Color embeddings, charge assignments, and proton stability in unified gauge theories*

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Three problems in hypothetical unified theories of electromagnetic, weak, and strong interactions are discussed here. First, the problem of embedding color in any simple gauge group is solved, and a complete classification of theories where the fermion color is restricted to $\mathbf{1}^c$, $\mathbf{3}^c$, and $\bar{\mathbf{3}}^c$ of SU_3^c is given. Generalizations are also indicated. Second, an unbroken U_1 generated by electric charge is embedded into each of the above theories and the charge assignments analyzed. Third, the general problem of stabilizing the proton by a suitable atomic mass number A is studied for the same set of theories. It is always possible to define A if the gauge group is not too small. In many of these theories there must be fermions with weird values of A: leptons with $A\neq 0$ and quarks with $A\neq 1/3$. Examples are discussed. Some future directions of research are indicated.

CONTENTS

I. INTRODUCTION

Quantum field theory with local gauge invariance appears to provide the appropriate framework for a dynamical theory of all elementary particle interactions. It is well known that electromagnetic interactions were the first to be described this way: the local phase invariance of the Lagrangian allows the construction of a renormalizable quantum field theory that agrees impressively well with experiment. Local phase invariance was generalized to non-Abelian gauge groups by Yang and Mills (1954) and Shaw (1955). [Also see Utiyama (1956), Gell-Mann and Glashow (1961), and Kibble (1961).] Although Yang-Mills theories were immediately seen to be mathematically beautiful and physically suggestive, it was not clear how to construct a sensible model in which the vector bosons (other than the photon) acquire masses without destroying its (then hoped for) renormalizability. Then it was discovered by Higgs (1964a, b) and others that the spontaneous breaking of a local symmetry does not imply a zero-mass Goldstone boson as it does in conventional theories.[1] When the quadratic part of the Lagrangian is rediagonalized by a gauge transformation, the degree of freedom that was expected to be the Goldstone boson becomes the longitudinal-spin degree of freedom of a massive vector boson. Later t'Hooft (1971) provided the crucial proof that spontaneously broken Yang-Mills theories are indeed renormalizable.

By extending the local gauge invariance beyond the U_1 of electrodynamics, it became possible to construct sensible models of weak and electromagnetic interactions in which the weak bosons acquire large masses (of order 100 GeV) through the Higgs mechanism (Weinberg, 1967; Salam, 1968). The experimental existence of the weak charged and neutral currents along with the electromagnetic current implies a local symmetry group at least as large as $SU_2\times U_1$ (Glashow, 1959). The basic strategy of using Yang-Mills Lagrangians to unify electromagnetic and weak interactions is very attractive. However, the $SU_2\times U_1$ theory is somewhat awkward. Besides the two gauge couplings of the two simple factors of the gauge group, there are choices of particle fields, their representations, and other arbitrary parameters in the $SU_2\times U_1$ invariant Lagrangian. It also ignores the strong interactions. Nevertheless, this model has provided a framework for organizing huge quantities of experimental data.

The formulation of a Yang-Mills theory of the strong interactions is greatly simplified once the fundamental role of quarks and gluons is recognized. The proposal subscribed to in this paper is that of quantum chromodynamics (QCD)[2]; the strong-interaction gauge group is called the color group. The gauged color symmetry must be at least as large as SU_3. (The choice of SU_3

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[1]The Goldstone theorem is discussed in Goldstone (1961); Nambu and Jona-Lasinio (1961); and Goldstone, Salam, and Weinberg (1962). For the Higgs mechanism, see Higgs (1964a, b); Englert and Brout (1964); and Guralnik, Hagen, and Kibble (1964).

[2]QCD was originally introduced by Nambu (1966) in a version based on the unconfined, integrally charged quarks of Han and Nambu (1965). Fractionally charged quarks were usually assigned parastatistics, as described by Greenberg (1964). Later it was shown by Fritzsch and Gell-Mann (1971) and Bardeen, Fritzsch, and Gell-Mann (1972) that the concept of color, with isolated particles assumed to be restricted to color singlets, had the same effect as parastatistics, with isolated particles assumed to be restricted to bosons and fermions. For early discussions of QCD with confined color see Fritzsch and Gell-Mann (1972); Fritzsch, Gell-Mann, and Leutwyler (1973); and Weinberg (1973a, b). Meanwhile, the asymptotic freedom of QCD was being pointed out by Politzer (1973) and by Gross and Wilczek (1973).

solves the statistics problem of the baryonic ground state, gives the correct π^0 decay rate with fractionally charged quarks, and yields rough agreement with the hadron-muon ratio R in e^+e^- annihilation.) In QCD the SU_3^c is unbroken, the theory is asymptotically free, and color is conjectured to be confined. (Color confinement may in fact be only approximate, but it is conceptually simplest for us to assume here that only color singlet states are observed in Nature.)

We neglect the effects of gravity in this paper, although there are some brief comments on supersymmetry and supergravity near the end of this section.

Any hypothetical unified Yang-Mills theory of electromagnetic, weak, and strong interactions must have a local symmetry group G at least as large as $G^w \times SU_3^c$:

$$G \supseteq G^w \times SU_3^c, \tag{1.1}$$

where $G^w \supseteq SU_2 \times U_1$ includes the low-mass bosons of the weak interactions and the photon. We assume that the photon couples to one of the G^w currents and, of course, that the corresponding U_1 is unbroken. In the minimal scheme $SU_2 \times U_1 \times SU_3$, there are three independent coupling constants and much arbitrariness in assigning fermions and other particles to representations of G. From a theoretical viewpoint, this looseness in the formulation of the theory would seem to be an unacceptable price to pay, for example, for economizing on the number of vector bosons. The simplest proposal for improving on this situation is to enlarge G so that there is only one gauge coupling constant (Georgi and Glashow, 1974; Fritzsch and Minkowski, 1975) and so that all the elementary fermions (quarks and leptons) are contained in a few irreducible representations of G. In the last of these references, the authors examined many of the cases treated here, but our discussion is somewhat more systematic.

It is assumed in this paper that G is a Lie group (and not, for example, a graded Lie group), that G contains $SU_2 \times U_1 \times SU_3^c$, and that there is only one independent gauge coupling. Thus G must be simple, or semisimple in the form $G \times G$ (G simple) where some global reflection symmetry constrains the (unrenormalized) gauge coupling constants to the two simple factors to be equal. The requirement that G be simple fixes many parameters and relationships that need to be determined experimentally in the $SU_2 \times U_1 \times SU_3^c$ model, but it also implies new interactions of Nature that have not been observed.

For the moment we assume the full gauge group G is simple, and leave until later the generalization to $G \times G$ theories. The flavor group G^{fl}, which is generated by the currents that couple to the color singlet bosons, must include G^w: $G^{\mathrm{fl}} \supseteq G^w$. Thus G has a maximal subgroup decomposition of the form,

$$G \supset G^{\mathrm{fl}} \times SU_3^c. \tag{1.2}$$

One purpose of this paper is to list the embeddings of SU_3^c in G and to classify the structure of G^{fl}. We note here that if G^{fl} is larger than $SU_2 \times U_1$, then the new generators are coupled to bosons that mediate interactions not yet observed. The supplementary flavor bosons are either of high mass or else not coupled significantly to transitions between light, familiar particles. Other new bosons implied by this kind of unification can include diotons, which are color octets that also carry flavor; leptoquarks, which change quarks into leptons; and diquarks, which change quarks into antiquarks. Clearly an unstable proton is often predicted in this kind of unified theory. Much of our paper is devoted to that problem.

The fields appearing in the Lagrangian are assigned to representations of G. The spin 1/2 fermion representation f must include leptons ($\mathbf{1}^c$), quarks ($\mathbf{3}^c$), and antiquarks ($\bar{\mathbf{3}}^c$): at present experiment does not suggest the existence of new fermions transforming as higher SU_3^c representations. Theoretically there is no objection to having fermions in more complicated color representations; indeed such fermions are commonplace in supersymmetric theories. Nevertheless, it is usually the case that some set of fermions (which might have spin 3/2, for example) belong to a representation containing only $\mathbf{1}^c$, $\mathbf{3}^c$ and $\bar{\mathbf{3}}^c$. In this article, we usually make the more limiting assumption that the spin 1/2 fermion colors are actually restricted to $\mathbf{1}^c$, $\mathbf{3}^c$ and $\bar{\mathbf{3}}^c$. The problem of finding all embeddings of SU_3^c in any simple G larger than $SU_2 \times U_1 \times SU_3^c$, with the proviso that there exists at least one nontrivial representation containing at most $\mathbf{1}^c$, $\mathbf{3}^c$ and $\bar{\mathbf{3}}^c$, is solved in Sec. II. We also discuss the structure of G^{fl} and find all other representations of G satisfying the same color restriction. These results will now be reviewed in detail; Sec. II may be regarded as a mathematical appendix.

The embedding procedure is greatly simplified by a theorem that is proved in the Appendix: if for a given embedding of SU_3^c in G, the color content of any nontrivial representation f is restricted to $\mathbf{1}^c$, $\mathbf{3}^c$, and $\bar{\mathbf{3}}^c$, then the "fundamental" representation of G must also contain at most, $\mathbf{1}^c$, $\mathbf{3}^c$, and $\bar{\mathbf{3}}^c$. By the fundamental representation we mean the defining representations of the classical groups and the lowest-dimensional nontrivial representations of the exceptional groups. Of course, the color restriction on f and the fundamental representation can be loosened to include $\mathbf{6}^c$, $\bar{\mathbf{6}}^c$, $\mathbf{8}^c$, etc., if desired, but this is not done here. The embedding is then identified by constructing the adjoint representation, which is always easily obtained from the fundamental one. Since the embedding is of the form $G \supset G^{\mathrm{fl}} \times SU_3^c$, the color singlet part of the adjoint representation then identifies the flavor group, and the remaining bosons are also listed. We construct all other irreducible representations of G with the color restriction and put the fermions into these representations or direct sums of them.

The structure of G^{fl} falls into one of four categories. We summarize the results of Sec. II in terms of those categories.

Class I: $G^{\mathrm{fl}} = G_l \times G_q \times U_1$, where G_l is a nontrivial simple factor that transforms only the color singlets, and G_q is another nontrivial simple factor that transforms only the color triplets (and antitriplets) of the fundamental representation.[3] The U_1 distinguishes $\mathbf{1}^c$ from $\mathbf{3}^c$ and (or) $\bar{\mathbf{3}}^c$. This embedding occurs only if G is a classical group, i.e., if G is SU_n (unitary), SO_n

[3] We often discuss $SO_4 \approx SU_2 \times SU_2$ as if it were a simple (sub) group.

(orthogonal), or Sp_{2n} (symplectic). Only the fundamental representation of G satisfies the color restriction: **n** of SU_n; **n** of SO_n (also called the vector representation **v**); or the **2n** of Sp_{2n}. Thus the color singlets of the fundamental representation may be identified with the leptons, and the color triplets with quarks. (See Cases 1, 4, and 6 in Sec. II.) Since the quarks and leptons have commuting flavor groups, the observed universality of leptonic and quark electromagnetic and weak charges must come from the symmetry-breaking mechanism. The **n** of SO_n and the **2n** of Sp_{2n} are self-conjugate representations, and therefore contain equal numbers of $\mathbf{3}^c$ and $\bar{\mathbf{3}}^c$. In this embedding the **n** of SU_n, which is complex, contains $\mathbf{1}^c$ and $\mathbf{3}^c$ only.

Class II: $G^{\mathrm{fl}} = G_l \times G_q \times G_{\bar{r}} \times U_1 \times U_1$. This is possible only for $G = SU_n$, with fermions assigned to the **n**, where **n** contains $\mathbf{3}^c$ q-type quarks, and $\bar{\mathbf{3}}^c$ $\bar{r}$-type antiquarks. The two U_1's distinguish among $\mathbf{1}^c$, $\mathbf{3}^c$, and $\bar{\mathbf{3}}^c$. This embedding is quite similar to Class I, but it contains some additional interest because there is a temptation to enlarge the color group to $SU_3 \times SU_3$. (See Sec. II, Case 3.)

Class III: $G^{\mathrm{fl}} = G_{q+l} \times U_1$, where G_{q+l} transforms the color singlet piece of the fundamental representation, but the fermions are in a different representation such that the same simple[3] factor G_{q+l} transforms both quarks and leptons. There are two cases:

If $G = SU_n$, then the fermions may be assigned to the irreducible representations constructed from $(\mathbf{n} \times \mathbf{n} \times \ldots \times \mathbf{n})_A$, where $(\ )_A$ means to antisymmetrize the Kronecker product. Except for $(\mathbf{n}^{n/2})_A$, n even, these representations are all complex. (See Case 2 in Sec. II.)

If $G = SO_m$, **f** may be a spinor representation. There is one self-conjugate spinor for SO_{2n+1} of dimension 2^n. SO_{4n} has two inequivalent self-conjugate spinors, each of dimension 2^{2n-1}, and the two spinors of SO_{4n+2} are complex and conjugate to one another. Each has dimension 2^{2n}. (See Case 5.)

In this class the universality of the weak charges is natural since there is only one non-Abelian factor in G^{fl}. However, the relation between quark and lepton electric charges has to come from the symmetry breaking.

Class IV: $G^{\mathrm{fl}} = G_{q+l}$. This embedding, which contains no U_1 factor that distinguishes $\mathbf{1}^c$ from $\mathbf{3}^c$, is possible only for the exceptional groups. Three of the five exceptional groups satisfy our assumptions, and in each case, only the fundamental representation satisfies the color restriction. We enumerate the results. (See Cases 7, 8, and 9.) For F_4, where $G^{\mathrm{fl}} = SU_3$, the fundamental representation is the **26**, which is self-conjugate. The flavor group for E_6 is $SU_3 \times SU_3$ and the fermions are assigned to the **27**, which is complex. For E_7, $G^{\mathrm{fl}} = SU_6$ and the fermions are assigned to the **56**, which is self-conjugate.

The restrictiveness of the exceptional groups, both in number and in internal structure, makes them quite attractive for model building. The universality of the quark and leptonic weak and electromagnetic charges is a consequence of the group structure, as is the 1/3 integral charge structure of the quarks if the leptons have charges ±1 and 0 only. The SU_3^c is naturally embedded in these groups (Günaydin and Gürsey, 1973; Gürsey, 1976).

The results of our classification are summarized in Table I. There are no other embeddings of SU_3^c in any simple G for which there is at least one representation satisfying the $\mathbf{1}^c$, $\mathbf{3}^c$, $\bar{\mathbf{3}}^c$ color restriction.

The classification of the fermion representations is not complete until we analyze their helicity structure. Fermions of a given chirality are transformed among themselves under G, which we continue to assume to be simple.

We first study the case where a scalar fermion number is defined, so that fermions and antifermions are initially distinguished. Suppose the left-handed fermions are assigned to $\mathbf{f}_L$ of G and the right-handed fermions to $\mathbf{f}_R$ of G. Then all left-handed states are in $\mathbf{f}_L + \bar{\mathbf{f}}_R$, and all right-handed states are in $\mathbf{f}_R + \bar{\mathbf{f}}_L$. If the quark-gluon couplings are to conserve parity, there must be a discrete symmetry that relates the quarks in $\mathbf{f}_L$ to those in $\mathbf{f}_R$, and also relates any antiquarks that may be in $\mathbf{f}_L$ to those in $\mathbf{f}_R$. This same discrete symmetry will relate the leptons in $\mathbf{f}_L$ and $\mathbf{f}_R$, if we ignore the possibility of adding G singlets to either $\mathbf{f}_L$ or $\mathbf{f}_R$. Consequently, $\mathbf{f}_L$ and $\mathbf{f}_R$ are either equivalent or else related by group conjugation. Theories in which $\mathbf{f}_R$ is equivalent to $\mathbf{f}_L$ are called vectorlike (Georgi and Glashow, 1973). If $\mathbf{f}_R$ is

TABLE I. Embedding of SU_3^c in G, representations with $1^c, 3^c, \bar{3}^c$ only of color. The bracket in case 5 means integer part.

Case	G	G^{fl}	f	Dimensionality
1.	SU_n	$SU_{n_1} \times SU_{n_3} \times U_1$	n	$n = n_1 + 3n_3$
2.	SU_n	$SU_{n-3} \times U_1$	$(\mathbf{n}^k)_A$	$\binom{n}{k}$
3.	SU_n	$SU_{n_1} \times SU_{n_3} \times SU_{n_{\bar{3}}} \times U_1 \times U_1$	n	$n = n_1 + 3n_3 + 3n_{\bar{3}}$
4.	SO_n	$SO_{n_1} \times SU_{n_3} \times U_1$	n	$n = n_1 + 6n_3$
5.	SO_n	$SO_{n-6} \times U_1$	σ, σ'	$2^{\left[\frac{n-1}{2}\right]}$
6.	Sp_{2n}	$Sp_{2n_1} \times SU_{n_3} \times U_1$	2n	$n = n_1 + 3n_3$
7.	F_4	SU_3	26	26
8.	E_6	$SU_3 \times SU_3$	27	27
9.	E_7	SU_6	56	56

equivalent to $\bar{f}_L$, we call the theory flavor chiral.[4]

We now examine the case where a scalar fermion number cannot be defined; f_L contains all the left-handed fermions and antifermions of the theory, and f_R contains all the right-handed ones. Parity conservation in the quark-gluon couplings then implies that the quarks in f_L and f_R are related by a discrete symmetry. As before the theory is either vectorlike or flavor chiral. In the latter case (f_R equivalent to $\bar{f}_L$), there exists a pseudoscalar quantum number that initially distinguishes f_L and f_R.

Further limitations on f_R given f_L follow from the renormalizability of the theory. The theory must not have divergences due to Adler (1969), Bell and Jackiw (1969) triangle anomalies. The fermion representation falls into one of three categories (Georgi and Glashow, 1973).

(1) If f_L is a self-conjugate representation, there will never be any problem with triangle anomalies. Such theories are always vectorlike.

(2) If f_L is a complex representation but G is not a unitary group, there is again no problem with triangle anomalies. These theories are based on $G=E_6$ (f = 27) or SO_{4n+2} (f = spinor),[5] and may be vectorlike or flavor chiral.

