ON THE K-WINNERS-TAKE-ALL NETWORK

E. Majani
Jet Propulsion Laboratory
California Institute of Technology

R. Erlanson, Y. Abu-Mostafa
Department of Electrical Engineering
California Institute of Technology

ABSTRACT

We present and rigorously analyze a generalization of the Winner-Take-All Network: the K-Winners-Take-All Network. This network identifies the K largest of a set of N real numbers. The network model used is the continuous Hopfield model.

I - INTRODUCTION

The Winner-Take-All Network is a network which identifies the largest of N real numbers. Winner-Take-All Networks have been developed using various neural networks models (Grossberg-73, Lippman-87, Feldman-82, Lazzaro-89). We present here a generalization of the Winner-Take-All Network: the K-Winners-Take-All (KWTA) Network. The KWTA Network identifies the K largest of N real numbers. The neural network model we use throughout the paper is the continuous Hopfield network model (Hopfield-84). If the states of the N nodes are initialized to the N real numbers, then, if the gain of the sigmoid is large enough, the network converges to the state with K positive real numbers in the positions of the nodes with the K largest initial states, and $N-K$ negative real numbers everywhere else.

Consider the following example: $N = 4$, $K = 2$. There are $6 = \binom{N}{K}$ stable states:$(++--)^T$, $(+-+-)^T$, $(+--+)^T$, $(--++)^T$, $(-+-+)^T$, and $(-++-)^T$. If the initial state of the network is $(0.3,\ -0.4,\ 0.7,\ 0.1)^T$, then the network will converge to $(v_1, v_2, v_3, v_4)^T$ where $v_1 > 0,\ v_2 < 0,\ v_3 > 0,\ v_4 < 0\ \ ((+-+-)^T)$.

In Section II, we define the KWTA Network (connection weights, external inputs). In Section III, we analyze the equilibrium states and in Section IV, we identify all the stable equilibrium states of the KWTA Network. In Section V, we describe the dynamics of the KWTA Network. In Section VI, we give two important examples of the KWTA Network and comment on an alternate implementation of the KWTA Network.

II - THE K-WINNERS-TAKE-ALL NETWORK

The continuous Hopfield network model (Hopfield-84) (also known as the Grossberg additive model (Grossberg-88)), is characterized by a system of first order differential equations which governs the evolution of the state of the network ($i = 1, \dots, N$) :

$$C\frac{du_i}{dt} = -\lambda_i u_i + \sum_{j=1}^{N} T_{ij} g(u_j) - t_i, \text{ where } \lambda_i = \sum_j |T_{ij}|, \; C > 0.$$

The sigmoid function $g(u)$ is defined by: $g(u) = f(G \cdot u)$, where $G > 0$ is the gain of the sigmoid, and $f(u)$ is defined by: 1. $\forall u, \; 0 < f'(u) \leq f'(0) = 1$, 2. $\lim_{u \to +\infty} f(u) = 1$, 3. $\lim_{u \to -\infty} f(u) = -1$.

The KWTA Network is characterized by mutually inhibitory interconnections $T_{ij} = -1$ for $i \neq j$, a self connection $T_{ii} = a$, $(|a| < 1)$ and an external input (identical for every node) which depends on the number K of winners desired and the size of the network N : $t_i = 2K - N$.

The differential equations for the KWTA Network are therefore:

$$\text{for all } i, \quad C\frac{du_i}{dt} = -\lambda u_i + (a+1)g(u_i) - \left(\sum_{j=1}^{N} g(u_j) - t\right), \tag{1}$$

where $\lambda = N - 1 + |a|$, $-1 < a < +1$, and $t = 2K - N$. Let us now study the equilibrium states of the dynamical system defined in (1). We already know from previous work (Hopfield-84) that the network is guaranteed to converge to a stable equilibrium state if the connection matrix (T) is symmetric (and it is here).

III - EQUILIBRIUM STATES OF THE NETWORK

The equilibrium states $\mathbf{u}^*$ of the KWTA network are defined by

$$\text{for all } i, \quad \frac{du_i}{dt} = 0,$$

i.e.,

$$\text{for all } i, \quad g(u_i^*) = \frac{\lambda}{a+1} u_i^* + \left(\frac{\sum_j g(u_j^*) - (2K - N)}{a+1}\right). \tag{2}$$

Let us now develop necessary conditions for a state $\mathbf{u}^*$ to be an equilibrium state of the network.

Theorem 1: For a given equilibrium state $\mathbf{u}^*$, every component u_i^* of $\mathbf{u}^*$ can be one of at most three distinct values.

Proof of Theorem 1.

