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ABSTRACT

In this paper the effect of an electromagnetic field on the propagation of a pure torsional elastic wave in a conducting circular cylinder is investigated. The general field equations and boundary conditions are linearized and the equations of motion for an infinitely long circular rod are obtained for the particular electromagnetic field configurations considered.

The torsional motion of a solid rod in a steady axial magnetic field with and without a steady electric field is considered. In the first case it is found that a pure torsional mode will not propagate. In the second case a pure torsional mode will propagate and its frequency equation is obtained. The results for a perfect conductor are compared to a real material.

The torsional motion of a hollow rod in a steady tangential magnetic field with and without a steady axial electric field is considered. Without the electric field the equations are completely uncoupled and the solution is the standard elastic one. The electric field introduces coupling via the induced magnetic field. The equations of motion are obtained, however the actual solutions are not obtained due to the mathematical complexity involved.

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LIST OF SYMBOLS

B	~ magnetic flux density
D	~ dielectric displacement
E	~ electric field intensity
H	~ magnetic field intensity
j	~ current density
u	~ particle displacement
v_m	~ magnetic velocity $= (\mu H_z/\rho)^{\frac{1}{2}}$
v_s	~ shear velocity $= (G/\rho)^{\frac{1}{2}}$
ϵ	~ inductive capacity
λ, G	~ Lame constants for an elastic solid
μ	~ magnetic permeability
ρ	~ material density
ρ_e	~ charge density
σ	~ conductivity
τ	~ skin depth; $1/\tau^2 = \sigma\omega\mu$
ω	~ frequency of torsional wave

SECTION A: INTRODUCTION AND STATEMENT OF PROBLEM

I. Introduction

In the past few years a considerable amount of research has been focused on extensions of dynamical theories of continuous media to cases where a magnetic field is present. The subject of magnetohydrodynamics is the most notable of these with an extensive research program now in progress. The field of magnetoelasticity had received little attention until 1956 and even now is not nearly as well studied as magnetohydrodynamics.

The early studies in magnetoelasticity were conducted by Knopoff [(1)], Chadwick [(2)], and Kaliski [(3, 4, 5)]. The motivation for the first two was to determine the effect of the terrestrial magnetic field on seismic waves. The results of these studies were to conclude that the effect was too small to be of importance in this area. On the other hand Kaliski viewed the problem as one in applied mathematics and proceeded to work out the equations of motion in general for an elastic and inelastic anisotropic solid. The majority of this work was for a perfectly conducting infinite solid with the exception of reference (5) which is concerned with Rayleigh waves in a half space. Percival [(6)] and Miles [(7)] concerned themselves with a thin shell in a magnetic field. Dunkin and Eringen's [(8)] study was quite similar in approach to that of Chadwick except that they included the effect of an electric field as well as a magnetic one. Also, in addition to considering an infinite medium, they solved the problem of a limited class of vibrations of an infinite flat plate.

The sole experimental investigation that appears in the literature was by Winnett [(9)], who considered the longitudinal vibration of a circular rod. His results were in fair agreement with the linearized theory and his experiment pointed out the difficulty in obtaining an accurate measurement of such a small effect.

In general most of the previous investigations were made either for a perfect conductor or an infinite medium, or both. The purpose of the present investigation was to include the effects of both finite conductivity and the boundary for one particular geometry. Therefore an effort was made to select a problem for which the elastic solution would be quite simple. The torsional problem of a circular cylinder was chosen. The investigation was further limited to pure torsional waves in an infinite rod.

The scope of the investigation was to obtain the equations of motion for the configurations selected and to investigate the coupling of the elastic waves and the electromagnetic fields. An underlying aim of this investigation was to choose the configuration most suitable for an experimental investigation. The approach presented is similar to that of Chadwick and Dunkin and Eringen, but differs from that of Knopoff and Miles.

II. General Equations and Boundary Conditions

In the problem of magnetoelasticity the coupling between an elastic wave propagating through a conducting medium and an electromagnetic field is studied. The equations governing the electromagnetic fields are the well known Maxwell's equations:

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \tag{1}$$

$$\nabla \times \vec{E} + \mu \frac{\partial \vec{H}}{\partial t} = 0 \tag{2}$$

$$\nabla \cdot \vec{H} = 0 \tag{3}$$

$$\nabla \cdot \vec{D} = \rho_e \tag{4}$$

where $\vec{D}$ is the dielectric displacement and $\vec{H}$ is the magnetic field intensity and these quantities are related to the electric field intensity and the magnetic flux density by the following relation

$$\vec{B} = \mu \vec{H}, \qquad \vec{E} = \frac{1}{\epsilon} \vec{D}$$

μ is the magnetic permeability and ϵ the inductive capacity. The current equation for a linear isotropic conductor is:

$$\vec{j} = \sigma(\vec{E} + \vec{v} \times \mu \vec{H}) + \rho_e \vec{v} \tag{5}$$

where σ is the conductivity, ρ_e the static charge density and $\vec{v}$ the particle velocity.

The equation of motion for a linear elastic solid with the electromagnetic force included is:

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + G) \nabla(\mathrm{div}\, \vec{u}) + G \nabla^2 \vec{u} + \vec{j} \times \mu \vec{H} + \rho_e \vec{H} \tag{6}$$

where λ and G are the Lamé constants for an elastic solid and ρ is the material density.

The system of units used is the m.k.s., or Giorgi system which is defined by Stratton [14]. This system appears to be accepted by most engineers and physicists although universal agreement certainly does not exist. The selection of a system of units is somewhat more important when two dynamical theories are combined.