(3) The complex representations of SU_n ($n \geq 3$) are unsafe, but may be used in a vectorlike theory, or in a nonvectorlike theory if there is an accidental cancellation of right- and left-handed anomalies separately. (In the latter case f_R is equivalent to $\bar{f}_L$, where f_R is reducible.) When the cancellation does take place, f_L often appears as the branching of a safe representation of a larger group. For example, the anomalies from the $\bar{\mathbf{5}}$ and $\mathbf{10}$ of SU_5 cancel, where the decomposition of the spinor of SO_{10} into SU_5 representations is given as $\mathbf{1}+\bar{\mathbf{5}}+\mathbf{10}$ (Georgi, 1975). (The singlet does not contribute to the anomaly.) Such accidental cancellation is somewhat artificial and we do not consider examples of it.

The results of this classification of the chiral structure of the fermion representations are summarized in Table II. Except for gauge groups permitting a flavor chiral theory, it is most natural to assume a vectorlike theory.

In Sec. III, which we also treat as a mathematical appendix to this introduction, we have studied the possible embeddings of the U_1 generated by the electric charge operator in G^{fl}. Quarks are assumed to have charges in the sequence (... 5/3, 2/3, −1/3, −4/3, ...), and leptons to have integral charges. (This is a fairly weak assumption for Class IV embeddings.) The results are organized around the above classification of the structure of G^{fl}.

In the Class I embeddings ($G^{fl}=G_l \times G_q \times U_1$), there is great freedom in defining the electric charge operator Q. The only constraint arises from the tracelessness of all G^{fl} generators: the sum of all fermion charges must be zero. There is no restriction on Q when the fermion representation is self-conjugate under antiparticle conjugation. Results for Class II embeddings are similar.

TABLE II. Classification of chiral fermion representations.

Type of representation	$f_R \sim f_L$ (vectorlike)	$f_R \sim \bar{f}_L$ (flavor chiral)
Real	Possible	Identical to vectorlike
Complex safe[a]	Possible	Possible
Complex unsafe[a]	Possible	Usually not possible

[a] Safe and unsafe from anomalies.

In Class III embeddings ($G^{fl}=G_{q+l} \times U_1$) the quarks and leptons transform under the same simple subgroup[3] of G^{fl}, but the extra U_1 distinguishes $\mathbf{1}^c$ from $\mathbf{3}^c$. Nevertheless, the quark charges, assumed to be in the −1/3 + integer sequence, determine the lepton charges. Usually the sum of the quark charges does not vanish. A general parametrization of the charge operator is given in Sec. III.

In Class IV embeddings ($G^{fl}=G_{q+l}$), the sum of the quark charges is zero and the quark charges determine the lepton charges. Possible charge assignments are easily listed.

We turn now to the question of proton decay, which can often occur in unified theories since quarks and leptons both appear in the same irreducible representations of G, and there must be bosons and possibly other fields that mediate quark-lepton transitions. In some models the leptoquarks also couple quarks to antiquarks that are assigned to the fermion multiplet, permitting the proton to decay to meson plus lepton in second order even in the absence of symmetry violation. After the local symmetry G is broken to $U_1 \times SU_3^c$, the proton will certainly decay unless there remains a conservation law that prohibits it. Thus, if the proton is stable, there must be a conserved quantum number A (A is a generalized atomic mass number) that agrees with the usual definition for ordinary matter, and such that the lowest-mass state with $A=1$ is the proton. The problem of defining A in a Yang-Mills theory of quarks and leptons based on a simple G is detailed in Sec. IV. The notation used there is that of a vectorlike theory, but we show in the examples at the end of this section that the extension to flavor chiral theories is trivial. Again we provide a full discussion here, and treat Sec. IV as a mathematical appendix.

The experimental bound on the decay rate of the proton into a muon is $\tau^{-1} < 10^{-30}$/year (Reines and Crouch, 1974), and the half-life of the proton is at least as long as 2×10^{28} years (Gurr *et al.*, 1967). Many theorists have simply accepted the instability of the proton in lowest order and supposed that the responsible bosons have such high effective masses (order of the Planck mass, 1.22×10^{19} GeV) that the amplitude is reduced to an experimentally acceptable value.[6] This may not be a

[4] The name "flavor chiral" is appropriate for E_6 and SO_{10}, where the flavor groups are $SU_3 \times SU_3$ and $SU_2 \times SU_2$, respectively, and for the simplest flavor chiral assignment the factors act chirally on the quarks. For SO_{14}, SO_{18}, ..., this name is less appropriate. Of course, when a scalar fermion number cannot be defined, the word vectorlike is not really appropriate either, since even in the absence of fermion masses such theories cannot be described using Dirac spinors with ordinary vector coupling.

[5] The proof that E_6 is safe is carried out in Gürsey, Ramond, and Sikivie (1975).

[6] S. Meshkov has nicknamed these bosons "Intermediate Vector Baseballs."

problem in principle, since the mass scale associated with a universal strength for the weak and strong interactions may be on the order of $10^{15}-10^{20}$ GeV (Georgi, Quinn, and Weinberg, 1974), which is not so different from the boson effective masses needed to reduce the rate of proton decay sufficiently even when it occurs in lowest order. Of course there may exist a mechanism that prevents these boson masses from being arbitrarily large compared to those of the weak intermediate bosons (Gildener, 1976). We should discuss briefly suggestions that have been made for ways in which each of these mass scales might be reduced. We first discuss the mechanisms of proton decay and how its rate might be retarded. We then turn to the question concerning the unification mass scale.

In a theory with proton decay, there is no atomic mass number (or baryon number) A that is exactly conserved. The case we have just mentioned, in which the proton decay proceeds in the lowest order of pertubation theory and the responsible bosons must have gigantic masses, is typically one in which there is no gauged quantum number X that changes in proton decay and there is also no exactly conserved fermion number. The leptoquarks are also antidiquarks, and the decay proceeds directly in second order in g^2. If there is an exactly conserved fermion number, but still no gauged quantum number that is changed by proton decay, then the decay can be put off to a somewhat higher order of perturbation theory, but quite large boson masses are still needed. A typical example of this situation is provided by the group E_6, with the fermions put into **27** and $\overline{\mathbf{27}}$ and with fermion number conserved; the decay amplitude is then of order g^4 rather than g^2.

Sufficiently slow proton decay without very large spin 1 boson masses is somewhat less difficult to achieve if we do have a generator X of the group G that must be broken (by an explicit Higgs mechanism or a hypothetical dynamical Higgs mechanism) in order for the decay to take place. Let us first consider the case in which fermion number is exactly conserved. In our discussion below of exact proton stability, we suppose that the Higgs violation, which gives a mass to the spin 1 boson gauging X, also breaks fermion number but preserves a linear combination, which is then A. Here we imagine that the Higgs violation leaves fermion number alone; A can be defined in the same way but is no longer conserved. If we look at models with explicit Higgs bosons ϕ that accomplish this violation, we can assign them to various reducible representations of G or else to a large irreducible one, in which some component has a nonzero vacuum expected value that gives a mass to the spin 1 boson gauging X and at the same time induces, through virtual spin 1 bosons, an unrenormalizable effective coupling, with a finite calculable coefficient, that causes proton decay. Typically the parameters of the Higgs potential in ϕ space can be adjusted to make this decay very slow. (When renormalizable effective couplings are induced, then the coefficients are, in principle, arbitrary, and can only be estimated if we assume that a cutoff imitates whatever mechanism will eliminate the infinities in an improved theory.)

Pati and Salam (1973, 1974) have discussed this kind of slow violation of A conservation, although their theory differs from the ones we consider in that their G is not simple and their quarks are not confined and not fractionally charged. We can, however, as an example, replace their theory of four quark flavors and four lepton flavors by one in which the gauge group G is SU_{16} and the spin 1/2 fermions are put into **16** and $\overline{\mathbf{16}}$, with conservation of fermions. The necessary Higgs violation to produce a slow proton decay can then be accomplished by ϕ fields in various reducible representations of G or else by ϕ's in an irreducible (299, 200-dimensional!) representation. The ϕ component with nonzero expected value in the vacuum can then couple to three gauge bosons that convert three quarks into three leptons, ultimately inducing proton decay.

We have seen that the treatment of slow proton decay with a conserved fermion number resembles the discussion below of proton stability in which A is defined as a linear combination of fermion number and X. We now consider the final case of retarded proton decay, where there is no fermion number but there is a generator X that would prohibit the decay if it were exactly conserved, which it is not. That case resembles the treatment below of proton stability when there is no fermion number and X coincides with A in the spin 1/2 sector of the theory. Here, instead of taking a conserved linear combination of X and some quantity that is nonzero outside the spin 1/2 sector, we just allow X to equal A and to be nonconserved. Again the dynamical or explicit Higgs bosons ϕ can be put into suitable representations of G that allow them to give a mass to the spin-1 boson gauging X and to induce the proton decay.

Although the coupling constants vary logarithmically with mass, the mass scale where the ratios of the electromagnetic and strong coupling constants become nearly equal to unity (that is, unification takes place) need not be anywhere near as large as the Planck mass. We give a brief review of the dependence of the unification mass on the theory. Let us first suppose that the symmetry violation occurs in only two stages, from G to $SU_2 \times U_1 \times SU_3^c$, and then to $U_1 \times SU_3^c$. The unification mass may be computed from the renormalization group equations. Georgi, Quinn, and Weinberg (1975) have found that it is sensitive to the electric charge and weak isospin assignments made in the theory; in some examples it may be as low as ~10^9 GeV, but in others it is much larger than the Planck mass.

Particularly low unification masses can be achieved when the symmetry breakdown is a multistage process. This possibility has been considered by Fritzsch and Minkowski (1975), although their examples have the defect of violating quark-lepton universality in the early stages of the symmetry breakdown. We avoid that difficulty in the following example. Suppose that in the first stage G breaks to $G_A \times G_B$, where the index of the adjoint representation of G_A is much larger than that of G_B. (The index is the Casimir operator of the group for a given representation times the dimensionality of the representation divided by the number of generators. The coefficient occurring in the renormalization group equations is 11/3 times the index of the adjoint minus 4/3 times the sum of the indices of all Dirac fermion representations.) G_B contains at least the minimal flavor group $SU_2 \times U_1$ and G_A contains SU_3^c. The coupling constants for these factors G_A and G_B will vary at different rates as the renormalization mass further decreases, with the growth of the coupling of G_A accelerated over that of SU_3^c alone. In due course G_A breaks to SU_3^c, but only after the coupling of G_A has increased to a value much larger than the coupling for G_B. Thus, in a $2n$-lepton, $2n$-quark model based on $G = SU_{8n}$, the first stage of symmetry violation could break SU_{8n} into $G_B = SU_n \times SU_2 \times U_1$ and $G_A = SU_{3n}$. At some smaller mass, the SU_{3n} could break to $SU_3^c \times SU_n$. At GeV energies, the two SU_n groups could be inconspicuous, coupling light to heavy quarks and light to heavy leptons, leaving $SU_2 \times U_1 \times SU_3^c$.

If the theory does contain mass scales of order of the Planck mass, then the unification procedure followed here, which omits gravitational processes, would be incomplete. It is sometimes suggested that it might be necessary to consider processes that appear to violate baryon number, such as the hypothetical radiation of all the energy of a small black hole by the Hawking (1975) effect. If quantum gravitation really permits such a process, then some kind of instability of the proton might be implied, but the whole matter is poorly understood. A more direct unification of Yang-Mills theory with Einsteinian gravitation is afforded by theories of extended supergravity, as discussed below.

Although there is no clear need for requiring an absolutely stable proton, one could try to impose exact proton stability and examine the consequences, pleasant or unpleasant. We do this here, ignoring the possibility that some kind of topological quantum number not apparent in the Lagrangian provides an atomic mass number. We examine the situations of dynamical symmetry breaking in some detail, and of explicit Higgs breaking briefly.

How may we obtain a conserved quantity A in a unified gauge theory? Suppose the Lagrangian contains only spin 1/2 and spin 1 fields, and the symmetry breakdown is dynamical. Then a single self-conjugate irreducible representation for the fermions precludes proton stability, since a conserved group generator X, if it survived the dynamical symmetry breakdown, would be associated with a massless vector boson and could not be used for A. Electric charge and color conservation are not enough to guarantee proton stability and we have already assumed there will be no accidental conservation laws. A pair of conjugate representations allows an ungauged conserved quantity Z to be defined. (For complex representations, this doubling is required by CPT.) The quantity Z cannot be A since the quarks and leptons would all have the same value of A. However we can have A conservation with Z and X (a local symmetry generator associated with a vector current) both broken and with some linear combination conserved. The boson becomes massive, and quarks and leptons can be distinguished. Since we do not explicitly calculate A from the broken theory, our choice of the local generator X presumes the appropriate symmetry-breaking patterns. If we had considered a reducible fermion representation with more complicated content, it would also be feasible to have, for example, exact lepton conservation. We do not analyze this possibility here. In any event, we assume that the eigenvalue spectrum of A corresponds to a sensible atomic mass number.

The ordinary quarks must have $A = 1/3$ so that the known baryons, which are composed of three of those quarks, all have $A = 1$. Similarly the known leptons must have $A = 0$. Then A will be the usual baryon number for known matter. Unified theories also often predict other quarks and leptons. It is natural to require that their A values are constrained so color singlet states have integral values of A. Otherwise, we would need to accept the more drastic prediction of observable fractional A, including new stable particles corresponding to the lowest mass state for each allowed fractional value. Consequently, the A value of a new quark falls in the sequence, 1/3 plus integer, and new leptons have integral A. We call quarks with $A = \ldots, -5/3, -2/3, 4/3, 7/3, \ldots$ weird quarks, and leptons with $A = \pm 1, \pm 2, \ldots$ weird leptons. These weird fermions should be heavy enough to have escaped being observed, but otherwise pose no problems other than lack of economy. Weird baryons would contain at least one weird quark, and have integral A, not equal to one. Weird mesons are $q\bar{q}$ states with A a nonzero integer.

The exchange of bosons carrying color and flavor would be needed in the mechanism for producing a single weird particle. The color singlet bosons and the octet of gluons all carry $A = 0$, but some of the bosons carry 1/3 integral values of A. Two ordinary quarks could exchange such a boson, transforming one quark into a lepton with $A = 0$ and the other quark into a weird antiquark with $A = 2/3$, which could then combine with an ordinary quark to form a weird meson with $A = 1$. If the boson masses are sufficiently large, this process would be suppressed. However, at sufficiently high energies pair production of weird particles would occur.

Weird particle decay does involve the exchange of a boson carrying $A = 1/3$ and would be extremely slow if its mass were large. As examples, a weird meson with $A = 1$ composed of a quark with $A = 1/3$ and a weird antiquark with $A = 2/3$ would have a lepton–baryon decay mode, and a weird baryon with $A = 0$, composed of two quarks with $A = 1/3$ and a weird quark with $A = -2/3$, might decay into three ordinary leptons, or into an ordinary meson and a lepton. We will be more specific in the examples.

With dynamical symmetry breaking there is only one case where proton stability does not imply weird particles: $G = SU_n$ and the fermions in the **n**. But A conservation may be imposed in any other gauge theory, as long as **f** is large enough to include weird fermions. (In this regard F_4 and E_6 are flavor poor, if the fermions are assigned to only a few irreducible representations.) Table III lists the A values that are implied by a simple choice of the local generator, as discussed in Sec. IV.

The situation of explicit Higgsism, although perhaps unattractive for nonsupersymmetric theories, does provide other solutions for the spectrum of A. Suppose the scalar and pseudoscalar fields imply new ungauged symmetries of the Lagrangian. As before A can be constructed by taking the linear combination of a broken local and global generator that is conserved. For simplicity we restrict our attention to fermions in a single irreducible self-conjugate representation of G, so that

TABLE III. Some examples of proton stabilization, with fermions doubled, and Majorana mass breaking. The general solutions are given in Sec. IV; these simple examples provide the most economical A spectra.

G	Representation	(Equation)	U_1 for X	A
SU_n	n	(2.4)	Explicit U_1	$\frac{1}{3}N_q$
SU_n	$(n^k)_A$	(2.11)	Explicit U_1	$\frac{1}{3}N_q-\frac{2}{3}N_r+N_L$
SO_n	n	(2.17)	Flavor raid ($SO_{2n'}\supset SU_{n'}\times U_1$) plus explicit U_1	$\frac{1}{3}N_q+\frac{4}{3}N_r+N_L$
SO_n	σ	(2.22) (2.24) (2.25)	Explicit U_1	$\frac{1}{3}N_q-\frac{2}{3}N_r+N_L$
Sp_{2n}	2n	(2.26)	Flavor raid ($Sp_{2n_1}\supset SU_{n_1}\times U_1$) plus explicit U_1	$\frac{1}{3}N_q+\frac{4}{3}N_r+N_L$
E_7	56	(2.36)	Flavor raid $SU_6\supset SU_5\times U_1$	$\frac{1}{3}N_q+\frac{1}{3}N_\sigma+\frac{4}{3}N_s-\frac{2}{3}N_r+N_L$

the global generator Z is zero for the fermions. Then the local generator alone is A for the fermions.

The phenomenology of weird particle production and decay is similar to before, except that the exchange of scalar particles that carry both flavor and color should be included. Weird particles can be completely avoided in the **n** of SO_n and the **2n** of Sp_{2n}, but otherwise some weirdness is also needed in this symmetry-breaking scheme. Again, the details are worked out in Sec.IV.

The embedding of SU_3^c, electromagnetic U_1, and the local U_1 for stabilizing the proton are all easily extended to $G\times G$ theories. Color can be embedded in $G\times G$ in essentially two ways.

SU_3^c may be explicitly contained in just one G factor, and the two G factors are transformed into one another by some abstract reflection principle. The $SU_4\times SU_4$ model of Pati and Salam (1974) is an example. The fermion representations are of the form, $(\mathbf{r}_1, \mathbf{r}_2)$. If the second G factor contains SU_3^c, then $\mathbf{r}_2$ must be one of the representations in Table I, and there is no restriction on $\mathbf{r}_1$. Such models are similar in spirit to the ones already discussed, so we do not consider them further.

The other possibility, to be considered in more detail, is that SU_3^c is generated by the sum of generators of SU_3 subgroups of each G factor. The color content of a $G\times G$ representation $(\mathbf{r}_1, \mathbf{r}_2)$ then contains the color representations in $\mathbf{r}_1\times\mathbf{r}_2$. The restriction that $(\mathbf{r}_1, \mathbf{r}_2)$ have at most $\mathbf{1}^c$, $\mathbf{3}^c$, and $\overline{\mathbf{3}}^c$ then implies that $\mathbf{r}_1$ or $\mathbf{r}_2$ must be an SU_3 singlet. This in turn imposes the requirement that $\mathbf{r}_1$ be a G singlet and $\mathbf{r}_2$ be one of the representations in Table I, or vice versa. The only possible fermion representations are then linear combinations of $(\mathbf{1}, \mathbf{r}_2)$ and $(\mathbf{r}_1, \mathbf{1})$.

