 x}{v}$ when

$\frac{k_{on}^0}{v\, N_{max}} \ll 1$. Hence, as long as $k_{on}^0 \ll v\, N_{max}$ our simple assumption of $k_{on}^0(s) =$
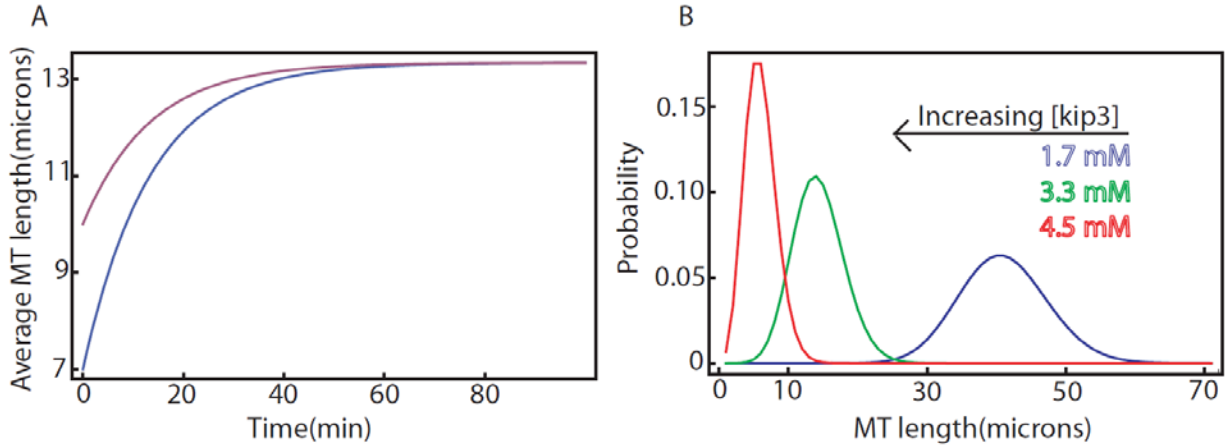
$k_{on}{'}\, [motors]$ is reasonable .

Figure 1: Predictions for the antenna model of depolymerizers (A) Time-dependent average length of a microtubule for $l_0 = 7\ \mu m$ (in blue) and $l_0 = 10 \mu m$ (in purple), for $r =1\ \mu m/min$ and $\gamma = 0.075/min$. (B) Prediction for the steady state length distribution for different concentrations of Kip3 motors i.e. [Kip3]= 1.7 mM(blue), [Kip3]= 3.3 mM(green) and [Kip3]= 4.5 mM(red). An increase in [Kip3] corresponds to an increase in disassembly rate of the microtubule, and thus we observe that the distributions shift towards lower microtubule length.

6. Severing

6.1 Solving the master equation

The master equation describing the evolution of $P(l,t)$ in time, for the mechanism of severing is given by

$$\frac{dP(l,t)}{dt} = rP(l-1) - rP(l) + s \sum_{i=l+1}^{\infty} P(i) - s \times (l-1)P(l) . \quad (45)$$

This equation is different from the ones described so far. As shown in Equation 41, in order to have a filament of length l, the filament can either grow from $l-1$ by adding a subunit with a rate r or shrink from any filament having a length larger than l by getting severed with a rate s. These terms add to the probability $P(l,t)$, that the filament has length l at time t. Alternatively, the filament of length l can either get severed $l-1$ ways with a rate s or add a subunit and become $l+1$ with a length dependent rate r. These terms reduce the probability $P(l,t)$. The first and third terms, thus represent the inflow of probability to and second and fourth terms represent the outflow of probability from the state l.

At steady state, the probability does not change with time. In other words, $\frac{dP(l,t)}{dt} = 0$. Hence from Equation 45, we get $P(l)(r + (l-1)s) = s\sum_{i=l+1}^{\infty} P(i) + P(l-1)r$. Using the normalization condition for the probability $P(l)$, $\sum_{i=1}^{\infty} P(i) = \sum_{i=1}^{l} P(i) + \sum_{i=l+1}^{\infty} P(i) = 1$, we obtain

$$P(l)(r + (l-1)s) = s\left(1 - \sum_{i=1}^{l} P(i)\right) + P(l-1)r . \quad (46)$$

Adding the term $sP(l)$ on both sides of the Equation 2, we obtain

$$P(l)(r + ls) = s\left(1 - \sum_{i=1}^{l-1} P(i)\right) + P(l-1)r . \quad (47)$$