The general electromagnetic boundary conditions for a moving medium are obtained in reference (8) and are summarized here. The boundary is shown in Fig. 1 with the system of unit vectors which will be used. The tangential boundary conditions are:

$$\|[\vec{E} + \vec{v} \times \vec{B}]\|_t = 0 \qquad (7)$$

$$\|[\vec{H} - \vec{v} \times \vec{D}]\|_t = i^s_m - \rho_e^s v_m \qquad (8)$$

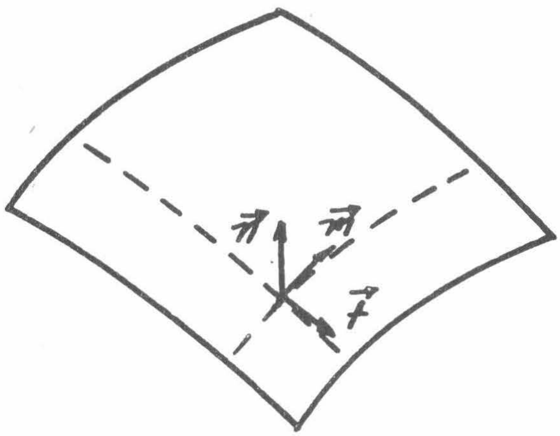

Figure (1): Boundary and system of coordinates.

where the subscripts m, n, and t refer to the unit vectors, the superscript s refers to a surface quantity and the symbol $\|[A]\|_t$ means the difference in the tangential component of A across the boundary. The conditions on the field components normal to the boundary are:

$$\|[\vec{B}]\|_n = 0 \qquad (9)$$

$$\|[\vec{D}]\|_n = \rho_e^s \qquad (10)$$

In most cases the normal component of the current must be continuous, hence

$$\left| \left[\sigma(\vec{E} + \vec{v} \times \vec{B}) \right] \right|_n = 0 \tag{11}$$

also serves as a normal boundary condition. If the boundary is between a solid medium and free space the velocity in all cases is that of the surface of the solid. The elastic boundary conditions are those of linear elastic theory.

III. Restricted Equations and Boundary Conditions

In general the total magnetic and electric fields will consist of a large statically applied field plus a small fluctuating part. Thus the total fields are given by:

$$\vec{H} = \vec{H}_o + \vec{h} \tag{12}$$

$$\vec{E} = \vec{E}_o + \vec{e} \tag{13}$$

These are inserted into equations (1) through (6) a system of nonlinear equations arises. To be consistent with linear elastic theory these equations are linearized by neglecting terms containing products of the small fluctuating quantities, such as $\vec{v} x \vec{h}$, $\vec{j} x \vec{h}$, etc. In addition we note that the elastic wave velocities are such that they are small compared to the propagation velocity of any electromagnetic quantities. Hence we ignore displacement currents, which is accomplished by dropping the last term in equation (1). Also the rod to be investigated will be infinite in length and initially without static charge density, therefore no charge density may be built up within the medium and ρ_e may be assumed to vanish. Therefore the governing equations now become:

$$\nabla \times \vec{H} = \vec{i} \tag{14}$$

$$\nabla \times \vec{E} + \mu \frac{\partial \vec{H}}{\partial t} = 0 \tag{15}$$

$$\vec{i} = \sigma(\vec{E} + \vec{v} \times \mu \vec{H}_0) \tag{16}$$

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + G)\nabla(\operatorname{div} \vec{u}) + G\nabla^2 \vec{u} + \vec{i} \times \mu \vec{H}_0 \tag{17}$$

with the conditions that the electric and magnetic fields must have zero divergence.

In the linearization of the boundary condition the surface currents are assumed to vanish so that equations (7) through (11) become

$$\|[\vec{E} + \mu \vec{v} \times \vec{H}_0]\|_t = 0 \tag{18}$$

$$\|[\vec{H}_0 + \vec{h}]\|_t = 0 \tag{19}$$

$$\|[\mu(\vec{H}_0 + \vec{h})]\|_n = 0 \tag{20}$$

$$(\vec{E} + \mu \vec{v} \times \vec{H}_0)_n = 0 \tag{21}$$

$$\|[\epsilon \vec{E} + \alpha \vec{v} \times \vec{H}_0]\|_n = \rho_e^s \tag{22}$$

where the last condition merely defines the surface charge and results from equation (10) with

$$\vec{D} = \epsilon\vec{E} + \alpha\,\vec{v}\times\vec{H} \qquad (23)$$

where

$$\alpha = \epsilon\mu - \epsilon_0\mu_0 \qquad (24)$$

with the subscript o referring to the quantities in free space. The last term in equation (23) results from consideration of electromagnetic theory of a moving medium.*

If these equations are to hold when the small fluctuating fields vanish then H_o and E_o must automatically satisfy their jump conditions, i.e.

$$[[H_0]]_t = 0\,, \quad \text{etc.}$$

Hence equations (18) through (22) reduce to

$$[[\vec{e} + \mu\vec{v}\times\vec{H}_0]]_t = [[\vec{h}]]_t = [[\mu\vec{h}]]_n$$
$$= (\vec{e} + \mu\vec{v}\times H_0)_n = 0 \qquad (25)$$

and

$$\rho_e^{s} = [[\epsilon\vec{e}]]_n + \alpha(\vec{v}\times\vec{H}_0)_n \qquad (26)$$

---

* Practically speaking $\alpha \approx 0$ and is included here only for the sake of completeness. This then becomes equivalent to the definition of the dielectric displacement given on page (59).

Since the charge density within the body is zero the current equation (16) reduces these to:

$$
\left.\begin{aligned}
\|[\vec{h}]\|_t &= \|[\mu \vec{h}]\|_n = 0 \\
\|[\tfrac{1}{\sigma}\vec{j}]\|_t &= (j)_n = 0
\end{aligned}\right\} \qquad (27)
$$

which are the linearized boundary conditions.

IV. Statement of Problem

The propagation of pure torsional waves in a circular cylinder under the influence of an electromagnetic field is to be studied. The rod is taken to be infinite in length and the applied field constant in time, hence as stated previously no possibility exists for a build-up of static charge density within the medium. Also surface currents are ignored. Four cases are to be studied:

Axial magnetic field:

I. With an axial electric field.

II. With vanishing electric field.

Tangential magnetic field:

III. With no electric field.

IV. With an axial electric field.

In cases I and II the applied magnetic field would be obtained from a solenoid of infinite length, hence the field is constant. The electric field is obtained by passing a steady state current through the rod. This electric field then induces a tangential magnetic field which must be considered. The rod in these two cases is solid and of radius b. The current density of the steady electric field is assumed uniform

In the third and fourth cases the applied magnetic field would be obtained by passing current down the outer conductor of a coaxial conductor and back along the center. This would necessitate a hollow rod and result in a magnetic field that varies as $1/r$. The electric field would again be generated by passing a current along the rod and as before this would induce an additional tangential magnetic field.