We first consider the chiral structure of the fermion representation for the case where the reflection symmetry between the two G factors is *not* parity. If $\mathbf{f}_L$ has the form $(\mathbf{r}_1, \mathbf{r}_2)$, then $\mathbf{f}_R$ transforms as $(\mathbf{r}_1, \mathbf{r}_2)$ or $(\overline{\mathbf{r}}_1, \overline{\mathbf{r}}_2)$. We simply obtain the vectorlike and flavor chiral theories, respectively, already summarized in Table II. In this case there is an $SU_3\times SU_3$ of vector currents and there is the temptation of enlarging SU_3^c to $SU_3\times SU_3$. Although this way of enlarging the color group differs from Case 3 in Sec.II, that discussion is still relevant. Otherwise, these $G\times G$ theories are similar to the G-simple theories already discussed.

Finally, we may suppose that parity is the reflection symmetry between the G factors, although this possibility raises several questions. If $\mathbf{f}_L$ is $(\mathbf{r}_1, \mathbf{r}_2)$, $\mathbf{f}_R$ must be $(\mathbf{r}_2, \mathbf{r}_1)$ or $(\overline{\mathbf{r}}_2, \overline{\mathbf{r}}_1)$: the first case is chiral, and the second case is conjugate chiral. Both $\mathbf{r}_1$ and $\mathbf{r}_2$ must be safe representations of G for the triangle anomalies to be absent, but safety through an accidental cancellation is a possibility as above. For simplicity we still ignore Class II theories in this connection.

The strong gauge group in these $G\times G$ theories is a chiral $SU_3\times SU_3$. There are several objections to considering such theories. First, the chiral quark-gluon theory by itself has anomalous divergences, which are cut off only at a very large energy by the unified theory. As always, we assume that the unbroken SU_3^c is generated by the sum of the corresponding SU_3 generators. The axial symmetries, generated by the differences, must be broken, and we may characterize the violation by a kind of effective mass-squared m_A^2 for the axial gluons inside the hadrons. This parameter m_A^2 must be at least several $(\mathrm{GeV})^2$ in order to avoid quark-quark spin-spin forces of the wrong sign and other unsuitable forces among quarks and antiquarks. When flavor radiative corrections are put in, there may also be parity-violating amplitudes of order $\alpha_{\mathrm{strong}}\,\alpha_{\mathrm{weak}}\,m_A^{-2}$ that require m_A^2 to be even larger. For all these reasons, we shall not pay any more attention to chiral $G\times G$ theories.

The enumeration of possible charge assignments for $G\times G$ theories is identical to that for the G-simple theories, since $\mathbf{r}_1$ or $\mathbf{r}_2$ must be a G singlet. The electric charge operator is simply the sum of generators of two corresponding U_1's from the G^{fl} subgroups of each G factor. The same argument holds for extracting X for stabilizing the proton. Thus we restrict our considerations to simple gauge groups, while recognizing that the results are valid for $G\times G$ theories.

Although some of the theories included in our classification are quite attractive, we find none so appealing that it should blind us to promising lines of development that have been excluded here. We consider briefly two of the more popular directions of research: (1) enlarging the global symmetry structure of the Lagrangian; (2) enlarging the local symmetry structure. In both cases, the Lie group symmetry of the Lagrangian is extended to a "graded" or "super" symmetry. The supersymmetry operators, which are adjoined to the Lie algebra, satisfy anticommutation relations with one another, and commutation relations with the members of the Lie algebra. They transform bosons into fermions, so that irreducible supermultiplets must contain both.

We first consider ordinary global supersymmetry (Golfand & Likhtman, 1971; Wess and Zumino, 1974) in direct product with a Yang-Mills internal symmetry (Salam and Strathdee, 1975; Ferrara and Zumino, 1974). The adjoint representation of gauge bosons is then accompanied by an adjoint representation of Majorana spin 1/2 fermions, which must include 8^c. Once color octet fermions are required, it is not so natural to demand that the fundamental representation of G be restricted to 1^c, 3^c, and $\bar{3}^c$ only. However, in any new embeddings, the flavor singlet set of generators would be larger than one or two 8's; thus there would be many new possibilities for the color group, and there would also be a further proliferation of fermions with new color representations. We conclude, therefore, that a globally supersymmetric theory with internal symmetry treated by means of a direct product would probably involve only embeddings treated in this article; the principal change would be to include a set of spin 1/2 Majorana fermions belonging to the adjoint representation, which is fully treated here for the description of the spin 1 bosons.

If the only supermultiplet considered is the one with spin 1 and spin 1/2 belonging to the adjoint representation of G, then the symmetry breakdown must be dynamical. However, it is not so ugly to introduce a supermultiplet of matter fields consisting of Majorana fermions, scalars, and pseudoscalars transforming as some representation of G, which could well be the adjoint and/or one of the representations discussed here for the spin 1/2 fermions. In that case, the Lagrangian is available for inspection, and in reality it is hard to break both the supersymmetry degeneracies and the internal symmetry degeneracies simultaneously as needed (Fayet and Iliopoulos, 1974). If these schemes do work, there will be a Goldstone fermion, which is not the neutrino since it decouples in the zero frequency limit (Freedman and de Wit, 1975; Bardeen, 1975).

The direct product is not the only way to combine "internal" symmetry with supersymmetry. We can have N-fold extended supersymmetry (Salam and Strathdee, 1974) in which there are N supersymmetry operators belonging to the vector representation of the "internal" group SO_N or SU_N. The supermultiplets then contain particles with more and more different spin values as N increases. If the group is made large enough to be the gauge group G of a unified Yang-Mills theory, then one is led to spins of 3/2 and higher and severe difficulties are then encountered in a theory with purely global extended supersymmetry.

We may consider, however, a case in which N is not very large and we do not attempt to gauge the SO_N or SU_N symmetry of extended supersymmetry by means of a Yang-Mills theory. Rather, we consider a direct product of extended supersymmetry and a Yang-Mills symmetry group G. Let us take the largest value of N that allows us to avoid spin 3/2 particles in the supermultiplet containing the Yang-Mills fields, namely $N=4$ (Fayet, 1976; Brink, Schwarz, and Scherk, 1977). One then has a set of vector particles in the adjoint representation of G, as well as four sets of Majorana spin 1/2 particles and three sets each of scalar and pseudoscalar particles, also in the adjoint representation. For such a case, as for the direct product of G with ordinary supersymmetry, our embeddings may well be sufficient and the adjoint representation, described in our work for the spin 1 bosons, can be used also for the spin 1/2 fermions and the spin 0 bosons, in the requisite number of copies. The renormalization group for this type of theory has the fascinating property that the leading g^4 term in the Gell-Mann–Low function $\psi(g^2)$ vanishes (Ferrara, 1976). It has been remarked (Abbott *et al.*, 1977; Curtright, 1977) that as a consequence the leading term in the anomalous divergence of the supercurrent (ignoring supergravity) also vanishes. Very recently (Jones, 1977, and Poggio and Pendleton, 1977) calculations of $\psi(g^2)$ have been extended to order g^6 using the method of Jones (1975) and it has been shown that the scale covariance $\psi(g^2)=0$ persists to that order, a fascinating result of unknown significance.

We shall return to this class of theories below, merely remarking here that they have a global $SU_4 \approx SO_6$ symmetry, and if such theories are to be useful, then we must learn how that symmetry can be spontaneously violated without contradicting experience.

Now let us discuss the treatment of gravitation in supersymmetry theory. The graviton must belong to a supermultiplet and one is immediately led to local supersymmetry or supergravity (Freedman, van Nieuwenhuizen, and Ferrara, 1976; Deser and Zumino, 1976) in which the graviton, which gauges the Poincaré group, is accompanied by one or more spin 3/2 particles that gauge supersymmetry or extended supersymmetry. (These massless spin 3/2 particles may eat the Goldstone fermions and become massive.) Plain $N=1$ supergravity has just a single spin 3/2 particle forming a supermultiplet with the graviton and is compatible with the introduction of direct-product internal symmetry as described above, with spin 1 and spin 1/2 particles in the adjoint representation of some group G and possibly spin 1/2, scalar, and pseudoscalar particles in some representation of G. However, such a theory is not renormalizable. (In any event, the applicability of our work would be the same as discussed above for the use of global supersymmetry.)

The most interesting type of theory so far proposed is probably extended supergravity (with $2 \leq N \leq 8$), especially the version with $N=8$, in which a single spin 2 graviton is accompanied by an SO_8 octet of spin 3/2 particles, a set of spin 1 particles belonging to the 28-dimensional adjoint representation of SO_8, a set of Majorana spin 1/2 particles belonging to the **56** of SO_8, and scalar and pseudoscalar multiplets in two 35-dimensional representations of SO_8.

The coupling involving Newton's constant is presumably supplemented by a dimensionless coupling that includes Yang-Mills behavior of the spin 1 bosons, which would thus gauge the group SO_8. The dimensionless coupling induces a cosmological term in Einstein's gravitational equation that is much too large; that is a difficulty unless some spontaneous violation of symmetry permits the cosmological term to be canceled almost completely. But the major problem with SO_8 supergravity is that SO_8 is too small a group to include color SU_3 times a sufficiently large flavor group as a subgroup.

One may speculate about a future unified field theory of all interactions and all elementary particles that would resemble SO_8 supergravity but involve sacrificing some principle now held sacred, so that the notion of extended supergravity could be generalized. In such a hypothetical theory, an internal symmetry group G larger than SO_8 would be gauged by spin 1 bosons, and both the spin 3/2 and spin 1/2 fermions would be assigned to representations of G. It is then very natural to suppose that the spin 3/2 fermions would belong to some basic representation of G and would include only color singlets, triplets, and antitriplets. In that case our embeddings of SU_3^c in G would be exhaustive and the representations that we study in this article for assignment to spin 1/2 fermions would be precisely relevant for the spin 3/2 fermions. (The spin 1/2 particles would presumably be assigned to more complicated representations.) The possibility that our review might be used in this way is an exciting one, although highly speculative, and has encouraged us to prepare it for publication after some years of delay.

Another fascinating possibility for a unified theory involves the direct product of $N=4$ extended supersymmetry with a gauged internal-symmetry group G, as described above in the global case. We introduce local $N=4$ supersymmetry; the graviton then belongs to a supermultiplet along with four spin 3/2 particles, six vector particles, four Majorana spin 1/2 particles, a scalar, and a pseudoscalar. This supermultiplet is coupled to itself with Newton's constant. The generalized Yang-Mills "matter field" discussed above is then introduced, containing one vector particle, four Majorana spin 1/2 particles, three scalars, and three pseudoscalars for each component of the adjoint representation of G. This "matter field" is then coupled to the gravity supermultiplet by Newton's constant and to itself by a dimensionless charge. The resulting theory is then no doubt unrenormalizable (like the $N=1$ supergravity theory with added matter multiplets mentioned earlier). However, a theory like the one we are discussing, at least for $G=SU_n$ for any n, can be obtained as an approximation, for energies small compared to the Planck mass, to the remarkable ten-dimensional string theory of Scherk and Schwarz (1974, 1975), in which six dimensions are "compactified" into a tiny ball so as to be physically insignificant at any reasonable energy, leaving effectively a four-dimensional theory. [For the original theory in 10 actual dimensions, see Ramond (1971) and Neveu and Schwarz (1971).] All elementary particles lie on Regge trajectories with a slope α' of the order of Newton's constant for the closed strings (giving the gravitational supermultiplet) and a slope $2\alpha'$ for the open strings (giving the particles of the matter field). The string version of the theory has been shown (Gliozzi *et al.*, 1977) to be free of ghosts and tachyons and may well be renormalizable. It does not have the difficulty of introducing a cosmological term into Einstein's gravitational equation, nor does it have to make use of an internal group G that is too small. Given G, our embeddings are probably the relevant ones, and we must utilize the adjoint representation, listed here for the spin 1 bosons, also for the spin 1/2 Majorana fermions (four times) and for the scalar and pseudoscalar bosons (three times each).

We conclude this introduction with some examples of unified models. Our main purpose is to illustrate the color embeddings, charge assignments, and the definition of an atomic mass number; the detailed phenomenology of many models is available in the literature and not reviewed here. There are five examples corresponding to Cases 1, 4, 2, 5, and 9, respectively (see Table I). In our examples we examine in some more detail the implications of the possible requirement that the proton be absolutely stable. Although we assume the gauge group is broken down to $U_1^{em}\times SU_3^c$, we look only at the details of the symmetry breaking needed for proton stabilization. We should emphasize that Secs. II and III also apply to proton unstable theories, and models like the one based on E_6 should not be discarded simply because the proton stable version must be fairly complicated. Much that we say here is summarized in Tables IV–VIII.

As a first example, we briefly discuss the familiar SU_n models, where the flavor group is identified in Eq. (2.7), the vector bosons are listed in Eq. (2.6), and the n_1 leptons and n_3 quarks are assigned to $\mathbf{n}$, with $n=n_1+3n_3$ in Eq. (2.4). The antifermions are assigned to $\bar{\mathbf{n}}$, and the Lagrangian has an additional global symmetry generated by fermion number. Since the $\mathbf{n}$ is a complex representation of SU_n, this theory must be vectorlike to avoid triangle anomalies; see, for example, Fritzsch, Gell-Mann, and Minkowski, 1975; Kingsley, Treiman, Wilczek, and Zee, 1975; de Rújula, Georgi, and Glashow, 1975; and Pakvasa, Simmons, and Tuan, 1975.

It is commonly assumed that quarks and leptons occur in SU_2 doublets with the standard charge assignment. Then for each lepton doublet with charges 0 and −1, there must be a quark doublet with charges 2/3 and −1/3 for the sum of the fermion charges to be zero. This version of the SU_n model has a purely vectorial weak neutral current. Of course many other SU_2 (and even charge) assignments are possible and may agree better with experiment.

Most of the physical content of this theory is bound up in the symmetry breakdown; not even the universality of the weak and electromagnetic couplings of quarks and leptons is present in the unbroken Lagrangian, since different sets of flavor bosons couple to quarks and to leptons. (See Table IV.) The proton is stable in the unbroken theory because of the conservation of the explicit U_1 in G^{f1} and of fermion number. But the explicit U_1 must be broken to avoid an unwanted massless boson. If we assume that the broken theory conserves only electric charge, fermion number and color, then the process $qq\to lll$ is allowed.

There is no reason *a priori* that the symmetry violation should not break fermion number. For example, a Majorana neutral lepton mass term breaks both fermion number and the explicit U_1 in G^{f1}, but does not break the

TABLE IV. Boson–fermion couplings in SU_n models with fermions in $\mathbf{n}$. The table lists the vector bosons in the unbroken theory that provide the $a \to b +$ boson transition, a and b are fermions. See Eqs. (2.4) and (2.6). The A values for the proton-stabilizing symmetry breaking discussed in the text are also given.

$a \backslash b$	l $(n_1, 1, 1^c)$ $A=0$	q $(1, n_3, 3^c)$ $A=1/3$
l $(n_1, 1, 1^c)$ $A=0$	$(n_1^2-1, 1, 1^c)$ $+ (1, 1, 1^c)$ $A=0$	$(n_1, \bar{n}_3, \bar{3}^c)$ $A=-1/3$
q $(1, n_3, 3^c)$ $A=1/3$	$(\bar{n}_1, n_3, 3^c)$ $A=1/3$	$(1, n_3^2-1, 1^c) + (1, 1, 1^c)$ $+ (1, 1, 8^c) + (1, n_3^2-1, 8^c)$ $A=0$

linear combination of the generators, $A = N_q/3$, which implies that A is conserved. The leptoquarks, assigned to $(\bar{n}_1, n_3, 3^c)$ of Eq. (2.6), then carry $A = 1/3$. The symmetry-breaking terms needed for $qqq \to lll$ must all vanish in this case, since they would carry nonzero values of A. This specific class of SU_n models is summarized in Table IV.

Our second example involves several choices. We assign the fermions to the $\mathbf{n}$ of SO_n, which is a self-conjugate representation, and the theory is vectorlike. Suppose that all fermions are assigned to a single $\mathbf{n}$, Eq. (2.17), that there is no additional global U_1, and that the explicit local U_1 is broken. As long as electric charge and color are the only conserved quantities, the 3^c boson-mass eigenstates will be mixtures of the $(n_1, n_3, 3^c)$ and $[1, \frac{1}{2}n_3(n_3+1), 3^c]$ bosons in Eq. (2.18). Then the leptoquark is also an antidiquark, and its exchange can, in general, cause the transition $qqq \to q\bar{q}l$, so that the proton is unstable in second order.

If we want to stabilize the proton, we must decide whether to obtain the global U_1 from some other sector of the theory or to double the fermion representation, thereby providing a fermion number. In the former case it is possible to obtain a scheme with no weird particles; see the discussion around Eq. (4.28). In the latter, the generator of the relevant local U_1 must have a nonzero value for leptons, which requires going beyond the explicit U_1. For simplicity we limit this example to n even and extract a second U_1 from $SO_{n1} \supset SU_{n'} \times U_1$ where $n' = n_1/2$. We refer to the procedure of taking part of the U_1 from a non-Abelian factor of G^{fl} as a "flavor raid." The local U_1, generated by the sum of this U_1 generator and the explicit U_1 generator, will be broken together with fermion number [see Eq. (4.12) and the discussion beneath]. The most economical scheme as far as weirdness is concerned has leptons with $A=0$ and 1, and quarks with $A=1/3$ and $-2/3$. The A values of the vector bosons are easily worked out from Eqs. (2.17) and (2.18), as is the phenomenology of weird decays. (The latter is similar to some examples worked out below.) This scheme is summarized in Table V.

As a third example, let us look at a Class III SU_n model (Case 2). For this example, we give a simple discussion of the phenomenology of weird fermions. The summary of this model for arbitrary n is contained in Table VI, but for definiteness we analyze a vectorlike SU_8 model where the fermions are assigned to the $\mathbf{56} = (\mathbf{8}^3)_A$, which is complex. The flavor group is $SU_5 \times U_1$, Eq. (2.10), and the Lagrangian possesses a global U_1 symmetry, which is generated by fermion number. The $SU_5 \times SU_3^c$ decomposition, Eq. (2.11), of the 56 of fermions is

$$56 = (\overline{10}, 1^c) + (10, 3^c) + (5, \bar{3}^c) + (1, 1^c),$$

while the vector bosons are assigned to the adjoint 63, Eq. (2.9)

$$63 = (24, 1^c) + (1, 1^c) + (1, 8^c) + (\bar{5}, 3^c) + (5, \bar{3}^c).$$

TABLE V. Boson–fermion couplings in the SO_{2n} model. Representations are decomposed under $SO_{2n} \supset SU_{n'} \times U_1 \times SU_{n_3} \times U_1 \times SU_3^c$, $n = n' + 3n_3$. The local generator used in defining A is a linear combination of the two U_1's explicit in this decomposition.