If we look at equation (2), we see that the last term of the righthandside expression is independent of i; let us denote this term by $H(\mathbf{u}^*)$. Therefore, the components u_i^* of the equilibrium state $\mathbf{u}^*$ must be solutions of the equation:

$$g(u_i^*) = \frac{\lambda}{a+1} u_i^* + H(\mathbf{u}^*).$$

Since the sigmoid function $g(u)$ is monotone increasing and $\lambda/(a+1) > 0$, then the sigmoid and the line $\frac{\lambda}{a+1} u_i^* + H(\mathbf{u}^*)$ intersect in at least one point and at most three (see Figure 1). Note that the constant $H(\mathbf{u}^*)$ can be different for different equilibrium states $\mathbf{u}^*$. ∎

The following theorem shows that the sum of the node outputs is constrained to being close to $2K - N$, as desired.

Theorem 2: If $\mathbf{u}^*$ is an equilibrium state of (1), then we have:

$$(a+1) \max_{u_i^* < 0} g(u_i^*) < \sum_{j=1}^{N} g(u_j^*) - 2K + N < (a+1) \min_{u_i^* > 0} g(u_i^*). \tag{3}$$

Proof of Theorem 2.

Let us rewrite equation (2) in the following way:

$$\lambda u_i^* = (a+1) g(u_i^*) - (\sum_j g(u_j^*) - (2K - N)).$$

Since u_i^* and $g(u_i^*)$ are of the same sign, the term $(\sum_j g(u_j^*) - (2K - N))$ can neither be too large (if $u_i^* > 0$) nor too low (if $u_i^* < 0$). Therefore, we must have

$$\begin{cases} (\sum_j g(u_j^*) - (2K - N)) < (a+1) g(u_i^*), & \text{for all } u_i^* > 0, \\ (\sum_j g(u_j^*) - (2K - N)) > (a+1) g(u_i^*), & \text{for all } u_i^* < 0, \end{cases}$$

which yields (3). ∎

Theorem 1 states that the components of an equilibrium state can only be one of at most three distinct values. We will distinguish between two types of equilibrium states, for the purposes of our analysis: those which have one or more components u_i^* such that $g'(u_i^*) \geq \frac{\lambda}{a+1}$, which we categorize as type I, and those which do not (type II). We will show in the next section that for a gain G large enough, no equilibrium state of type II is stable.

IV - ASYMPTOTIC STABILITY OF EQUILIBRIUM STATES

We will first derive a necessary condition for an equilibrium state of (1) to be asymptotically stable. Then we will find the stable equilibrium states of the KWTA Network.

IV-1. A NECESSARY CONDITION FOR ASYMPTOTIC STABILITY

An important necessary condition for asymptotic stability is given in the following theorem.

Theorem 3: Given any asymptotically stable equilibrium state $\mathbf{u}^*$, at most one of the components u_i^* of $\mathbf{u}^*$ may satisfy:

$$g'(u_i^*) \geq \frac{\lambda}{a+1}.$$

Proof of Theorem 3.

Theorem 3 is obtained by proving the following three lemmas.

Lemma 1: Given any asymptotically stable equilibrium state $\mathbf{u}^*$, we always have for all i and j such that $i \neq j$:

$$\lambda > a\frac{g'(u_i^*) + g'(u_j^*)}{2} + \frac{\sqrt{a^2\left(g'(u_i^*) - g'(u_j^*)\right)^2 + 4g'(u_i^*)g'(u_j^*)}}{2}. \tag{4}$$

Proof of Lemma 1.

System (1) can be linearized around any equilibrium state $\mathbf{u}^*$:

$$\frac{d(\mathbf{u}-\mathbf{u}^*)}{dt} \approx L(\mathbf{u}^*)(\mathbf{u}-\mathbf{u}^*), \text{ where } L(\mathbf{u}^*) = T \cdot \operatorname{diag}\left(g'(u_1^*), \ldots, g'(u_N^*)\right) - \lambda I.$$

A necessary and sufficient condition for the asymptotic stability of $\mathbf{u}^*$ is for $L(\mathbf{u}^*)$ to be negative definite. A necessary condition for $L(\mathbf{u}^*)$ to be negative definite is for all 2×2 matrices $L_{ij}(\mathbf{u}^*)$ of the type

$$L_{ij}(\mathbf{u}^*) = \begin{pmatrix} ag'(u_i^*) - \lambda & -g'(u_j^*) \\ -g'(u_i^*) & ag'(u_j^*) - \lambda \end{pmatrix}, \quad (i \neq j)$$

to be negative definite. This results from an infinitesimal perturbation of components i and j only. Any matrix $L_{ij}(\mathbf{u}^*)$ has two real eigenvalues. Since the largest one has to be negative, we obtain:

$$\frac{1}{2}\left(ag'(u_i^*) - \lambda + ag'(u_j^*) - \lambda + \sqrt{a^2\left(g'(u_i^*) - g'(u_j^*)\right)^2 + 4g'(u_i^*)g'(u_j^*)}\right) < 0. \blacksquare$$

Lemma 2: Equation (4) implies:

$$\min\left(g'(u_i^*), g'(u_j^*)\right) < \frac{\lambda}{a+1}. \tag{5}$$

Proof of Lemma 2.