Since this investigation is limited to pure torsional waves the displacement field of the rod is specified to contain only a tangential component, i.e. u_θ is the only nonvanishing component. This may be a function of the radial coordinate r, and the axial coordinate z, but the symmetry in the θ coordinate rules out any dependence on this variable. This places rather strict restrictions on the modes that may be studied; however, it is felt that this method of approach will be more fruitful than a direct attack on the general problem which might well prove unmanageable in the end.

In general even though the equations have been linearized they are still a system of partial differential equations, the straightforward solution to which might prove a formidable task. Hence a product solution with harmonic variation in time will be sought. This reduces the governing equations to a system of linear ordinary differential equations. In particular, the small fluctuating quantities such as $\vec{j}$, $\vec{h}$, $\vec{u}$, etc., will be assumed to be propagating in the z, or axial direction with harmonic time dependence. Therefore a general quantity will have the form

$$\phi(r, z, t) = \phi(r)\, e^{i(\omega t - \gamma z)} \qquad (28)$$

where we choose to study the case of the frequency ω real and the propagation paramenter γ may in general be complex. This approach was chosen by Chadwick [2] and others [8]; however, several authors [1, 7, 9] have chosen the alternate approach of complex frequency.

SECTION B: AXIAL MAGNETIC FIELD

I. Case I: Axial Magnetic Field with Axial Electric Field

The geometry and coordinate system to be used are shown in Fig. 2. The applied fields H_z and E_z are constant and the tangential magnetic field induced by the electric field is given by:

$$\left.\begin{aligned} H_\theta &= \frac{\sigma b^2}{2r} E_z \qquad \text{outside} \\ H_\theta &= \frac{\sigma r}{2} E_z \qquad \text{inside} \end{aligned}\right\} \quad (31)$$

Hence the applied fields within the rod are:

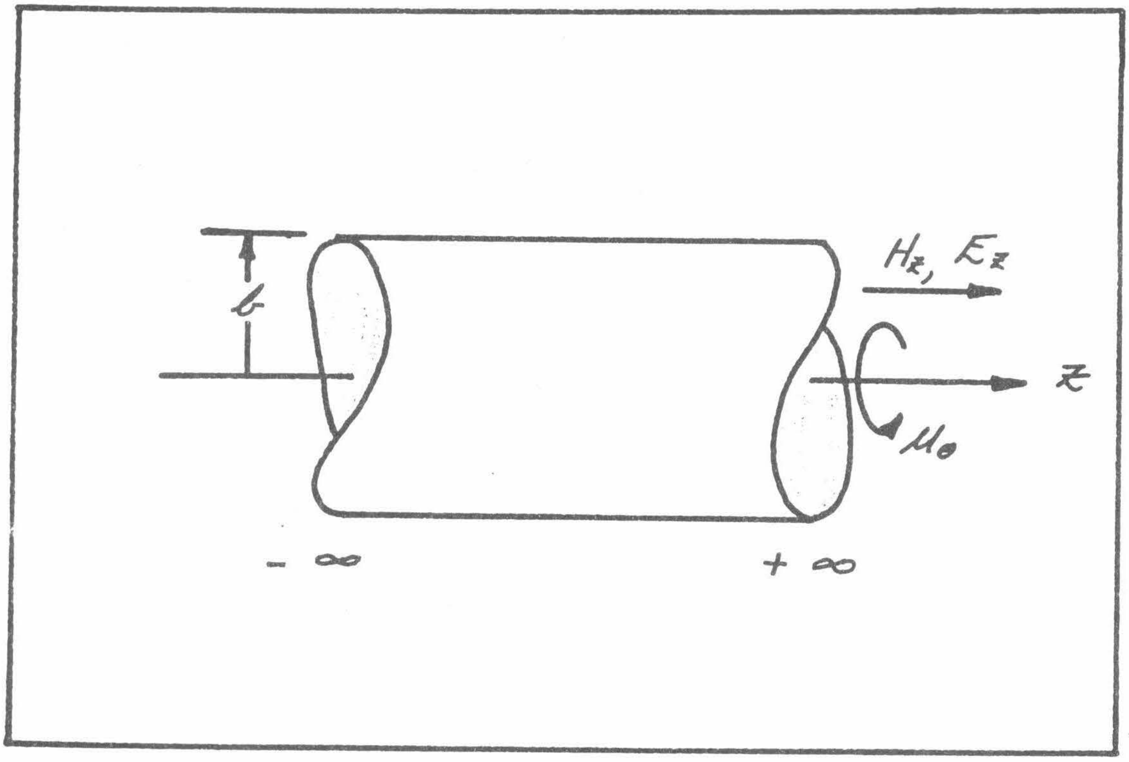

Fig. 2. Solid rod with axial magnetic and electric fields.

$$\vec{H}_0 = H_z \hat{i}_z + \frac{\sigma r}{2} E_z \hat{i}_\theta \qquad (32)$$

$$\vec{E}_0 = E_z \hat{i}_z \qquad (33)$$

where $\hat{i}_r$, $\hat{i}_z$ and $\hat{i}_e$ are the unit vectors.

In addition the current must also have a static z-component:

$$\vec{j}_T = \sigma E_z \hat{i}_z + \vec{j} \qquad (34)$$

where j_T is the total current and $\vec{j}$ represents the fluctuating current.

If equations (32) through (34) plus the assumed displacement field are introduced into the governing equations (14) through (17), and equation (28) is utilized the equations may be written out in component form as follows:

$$\left.\begin{aligned} & i\gamma h_\theta - j_r = 0 \\ & -i\gamma h_r - \frac{dh_z}{dr} - j_\theta = 0 \\ & \frac{1}{r}\frac{d}{dr}(r h_\theta) - j_z = 0 \end{aligned}\right\} \qquad (35)$$

$$\left.\begin{aligned} & i\gamma e_\theta + i\omega\mu h_r = 0 \\ & -i\gamma e_r - \frac{de_z}{dr} + i\omega\mu h_\theta = 0 \\ & \frac{1}{r}\frac{d}{dr}(r e_\theta) + i\omega\mu h_z = 0 \end{aligned}\right\} \qquad (36)$$

$$\left.\begin{aligned} & j_r = \sigma(e_r + i\omega\mu H_z u_\theta) \\ & j_\theta = \sigma e_\theta \\ & j_z = \sigma e_z \end{aligned}\right\} \qquad (37)$$

$$\left.\begin{aligned} & \mu\left[H_z j_\theta - \frac{\sigma r}{2} E_z j_z - \sigma E_z h_\theta - \frac{\sigma^2 E_z^2}{2} r\right] = 0 \\ & G\left[\nabla^2 u_\theta - \frac{1}{r^2} u_\theta\right] + \rho\omega^2 u_\theta + \mu\left[\sigma E_z h_r - H_z j_r\right] = 0 \\ & \frac{1}{2}\mu\sigma E_z r j_r = 0 \end{aligned}\right\} \qquad (38)$$

From the third equation or z component of (38) for a non-vanishing electric field it is evident that j_r must vanish. This condition naturally arises from the assumed displacement field of a pure torsional wave, that is, $u_z = 0$. From equation (35-r) this condition also implies that h_θ must be zero, this in turn through equation (35-z) implies that j_z must vanish. Finally the condition is reached where all of the induced quantities must vanish in order that the equations remain consistant. Hence we conclude that a pure torsional wave cannot propagate under the specified arrangement of fields.