$a \backslash b$	$\bar{l}$ $(\bar{n}', 1, 1^c)$ $A=0$	q $(1, n_3, 3^c)$ $A=1/3$	$\bar{r}$ $(1, \bar{n}_3, \bar{3}^c)$ $A=2/3$	L $(n', 1, 1^c)$ $A=1$
$\bar{l}$ $(\bar{n}', 1, 1^c)$ $A=0$	$(n'^2-1, 1, 1^c) + 2(1, 1, 1^c)$ $A=0$	$(\bar{n}', \bar{n}, \bar{3}^c)$ $A=-1/3$	$(\bar{n}', \bar{n}_3, 3^c)$ $A=-2/3$	$\overline{[\frac{1}{2}n'(n'-1), 1, 1^c]}$ $A=-1$
q $(1, n_3, 3^c)$ $A=1/3$	$(n', \bar{n}_3, 3^c)$ $A=1/3$	$(1, n_3^2-1, 1^c) + (1, 1, 1^c)$ $+ (1, 1, 8^c)$ $A=0$	$[1, \frac{1}{2}n_3(n_3+1), \bar{3}^c]$ $+ [1, \frac{1}{2}n_3(n_3-1), 6^c]$ $A=-1/3$	$(\bar{n}', n_3, 3^c)$ $A=-2/3$
$\bar{r}$ $(1, \bar{n}_3, \bar{3}^c)$ $A=2/3$	$(n', \bar{n}_3, \bar{3}^c)$ $A=2/3$	$\overline{[1, \frac{1}{2}n_3(n_3+1), 3^c]}$ $+ \overline{[1, \frac{1}{2}n_3(n_3-1), \bar{6}^c]}$ $A=1/3$	$(1, n_3^2-1, 1^c)$ $+ (1, 1, 1^c) + (1, 1, 8^c)$ $A=0$	$(\bar{n}', \bar{n}_3, \bar{3}^c)$ $A=-1/3$
L $(n', 1, 1^c)$ $A=1$	$[\frac{1}{2}n'(n'-1), 1, 1^c]$ $A=1$	$(n', \bar{n}_3, \bar{3}^c)$ $A=2/3$	$(n', n_3, 3^c)$ $A=1/3$	$(n'^2-1, 1, 1^c)$ $+ 2(1, 1, 1^c)$ $A=0$

TABLE VI. Boson–fermion couplings in Class III SU_n models. The A assignments are made assuming that X generates the explicit U_1 in Eq. (2.10).

$a \setminus b$	$\bar{l}$ $((\mathbf{n}_1^k)_A, 1^c)$ $A=0$	q $((\mathbf{n}_1^{k-1})_A, 3^c)$ $A=1/3$	$\bar{r}$ $((\mathbf{n}_1^{k-2})_A, \bar{3}^c)$ $A=2/3$	L $((\mathbf{n}_1^{k-3})_A, 1^c)$ $A=1$
$\bar{l}$ $[(\mathbf{n}_1^k)_A, 1^c]$ $A=0$	$(\mathbf{n}_1^2-1, 1^c)+(1,1^c)$ $A=0$	$(\mathbf{n}_1, \bar{3}^c)$ $A=-1/3$	None	None
q $[(\mathbf{n}_1^{k-1})_A, 3^c]$ $A=1/3$	$(\bar{\mathbf{n}}_1, 3^c)$ $A=1/3$	$(\mathbf{n}_1^2-1, 1^c)+(1,1^c)+(1,8^c)$ $A=0$	$(\mathbf{n}_1, \bar{3}^c)$ $A=-1/3$	None
$\bar{r}$ $[(\mathbf{n}_1^{k-2})_A, \bar{3}^c]$ $A=2/3$	None	$(\bar{\mathbf{n}}_1, 3^c)$ $A=1/3$	$(\mathbf{n}_1^2-1, 1^c)+(1,1^c)+(1,8^c)$ $A=0$	$(\mathbf{n}_1, \bar{3}^c)$ $A=-1/3$
L $[(\mathbf{n}_1^{k-3})_A, 1^c]$ $A=1$	None	None	$(\bar{\mathbf{n}}_1, 3^c)$ $A=1/3$	$(1, 1^c)$ $A=0$

There are 25 flavor bosons, including the photon, and three known weak bosons. The other bosons, which transform as $(\bar{5}, 3^c)$ and $(5, \bar{3}^c)$, mediate transitions between $\bar{l}$ in $(\overline{10}, 1^c)$ and q in $(10, 3^c)$, between q and $\bar{r}$ in $(5, \bar{3}^c)$, and between $\bar{r}$ and the singlet "L" lepton. (See Table VI.)

The electric charge assignments are easily cataloged from Eqs. (3.3) and (3.4). It is rather natural to choose the standard charges since it is the only assignment in which the charge spread of the l's and q's is less than 4, the lepton charges are less than 3, and there is more than one neutral lepton. Then six l leptons have $Q=-1$ and the remaining four have $Q=0$; four of the q quarks have $Q=2/3$, and six have $Q=-1/3$; four r quarks have $Q=2/3$ and the other one has $Q=-1/3$; and the L lepton has $Q=-1$. Note that the L and $\bar{l}$ leptons cannot be mixed by symmetry breaking, since they have no charges in common.

If there is an $\bar{r}q$ meson less massive than the proton, then the proton may decay in second order into this meson plus a lepton. If the L is light enough, the proton can decay into three leptons, $\bar{l}\bar{l}L$. But if the light quarks are all of q-type, and the light leptons are of l-type, then it is perhaps natural to suppose that the $(\bar{r}q)$ mesons and also L leptons would be heavier than the proton. In this case, the proton is guaranteed to be stable until the explicit U_1 of Eq. (2.10) is broken. Moreover, if fermion number is not finally broken, the proton decay to three light leptons does not violate the remaining conservation laws, although this process could be very slow.

The proton can, of course, be stabilized by breaking fermion number together with a U_1 in G^{fl}. The simplest solution uses the explicit U_1; the general solution is given in Eqs. (4.11) and (4.12). We avoid making a flavor raid in this example, so that the $\bar{l}$ leptons then have $A=0$; the q quarks have $A=1/3$, the r quarks have $A=-2/3$; and the L lepton has $A=1$. [See Eqs. (4.7)–(4.10).] The model, A assignments, and boson couplings are summarized in Table VI.

In order to effect transitions from weird to ordinary fermions, a boson carrying A must be involved. The bosons in the $(\bar{5}, 3^c)$ carry $A=1/3$, those in the $(5, \bar{3}^c)$ carry $A=-1/3$, and the remaining bosons have $A=0$. The $(\bar{5}, 3^c)$ bosons mediate the transition $r \to \overline{lqq}$ in second order. Thus a weird baryon with $A=0$ composed of rqq could decay into a lepton and two mesons. Similarly a weird meson with $A=1$ composed of $\bar{r}q$ would decay into a baryon and a lepton. The only second-order transitions from L into other fermions are $L \to \overline{rrq}$ and $L \to \bar{r}ql$, so the L decay may be a two-step process. L's produced in pairs in e^+e^- annihilation could be nearly stable.

Our fourth example is again a model with a Class III embedding, but the theory may be either vectorlike or flavor chiral. We assign the fermions to the spinor representation of SO_{14}, with color embedded as $\mathrm{SO}_{14} \supset \mathrm{SO}_8 \times \mathrm{U}_1 \times \mathrm{SU}_3^c$. Models of this kind have been studied by Fritzsch and Minkowski, 1975. The SO_{14} spinor is complex and 64-dimensional, and in the vectorlike model the quark and lepton content is given by Eq. (2.24)

$$64 = (8, 1^c) + (8, 3^c) + (8', 1^c) + (8', \bar{3}^c),$$

where 8 and 8′ are the two real inequivalent spinor representations of SO_8. The antifermions are, of course, assigned to the $\overline{64}$. Thus all the left-handed spin 1/2 states are classified by $64 + \overline{64}$, as are the right-handed states in the vectorlike model.

Among the solutions for the electric charge assignments given in Eq. (3.5)–(3.8), there are two interesting ones: the standard charge assignment is recovered with $n_1=1$ and $n_2=n_3=n_4=0$. Half of the leptons have $Q=-1$, and the other half have $Q=0$; half the quarks have $Q=2/3$, the other half have $Q=-1/3$. For the solution, $n_1=n_2=n_3=1$ and $n_4=0$, the leptons in $(8', 1^c)$ have the charges $\{2,1,1,1,0,0,0,-1\}$, which are minus the charges of the leptons in $(8, 1^c)$. The quarks in $(8, 3^c)$ have charges $\{5/3, 2/3, 2/3, 2/3, -1/3, -1/3, -1/3, -4/3\}$, and the antiquarks in $(8', \bar{3}^c)$ have minus those charges. The bosons are listed in Eq. (2.18) with $n_3=1$ and $n_1=8$:

$$91 = (28, 1^c) + (1, 1^c) + (1, 8^c) + (8_v, 3^c) + (8_v, \bar{3}^c) + (1, 3^c) + (1, \bar{3}^c),$$

where 8_v is the vector representation of SO_8. From Eqs. (2.15) and (2.16), we see that the $(8_v, 3^c)$ and $(8_v, \bar{3}^c)$ bo-

TABLE VII. Boson–fermion couplings in SO_n model where fermions are assigned to the spinor representation. The local generator used in defining A generates the explicit U_1 in Eq. (2.20) or Eq. (2.23). A is the adjoint representation of SO_{n-6}, the $(n-6)(n-7)/2$.

a \ b	$\bar{l}$ $(\xi', 1^c)$ $A=0$	q $(\xi, 3^c)$ $A=1/3$	$\bar{r}$ $(\xi', \bar{3}^c)$ $A=2/3$	L $(\xi, 1^c)$ $A=1$
$\bar{l}$ $(\xi', 1^c)$ $A=0$	$(A, 1^c)+(1, 1^c)$ $A=0$	$(n-6, \bar{3}^c)$ $A=-1/3$	$(1, 3^c)$ $A=-2/3$	None
q $(\xi, 3^c)$ $A=1/3$	$(n-6, 3^c)$ $A=1/3$	$(A, 1^c)+(1, 1^c)+(1, 8^c)$ $A=0$	$(n-6, \bar{3}^c)$ $A=-1/3$	$(1, 3^c)$ $A=-2/3$
$\bar{r}$ $(\xi', \bar{3}^c)$ $A=2/3$	$(1, \bar{3}^c)$ $A=2/3$	$(n-6, 3^c)$ $A=1/3$	$(A, 1^c)+(1, 1^c)+(1, 8^c)$ $A=0$	$(n-6, \bar{3}^c)$ $A=-1/3$
L $(\xi, 1^c)$ $A=1$	None	$(1, \bar{3}^c)$ $A=2/3$	$(n-6, 3^c)$ $A=1/3$	$(A, 1^c)+(1, 1^c)$ $A=0$

sons provide the transitions, $\bar{l} \leftrightarrow q$, $q \leftrightarrow r$, and $r \leftrightarrow L$, where L are leptons assigned to $(8, 1^c)$, q to $(8, 3^c)$, $\bar{l}$ to $(8', 1^c)$, and $\bar{r}$ to $(8', \bar{3}^c)$. Those bosons act as the $(5, 3^c)$ and $(5, \bar{3}^c)$ bosons do in the SU_8 example. There is a difference however, because the SO_{14} example has $(1, 3^c)$ and $(1, \bar{3}^c)$ bosons in addition, and those connect $L \leftrightarrow q$ and $\bar{l} \leftrightarrow \bar{r}$.

The proton is rather naturally unstable in this scheme because both L and $\bar{l}$ lepton multiplets may have neutral members, and, for example, an $(8_v, 1^c)$ breaking term would mix the neutral l and L members, and the 3 "light" lepton decay mode would be opened.

In the most economical construction of A so far as weirdness is concerned (see Case 5 of Sec. IV), the broken local generator generates the explicit U_1. Then the weird L leptons have $A=1$ and the weird r quarks have $A=-2/3$. This possibility precludes L-$\bar{l}$ mixing, and a $(8_v, 1^c)$ breaking term must be absent. The color triplet bosons again carry fractional values of $A (A=\pm 1/3, \pm 2/3)$, but the decay processes for the weird particles are similar to those of the SU_8 model. The model is summarized in Table VII.

The flavor chiral theory places the left-handed fermions in the **64** and the right-handed fermions in the $\overline{\mathbf{64}}$, where both of these multiplets have fermion number +1. Thus all the left-handed spin 1/2 states are classified by $\underline{\mathbf{64}}+\mathbf{64}$, and all the right-handed spin 1/2 states are in $\overline{\mathbf{64}}+\overline{\mathbf{64}}$. (The U_1 symmetry connecting **64** and $\overline{\mathbf{64}}$ is generated by a pseudoscalar charge, and cannot be used to construct A). If q_L are assigned to the $(8, 3^c)$, then q_R must be assigned to the $(8', 3^c)$ in the $\overline{\mathbf{64}}$ in order for q_L and q_R to have the same charge, color, and baryon number. At this point, the analysis of proton stability becomes identical to the vectorlike case, where q_L and q_R both transform as $(8, 3^c)$. However, one expects the analysis of the full symmetry breakdown of vectorlike and flavor chiral theories to differ in many details.

As our final example, we look at an E_7 model. The exceptional group models without fermion number predict proton decay in second order, even before the symmetries are violated by the process $(qqq) \rightarrow (\bar{q}ql)$. For a proton-stable E_7 model, we double the fermion representation. There is no natural local U_1. Instead the U_1 must be raided from $G^{fl}=SU_6$. Of course such a flavor raid may be made in nonexceptional group models, but here it is mandatory. As discussed in Sec. IV, we take the U_1 from $SU_6 \supset SU_5 \times U_1$, and obtain a rich spectrum of weird particles, Eq. (4.20). This scheme has six $A=1/3$ quarks, of which one $Q=-1/3$ quark is separated from the other three with that charge. (See Table VIII.) The phenomenology is expected to be quite similar to that of the proton unstable theory (Gürsey and Sikivie, 1976, 1977; Ramond, 1976, 1977) based on a single **56** of fermions. We note the similarities of this model and the SU_8 model where the fermions are assigned to the **28**, and the embedding is $SU_8 \supset SU_5 \times U_1 \times SU_3^c$. The **56** of E_7 branches to $\mathbf{28}+\overline{\mathbf{28}}$ of SU_8. The results are summarized in Table VIII.

II. EMBEDDING SU_3^c IN SIMPLE LIE GROUPS

The object in this section is to solve the mathematical problem that arose in the introduction: given a simple Lie group G larger than SU_3, find all embeddings of SU_3 (identified physically as the color group SU_3^c of the strong interactions) for which at least one representation of G contains at most color singlets, triplets, and antitriplets. Then, for each embedding, find all other representations (if there are any) that satisfy the same color restriction. In this article we assign the fermions to such representations, but we recognize that the embeddings may be useful for theories in which the fermions are assigned to other representations.

For any embedding of SU_3^c in G, the color content of the generators of G is specified, and we may therefore identify the flavor subgroup G^{fl}, which is generated by all the color singlet generators of G. The subgroup decomposition is

$$G \supset G^{fl} \times SU_3^c . \tag{2.1}$$

The generators of G^{fl} include, of course, the weak

TABLE VIII. Boson–fermion couplings in the E_7 model, fermions in the 56. The $SU_6 \times SU_3^c$ decompositions of the fermions and bosons are contained in the boxes. We assume X used in stabilizing the proton generates the U_1 in $SU_6 \supset SU_5 \times U_1$. Thus the subdivision of eigenstates of A corresponds to the $SU_5 \times SU_3^c$ decomposition of the 56 and 133.

$a \backslash b$	$\boxed{(20,1^c)}$		$\boxed{(6,3^c)}$		$\boxed{(\overline{6},\overline{3}^c)}$	
	$\overline{l}$ $(\overline{10},1^c)$ $A=0$	L $(10,1^c)$ $A=1$	q $(5,3^c)$ $A=1/3$	s $(1,3^c)$ $A=4/3$	$\overline{r}$ $(\overline{5},\overline{3}^c)$ $A=2/3$	$\overline{\sigma}$ $(1,\overline{3}^c)$ $A=-1/3$
$\overline{l}$ $(\overline{10},1^c)$ $A=0$	$(24,1^c)+(1,1^c)$ $A=0$	$(5,1^c)$ $A=-1$	$(10,\overline{3}^c)$ $A=-1/3$	None $A=-4/3$	$(\overline{5},3^c)$ $A=-2/3$	$(\overline{10},3^c)$ $A=1/3$
$\boxed{(20,1^c)}$	$\boxed{(35,1^c)}$		$\boxed{(15,\overline{3}^c)}$		$\boxed{(\overline{15},3^c)}$	
L $(10,1^c)$ $A=1$	$(\overline{5},1^c)$ $A=1$	$(24,1^c)+(1,1^c)$ $A=0$	$(5,\overline{3}^c)$ $A=2/3$	$(10,\overline{3}^c)$ $A=-1/3$	$(\overline{10},3^c)$ $A=1/3$	None
q $(5,3^c)$ $A=1/3$	$(\overline{10},3^c)$ $A=1/3$	$(\overline{5},3^c)$ $A=-2/3$	$(24,1^c)+(1,1^c)$ $+(1,8^c)$ $A=0$	$(5,1^c)$ $A=-1$	$(10,\overline{3}^c)$ $A=-1/3$	$(5,\overline{3}^c)$ $A=2/3$
$\boxed{(6,3^c)}$	$\boxed{(\overline{15},3^c)}$		$\boxed{(35,1^c)+(1,8^c)}$		$\boxed{(15,\overline{3}^c)}$	
s $(1,3^c)$ $A=4/3$	None	$(\overline{10},3^c)$ $A=1/3$	$(\overline{5},1^c)$ $A=1$	$(1,1^c)+(1,8^c)$ $A=0$	$(5,\overline{3}^c)$ $A=2/3$	None
$\overline{r}$ $(\overline{5},\overline{3}^c)$ $A=2/3$	$(5,3^c)$ $A=2/3$	$(10,\overline{3}^c)$ $A=-1/3$	$(\overline{10},3^c)$ $A=1/3$	$(\overline{5},3^c)$ $A=-2/3$	$(24,1^c)+(1,1^c)$ $+(1,8^c)$ $A=0$	$(\overline{5},1^c)$ $A=1$
$\boxed{(\overline{6},\overline{3}^c)}$	$\boxed{(15,\overline{3}^c)}$		$\boxed{(\overline{15},3^c)}$		$\boxed{(35,1^c)+(1,8^c)}$	
$\overline{\sigma}$ $(1,\overline{3}^c)$ $A=-1/3$	$(10,\overline{3}^c)$ $A=-1/3$	None	$(\overline{5},3^c)$ $A=-2/3$	None	$(5,1^c)$ $A=-1$	$(1,1^c)+(1,8^c)$ $A=0$

charges and the exactly conserved electric charge Q. We are also assuming that the color charges, the generators of SU_3^c, are exactly conserved. (Occasionally, we shall toy with the idea of enlarging the exactly conserved Yang-Mills group of the strong interactions to the direct product of SU_3 and another group, the whole being a subgroup of G.)