Consider the function h of three variables:

$$h\left(a, g'(u_i^*), g'(u_j^*)\right) = a\frac{g'(u_i^*) + g'(u_j^*)}{2} + \frac{\sqrt{a^2\left(g'(u_i^*) - g'(u_j^*)\right)^2 + 4g'(u_i^*)g'(u_j^*)}}{2}.$$

If we differentiate h with respect to its third variable $g'(u_j^*)$, we obtain:

$$\frac{\partial h\left(a, g'(u_i^*), g'(u_j^*)\right)}{\partial g'(u_j^*)} = \frac{a}{2} + \frac{a^2 g'(u_j^*) + (2-a^2)g'(u_i^*)}{2\sqrt{a^2\left(g'(u_i^*) - g'(u_j^*)\right)^2 + 4g'(u_i^*)g'(u_j^*)}}$$

which can be shown to be positive if and only if $a > -1$. But since $|a| < 1$, then if $g'(u_i^*) \leq g'(u_j^*)$ (without loss of generality), we have:

$$h\left(a, g'(u_i^*), g'(u_j^*)\right) \geq h\left(a, g'(u_i^*), g'(u_i^*)\right) = (a+1)g'(u_i^*),$$

which, with (4), yields:

$$g'(u_i^*) < \frac{\lambda}{a+1},$$

which yields Lemma 2. ∎

Lemma 3: If for all $i \neq j$,

$$\min\left(g'(u_i^*), g'(u_j^*)\right) < \frac{\lambda}{a+1},$$

then there can be at most one u_i^* such that:

$$g'(u_i^*) \geq \frac{\lambda}{a+1}.$$

Proof of Lemma 3.

Let us assume there exists a pair (u_i^*, u_j^*) with $i \neq j$ such that $g'(u_i^*) \geq \frac{\lambda}{a+1}$ and $g'(u_j^*) \geq \frac{\lambda}{a+1}$, then (5) would be violated. ∎

IV-2. STABLE EQUILIBRIUM STATES

From Theorem 3, all stable equilibrium states of type I have exactly one component γ (at least one and at most one) such that $g'(\gamma) \geq \frac{\lambda}{a+1}$. Let N_+ be the number of components α with $g'(\alpha) < \frac{\lambda}{a+1}$ and $\alpha > 0$, and let N_- be the number of components β with $g'(\beta) < \frac{\lambda}{a+1}$ and $\beta < 0$ (note that $N_+ + N_- + 1 = N$). For a large enough gain G, $g(\alpha)$ and $g(\beta)$ can be made arbitrarily close to $+1$ and -1 respectively. Using Theorem 2, and assuming a large enough gain, we obtain: $-1 < N_+ - K < 0$. N_+ and K being integers, there is therefore no stable equilibrium state of type I.

For the equilibrium states of type II, we have for all i, $u_i^* = \alpha(> 0)$ or $\beta(< 0)$ where $g'(\alpha) < \frac{\lambda}{a+1}$ and $g'(\beta) < \frac{\lambda}{a+1}$. For a large enough gain, $g(\alpha)$ and $g(\beta)$ can be made arbitrarily close to $+1$ and -1 respectively. Using theorem 2 and assuming a large enough gain, we obtain: $-(a+1) < 2(N_+ - K) < (a+1)$, which yields $N_+ = K$.

Let us now summarize our results in the following theorem:

Theorem 4: For a large enough gain, the only possible asymptotically stable equilibrium states $\mathbf{u}^*$ of (1) must have K components equal to $\alpha > 0$ and $N - K$ components equal to $\beta < 0$, with

$$\begin{cases} g(\alpha) = \frac{\lambda}{a+1}\alpha + \frac{K(g(\alpha)-g(\beta)-2)+N(1+g(\beta))}{a+1}, \\ g(\beta) = \frac{\lambda}{a+1}\beta + \frac{K(g(\alpha)-g(\beta)-2)+N(1+g(\beta))}{a+1}. \end{cases} \tag{7}$$

Since we are guaranteed to have at least one stable equilibrium state (Hopfield-84), and since any state whose components are a permutation of the components of a stable equilibrium state, is clearly a stable equilibrium state, then we have:

Theorem 5: There exist at least $\binom{N}{K}$ stable equilibrium states as defined in Theorem 4. They correspond to the $\binom{N}{K}$ different states obtained by the $N!$ permutations of one stable state with K positive components and $N - K$ positive components.