II. Case II: Vanishing Electric Field

The other alternative left in equation (38-z) is to assume that the electric field vanishes, which is the condition specified for case II. Under this assumptions the equations become:

$$\left.\begin{aligned} & i\gamma h_\theta - j_r = 0 \\ & -i\gamma h_r - \frac{dh_z}{dr} - j_\theta = 0 \\ & \frac{1}{r}\frac{d}{dr}(r h_\theta) - j_z = 0 \end{aligned}\right\} \qquad (35')$$

$$\left.\begin{aligned} & i\gamma e_\theta + i\omega\mu h_r = 0 \\ & -i\gamma e_r - \frac{de_z}{dr} + i\omega\mu h_\theta = 0 \\ & \frac{1}{r}\frac{d}{dr}(r e_\theta) + i\omega\mu h_z = 0 \end{aligned}\right\} \qquad (36')$$

$$\left.\begin{aligned} & j_r = \sigma(e_r + i\omega\mu H_z u_\theta) \\ & j_\theta = \sigma e_\theta \\ & j_z = \sigma e_z \end{aligned}\right\} \qquad (37')$$

$$\left.\begin{aligned} & \mu H_z j_\theta = 0 \\ & G\left[\nabla^2 u_\theta - \frac{1}{r^2} u_\theta\right] + \rho\omega^2 u_\theta - \mu H_z j_r = 0 \\ & 0 = 0 \end{aligned}\right\} \qquad (38')$$

In equation (38'-r) if the magnetic field does not vanish, then $j_\theta = 0$, which leads through the remainder of the equations to the following conditions:

$$
\begin{array}{llll}
u_r = 0 & j_r & h_r = 0 & e_r \\
u_\theta & j_\theta = 0 & h_\theta & e_\theta = 0 \\
u_z = 0 & j_z & h_z = 0 & e_z
\end{array}
$$

and the remaining equations are:

$$
\left.\begin{array}{l}
i\gamma h_\theta - j_r = 0 \\
\frac{1}{r}\frac{d}{dr}(r h_\theta) - j_z = 0
\end{array}\right\} \quad (35')
$$

$$
i\gamma e_r - \frac{de_z}{dr} + i\omega\mu h_\theta = 0 \qquad (36')
$$

$$
\left.\begin{array}{l}
j_r = \sigma(e_r + i\omega\mu H_z u_\theta) \\
j_z = \sigma e_z
\end{array}\right\} \quad (37')
$$

$$
G\left[\nabla^2 u_\theta - \frac{1}{r^2}u_\theta\right] + \rho\omega^2 u_\theta - \mu H_z j_r = 0 \qquad (38')
$$

If the remaining components of the magnetic field and electric field are eliminated as well as j_z, these equations may be reduced to the two equations

$$
\nabla^2 u_\theta - \frac{1}{r^2}u_\theta + \frac{\rho\omega^2}{G}u_\theta - \frac{\mu H_z}{G} j_r = 0 \qquad (39)
$$

$$
\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r j_r)\right) - \gamma^2 j_r - i\omega\mu\sigma j_r + i\gamma^2\omega\mu\sigma H_z u_\theta = 0 \quad (40)
$$

which is a system of coupled second order, linear ordinary differential equations with non-constant coefficients. We also note that the two differential operators:

$$
\nabla^2\phi - \frac{1}{r^2}\phi = \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \gamma^2\phi - \frac{1}{r^2}\phi
$$

and

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r\phi)\right) - \gamma^2\phi = \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \gamma^2\phi - \frac{1}{r^2}\phi$$

are equivalent and hence denote it as

$$L\phi = \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \gamma^2\phi - \frac{1}{r^2}\phi \qquad (41)$$

Therefore the above equations may be rewritten in the form

$$L u_\theta + \left(\frac{\omega}{v_s}\right)^2 u_\theta - \frac{1}{H_z}\left(\frac{v_H}{v_s}\right)^2 j_r = 0 \qquad (42)$$

$$L j_r - i\frac{1}{\delta^2} j_r + i\left(\frac{1}{\delta}\right)^2 H_z u_\theta = 0 \qquad (43)$$

where

$$\frac{1}{\delta^2} = \sigma\omega\mu \qquad (44)$$

$$v_s^2 = G/\rho \qquad (45)$$

$$v_H^2 = \frac{\mu H_z^2}{\rho} \qquad (46)$$

Equation (44) defines the "skin depth" of a conductor in an electromagnetic field; (45) the shear velocity of an elastic wave, and (46) a magnetic velocity. The latter may be shown to correspond to the Alfven wave speed in a fluid. It is also of interest to note that the electromagnetic energy density in this case is given by

$$U = \frac{1}{2}\frac{B^2}{\mu} = \frac{1}{2}\mu H^2$$

if the small contribution due to the fluctuating fields is ignored.

Hence v_m^2 is the ratio of twice the electromagnetic energy density to the material density.

Now if the magnetic field vanishes in equation (42) it reduces to the elastic wave equation:

$$L u_\theta + \left(\frac{\omega}{v_s}\right)^2 u_\theta = 0 \qquad (47)$$

which has the solution

$$u_\theta = A J_1(\lambda_1 r) + B Y_1(\lambda_1 r) \qquad (48)$$

with

$$\lambda_1^2 = \gamma^2 - (\omega/v_s)^2$$

where $J_1(\lambda_1 r)$ and $Y_1(\lambda_2 r)$ are Bessel functions of the first and second kind of the first order.