The restriction on the color content of the fermion representation permits a very simple embedding procedure. We prove in the Appendix the crucial theorem: if any representation f of G decomposed according to Eq. (2.1) contains at most 1^c, 3^c, and $\overline{3}^c$, then the fundamental representation also contains at most 1^c, 3^c, and $\overline{3}^c$. (The fundamental representations of the simple Lie groups are: n of SU_n; n of SO_n; 2n of Sp_{2n}; 7 of G_2; 26 of F_4; 27 of E_6; 56 of E_7; and 248 of E_8.) Thus we may study the embedding in terms of the fundamental representation n, which is not necessarily the fermion representation, with a $G^{fl} \times SU_3^c$ decomposition of the form

$$\mathrm{n} = (\mathrm{n}_1, 1^c) + (\mathrm{n}_3, 3^c) + (\mathrm{n}_{\overline{3}}, \overline{3}^c)\,, \tag{2.2}$$

where n_1, n_3 and $\mathrm{n}_{\overline{3}}$ are representations of G^{fl}.

The generators of G belong to the adjoint representation Adj(G), and its color content is easily obtained from that of the fundamental representation. With the $G^{fl} \times SU_3^c$ embedding, Adj(G) has the form

$$\mathrm{Adj}(G) = (\mathrm{Adj}(G^{fl}), 1^c) + (1, 8^c) + \text{cross terms}\,, \tag{2.3}$$

where the cross terms correspond to the generators of G that mix flavor and color, except in one case for SU_n. With n_3 and $n_{\overline{3}}$ both different from zero, Adj(G) includes two $(1,8^c)$'s, which generate an $SU_3 \times SU_3$ subgroup of SU_n. The sum of the corresponding generators generates SU_3^c. There is then a temptation to enlarge the color group to $SU_3 \times SU_3$, although doing so is optional. This situation is discussed under Case 3 below. We identify G^{fl} by explicit examination. It is then straightforward to determine the other representations of G that satisfy our color restrictions.

The whole procedure can be generalized by considering embeddings $G \supset G^{fl} \times SU_3^c$ with progressively more and more relaxed color restrictions on the fundamental representations and on the others to be used for the fermions.

We carry out this procedure for all simple groups in this section. The results are summarized in Table I, where the group, embeddings, and representations are tabulated. The derivation of these results occupies the remainder of this section. We have also included a short review of each of the Lie algebras. For a more complete review, see, for example, Dynkin (1957), Wybourne (1974), or Gilmore (1974); the tables of Patera and Sankoff (1973) contain useful group theoretic results.

A. Unitary groups

The special unitary group SU_n is generated by an algebra (called A_{n-1} in the mathematics literature) of rank $n-1$ and order n^2-1. The geometrical interpretation of SU_n is that it leaves invariant the scalar products of vectors in an n-dimensional complex vector space. Its representations are in general complex. The $n-1$ representations obtained by taking totally antisymmetric k-fold products $(\mathrm{n}^k)_A$ are irreducible and of dimension $\binom{n}{k}$, a binomial coefficient. The conjugate representation of $(\mathrm{n}^k)_A$ is equivalent to $(\mathrm{n}^{n-k})_A$. All other representations are obtained from partially symmetrized products of n with itself. The adjoint representation n^2-1 is contained in the product $\mathrm{n}\times\overline{\mathrm{n}}=(\mathrm{n}^2-1)+1$.

Case 1:

Because n is complex, the simplest form of Eq. (2.2) is

$$\mathrm{n}=(\mathrm{n}_1,1,1^c)+(1,\mathrm{n}_3,3^c)\,, \tag{2.4}$$

where $n=n_1+3n_3$, and n_1 and n_3 are integers greater than 1. The notation in Eq. (2.4) reflects the fact that this will be a Class I embedding,

$$G^{\mathrm{fl}}=G_l\times G_q\times \mathrm{U}_1\,, \tag{2.5}$$

as defined in the Introduction. Since this is a special example of Eq. (2.2) with $n_{\overline{3}}=0$, it will not yield the only embedding. However, as we shall see, the general case has new features that should be discussed separately. The adjoint representation of SU_n provides the list of the generators needed to identify the embedding

$$\begin{aligned}\mathrm{n}^2-1=&(\mathrm{n}_1^2-1,1,1^c)+(1,\mathrm{n}_3^2-1,1^c)+(1,1,1^c)+(1,1,8^c)\\&+(\mathrm{n}_1,\overline{\mathrm{n}}_3,\overline{3^c})+(\overline{\mathrm{n}}_1,\mathrm{n}_3,3^c)+(1,\mathrm{n}_3^2-1,8^c)\,.\end{aligned} \tag{2.6}$$

Note that Eq. (2.6) includes no flavor cross terms that are color singlets, so that G^{fl} of Eq. (2.1) is indeed given by

$$G^{\mathrm{fl}}=\mathrm{SU}_{n1}\times\mathrm{SU}_{n3}\times\mathrm{U}_1\,. \tag{2.7}$$

The U_1 distinguishes 1^c and 3^c in the fundamental representation.

For $n_3>1$, only the n of SU_n satisfies the restriction that no more than 1^c, 3^c, and $\overline{3}^c$ occur in the fermion representation; the leptons are assigned to $(\mathrm{n}_1,1,1^c)$ and the quarks to $(1,\mathrm{n}_3,3^c)$ of Eq. (2.4). The assignment must be vectorlike in order to avoid divergences from triangle anomalies.

Case 2:

We may ignore the trivial case $n_1=1$ in Eq. (2.4) for which all leptons have the same electric charge, but an interesting special case occurs for $n_3=1$ and $n_1>1$. Then n becomes

$$\mathrm{n}=(\mathrm{n}_1,1^c)+(1,3^c)\,, \tag{2.8}$$

with $n_1=n-3$. The adjoint representation is

$$\mathrm{n}^2-1=(\mathrm{n}_1^2-1,1^c)+(1,1^c)+(1,8^c)+(\overline{\mathrm{n}}_1,3^c)+(\mathrm{n}_1,\overline{3}^c)\,, \tag{2.9}$$

which implies the Class III embedding,

$$G^{\mathrm{fl}}=\mathrm{SU}_{n1}\times\mathrm{U}_1\,. \tag{2.10}$$

Equation (2.8) by itself is not an interesting candidate for the fermions. However, our color restriction is satisfied for the representations of dimension $\binom{n}{k}$, obtained by antisymmetrizing n k times,

$$\begin{aligned}(\mathrm{n}_1^k)_A=&[(\mathrm{n}_1^k)_A,1^c]+[(\mathrm{n}_1^{k-1})_A,3^c]+[(\mathrm{n}_1^{k-2})_A,\overline{3}^c]\\&+[(\mathrm{n}_1^{k-3})_A,1^c]\,.\end{aligned} \tag{2.11}$$

The last term is omitted for $k=2$. If n is even and $k=n/2$, $(\mathrm{n}^k)_A$ is self-conjugate. [For example, the 20 of SU_6 is $(6^3)_A$, which is equivalent to $(\overline{6}^3)_A$.] Otherwise, each of these representations is complex and unsafe from triangle anomaly divergences. There are no other representations of SU_n that satisfy our color restriction.

Case 3:

If both n_3 and $n_{\overline{3}}$ of Eq. (2.2) are nonzero, the adjoint contains two color octets and the most natural embedding is not really of the form Eq. (2.1) since the two 8's are generators of separate SU_3's. The embedding should be written initially as

$$\mathrm{SU}_n\supset(\mathrm{SU}_{n1}\times\mathrm{SU}_{n3}\times\mathrm{SU}_{n\overline{3}}\times\mathrm{U}_1\times\mathrm{U}_1)\times\mathrm{SU}_3^{c'}\times\mathrm{SU}_3^{c''}, \tag{2.12}$$

where the flavor group has the structure of a Class II embedding

$$G^{\mathrm{fl}}=G_l\times G_q\times G_{\overline{r}}\times\mathrm{U}_1\times\mathrm{U}_1\,. \tag{2.13}$$

Only the fundamental representation satisfies the color restrictions. The n contains n_1 leptons, n_3 q quarks and $n_{\overline{3}}$ $\overline{r}$ antiquarks, with $n=n_1+3n_3+3n_{\overline{3}}$. Since n of SU_n is unsafe from triangle anomalies, the fermion assignment must be vectorlike; both $\mathrm{SU}_3^{c'}$ and $\mathrm{SU}_3^{c''}$ are generated by vector currents. Consequently there is a temptation to enlarge the color group.

So that both q and r quarks be confined, the conventional color generators must be sums of the corresponding $\mathrm{SU}_3^{c'}$ and $\mathrm{SU}_3^{c''}$ generators. The eight SU_3^c generators are conserved, but there are two distinct possibilities for the remaining eight $\mathrm{SU}_3^{c'}\times\mathrm{SU}_3^{c''}$ generators: either they are all broken, or they are all conserved. If only SU_3^c is conserved we obtain the usual strong interaction gauge group: the q and r quarks would be confined by the same set of gluons, and the hadron spectrum would include qqr and $q\overline{r}$ states. If the unbroken strong gauge group were the full $\mathrm{SU}_3\times\mathrm{SU}_3$, then the q and r quarks would be bound together by different sets of gluons. Consequently $q\overline{r}$ states, which transform as $(3,\overline{3})$ of color, would be confined; similar considerations apply to the qqr states. The qqq and $q\overline{q}$ hadrons would be quite distinct from the rrr and $r\overline{r}$ ones. Bosons of the unified theory transforming as $(3,3)$ and $(\overline{3},\overline{3})$ of color would couple q to r.

B. Orthogonal groups

The special orthogonal group SO_m is the group of transformations that leaves invariant the scalar products of vectors in an m-dimensional real-vector space. The defining or vector representation is denoted by **v** or m, and is self-conjugate. The adjoint representation is obtained irreducibly from $(\mathrm{m}\times\mathrm{m})_A$, and has dimension $\frac{1}{2}m(m-1)$. The spinor spaces associated with the algebras differ for m even and m odd.

For an odd-dimensional defining space, the algebra (called B_n) of the group SO_{2n+1} has rank n and order $n(2n+1)$. There is a single real spinor representation σ of dimension 2^n, and it cannot be obtained from Kronecker products of **v** with itself. However, σ is determined up to phases by **v**, as can be seen from the decomposition of the direct product

$$\sigma\times\sigma=\sum_{k=0}^{n}(\mathbf{v}^k)_A\,, \tag{2.14}$$

where $(\mathbf{v}^k)_A$ are all irreducible SO_{2n+1} representations. Note that both the vector and adjoint representations appear in $\sigma\times\sigma$. The assignment of fermions to σ obeys our color restriction only for $n_3=1$, and so $n_3=1$ and $n_3>1$ will be considered as separate cases.

In an even number of dimensions, the algebra (called D_n) of the group SO_{2n} has rank n and order $n(2n-1)$. There are two inequivalent spinors, σ and σ', each of dimension 2^{n-1}. The Kronecker products of σ and σ' are

$$\sigma\times\sigma'=(\mathbf{v}^{n-1})_A+(\mathbf{v}^{n-3})_A+\dots \tag{2.15}$$

$$\sigma\times\sigma=\tilde{\sigma}+(\mathbf{v}^{n-2})_A+(\mathbf{v}^{n-4})_A+\dots\,, \tag{2.16}$$

where $\tilde{\sigma}$ is a representation of dimension $\frac{1}{2}\binom{2n}{n}$, and a formula similar to Eq. (2.16) holds for $\sigma'\times\sigma'$. For n even, both σ and σ' are self-conjugate, the vector appears in $\sigma\times\sigma'$, and the adjoint in $\sigma\times\sigma$ (and $\sigma'\times\sigma'$). For n odd, $\sigma'=\overline{\sigma}$ is the conjugate of σ, which is complex, and $\sigma\times\overline{\sigma}$ contains the adjoint representation.

We should mention the equivalence of the algebras of SO_6 and SU_4, of SO_5 and Sp_4, and also of SU_2, SO_3, and Sp_2.

Case 4:

The fundamental representation of SO_n is the vector representation n, where n may be even or odd. Since n is self-conjugate, 3^c and $\overline{3}^c$ must appear symmetrically, and Eq. (2.2) must be of the form

$$\mathrm{n}=(\mathrm{n}_1,1,1^c)+(1,\mathrm{n}_3,3^c)+(1,\overline{\mathrm{n}}_3,\overline{3}^c)\,, \tag{2.17}$$

where $n=n_1+6n_3$ and $(\mathrm{n}_1,1)$ is a self-conjugate representation of G^{fl}. Here we consider n_1 and n_3 greater than 1; as our notation indicates, this is a Class I embedding. G^{fl} is identified from the adjoint representation

$$\begin{aligned}(\mathrm{n}\times\mathrm{n})_A=&[\tfrac{1}{2}\mathrm{n}_1(\mathrm{n}_1-1),1,1^c]+(1,\mathrm{n}_3^2-1,1^c)+(1,1,1^c)\\&+(1,1,8^c)+(\mathrm{n}_1,\mathrm{n}_3,3^c)+(\mathrm{n}_1,\overline{\mathrm{n}}_3,\overline{3}^c)\\&+[1,\tfrac{1}{2}\mathrm{n}_3(\mathrm{n}_3+1),\overline{3}^c]+[1,\overline{\tfrac{1}{2}\mathrm{n}_3(\mathrm{n}_3+1)},3^c]\\&+[1,\tfrac{1}{2}\mathrm{n}_3(\mathrm{n}_3-1),6^c]+[1,\overline{\tfrac{1}{2}\mathrm{n}_3(\mathrm{n}_3-1)},\overline{6}^c]\,,\end{aligned} \tag{2.18}$$

and the flavor group is

$$G^{fl}=SO_{n_1}\times SU_{n_3}\times U_1\,. \tag{2.19}$$

The explicit U_1 in Eq. (2.19) counts 3^c's minus $\overline{3}^c$'s in Eq. (2.17), and has zero eigenvalue for the color singlet part of n. There are no other representations of SO_n $(n_3>1)$ that satisfy our color restriction.

Case 5:

Although the structure of the spinor representations of SO_m differs for m even and odd, the characterization of their color content is similar enough to treat them together. Recall that we embed SU_3^c through the fundamental representation, Eq. (2.17). The requirement that the spinor representation contain 1^c, 3^c, and $\overline{3}^c$ only implies the same for m, as proved in the Appendix. We first show that $n_3=1$ in Eq. (2.17), so $G^{fl}=SO_{m-6}\times U_1$; we then give the $SO_{m-6}\times SU_3^c$ decomposition of the spinors.

First consider SO_{2n+1}. Since the spinor of SO_{2n+1} has only $1^c, 3^c, \overline{3}^c$, the color content of $\sigma\times\sigma$ cannot go beyond $1^c, 3^c, \overline{3}^c, 6^c, \overline{6}^c, 8^c$. However, the color content of $\sigma\times\sigma$ for the embedding can be computed directly from Eqs. (2.14) and (2.17). Specifically, if $n_3>1$, $(\mathbf{v}^3)_A$ contains a 15^c, so σ can satisfy the color restriction only if $n_3=1$ in Eq. (2.17). Since the factor SU_{n_3} disappears for $n_3=1$, the flavor group is

$$G^{fl}=SO_{2n-5}\times U_1\,. \tag{2.20}$$

The decomposition of the SO_{2n+1} spinor into $SO_{2n+1-2j}\times SO_{2j}$ representations is particularly simple

$$\sigma=(\xi,\rho)+(\xi,\rho')\,, \tag{2.21}$$

where ξ is the $SO_{2n+1-2j}$ spinor, and ρ and ρ' are the two SO_{2j} spinors. We set $j=3$, so that ρ and ρ' are the 4 and $\overline{4}$ of $SO_6\approx SU_4$. When SU_4 is broken down to $U_1\times SU_3^c$, 4 decomposes into 1^c+3^c and the decomposition of σ is

$$\sigma=(\xi,1^c)+(\xi,3^c)+(\xi,1^c)+(\xi,\overline{3}^c)\,. \tag{2.22}$$

The proof that the color restriction on σ requires $n_3=1$ works also for the spinors of SO_{2n}. For n odd, σ is complex and $\sigma'=\overline{\sigma}$, so that neither $\sigma\times\overline{\sigma}$ nor $\sigma\times\sigma$ can have color representations of dimension greater than 8. Equations (2.15) and (2.16) then imply that $n_3=1$ in Eq. (2.17). The same argument applies if n is even, although σ and σ' are then self-conjugate spinors and only $\sigma\times\sigma$ needs to satisfy the color restriction. The flavor group is then

$$G^{fl}=SO_{2n-6}\times U_1\,, \tag{2.23}$$

and we have a Class III embedding.

The $SO_{2n-6}\times SO_6$ decompositions of the SO_{2n} spinors are

$$\sigma=(\xi,4)+(\xi',\overline{4})\,,$$
$$\sigma'=(\xi'4)+(\xi,\overline{4})\,,$$

and the $G^{fl}\times SU_3^c$ decompositions are

$$\sigma=(\xi,1^c)+(\xi,3^c)+(\xi',1^c)+(\xi',\overline{3}^c)\,, \tag{2.24}$$

$$\sigma'=(\xi',1^c)+(\xi',3^c)+(\xi,1^c)+(\xi,\overline{3}^c)\,, \tag{2.25}$$

where ξ and ξ' are SO_{2n-6} spinors. As before, the U_1 of Eq. (2.23) appears in the decomposition $SU_4\supset SU_3^c\times U_1$.