For $l = 1$, $P(1)(r + s) = s + P(0)r$. Using the convention $P(0) = 0$, in Equation 43 we find, $P(1) = \frac{s}{r+s}$. For $l = 2$, similarly we find

$$P(2) = \frac{1}{r+2s} s(1 - P(1)) + P(1)r . \quad (48)$$

Substituting $P(1)$ in Equation 49, we obtain $P(2) = \frac{2\,rs}{r+2s}\left(\frac{1}{r+s}\right)$. Dividing the numerator

and denominator by s^2, we get $P(2) = \left(\frac{2\frac{r}{s}}{\frac{r}{s}+2}\right)\left(\frac{1}{\frac{r}{s}+1}\right)$. Continuing with the same logic,

for $l = 3$ we find $P(3) = \frac{1}{r+3s}s\left(1 - \big(P(1) + P(2)\big)\right) + P(2)r$. Substituting $P(1)$ and $P(2)$

in this equation, we obtain $P(3) = \frac{3\,r^2 s}{(r+3s)(r+2s)(r+s)}$. Dividing the numerator and

denominator by s^3, we obtain $P(3) = \frac{3\left(\frac{r}{s}\right)^2}{\left(\frac{r}{s}+3\right)\left(\frac{r}{s}+2\right)\left(\frac{r}{s}+1\right)}$. Thus the master equations in

Equation 45 can be recursively solved to give a closed form solution of the form

$$P(l) = \frac{l\,s\,r^{l-1}}{(r+s)(r+2s)(r+3s)\dots(r+ls)}, \qquad (49)$$

6.2 Estimation of mean length compared to exact result

In the case of severing, the disassembly rate (i.e., rate of subunit loss) is length dependent,

and can be approximated as $\gamma(l) \sim sl \times \frac{l}{2} = \frac{sl^2}{2}$ (See main text Section 2.3.2). In other

words, severing leads to a quadratic dependence of the disassembly rate on the length of

the filament (Figure 10C). It is possible to get a rough estimate of the average steady state

filament length by equating the disassembly rate $\gamma(l)$ with the assembly rate r, i.e.

$\langle l \rangle = \sqrt{\frac{2\,r}{s}}$ (Figure S2 in purple). The exact solution of the mean filament length is given by

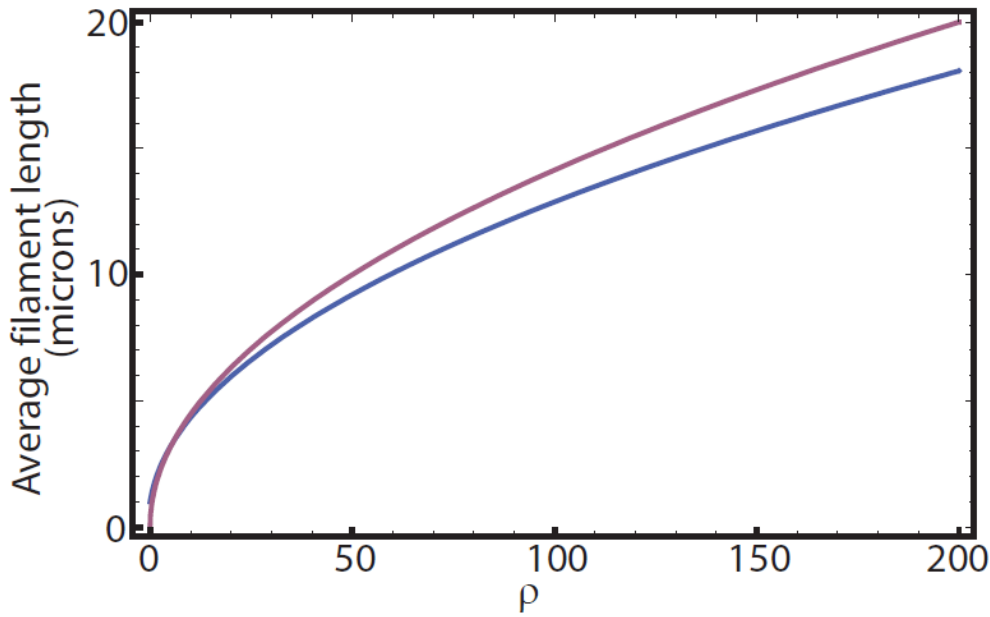the Equation 29 in Section 2.3.2.1 (Figure S2 in blue).

Figure 2: Average filament length from severing. This plot compares the average filament length obtained exactly by solving for the steady state distribution (in blue) and by assuming a disassembly rate having a quadratic dependence on the length of the filament (in purple) as a function the severing parameter $\rho = r/s$. The rough estimate using the expression $\langle l \rangle = \sqrt{2r/s}$ is larger than the exact mean evaluated by Equation 43 after $\rho \sim$ 30.