Likewise, equation (43) reduces to

$$L \dot{j}_r - i \frac{1}{c^2} \dot{j}_r = 0 \qquad (49)$$

which has the solution

$$\dot{j}_r = C I_1(\lambda_2 r) + D K_1(\lambda_2 r) \qquad (50)$$

with

$$\lambda_2^2 = \gamma^2 + i\sigma\omega\mu$$

where $I_1(\lambda_2 r)$ and $K_1(\lambda_2 r)$ are the modified Bessel functions of the first and second kind and of the first order.

Hence we expect that, in general, equations (42) and (43) would have solutions of a similar form, that is, the entire character of the solutions would not be expected to change for a particular value of the parameter H_z. (It is realized that this constitutes an assumption, the validity of which will be shown.)

Suppose the solution to equations (42) and (43) are some combination of Bessel Type functions. The only restriction that is placed on this function is that it obeys the Bessel function recursion relations:

$$\left.\begin{aligned} \frac{2\nu}{x} Z_\nu(x) &= Z_{\nu-1}(x) - Z_{\nu+1}(x) \\ 2 Z_\nu'(x) &= Z_{\nu-1}(x) + Z_{\nu+1}(x) \end{aligned}\right\} \qquad (51)$$

where $Z_\nu(x)$ is some Bessel Type function and the prime denotes differentiation with respect to the variable x. It is shown[10] that this is true for linear combinations and products of both ordinary and modified Bessel functions. Therefore the form of the solutions is assumed to be:

$$\left.\begin{aligned} u_\theta &= K_1 Z_1(\lambda_1 r) \\ j_r &= K_2 Z_1(\lambda_2 r) \end{aligned}\right\} \qquad (52)$$

Now, using the recursion relations (51),

$$L Z_1(\lambda_1 r) = -(\lambda_1^2 + \gamma^2) Z_1(\lambda_1 r) \qquad (53)$$

so that the system of differential equations may be reduced to the following system of algebraical ones:

$$\begin{pmatrix} -(\lambda_1^2+\gamma^2)+(\omega/v_s)^2 & -\frac{1}{H_z}(v_H/v_s)^2 \\ i(\gamma/c)^2 H_z & -(\lambda_2^2+\gamma^2)-i\frac{1}{\epsilon c^2} \end{pmatrix} \begin{Bmatrix} K_1 Z_1(\lambda_1 \Pi) \\ K_2 Z_1(\lambda_2 \Pi) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (5A)$$

and the necessary and sufficient condition that a nontrivial solution exists is the vanishing of the determinant.

$$\begin{vmatrix} -(\lambda_1^2+\gamma^2)+(\omega/v_s)^2 & -\frac{1}{H_z}(v_H/v_s)^2 \\ -i(\gamma/c)^2 H_z & -(\lambda_2^2+\gamma^2)-i\frac{1}{\epsilon c^2} \end{vmatrix} = 0 \qquad (55)$$

This can be rewritten, using the knowledge that the elastic boundary conditions may be satisfied by setting $\lambda_1 = 0$, in the following form:

$$\gamma^4 + \left[\lambda_2^2 - (\omega/v_s)^2 + i\frac{1}{\epsilon c^2}\left\{1+(v_H/v_s)^2\right\}\right]\gamma^2 - \frac{1}{\epsilon c^2}(\omega/v_s)^2 = 0 \qquad (55)$$

This then constitutes the frequency equation for this case. The value of λ_2, as well as λ_1, are determined through the boundary conditions.

III. Restriction to a Perfect Conductor

The limiting case of a perfect conductor will be introduced as it results in some rather simple expressions for the wave speed and will provide a means of comparing these results with those presented in the literature. However, a word of caution must be injected as, if a perfect conductor is assumed, this physically implies that the induced currents become large, thereby inducing large magnetic fields and the entire process of linearization may no longer be valid. With this possible restriction in mind equation (55) is rewritten by letting $\sigma \to \infty$, which implies $\tilde{c} \to 0$ in the following form:

$$\gamma^2 = \frac{(\omega/v_s)^2}{1+(v_H/v_s)^2} \qquad (56)$$

Since all quantities on the right hand side are real, $\gamma^2 = p^2$ and the velocity of propagation is given by

$$v_p^{\,2} = v_s^{\,2}\left[1+(v_H/v_s)^2\right] \qquad (57)$$

and the damping coefficient g is zero. This is the same result that Chadwick [2] obtained for a shear wave traveling in an infinite medium. Hence just as in the pure elastic case the torsional wave in a solid rod travels with the shear velocity and is neither damped nor dispersed.

IV. Boundary Conditions

If the finite conductivity is retained, the boundary conditions must be applied to obtain further information from equation (55). It may easily be shown that for this displacement field the only remaining elastic boundary condition of a stress free surface to be satisfied is:

$$\tau_{r\theta} = G\left\{r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right\}\Big|_{r=b} = 0 \qquad (58)$$

If the form of assumed solution as given by equation (48) is retained, only the part containing $J_1(\lambda r)$ remains, since the condition of finite displacement at the center of the rod requires that the coefficient of $Y_1(\lambda r)$ vanishes. Hence equation (58) may be written:

$$AG\lambda_1^2 \frac{d}{d(\lambda_1 r)}\left(\frac{J_1(\lambda_1 r)}{\lambda_1 r}\right)\bigg|_{r=b} = 0 \qquad (58')$$

which may be satisfied by

$$\lambda_1 = 0 \qquad (59)$$

As in the elastic case, a possible solution for this value of λ_1 is

$$u_\theta = Ar \qquad (60)$$

which leaves equation (53) and hence (55) unchanged.

The boundary condition on the radial induced current is given by equation (27) as

$$j_r\Big|_{r=b} = 0 \qquad (61)$$

and, as in the displacement solution, only the $I_1(\lambda_2 r)$ term of equation (50) is retained. Therefore the boundary condition becomes:

$$I_1(\lambda_2 r)\Big|_{r=b} = 0 \qquad (61')$$

but $$I_1(\lambda_2 r) = -iJ_1(i\lambda_2 r)$$

for

$$-\pi < \arg(\lambda_2 r) \leq \frac{\pi}{2}$$

$$\frac{\pi}{2} < \arg(\lambda_2 r) \leq \pi$$

and equation (61') is:

$$J_1(i\lambda_2 b) = 0$$

The first root of this equation is the trivial one of zero which implies that the induced current vanishes. The next root is [13]:

$$i\lambda_2 b = 3.832 \qquad (62)$$

This corresponds to the lowest mode possible which is coupled to the electromagnetic fields and hence the first mode of interest. Selection of higher roots of equation (58') corresponds to modes that are highly damped and do not become important until the wavelength is much smaller than the radius of the rod. Likewise higher roots of equation (61') become important also under these conditions. These higher modes are equivalent to the second torsional mode in a solid elastic rod. The physical observation of this mode is extremely difficult.