There are no other representations of SO_m that satisfy the color restriction, since σ, σ' and v each contain both 3^c and $\overline{3}^c$.

C. Symplectic groups

The symplectic algebra (called C_n) generates the group Sp_{2n} of transformations that leave invariant a skew-symmetric quadratic form in a real $2n$-dimensional vector space. It is of rank n and of order $n(2n+1)$. All of its representations are self-conjugate and may be obtained from Kronecker products of the $2n$-dimensional defining representation with itself. The adjoint is the symmetric product of $\mathbf{2n}$ with itself. The products $(2\mathrm{n}^k)_s$ are irreducible, while $(2\mathrm{n}^k)_A$ is a sum of two irreducible representations, one of dimension $\binom{2n}{k}-\binom{n}{k-2}$ and the other of dimension $\binom{2n}{k-2}$. Since $2n$ is real, 3^c and $\overline{3}^c$ must appear symmetrically.

Case 6:

The most general form of 2n consistent with Eq. (2.2) is

$$\mathbf{2n} = (\mathbf{2n_1}, \mathbf{1}, \mathbf{1}^c) + (\mathbf{1}, \mathbf{n_3}, \mathbf{3}^c) + (\mathbf{1}, \overline{\mathbf{n}}_3, \overline{\mathbf{3}}^c), \tag{2.26}$$

where $n = n_1 + 3n_3$. Because both $\mathbf{3}$ and $\overline{\mathbf{3}}^c$ appear in $\mathbf{2n}$, all higher representations have at least $\mathbf{8}^c$ and are excluded for fermions by our color restriction. The adjoint representation is

$$\begin{aligned}\mathbf{n(2n+1)} = {} & [\mathbf{n_1(2n_1+1)}, \mathbf{1}, \mathbf{1}^c] + (\mathbf{1}, \mathbf{n_3^2-1}, \mathbf{1}^c) + (\mathbf{1}, \mathbf{1}, \mathbf{1}^c) \\ & + (\mathbf{1}, \mathbf{1}, \mathbf{8}^c) + (\mathbf{2n_1}, \mathbf{n_3}, \mathbf{3}^c) + (\mathbf{2n_1}, \overline{\mathbf{n}}_3, \overline{\mathbf{3}}^c) \\ & + [\mathbf{1}, \tfrac{1}{2}\mathbf{n_3(n_3+1)}, \mathbf{6}^c] + [\mathbf{1}, \overline{\tfrac{1}{2}\mathbf{n_3(n_3+1)}}, \overline{\mathbf{6}}^c] \\ & + [\mathbf{1}, \tfrac{1}{2}\mathbf{n_3(n_3-1)}, \overline{\mathbf{3}}^c] + [\mathbf{1}, \overline{\tfrac{1}{2}\mathbf{n_3(n_3-1)}}, \mathbf{3}^c] \\ & + (\mathbf{1}, \mathbf{n_3^2-1}, \mathbf{8}^c),\end{aligned} \tag{2.27}$$

which implies the Class I embedding,

$$G^{f1} = Sp_{2n_1} \times SU_{n_3} \times U_1 . \tag{2.28}$$

For $n_3 = 1$ this embedding is formally of Class III, but physically trivial since the $\mathbf{2n}$ only obeys the color restriction.

D. Exceptional groups

Besides the four infinite sequences of simple classical Lie groups, there are five exceptional groups, which do not have the simple geometrical interpretations reviewed above. Instead each is the invariance group of the multiplication table of certain matrices with elements that belong to the nonassociative octonion algebra. It is difficult to build Lie algebras from underlying nonassociative systems, and so their number is limited. These groups have some apparent advantages for particle physics. Quark–lepton universality is assured since the flavor group, which is in Class IV, contains no U_1 factor to distinguish color triplets from singlets. Also, there is a natural SU_3^c, which is the subgroup of the automorphism group G_2 of the underlying octonion algebra that leaves one of its elements invariant. (Of course SU_3^c might be a different subgroup of G, and we consider that possibility also.)

Two of the exceptional groups fall outside our assumptions. G_2 has rank 2, and SU_3 alone is a maximal subgroup; thus G^{f1} is trivial, lacking even a U_1 for electromagnetism. E_8 has rank 8 and 248 generators. It is the only Lie group for which the smallest representation is the adjoint; there are no representations satisfying our color restriction.

We have listed for reference the Kronecker products of the fundamental and adjoint representations of the exceptional groups in Table IX.

Case 7:

F_4 has rank 4 and 52 generators. SU_3^c is embedded by

$$F_4 \supset SU_3 \times SU_3^c , \tag{2.29}$$

so that $G^{f1} = SU_3$. Only the smallest nontrivial representation satisfies the color restrictions. Like all other representations of F_4, it is self-conjugate:

$$\mathbf{26} = (\mathbf{8}, \mathbf{1}^c) + (\mathbf{3}, \mathbf{3}^c) + (\overline{\mathbf{3}}, \overline{\mathbf{3}}^c). \tag{2.30}$$

If the other SU_3 were the color group, there would be no F_4 representations satisfying our color constraint. (This is proved in the Appendix.) The $SU_3 \times SU_3^c$ decomposition of the adjoint representation is

$$\mathbf{52} = (\mathbf{8}, \mathbf{1}^c) + (\mathbf{1}, \mathbf{8}^c) + (\overline{\mathbf{6}}, \mathbf{3}^c) + (\mathbf{6}, \overline{\mathbf{3}}^c). \tag{2.31}$$

TABLE IX. Decomposition of products of fundamental and adjoint representations of G_2, F_4, E_6, E_7, and E_8.

G_2:	$\mathbf{7} \times \mathbf{7} = \mathbf{7}_A + \mathbf{14}_A + \mathbf{1}_s + \mathbf{27}_s$
	$\mathbf{7} \times \mathbf{14} = \mathbf{7} + \mathbf{27} + \mathbf{64}$
	$\mathbf{14} \times \mathbf{14} = \mathbf{14}_A + \mathbf{77}'_A + \mathbf{1}_s + \mathbf{27}_s + \mathbf{77}_s$
F_4:	$\mathbf{26} \times \mathbf{26} = \mathbf{1}_A + \mathbf{52}_A + \mathbf{273}_A + \mathbf{26}_s + \mathbf{324}_s$
	$\mathbf{26} \times \mathbf{52} = \mathbf{26} + \mathbf{273} + \mathbf{1053}'$
	$\mathbf{52} \times \mathbf{52} = \mathbf{52}_A + \mathbf{1274}_A + \mathbf{1}_s + \mathbf{324}_s + \mathbf{1053}_s$
E_6:	$\mathbf{27} \times \mathbf{27} = \overline{\mathbf{351}}'_A + \mathbf{27}_s + \mathbf{351}_s$
	$\mathbf{27} \times \overline{\mathbf{27}} = \mathbf{1} + \mathbf{78} + \mathbf{650}$
	$\mathbf{27} \times \mathbf{78} = \mathbf{27} + \mathbf{351}' + \mathbf{1728}$
	$\mathbf{78} \times \mathbf{78} = \mathbf{78}_A + \mathbf{2925}_A + \mathbf{1}_s + \mathbf{650}_s + \mathbf{2430}_s$
E_7:	$\mathbf{56} \times \mathbf{56} = \mathbf{1}_A + \mathbf{1539}_A + \mathbf{133}_s + \mathbf{1463}_s$
	$\mathbf{56} \times \mathbf{133} = \mathbf{56} + \mathbf{912} + \mathbf{6480}$
	$\mathbf{133} \times \mathbf{133} = \mathbf{133}_A + \mathbf{8645}_A + \mathbf{1}_s + \mathbf{1539}_s + \mathbf{7371}_s$
E_8:	$\mathbf{248} \times \mathbf{248} = \mathbf{248}_A + \mathbf{30380}_A + \mathbf{1}_s + \mathbf{3875}_s + \mathbf{27000}_s$

Case 8:

E_6 has rank 6 and 78 generators. Its fundamental representation is complex, and decomposes as

$$\mathbf{27} = (\mathbf{3}, \overline{\mathbf{3}}, \mathbf{1}^c) + (\mathbf{1}, \mathbf{3}, \mathbf{3}^c) + (\overline{\mathbf{3}}, \mathbf{1}, \overline{\mathbf{3}}^c) \tag{2.32}$$

under the maximal subgroup decomposition

$$E_6 \supset (SU_3 \times SU_3) \times SU_3^c . \tag{2.33}$$

Either of the other SU_3's could be identified as color, and the results of this paper would be unchanged. The adjoint representation is

$$\begin{aligned}\mathbf{78} = {} & (\mathbf{8}, \mathbf{1}, \mathbf{1}^c) + (\mathbf{1}, \mathbf{8}, \mathbf{1}^c) + (\mathbf{1}, \mathbf{1}, \mathbf{8}^c) \\ & + (\mathbf{3}, \mathbf{3}, \overline{\mathbf{3}}^c) + (\overline{\mathbf{3}}, \overline{\mathbf{3}}, \mathbf{3}^c).\end{aligned} \tag{2.34}$$

The $\mathbf{27}$ and $\overline{\mathbf{27}}$ are the only representations with $\mathbf{1}^c$, $\mathbf{3}^c$, and $\overline{\mathbf{3}}^c$ only.

Case 9:

E_7 has rank 7 and 133 generators. The color can be embedded by

$$E_7 \supset SU_6 \times SU_3^c . \tag{2.35}$$

Only the $\mathbf{56}$ satisfies the color restrictions: its $SU_6 \times SU_3^c$ decomposition is

$$\mathbf{56} = (\mathbf{20}, \mathbf{1}^c) + (\mathbf{6}, \mathbf{3}^c) + (\overline{\mathbf{6}}, \overline{\mathbf{3}}^c). \tag{2.36}$$

The decomposition of the adjoint representation is

$$\mathbf{133} = (\mathbf{35}, \mathbf{1}^c) + (\mathbf{1}, \mathbf{8}^c) + (\overline{\mathbf{15}}, \mathbf{3}^c) + (\mathbf{15}, \overline{\mathbf{3}}^c). \tag{2.37}$$

It is also possible that the SU_3^c is embedded in the SU_6 subgroup of E_7. This could happen in two ways: (1) If $SU_6 \supset SU_3^c \times SU_3 \times U_1$, then the $\mathbf{56}$ decomposes to $\mathbf{27} + \overline{\mathbf{27}} + \mathbf{1} + \mathbf{1}$ of E_6. Only the $\mathbf{56}$ satisfies our color restrictions; (2) If $SU_6 \supset SU_3^c \times SU_2$, then no E_7 representation satisfies our color restrictions, which is proven in the Appendix.

III. THE ELECTRIC CHARGE OPERATOR

The vector bosons responsible for mediating the electromagnetic (and weak) interactions are coupled to generators of the flavor group G^{fl}; the electric charge Q generates a U_1 subgroup of G^{fl}. The quarks have fractional electric charges in the sequence $(\ldots 5/3, 2/3, -1/3, -4/3, \ldots)$ and the leptons have integral charges $(\ldots, 1, 0, -1, \ldots)$. In this section we construct all possible charge operators consistent with these restrictions. The "standard charge assignment," where the quark charges are restricted to 2/3 and −1/3 and the lepton charges to 0 and −1, is always possible, although the number of fermions of each charge depends on G^{fl}, but of course other charge assignments are also possible. We proceed by constructing the charge operator for each of the four classes of G^{fl} identified in the Introduction. The charge operator in Class I and II theories is only slightly constrained; little discussion is needed. The restrictions for exceptional groups are so tight that possible charge assignments may be listed. Only Class III theories require any effort.

Class I: $G^{fl} = G_l \times G_q \times U_1$ (Cases 1, 4, and 6 of Table I and Sec. II).

Since the quarks and leptons are transformed by different factors of G^{fl}, the observed universality of their weak and electromagnetic charges must result from the mechanism that breaks down G to $SU_3^c \times U_1$. This is reflected in the freedom in defining Q, which may be written in the general form

$$Q = \alpha I_l + \beta I_q + \gamma I_0 , \tag{3.1}$$

where I_l generates a U_1 in G_l, I_q generates a U_1 in G_q, and I_0 generates the explicit U_1. We shall simply enumerate all the constraints on the eigenvalues of Q.

Case 1:

There is only one constraint on the charges of the fermions in the $\mathbf{n}$ of SU_n: the sum of all quark charges (of all colors) is minus the sum of lepton charges, which is proportional to γ in Eq. (3.1). For the standard charge assignment, this constraint becomes the number of $Q = -1$ leptons, which equals the number of $Q = 2/3$ quarks minus the number of $Q = -1/3$ quarks; in this special case, the number of quark flavors is equal to the number of lepton flavors.

Cases 4 and 6:

The vector representations $\mathbf{n}$ of SO_n and $\mathbf{2n}$ of Sp_{2n} are self-conjugate. If a single $\mathbf{n}$ of SO_n, say, contains both fermions and their antiparticles, then there is complete freedom in defining the fermion charges. It is also possible that $\mathbf{n}$ [Eqs. (2.17) or (2.26)] contains quarks and antiquarks (q and $\bar{r}$) such that q and r are not equivalent. CPT invariance then requires another fermion representation $\mathbf{n}$ containing the $\bar{q}$ and r. I_0 counts the number of q's minus the number of $\bar{r}$'s. The constraints on the charges are: the sum of the charges in $(\mathrm{n}, \mathbf{1}, \mathbf{1}^c)$ of $\mathbf{n}$ is zero, and the sum of q charges is proportional to γ and is equal to the sum of r charges.

Class II: $G^{fl} = G_l \times G_q \times G_{\bar{r}} \times U_1 \times U_1$ (Case 3).

The only constraint on the electric charges is that the sum over all fermion charges in $\mathbf{n}$ must be zero, as in Case 1 of Class I.

Class III: $G^{fl} = G_{l+q} \times U_1$ (Cases 2 and 5).

The generators of G_{l+q} act both on quarks and leptons, and the U_1 distinguishes $\mathbf{1}^c$ from $\mathbf{3}^c$. Thus the quark charges determine the lepton charges, although quarks and leptons sometimes belong to different representations of G_{q+l} so the patterns need not be identical. The most general form of Q is

$$Q = \alpha I + \beta I_0 , \tag{3.2}$$

where αI generates a U_1 in G_{q+l}, and I_0 generates the explicit U_1 factor. The value of β is determined largely by the fractional nature of the quark charges, and the integer spacing of the charge values is controlled by α.

Case 2:

The charge operator for Class III SU_n theories is a linear combination of an SU_{n_1} generator ($n_1 = n-3$) and the generator I_0 of the explicit U_1. When acting on $\mathbf{n}$ in Eq. (2.8), I_0 has eigenvalue $1/n_1$ for $(\mathrm{n}_1, \mathbf{1}^c)$ and $-1/3$ for $(\mathbf{1}, \mathbf{3}^c)$. The eigenvalues of the SU_{n_1} generator may be parametrized by a set of n_1 integers, $\{\mathfrak{M}_\alpha^1\} \equiv \{m_\alpha\} = \{m_1, m_2, \ldots, m_{n_1}\}$, because of the integral spacing of the electric charges. We define $M = \sum_{\alpha=1}^{n_1} m_\alpha$, and the eigenvalues of this generator are then $m_\alpha - (1/n_1)M$ when acting on $(\mathrm{n}_1, \mathbf{1}^c)$ and are 0 for $(\mathbf{1}, \mathbf{3}^c)$. There is no loss of generality if we require $0 \le M < n_1$.

The fermions are assigned to $(\mathbf{n}^k)_A$, Eq. (2.11). Thus it is helpful to define $\{\mathfrak{M}_\alpha^k\}$ as the set of $\binom{n_1}{k}$ integers obtained by summing k different m's in all different ways. If we denote the charges in $[(\mathrm{n}_1^k)_A, \mathbf{1}^c]$ by $Q_\alpha[(\mathrm{n}_1^k)_A, \mathbf{1}^c]$, etc., then the electric charges of the fermions may be parametrized as

$$\begin{aligned} &Q_\alpha[(\mathrm{n}_1^k)_A, \mathbf{1}^c] = \mathfrak{M}_\alpha^k + r , \quad \alpha = 1, \ldots, \tbinom{n_1}{k} , \\ &Q_\alpha[(\mathrm{n}_1^{k-1})_A, \mathbf{3}^c] = \mathfrak{M}_\alpha^{k-1} + s - 1/3 , \quad \alpha = 1, \ldots, \tbinom{n_1}{k-1} , \\ &Q_\alpha[(\mathrm{n}_1^{k-2})_A, \bar{\mathbf{3}}^c] = \mathfrak{M}_\alpha^{k-2} + 2s - r - 2/3 , \quad \alpha = 1, \ldots, \tbinom{n_1}{k-2} , \\ &Q_\alpha[(\mathrm{n}_1^{k-3})_A, \mathbf{1}^c] = \mathfrak{M}_\alpha^{k-3} + 3s - 2r - 1 , \quad \alpha = 1, \ldots, \tbinom{n_1}{k-3} , \end{aligned} \tag{3.3}$$

where r and s must be integers, and must also satisfy the constraint

$$(3k - n) r = (M + 3s - 1) k . \tag{3.4}$$

[The parameter β of Eq. (3.2) is determined in terms of r and also in terms of s, thus giving rise to Eq. (3.4).]

Let us examine some examples of Eqs. (3.3) and (3.4). If we require that $\{\mathfrak{M}_\alpha^k\}$ contain two values only, then there are two possibilities for $\{m_\alpha\}$:

(a) $\{m_\alpha\} = \{1, 0, \ldots, 0\}$. If $\{Q_\alpha[(\mathbf{n}^k)_A, \mathbf{1}^c]\}$ contains charges $Q = 0$ and $Q = 1$ (antileptons), then $r = s = 0$, and the standard charge assignment is recovered. The particles are as follows:

$[(\mathrm{n}_1^k)_A, \mathbf{1}^c]$ antileptons, $\binom{n-4}{k-1}$ with $Q = 1$ and $\binom{n-4}{k}$ with $Q = 0$;

$[(\mathrm{n}_1^{k-1})_A, \mathbf{3}^c]$ quarks,

$\binom{n-4}{k-2}$ with $Q = 2/3$ and $\binom{n-4}{k-1}$ with $Q = -1/3$;

$[(\mathrm{n}_1^{k-2})_A, \bar{\mathbf{3}}^c]$ antiquarks,

$\binom{n-4}{k-3}$ with $Q = 1/3$ and $\binom{n-4}{k-2}$ with $Q = -2/3$;

$[(\mathrm{n}_1^{k-3})_A, \mathbf{1}^c]$ leptons, $\binom{n-4}{k-4}$ with $Q = 0$ and $\binom{n-4}{k-3}$ with $Q = -1$.