V - THE DYNAMICS OF THE KWTA NETWORK

Now that we know the characteristics of the stable equilibrium states of the KWTA Network, we need to show that the KWTA Network will converge to the stable state which has $\alpha > 0$ in the positions of the K largest initial components. This can be seen clearly by observing that for all $i \neq j$:

$$C\frac{d(u_i - u_j)}{dt} = \lambda(u_i - u_j) + (a+1)(g(u_i) - g(u_j)).$$

If at some time T, $u_i(T) = u_j(T)$, then one can show that $\forall t$, $u_i(t) = u_j(t)$. Therefore, for all $i \neq j$, $u_i(t) - u_j(t)$ always keeps the same sign. This leads to the following theorem.

Theorem 6: (Preservation of order) For all nodes $i \neq j$,

$$(u_i(0) < u_j(0)) \Leftrightarrow (\forall t,\ u_i(t) < u_j(t)), \text{ and } (u_i(0) = u_j(0)) \Leftrightarrow (\forall t,\ u_i(t) = u_j(t)).$$

We shall now summarize the results of the last two sections.

Theorem 7: Given an initial state $\mathbf{u}^*(0)$ and a gain G large enough, the KWTA Network will converge to a stable equilibrium state with K components equal to a positive real number ($\alpha > 0$) in the positions of the K largest initial components, and $N - K$ components equal to a negative real number ($\beta < 0$) in all other $N - K$ positions.

This can be derived directly from Theorems 4, 5 and 6: we know the form of all stable equilibrium states, the order of the initial node states is preserved through time, and there is guaranteed convergence to an equilibrium state.

VI - DISCUSSION

The well-known Winner-Take-All Network is obtained by setting K to 1.

The N/2-Winners-Take-All Network, given a set of N real numbers, identifies which numbers are above or below the median. This task is slightly more complex computationally ($\approx O(N \log(N))$ than that of the Winner-Take-All ($\approx O(N)$). The number of stable states is much larger,

$$\binom{N}{N/2} \approx \frac{2^N}{\sqrt{2\pi N}},$$

i.e., asymptotically exponential in the size of the network.

Although the number of connection weights is N^2, there exists an alternate implementation of the KWTA Network which has $O(N)$ connections (see Figure 2). The sum of the outputs of all nodes and the external input is computed, then negated and fed back to all the nodes. In addition, a positive self-connection ($a + 1$) is needed at every node.

The analysis was done for a "large enough" gain G. In practice, the critical value of G is $\frac{\lambda}{a+1}$ for the $N/2$-Winners-Take-All Network, and slightly higher for $K \neq N/2$. Also, the analysis was done for an arbitrary value of the self-connection weight a ($|a| < 1$). In general, if a is close to $+1$, this will lead to faster convergence and a smaller value of the critical gain than if a is close to -1.

VII - CONCLUSION

The KWTA Network lets all nodes compete until the desired number of winners (K) is obtained. The competition is ibatained by using mutual inhibition between all nodes, while the number of winners K is selected by setting all external inputs to $2K - N$. This paper illustrates the capability of the continuous Hopfield Network to solve exactly an interesting decision problem, i.e., identifying the K largest of N real numbers.

Acknowledgments

The authors would like to thank John Hopfield and Stephen DeWeerth from the California Institute of Technology and Marvin Perlman from the Jet Propulsion Laboratory for insightful discussions about material presented in this paper. Part of the research described in this paper was performed at the Jet Propulsion Laboratory under contract with NASA.

References

J.A. Feldman, D.H. Ballard, "Connectionist Models and their properties," *Cognitive Science*, Vol. **6**, pp. 205-254, 1982

S. Grossberg, "Contour Enhancement, Short Term Memory, and Constancies in Reverberating Neural Networks," *Studies in Applied Mathematics,* Vol. LII (**52**), No. 3, pp. 213-257, September 1973

S. Grossberg, "Non-Linear Neural Networks: Principles, Mechanisms, and Architectures," *Neural Networks,* Vol. **1**, pp. 17-61, 1988

J.J. Hopfield, "Neurons with graded response have collective computational properties like those of two-state neurons," *Proc. Natl. Acad. Sci. USA*, Vol. **81**, pp. 3088-3092, May 1984