V. A Numerical Example

For the purpose of illustrating the magnitude of the magnetoelastic effect a numerical example will be presented. Consider the following case which corresponds to an aluminum rod.

$$v_s = 3 \times 10^3 \ \mathrm{M/sec}$$

$$b = 1 \ \mathrm{cm}$$

$$1/\zeta^2 = \mu\sigma\omega = 40\omega$$

$$B = 10 \ \mathrm{webers/m^2}$$

$$\therefore \quad v_M^2 = 2.84 \times 10^4 \ \mathrm{M/sec}$$

and

$$(v_M/v_s)^2 = 4.39 \times 10^{-3}$$

First, the case of a perfect conductor will be considered. From equation (57) the change in the velocity of propagation is

$$v_p/v_s = 1 + 0.57 \times 10^{-3}$$

While this is capable of being measured experimentally a word of caution should be injected, namely that the assumed magnetic field is quite high and could only be obtained for a very short time. A more realistic value might be 1 weber/meter^2 which would yield

$$v_p/v_s = 1 + 0.57 \times 10^{-5}$$

the experimental measurement of which might justifiably be questioned.

Introduction of finite conductivity results in the velocity of propagation becoming a transcendental function of not only the magnetic field but also the frequency of the torsional wave. Therefore this mode will be dispersive as well as damped. This provides an excellent means of checking the theory by an experiment. If the same rod is considered and the numerical values are inserted into equation (55), the following results are obtained. Two values of γ^2 result; one corresponds to the mode of propagation that has been chosen for consideration; the other is a highly damped case with a very low velocity of propagation. For each of the γ^2 there are two roots which correspond to the righward and leftward traveling waves, hence the four roots of equation (55) are accounted for. The results of numerical calculation of the wave speed are shown in Fig. 3. It should be noted that these calculations involve taking the difference of large complex numbers which are approximately equal; therefore the values presented are for the purpose of showing the trend only. Upon first inspection the effect of the electromagnetic field seems quite large, however, this plot is for the phase velocity and the following must be considered. The signal, or information in a wave will be propagated with the group velocity which is given by

$$v_g = \frac{d\omega}{d\gamma}$$

Now if a plot of γ versus ω is made the group velocity may be determined. Upon investigation for this case the group velocity is found to be given by the shear velocity for frequencies at or above 100 kc and to fall off very rapidly to zero at approximately 50 kc. Therefore a lower limit on frequency is determined. Below this limit this mode will not propagate; most likely the pure elastic torsional wave would result. From 100 kc to 1 Mc the group as well as phase velocity is given by the shear velocity. Above 1 Mc the wavelength is shorter than the radius of the rod and higher modes would result. Hence if experiment measurements are to be made the range defined by Fig. 3 would be most useful. This type of behavior is quite similar to that found in waveguide problems.

In passing it is worthwhile noting that the experimental measurements of reference (9) were made in the range of frequencies defined by Fig. 3.

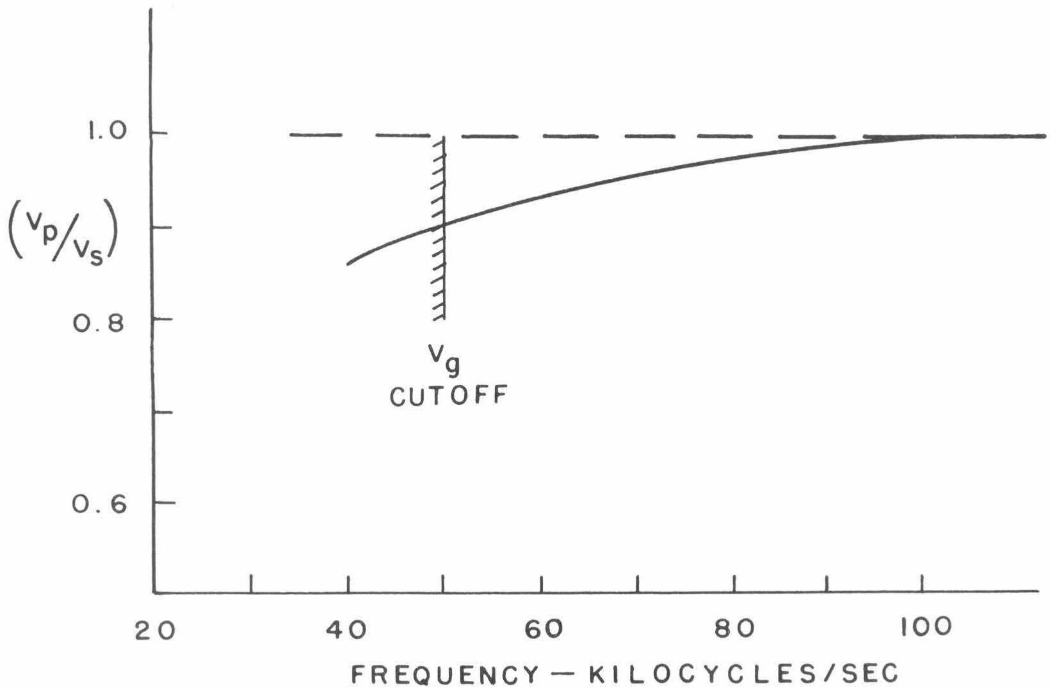

Fig. 3. Variation of phase velocity with frequency.

SECTION C: TANGENTIAL MAGNETIC FIELD

I. Case III: Magnetic Field Only

In the remaining two cases the applied magnetic field is tangential and is given by:

$$\vec{H_o} = \frac{I}{2\pi r} \hat{i}_\theta = \frac{1}{r} H_\theta \hat{i}_\theta \qquad (63)$$

where I is the total current being carried by the coaxial conductor which supplies the magnetic field.

The configuration to be considered and the coordinate system to be employed is shown in Fig. 4.