(b) The standard charge assignment may also be obtained in a different way from $\{m_\alpha\}=\{0,1,\ldots,1\}$ but only for n even and $k=n/2$.

Larger spreads of electric charge are always possible, although Eq. (3.4) helps to limit the number of cases. As an interesting example, consider $\{m_\alpha\}=\{1,1,0,0,\ldots,0\}$, with the requirement that the charges $Q_\alpha[(\mathbf{n}_1^k)_A, \mathbf{1}^c]$ have the values $1, 0, -1$. Then $r=-1$ and $s=1/3[(n/k)-4]$, which implies that $n=4k, 7k, 10k$, etc. For the case $n=8$ and $k=2$, there is a quark singlet with $Q=-1/3$. This charge assignment for the **28** of SU_8 is related to the standard one for the **56** of E_7 (see Case 9 below); the SU_8 decomposition of the **56** of E_7 is $\mathbf{28}+\overline{\mathbf{28}}$.

Case 5:

The charge operator of a Class III SO_n theory is a linear combination of an SO_{n-6} generator and the generator of the explicit U_1 in Eq. (2.20) or (2.23). (We treat even and odd n together.) The eigenvalues of the SO_{n-6} generator depend on $m\equiv[(n-6)/2]$ parameters, which may be chosen to be integers because of the spacing of the electric charges. The eigenvalues of this generator acting on the $(\mathbf{n-6}, \mathbf{1}^c)$ are $\pm n_1, \pm n_2, \ldots, \pm n_m$, (0), where $n_1,\ldots,n_m$ are non-negative integers and the extra zero eigenvalue is present if n is odd. The eigenvalues of this generator when acting on the spinor representation(s) are then given by the set $\mathfrak{M}$, where

$$\mathfrak{M}=\{\tfrac{1}{2}(\pm n_1 \pm n_2 \pm \cdots \pm n_m)\}. \tag{3.5}$$

Equation (3.5) is the solution of Eq. (2.14) or Eqs. (2.15) and (2.16). It is most easily derived from the Clifford algebra structure of the SO_{n-6} spinors, but it may also be constructed using the $SU_m \times U_1$ decomposition of the SO_{n-6} spinors. For n odd there is only one spinor of dimension 2^m, and the eigenvalues are just the set $\mathfrak{M}$. When n is even there are two SO_{n-6} spinors, each of dimension 2^{m-1}: one has eigenvalues in the subset of $\mathfrak{M}$ with an even number of minus signs, which we call $\mathfrak{M}_+$; and the other has eigenvalues in the subset of $\mathfrak{M}$ with an odd number of minus signs, $\mathfrak{M}_-$. We also define

$$N=\tfrac{1}{2}\sum_{i=1}^{m} n_i. \tag{3.6}$$

All elements of $\mathfrak{M}$ are integer (or half integer) if N is. The sum of the elements of $\mathfrak{M}_\pm$ is zero, since αI is traceless in any representation. If m and n are even, then the set $\mathfrak{M}_\pm$ is identical to the set $-\mathfrak{M}_\pm$, and for n even and m odd, $\mathfrak{M}_-$ is identical to $-\mathfrak{M}_+$.

The explicit U_1 is obtained from the decomposition $SO_6 \approx SU_4 \supset U_1 \times SU_3^c$. Finding the eigenvalues of the generator of the U_1 is a simple application of the SO_{n-6} results that we just obtained. The **6** of SO_6 breaks into $\mathbf{3}^c+\overline{\mathbf{3}}^c$ of SU_3^c, and the eigenvalues of the generator of the U_1 are $\{1,1,1,-1,-1,-1\}$. The eigenvalues of the **4** of SO_6, computed from Eq. (3.5), are then $\frac{1}{2}\{3,-1,-1,-1\}$. We adjust β in Eq. (3.2) so that quarks and leptons occur in the appropriate charge sequences. In the case that N in Eq. (3.6) is integral, the sets of electric charges for the fermions in the SO_n spinor representations, Eqs. (2.22) or (2.24) and (2.25), are

$$\begin{aligned}
Q_\alpha(\xi,\mathbf{1}^c)&=\mathfrak{M}_{+\alpha}+3k+1\,,\\
Q_\alpha(\xi,\mathbf{3}^c)&=\mathfrak{M}_{+\alpha}-k-1/3\,,\\
Q_\alpha(\xi',\mathbf{1}^c)&=\mathfrak{M}_{-\alpha}-3k-1\,,\\
Q_\alpha(\xi',\overline{\mathbf{3}}^c)&=\mathfrak{M}_{-\alpha}+k+1/3\,,
\end{aligned} \tag{3.7}$$

where the $3k+1$ is the average charge of the $(\xi,\mathbf{1}^c)$ leptons, etc., and k is an integer. For n odd, $\mathfrak{M}_+=\mathfrak{M}_-=\mathfrak{M}$ in Eq. (3.7). The half-integer N case is obtained by replacing k by $k-1/2$ in Eq. (3.7) so that

$$\begin{aligned}
Q_\alpha(\xi,\mathbf{1}^c)&=\mathfrak{M}_{+\alpha}+3k-1/2\,,\\
Q_\alpha(\xi,\mathbf{3}^c)&=\mathfrak{M}_{+\alpha}-k+1/6\,,\\
Q_\alpha(\xi',\mathbf{1}^c)&=\mathfrak{M}_{-\alpha}-3k+1/2\,,\\
Q_\alpha(\xi',\overline{\mathbf{3}}^c)&=\mathfrak{M}_{-\alpha}+k-1/6\,.
\end{aligned} \tag{3.8}$$

In both cases, the average lepton charge in $(\xi,\mathbf{1}^c)$ [or $(\xi',\mathbf{1}^c)$] is minus three times the average quark charge in $(\xi,\mathbf{3}^c)$ [or $(\xi',\mathbf{3}^c)$], and the spread in the charges is $2N$.

The standard charge assignment is recovered for $k=0$ in Eq. (3.8), and $\{n_\alpha\}=\{n_1,\ldots n_m\}=\{1,0,\ldots,0\}$ Then half the leptons have $Q=0$ and the other half have $Q=-1$; half the quarks have $Q=2/3$ and the other half have $Q=-1/3$. In all these four cases of charge and color, there is the same even number of particles. This corresponds to αI being the generator of the U_1 in the embedding $SO_{n-6}\supset SO_{n-8}\times SO_2=SO_{n-8}\times U_1$. There are no models with the average lepton charge of $-1/2$ and a spread of 2 units of charge. There are three cases if a spread of three units of charge is desired with average lepton charge $-1/2$. These correspond to: (a) $\{n_\alpha\}=\{3,0,\ldots,0\}$, which has charged currents with $\Delta Q=3$ only; (b) $\{n_\alpha\}=\{2,2,0,\ldots,0\}$; and (c) $\{n_\alpha\}=\{1,1,1,0,\ldots,0\}$. As a final example, if the average lepton charge is 1, then the smallest charge spread is 2, and the only interesting case is $\{n_\alpha\}=\{1,1,0,\ldots,0\}$. Then leptons have charges 0, 1, and 2; and quarks have charges $2/3$, $-1/3$, and $-4/3$, as can be seen from Eq. (3.7).

Class IV: $G^{f1}=G_{l+q}$ (Cases 7, 8, and 9)

In the exceptional groups, G^{f1} acts both on quarks and leptons as a simple flavor group. There are no U_1 factors that distinguish color. The sum of the quark charges vanishes. Quark-lepton universality is imposed at the level of the gauge group, and the quarks and leptons transform as different representations of G^{f1}, which are related in such a way that 1/3 integral charge assignments for quarks imply integral charge assignments for leptons.

Case 7:

The flavor group for F_4 is SU_3, and the fermions are assigned to the **26**, Eq. (2.30). The eigenvalues of the electric charge operator when acting on the **3** can be parametrized by $(n_1-1/3,\ n_2-1/3,\ -n_1-n_2+2/3)$, where n_1 and n_2 are integers. The leptons in $(\mathbf{8},\mathbf{1}^c)$ have charges $\{\pm(n_1-n_2), \pm(2n_1+n_2-1), \pm(2n_2+n_1-1), 0, 0\}$. The standard charge assignment, which in this case has $Q=\pm1$ leptons, is recovered with $n_1=n_2=0$.

Case 8:

The generator of the $U_1 \subset SU_3 \times SU_3$, which is G^{f1} for E_6, must possess eigenvalues $(n_1 - 1/3, n_2 - 1/3, -n_1 - n_2 + 2/3)$ in the $(\mathbf{1}, \mathbf{3})$ representation, and $(m_1 - 1/3, m_2 - 1/3, -m_1 - m_2 + 2/3)$ in the $(\mathbf{3}, \mathbf{1})$ representation. Then the eigenvalues of Q for the $(\mathbf{3}, \overline{\mathbf{3}}, \mathbf{1}^c)$ of Eq. (2.32) are $\{m_1 - n_1, m_1 - n_2, m_2 - n_1, m_2 - n_2, m_1 + n_1 + n_2 - 1, -m_1 - m_2 - n_1 + 1, m_2 + n_1 + n_2 - 1, -m_1 - m_2 - n_2 + 1, n_1 + n_2 - m_1 - m_2\}$. The standard charge assignment, which is the only one that looks interesting, is recovered with $n_1 = n_2 = m_1 = m_2 = 0$.

Case 9:

The generator of an arbitrary $U_1 \subset SU_6$ is specified by five parameters. The quarks are in a $\mathbf{6}$ of SU_6, and the charge eigenvalues are $n_i - 1/3$, $i = 1, \ldots, 6$ with $N = \sum_{i=1}^{6} n_i = 2$. The charges in the $(\mathbf{20}, \mathbf{1}^c)$ are then the set $\mathfrak{M}_3 - 1$, where $\mathfrak{M}_3$ is the set of all different threefold sums of n_i. Only if $\{n_\alpha\} = \{1, 1, 0, 0, 0, 0\}$ do we recover the standard charge assignment. A charge $-4/3$ quark occurs in the model with $\{n_\alpha\} = \{1, 1, 1, 0, 0, -1\}$.

IV. BARYON NUMBER CONSERVATION

In unified theories, where both quarks ($\mathbf{3}^c$) and leptons ($\mathbf{1}^c$) are assigned to the same irreducible representation of G, there exist vector bosons (leptoquarks) that transform leptons into quarks. This opens the way for proton decay unless there exists a conserved quantum number A for which the proton is the lowest mass state with $A = 1$. As emphasized in the Introduction, exact proton stabilization is not required, since there are many ways to increase the proton lifetime beyond the experimental limit. Still, a stable proton does not contradict the data, and it is of interest to examine its implications for the unified theories discussed here.

Empirically, conservation of A does not have a long-range force associated with it. Thus, even though A is an additive quantum number like Q, it cannot be a generator of a local U_1. It is therefore necessary for the Lagrangian to possess more symmetry than the local gauge group G. An additional global U_1 symmetry alone is not adequate either, since its generator can merely count representations of G, and cannot distinguish between quarks and leptons contained in one representation. It can help to put off proton decay to higher order, but it cannot make the proton absolutely stable. However, if this global U_1 and some local U_1 are both broken in such a way that a linear combination of the generators is conserved, the vector boson acquires a mass (leaving no physical Goldstone boson) and the unbroken linear combination gives an exact conservation law. The emphasis on U_1's reflects the fact that A is an additive quantum number. The Lagrangian should possess at least a global U_1 symmetry in addition to G. (A possible exception is the generation of conserved topological quantum numbers, but no examples with a short-range force in three spatial dimensions have yet been given.) As a matter of notation, let X be the generator of the local U_1 and Z be the generator of the global U_1. In carrying out this study we have neglected possible breaking of Z caused by tunneling effects in the vacuum (instantons). Even in theories where there is violation, the proton decay due to this mechanism is very slow ('t Hooft, 1976).

We now discuss ways of obtaining Z. If the Lagrangian contains only an adjoint representation of vector bosons and one self-conjugate irreducible representation of fermions, there are no global symmetries. Extra multiplets must be added before the Lagrangian acquires an additional global U_1 symmetry. There are two possibilities:

(1) The Lagrangian acquires an extra phase invariance if the fermion representation is doubled; this global symmetry is generated by fermion number. The fermions ($Z_f = +1$) are in f (some of the fermions may be antiquarks or antileptons), and their antiparticles ($Z_f = -1$) are in $\bar{\text{f}}$. Of course doubling f is necessary if f is a complex representation of SU_n, SO_{2n}, or E_6. In the first set of solutions for A, derived later in this section, we assume that Z is fermion number. The simplest prototype is a dynamically broken theory with spin 1 and spin 1/2 fields only. The bosons carry $Z = 0$, and the fermions belong to an irreducible f with $Z = 1$, and the antifermions to $\bar{\text{f}}$ with $Z = -1$. Fermion number and X are broken by a Majorana lepton mass term, which has the form $ll + \overline{ll}$. For simplicity we do not consider Lagrangians with additional global symmetry, arising, say, from additional fields in the Lagrangian or from f being reducible. This generalization, which is straightforward, can provide both a conserved baryon number and lepton number, if desired. We then seek an appropriate X so that

$$A = Z + X \tag{4.1}$$

is a suitable atomic mass number. We find that only for Cases 1 and 3 (see Table I) is it possible for all leptons to have $A = 0$ and all quarks to have $A = 1/3$; other cases always predict weird particles. Simple examples are listed in Table III.

(2) The global U_1 is a symmetry outside the fermion sector so that Z for the fermions is zero. The prototype theory here is one in which all fermions are assigned to a single irreducible self-conjugate f, and explicit Higgs fields have nonzero values of Z. If Z and X are broken there with their sum conserved, some of the spin zero fields will acquire nonzero A (the vacuum expectation of these fields must vanish), the zero-mass boson is made heavy, and the atomic mass number in the fermion sector is simply

$$A = X\,. \tag{4.2}$$

Again for simplicity we allow just one global U_1, so the fermions are assigned to self-conjugate representations. (Cases 1, 3, and 8 are absent.) It is possible to formulate SO_n and Sp_{2n} theories of Class I with a stable proton, but no weird fermions; otherwise weird fermions are required.

Whether the symmetry breaking in an actual theory follows the pattern prescribed here will depend on the specific Lagrangian. Some models might not provide a suitable atomic mass number even though the possibility appears in our classification. However, there is often a wide choice of X's that are satisfactory.

Since X is violated it must be a generator of G^{fl}. In theories based on the classical groups, G^{fl} contains an explicit U_1 subgroup, and usually its generator is a suitable X. (The one exception is $\mathbf{f}=\mathbf{n}$ of SO_n with fermion doubling.) It is also possible to extract part of X from the non-Abelian subgroups of G^{fl}; we call this procedure a flavor raid. At present, flavor raids for classical groups appear somewhat academic, but we indicate how they may be systematically pursued. For the exceptional groups, X must result from a flavor raid since there is no explicit U_1 factor. Just as in Sec. III, we must construct the eigenvalues of a generator of $\mathrm{U}_1 \subset G^{\mathrm{fl}}$, and many of the techniques needed there will again be useful.

A. Doubled f, Majorana mass breaking

Case 1:

We use this case to set our notation. Let the fermions be in the $\mathbf{n}$ of SU_n, so that

$$\mathbf{f}=(\mathbf{1},\mathbf{1}^c)+(\mathbf{q},\mathbf{3}^c)\,. \tag{4.3}$$

Comparing this with Eq. (2.4), we see that $\mathbf{l}=(\mathbf{n}_1,\mathbf{1})$ and $\mathbf{q}=(\mathbf{1},\mathbf{n}_3)$ are representations of G^{fl}. The global U_1 is generated by the fermion number operator

$$Z=N_l+N_q\,, \tag{4.4}$$

where N_q means number of q's minus number of anti-q's. There are n_1+n_3-1 local U_1's in $\mathrm{SU}_{n_1}\times\mathrm{SU}_{n_3}\times\mathrm{U}_1$, and the only restriction is that X not be the electric charge operator. However, the only choice of X that does not imply weird fermions is the generator of the explicit U_1 in Eq (2.7):

$$X=N_l-(n_1/3n_3)N_q\,. \tag{4.5}$$

The Majorana mass term, which breaks both X and Z, has the schematic form $ll+\overline{ll}$ so that A cannot contain N_l. Thus A is proportional to $Z-X$, and after proper normalization

$$A=\tfrac{1}{3}N_q\,, \tag{4.6}$$

which is readily identified with baryon number, since all leptons have $A=0$, and the proton, which is the lowest-mass 3-quark state, has $A=1$. Lepton number is violated because of the Majorana lepton mass, but this can be avoided in more complicated examples. Any other X, which would be obtained from a flavor raid, would have to distinguish different types of quarks and (or) leptons, and therefore would necessarily imply weird particles. The elaboration of these cases is straightforward, but not very instructive.

Case 2:

The k-times antisymmetrized $\mathbf{n}$ of SU_n of Eq. (2.11) may be written

$$\mathbf{f}=(\bar{\mathbf{l}},\mathbf{1}^c)+(\mathbf{q},\mathbf{3}^c)+(\bar{\mathbf{r}},\bar{\mathbf{3}}^c)+(\mathbf{L},\mathbf{1}^c)\,. \tag{4.7}$$

In the simplest scheme X generates the explicit U_1 of Eq. (2.10), and may be normalized to

$$X=Z-\frac{n}{k}\left(\frac{1}{3}N_q-\frac{2}{3}N_r+N_L\right), \tag{4.8}$$

where the fermion number operator Z is

$$Z=-N_l+N_q-N_r+N_L\,. \tag{4.9}$$

With the Majorana mass and the known leptons in $(\mathbf{1},\mathbf{1}^c)$, we obtain

$$A=\tfrac{1}{3}N_q-\tfrac{2}{3}N_r+N_L. \tag{4.10}$$

Equation (4.10) implies, for example, the existence of heavy L leptons that decay into ordinary baryons, and three-weird-quark ($3r$) states with $A=-2$ that could decay into an antideuteron plus a light lepton. These weird particles must be sufficiently massive to avoid conflict with experiment.