J. Lazzaro, S. Ryckebusch, M.A. Mahovald, C.A. Mead, "Winner-Take-All Networks of $O(N)$ Complexity," in this volume, 1989

R.P. Lippman, B. Gold, M.L. Malpass, "A Comparison of Hamming and Hopfield Neural Nets for Pattern Classification," MIT Lincoln Lab. Tech. Rep. TR-769, 21 May 1987

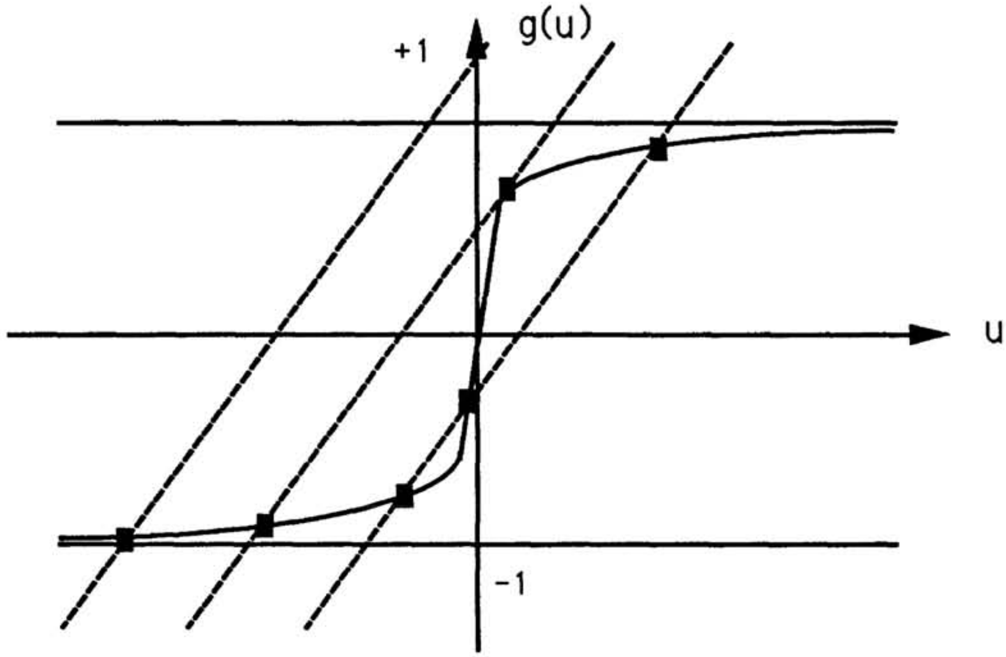

Figure 1: Intersection of sigmoid and line.

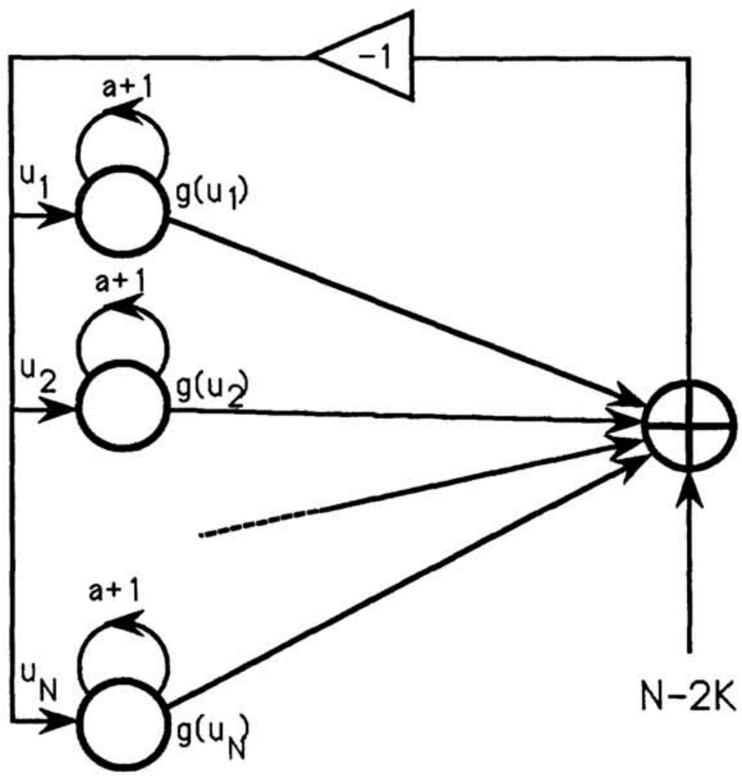

Figure 2: An Implementation of the KWTA Network.