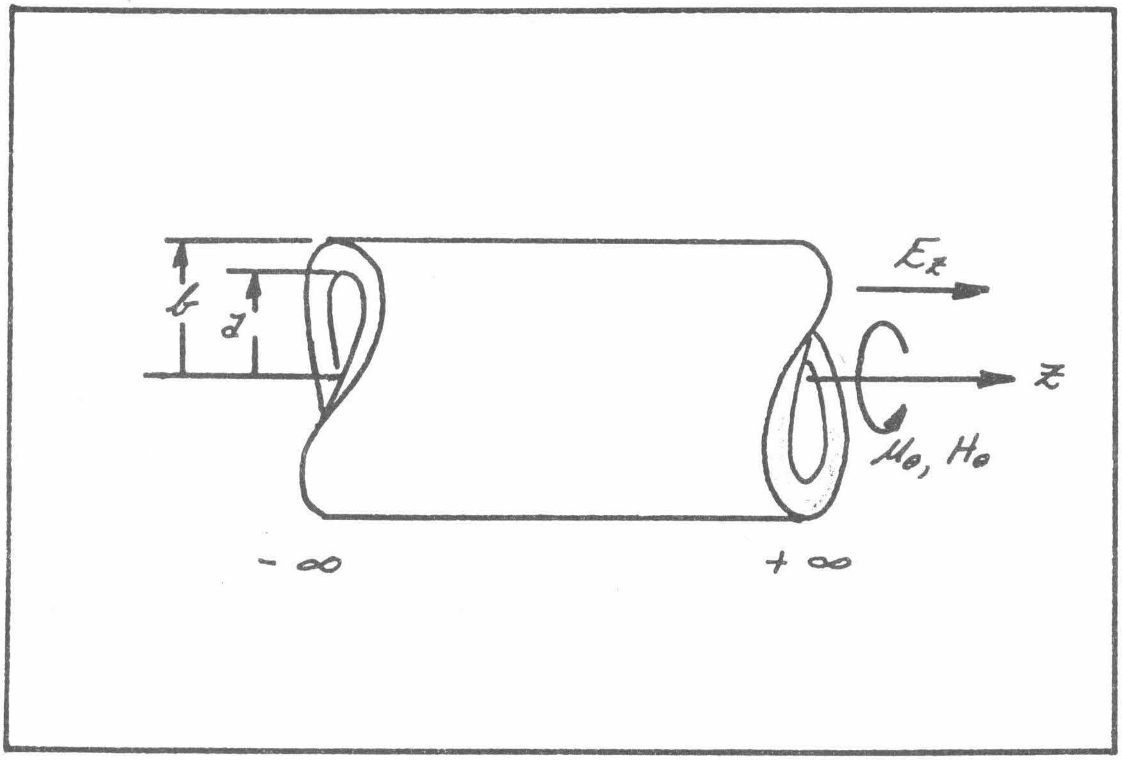

Fig. 4: Hollow rod with tangential magnetic field.

The governing equations are still given by equations (14) through (17) and the displacement field is completely specified by the tangential component u_θ. Upon substitution of this plus equation (63) into the governing equations, they become:

$$\left.\begin{aligned} i\gamma h_\theta - j_r &= 0 \\ -i\gamma h_r - \frac{dh_z}{dr} - j_\theta &= 0 \\ \frac{1}{r}\frac{d}{dr}(r h_\theta) - j_z &= 0 \end{aligned}\right\} \qquad (64)$$

$$\left.\begin{aligned} i\gamma e_\theta + i\omega\mu h_r &= 0 \\ -i\gamma e_r - \frac{de_z}{dr} + i\omega\mu h_\theta &= 0 \\ \frac{1}{r}\frac{d}{dr}(r e_\theta) + i\omega\mu h_z &= 0 \end{aligned}\right\} \qquad (65)$$

$$\vec{j} = \sigma \vec{e} \qquad (66)$$

$$\left.\begin{aligned} \frac{1}{r} H_\theta j_z &= 0 \\ G\left[\nabla^2 u_\theta - \frac{1}{r^2} u_\theta\right] + \rho\omega^2 u_\theta &= 0 \\ \frac{1}{r} H_\theta j_r &= 0 \end{aligned}\right\} \qquad (67)$$

The first and third equations of (67) imply that $j_r = j_z = 0$. This in turn through the remaining equations leads to the following conditions which must be satisfied in order that the equations will be consistent.

$u_r = 0$	h_r	$e_r = 0$	$j_r = 0$
u_θ	$h_\theta = 0$	e_θ	j_θ
$u_z = 0$	h_z	$e_z = 0$	$j_z = 0$

and the remaining equations are:

$$-i\gamma h_r - \frac{dh_z}{dr} - j_\theta = 0 \qquad (64)$$

$$\left.\begin{aligned} i\gamma e_\theta + i\omega\mu h_r &= 0 \\ \frac{1}{r}\frac{d}{dr}(r e_\theta) + i\omega\mu h_z &= 0 \end{aligned}\right\} \qquad (65)$$

$$j_\theta = \sigma e_\theta \qquad (66)$$

$$\nabla^2 u_\theta - \frac{1}{r^2} u_\theta + \frac{\rho\omega^2}{G} u_\theta = 0 \qquad (67)$$

Hence it is seen that the equations are not coupled in this case. In particular, equation (67) yields a purely elastic wave in a hollow cylinder, a problem which has been solved by Gazis[11, 12]. This case in itself hence is of no further interest. However, it will be instructive in the next case if the equation governing the electromagnetic wave is obtained. This is done exactly as before by elimination of the other variables and the equation is:

$$L h_r - i\sigma\omega\mu h_r = 0 \qquad (68)$$

which has the solution

$$h_r = a_1 I_1(\lambda_2 r) + b_1 K_1(\lambda_2 r) \qquad (69)$$

where

$$\lambda_2^2 = \gamma^2 + i\sigma\omega\mu \qquad (70)$$

The boundary condition as given by equation (27) is:

$$h_r \Big|_{\substack{r=a \\ r=b}} = 0 \qquad (71)$$

This is quite similar to the problem solved by Gazis except this involves modified Bessel functions with complex arguments. In his analysis of the simpler problem Gazis was forced to resort to the use of a digital computer, hence this case will not be pursued further, especially in view of the uncoupling of the elastic wave.