The most general A assignment is obtained from a flavor raid on SU_{n-3}. The calculation and solution of the spectrum of A is almost identical to the one for the electric charge in Case 2 of Sec. III. Following the notation defined there, we obtain the solution,

$$\begin{aligned}
&A_\alpha[(\mathbf{n}_1^k)_A,\ \mathbf{1}^c]=\mathfrak{M}_\alpha^k+r,\\
&A_\alpha[(\mathbf{n}_1^{k-1})_A,\ \mathbf{3}^c]=\mathfrak{M}_\alpha^{k-1}+s+1/3,\\
&A_\alpha[(\mathbf{n}_1^{k-2})_A,\ \bar{\mathbf{3}}^c]=\mathfrak{M}_\alpha^{k-2}+2s-r+2/3,\\
&A_\alpha[(\mathbf{n}_1^{k-3})_A,\ \mathbf{1}^c]=\mathfrak{M}_\alpha^{k-3}+3s-2r+1\,,
\end{aligned} \tag{4.11}$$

where r and s are integers satisfying

$$(3k-n)r=(M+3s+1)k-nZ\,, \tag{4.12}$$

and Z is the fermion number, which must be nonzero. Cataloging A assignments is similar to cataloging electric charge assignments, and is easily carried out for specific cases.

Case 3:

We write the fermion representation of SU_n in Case 3 as

$$\mathbf{f}=(\mathbf{1},\mathbf{1}^c)+(\mathbf{q},\mathbf{3}^c)+(\bar{\mathbf{r}},\bar{\mathbf{3}}^c)\,, \tag{4.13}$$

where the generators of the unbroken SU_3^c are the sums of those of the two SU_3's of Eq. (2.12). Then with $Z=N_l+N_q-N_r$ and $X=-(n_3-n_1)N_l+(2n_3+n_1)N_q+(n_1+2n_3)N_r$, we find

$$A=\tfrac{1}{3}(N_q+N_r)\,. \tag{4.14}$$

The choice of X is made from linear combinations of the generators of the two explicit U_1's in such a way as to eliminate weird particles. X can be broken by a Majorana mass only if $n_3\neq n_{\bar{3}}$. The situation is similar if the two SU_3's are both conserved and confining (i.e., the color group is enlarged to $\mathrm{SU}_3^{c'}\times\mathrm{SU}_3^{c''}$).

Case 4:

Let us label the vector representation of SO_n, Eq. (2.17), as

$$\mathbf{f}=(\mathbf{1},\mathbf{1}^c)+(\mathbf{q},\mathbf{3}^c)+(\bar{\mathbf{r}},\bar{\mathbf{3}}^c)\,, \tag{4.15}$$

where $\mathbf{r}$ and $\mathbf{q}$ each belong to the $\mathbf{n}_3$ representation of SU_{n_3}. The explicit U_1 in Eq. (2.19) does not count $\mathbf{l}$'s, and cannot be broken by a Majorana mass. If the breaking of this U_1 were to appear elsewhere in the theory, then it would be possible to obtain $A=\frac{1}{3}(N_q+N_r)$, but in this set of models, part of X must come from a flavor raid on SO_n; for example, decompose $\mathrm{SO}_{n_1}\supset\mathrm{SU}_{n'}\times\mathrm{U}_1$ for $n_1=2n'$. Then $\mathbf{n}_1$ of SO_{n_1} decomposes into $\mathbf{n}'+\bar{\mathbf{n}}'$ of $\mathrm{SU}_{n'}$ and the U_1 generator distinguishes two kinds of leptons, which we call l and L. There is then a linear combination of this generator and the explicit U_1 generator such that

$$A = N_L + \tfrac{1}{3} N_q + \tfrac{4}{3} N_r . \qquad (4.16)$$

For the case $n_1 = 2n' + 1$, there are 2 types of weird leptons. Of course more complicated flavor raids are possible, but enumerating them is probably academic.

Case 5:

The spinor representations of SO_n take the form, Eqs. (2.22) or (2.24) and (2.25)

$$\mathbf{f} = (\mathrm{L}, 1^c) + (\mathrm{q}, 3^c) + (\bar{\mathrm{r}}, \bar{3}^c) + (\bar{\mathrm{l}}, 1^c) , \qquad (4.17)$$

where $\bar{\mathrm{l}}$ and $\bar{\mathrm{r}}$ transform as the ξ spinor of SO_{n-6}, and q and L as the ξ' spinor. (We recall that ξ and ξ' are equivalent for n odd and conjugates of each other when n is twice an even number.) The explicit U_1's in Eqs. (2.20) and (2.23) are generated by

$$X = N_L - \tfrac{1}{3} N_q + N_l - \tfrac{1}{3} N_r , \qquad (4.18)$$

and combining Eq. (4.18) with $Z = N_L + N_q - N_r - N_l$, we find

$$A = N_L + \tfrac{1}{3} N_q - \tfrac{2}{3} N_r . \qquad (4.19)$$

A flavor raid requires computing the eigenvalue spectrum of an arbitrary U_1 in SO_{n-6}, as we already have for the electric charge assignments in Case 5 of Sec. III. Following the notation established there the general solution for N integer is

$$\begin{aligned} A_\alpha(\xi,\ 1^c) &= \mathfrak{M}_{+\alpha} + r + 1 , \\ A_\alpha(\xi,\ 3^c) &= \mathfrak{M}_{+\alpha} + s + 1/3 , \\ A_\alpha(\xi',\ 1^c) &= \mathfrak{M}_{-\alpha} + (3s - r)/2 , \\ A_\alpha(\xi',\ \bar{3}^c) &= \mathfrak{M}_{-\alpha} + 2/3 + (r + s)/2 , \end{aligned} \qquad (4.20)$$

where r and s must both be even integers or odd integers, and the fermion number

$$Z = (r + 3s)/4 + 1/2 \qquad (4.21)$$

must be nonzero. The half-integer N solution is obtained by replacing r by $r - 1/2$ and s by $s - 1/2$ everywhere in Eqs. (4.20) and (4.21).

Case 6:

The analysis of the vector representation of Sp_{2n} is identical to Case 4, except that n_1 cannot be odd.

Case 7:

X must generate a U_1 in G_{q+l} for exceptional gauge groups. For any choice of this U_1, the proton stable F_4 theory that satisfies the restrictions we have made is flavor poor. The U_1 generator X is a flavor SU_3 generator, which is easily parametrized in a general way consistent with the constraints on A. The **26**, which is doubled, has integer fermion number so that all the leptons in $(\mathbf{8}, 1^c)$ have integer A. Although this model has six quarks and eight leptons, it is straightforward to prove that if one (or more) lepton has $A = 0$, then at least five of the six quarks are weird. Thus this F_4 model can only have one quark flavor.

Case 8:

The **27** of E_6 is complex. A general parametrization is

$$\begin{aligned} A_\alpha(1, 3, 3^c) &= n_\alpha - k + (M - 2N)/3 , \\ A_\alpha(\bar{3}, 1, \bar{3}^c) &= -m_\alpha - k - (N - 2M)/3 , \\ A_\alpha(3, \bar{3}, 1^c) &= m_i - n_j - k, \quad i, j = 1, 2, 3 , \end{aligned} \qquad (4.22)$$

where n_α, m_α, and k are integers, $N = n_1 + n_2 + n_3$, $M = m_1 + m_2 + m_3$, and fermion number Z is

$$Z = (M - N)/3 - k . \qquad (4.23)$$

The sums M and N are restricted to:
(1) $M = 0$, $N = 1$ or $M = 1$, $N = 0$; and
(2) $M = N = 2$ and $k \neq 0$.

The richest arrangement has six leptons with $A = 0$ and four quarks with $A = 1/3$ (three from one triplet). This is the solution with $n_1 = n_2 = n_3 = 0$, so that X generates a U_1 in just one of the SU_3 factors. Other solutions with at least one nonweird lepton have at most three quarks with $A = 1/3$.

Case 9:

The E_7 doubled **56** of fermions, Eq. (2.36), has **12** quarks; X must generate a $U_1 \subset SU_6$. As in Case 9 of Section III, we parametrize the U_1 in terms of six integers n. Then there are two solutions

$$\begin{aligned} A_\alpha(6, 3^c) &= n_\alpha + k + 1/3 , \\ A_\alpha(\bar{6}, \bar{3}^c) &= -n_\alpha + k + 2/3 , \\ A_\alpha(20, 1^c) &= \mathfrak{M}_{3\alpha} + k , \end{aligned} \qquad (4.24)$$

where k is any integer, $N = \sum n_\alpha = 1$, and $\mathfrak{M}_3$ is the set of 20 threefold sums of n_α. The other solution is

$$\begin{aligned} A_\alpha(6, 3^c) &= n_\alpha + k - 2/3 , \\ A_\alpha(\bar{6}, \bar{3}^c) &= -n_\alpha + k + 2/3 , \\ A_\alpha(20, 1^c) &= \mathfrak{M}_{3\alpha} + k - 2 , \end{aligned} \qquad (4.25)$$

where k, which is fermion number, is a nonzero integer and $N = 4$. Thus we see that X cannot generate the U_1 in $SU_6 \supset SU_3 \times SU_3 \times U_1$, and that the $SU_6 \supset SU_4 \times SU_2 \times U_1$ theory has only two nonweird quarks. The solution with the least number of weird fermions is Eq. (4.24) with $\{n_\alpha\} = \{1, 0, 0, 0, 0, 0\}$ and $k = 0$. This corresponds to X generating the U_1 in $SU_6 \supset SU_5 \times U_1$. There are then five nonweird q quarks and one weird s quark in the $(6, 3^c)$, and five weird $\bar{r}$ antiquarks and one nonweird $\bar{\sigma}$ in the $(\bar{6}, \bar{3}^c)$. Half the leptons are weird. A is given by

$$A = N_L + N_q/3 + N_\sigma/3 + 4N_s/3 - 2N_r/3 . \qquad (4.26)$$

This is the only example (without introducing more fermion representations) of a proton-preserving exceptional gauge theory with enough flavors for good phenomenology.

B. Single f, doubling and breaking elsewhere

In this set of solutions we assign all the fermions to a single irreducible self-conjugate representation of G, such as can be found in Cases 2, 4, 5, 6, 7, and 9 of Table I. (We continue to follow the numbering system of Sec. II.) The global symmetry operator Z has zero eigenvalue for the fermion representation, and Z and X are broken in some other sector of the theory such as the scalar-pseudoscalar sector, so $A=X$ for the fermions. The problem is reduced to studying local U_1's that are not generated by electric charge.

Case 2:

The self-conjugate SU_n representation $(2k^k)_A$, Eq. (2.11) with $n=2k$, need not be doubled, but a flavor raid is needed if there are to be any $A=0$ leptons. Equation (4.11) is the general solution for A, and for this case, Eq. (4.12) is modified to read

$$r=M+3s+1\,. \tag{4.27}$$

No further constraints arise from particle conjugation in f. A simple example is obtained with $\{n_\alpha\}=\{1,0,\ldots,0\}$, $M=1$. The existence of leptons with $A=0$ imposes $r=s=-1$. Here X is the generator of the U_1 in the embedding $SU_{n-3}\supset SU_{n-4}\times U_1$.

Cases 4 and 6:

The discussions of the self-conjugate vector representations of SO_n and Sp_{2n} are identical. The generator of the explicit U_1, Eqs. (2.19) and (2.28), simply counts the number of 3^c's minus the number of $\overline{3}^c$'s. Therefore

$$A=N_q/3\,. \tag{4.28}$$

When it is not possible to break this U_1 in another sector of the theory, then a flavor raid is necessary and implies weird particles.

Case 5:

The generator of the natural U_1 counts both 1^c and 3^c in the spinor representations of SO_n, so it must be combined with an SO_{n-6} generator to obtain some $A=0$ leptons. (Only the SO_{2n+1}, all n, and SO_{2n}, n even, spinors are self-conjugate.) Fermion number is zero, so $r=-2-3s$ from Eq. (4.21), which automatically satisfies the antiparticle conjugation constraints. A simple example of this general solution, given by $\{n_\alpha\}=\{1,0,\ldots,0\}$, corresponds to X generating the U_1 in $SO_{n-6}\supset SO_{n-8}\times U_1$. There are equal numbers of leptons with $A=0$ and $A=1$, and quarks with $A=1/3$ and $A=-2/3$.

Case 7:

There are only three quark flavors in the particle-antiparticle self-conjugate 26 of F_4, and X must generate a U_1 in the flavor SU_3. Consequently there can be at most two nonweird quarks.

Case 9:

There are six quarks and ten leptons in the 56 of E_7. There is only one solution with more than three nonweird leptons or more than three nonweird quarks: X generates the U_1 in $SU_6\supset SU_4\times SU_2\times U_1$. Then there are four quarks with $A=1/3$ and six leptons with $A=0$.

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APPENDIX

We state and prove here the theorem that justifies the embedding procedure followed in Sec. II. Consider any embedding of SU_3^c in a simple Lie group G for which there is at least one representation f with color content restricted to 1^c, 3^c, and $\overline{3}^c$. Then the fundamental representation of G must also be limited to 1^c, 3^c, and $\overline{3}^c$. In other words, the condition that the fermion representation contains color singlets, triplets, and possibly antitriplets *implies* Eq. (2.2) for the fundamental representation. The fundamental representations of the simple Lie groups are: n of SU_n; n of SO_n; 2n of Sp_{2n}; 7 of G_2; 26 of F_4; 27 of E_6; 56 of E_7; and 248 of E_8.

The proof for the classical groups merely requires finding the color content of the group generators, which is explicitly displayed by the adjoint representation. Let c be a set of generators forming an irreducible representation of SU_3^c. Since each group generator must transform f within the representation, it is necessary that c acting on any SU_3^c representation in f contain at least one of the color representations in f. If f has only color singlets, triplets, and antitriplets, then $c\times 1^c$, $c\times 3^c$, *or* $c\times\overline{3}^c$ must contain a 1^c, 3^c, *or* $\overline{3}^c$. This is true *only if* c is 1^c, 3^c, $\overline{3}^c$, 6^c, $\overline{6}^c$, or 8^c. Thus f can satisfy the color restriction only if each of the color representations in the adjoint G has dimension less than or equal to 8. The proof is completed by constructing the adjoint representation, which must satisfy this condition, from the fundamental representation.

Suppose the n of SU_n violates our color restriction, so that it contains a set of operators d transforming as some higher representation (dimension greater than 3) of SU_3^c. The adjoint representation of SU_n is $n\times\overline{n}-1$, so it includes generators transforming under SU_3^c as the representations in $d\times\overline{d}$, which always contains a 27^c. The theorem then follows for SU_n. The proof is similar for the orthogonal and symplectic groups. If the n (vector representation) of SU_n contains a set d as defined above, then the adjoint representation, constructed from $(n\times n)_A$, must include sets of generators that transform as the representations in $(d\times d)_A$ under SU_3^c. The theorem follows since $(d\times d)_A$ always has at least one representation of dimension greater than 8. The adjoint representation of Sp_{2n} is constructed from $(2n\times 2n)_s$, and includes color operators in $(d\times d)_s$. As before, d must be empty if $(2n\times 2n)_s$ is to have no sets of generators that transform under SU_3^c as a representation of dimension greater than 8.

There exist embeddings of SU_3^c in the exceptional groups where the fundamental representation is not restricted to 1^c, 3^c, and $\overline{3}^c$, but where the adjoint contains no color representations of dimension greater than 8. The previous proof must then be supplemented with some information about the commutation relations of the group generators. We consider each exceptional group individually.

Three cases are trivial. Since the 7 of G_2 is self-conjugate, it must decompose to $1^c+3^c+\overline{3}^c$. The decomposition of E_6 into $SU_3\times SU_3\times SU_3$ is essentially symmetrical, and any SU_3 may be color, as is clear from Eqs. (2.32) and (2.34). E_8 is hopeless since the fundamental representation is the adjoint, which must have an 8^c.

We might use the other SU_3 of Eq. (2.29) as the color subgroup of F_4. The adjoint representation is then

$$52=(1,8^c)+(8,1^c)+(3,\overline{6}^c)+(\overline{3},6^c)\,,$$

as can be seen from Eq. (2.31). A color octet does appear in the fundamental representation, Eq. (2.30), for this embedding, so we must prove that no higher representations satisfy our color restrictions. Consider the action of the generators transforming as $(3,\overline{6}^c)$ on the supposed higher representation of the form, $(\mathbf{y},1^c)+(\mathbf{x},3^c)+(\overline{\mathbf{x}},\overline{3}^c)$, $\mathbf{x}$ and $\mathbf{y}$ being any representations of the flavor SU_3. Since these generators annihilate the pieces transforming as $(\mathbf{y},1^c)$ and $(\overline{\mathbf{x}},\overline{3}^c)$, they must not annihilate $(\mathbf{x},3^c)$. However, we prove that they do. The generators in $(\overline{3},6^c)$ must annihilate $(\mathbf{x},3^c)$. All the commutation relations among these generators yield generators in $(3,\overline{6}^c)$. Therefore the $(3,\overline{6}^c)$ generators must also annihilate $(\mathbf{x},3^c)$ and the representation cannot be of the supposed form. This completes the theorem for F_4.

Suppose SU_3^c is embedded in the SU_6 subgroup of E_7, Eq. (2.35). If the color is identified with one of the SU_3's in $SU_6\supset SU_3\times SU_3\times U_1$, then the fundamental **56** of E_7, under the decomposition of $E_7\supset E_6\times U_1$, is $56=27+\overline{27}+1+1$. This has no higher color representations. The other possibility is to obtain the SU_3^c from $SU_6\supset SU_2\times SU_3^c$. Here, higher color representations do occur in the **56**, since the **20** of SU_6 decomposes as $(2,8^c)+(4,1^c)$. Again, we must prove that no other E_7 representation satisfies the color restrictions. Under $SU_6\supset SU_2\times SU_3^c$, the only SU_6 representations restricted to 1^c, 3^c, $\overline{3}^c$ are **1**, **6**, and $\overline{\mathbf{6}}$; the $SU_3\times SU_6$ content of the supposed higher representation must be $(\mathbf{y},1)+(\mathbf{x},6)+(\overline{\mathbf{x}},\overline{6})$. The generators of E_7 include a set $(\overline{3},15)$, which annihilates $(\mathbf{y},1)$ and $(\mathbf{x},6)$, since 15×6 does not have **1**, **6**, or $\overline{\mathbf{6}}$. Moreover the set of generators formed by the cummutators in $(3,\overline{15})$ with itself also falls into the $(\overline{3},15)$ class, so that $(\overline{3},15)$ also annihilates $(\overline{\mathbf{x}},\overline{6})$. Thus there is no faithful representation of the supposed type.

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