II. Case IV: Axial Electric Field

The configuration for this case is the same as that shown in Fig. 4 except an axial electric field is applied. This field is given by:

$$\vec{E_0} = E_z \hat{i}_z \qquad (72)$$

and induces the following magnetic field within the conductor.

$$(H_\theta)_E = \frac{\sigma E_z}{2}\left(r - \frac{b^2}{r}\right)$$

Hence the total magnetic field is given by

$$\vec{H_0} = \left[\frac{\sigma E_z}{2}\left(r - \frac{b^2}{r}\right) + \frac{1}{r} H_\theta\right] \hat{i}_\theta \qquad (73)$$

If these fields are introduced into the governing equations they become

$$\nabla \times \vec{h} = \vec{j} \qquad (74)$$

$$\nabla \times \vec{e} + \mu \frac{\partial \vec{h}}{\partial t} = 0 \qquad (75)$$

$$\vec{j} = \sigma \vec{e} \qquad (76)$$

$$G \nabla^2 \vec{u_0} - \rho \frac{\partial^2 \vec{u_0}}{\partial t^2} + \mu \left\{ -\left[\tilde{H}_\theta j_z + \sigma E_z h_\theta\right] \hat{i}_r + \sigma E_z h_r \hat{i}_\theta + \tilde{H}_\theta j_r \hat{i}_z \right\} = \mu \sigma E_z \tilde{H}_\theta \qquad (77)$$

where

$$\tilde{H}_\theta = \left[\frac{\sigma E_z}{2}\left(r - \frac{b^2}{r}\right) + \frac{1}{r} H_\theta\right] \qquad (78)$$

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$$G\,\nabla^2 \vec{u}_\theta - \rho \frac{\partial^2 \vec{u}_\theta}{\partial t^2} + \mu\Big\{-\left[\tilde{H}_\theta j_z + \sigma E_z h_\theta\right]\hat{i}_r + \sigma E_z h_r \hat{i}_\theta + \tilde{H}_\theta j_r \hat{i}_z\Big\} = \mu\sigma E_z \tilde{H}_\theta \tag{77}$$

where

$$\tilde{H}_\theta = \left[\frac{\sigma E_z}{2}\left(r - \frac{b^2}{r}\right) + \frac{1}{r}H_\theta\right] \tag{78}$$

The term on the left hand side of equation (77) represents a steady state stress due to the combined effects of the electromagnetic fields. Since the problem is linear it may be ignored in seeking the time dependent solution. Doing so, the equations are written in component form as:

$$\left.\begin{aligned} i\gamma h_\theta - j_r &= 0 \\ -i\gamma h_r - \frac{dh_z}{dr} - j_\theta &= 0 \\ \frac{1}{r}\frac{d}{dr}(r h_\theta) - j_z &= 0 \end{aligned}\right\} \qquad (74)$$

$$\left.\begin{aligned} i\gamma e_\theta + i\omega\mu h_r &= 0 \\ -i\gamma e_r - \frac{de_z}{dr} + i\omega\mu h_\theta &= 0 \\ \frac{1}{r}\frac{d}{dr}(r e_\theta) + i\omega\mu h_z &= 0 \end{aligned}\right\} \qquad (75)$$

$$\vec{j} = \sigma\vec{e} \qquad (76)$$

$$\left.\begin{aligned} \tilde{H}_\theta j_z + \sigma E_z h_\theta &= 0 \\ G\left[\nabla^2 u_\theta - \frac{1}{r^2}u_\theta\right] + \rho\omega^2 u_\theta + \mu\sigma E_z h_r &= 0 \\ \tilde{H}_\theta j_r &= 0 \end{aligned}\right\} \qquad (77')$$

The last of equation (77') implies $j_r = 0$, which leads as before to the following set of conditions:

$u_r = 0$	h_r	$j_r = 0$	$e_r = 0$
u_θ	$h_\theta = 0$	$j_\theta =$	e_θ
$u_z = 0$	h_z	$j_z = 0$	$e_z = 0$

The remaining equations are:

$$i\gamma h_r + \frac{dh_z}{dr} + j_\theta = 0 \tag{74}$$

$$\left.\begin{aligned} i\gamma e_\theta + i\omega\mu h_r &= 0 \\ \frac{1}{r}\frac{d}{dr}(r e_\theta) + i\omega\mu h_z &= 0 \end{aligned}\right\} \tag{75}$$

$$j_\theta = \sigma e_\theta \tag{76}$$

$$G\left[\nabla^2 u_\theta - \frac{1}{r^2} u_\theta\right] + \rho\omega^2 u_\theta + \mu\sigma E_z h_r = 0 \tag{77'}$$

Note that equations (74) through (76) are identical to the corresponding ones of the previous case, hence the equations of motion may be written down immediately.

$$L h_r - i\frac{1}{\delta^2} h_r = 0 \tag{78}$$

$$L u_\theta + (\omega/v_s)^2 + \frac{\mu\sigma E_z}{G} h_r = 0 \tag{79}$$

Equation (78) is identical to the previous case and may be solved directly. The solution as given by equation (69) is

$$h_r = A_1 I_1(\lambda_2 r) + B_1 K_1(\lambda_2 r) \tag{69}$$

with

$$\lambda_2^2 = \gamma^2 + i\frac{1}{\delta^2}$$

The coupling term of equation (79) is now determined and this equation may also be solved. The homogeneous solution is given by equation (48) as

$$(u_\theta)_o = A J_1(\lambda_1 r) + B Y_1(\lambda_1 r) \qquad (78)$$

with

$$\lambda_1^2 = \gamma^2 - (\omega / v_s)^2$$

The particular solution is obtained by means of the general expressions for a particular solution of an ordinary differential equation, i.e.

$$(u_\theta)_p = v_1(\lambda_1 r) J_1(\lambda_1 r) + v_2(\lambda_1 r) Y_1(\lambda_1 r) \qquad (80)$$

where if $\lambda_1 r = x$

and $\lambda_2 r = y$

the variable coefficients are given after some algebra by

$$\frac{d v_1}{d x} = - \frac{\pi x}{2} Y_1(x) \left[P I_1(y) + Q K_1(y) \right] \qquad (81)$$

$$\frac{d v_2}{d x} = \frac{\pi x}{2} J_1(x) \left[P I_1(y) + Q K_1(y) \right] \qquad (82)$$

with

$$P = - \frac{\mu \sigma E_z}{G} A$$

$$Q = - \frac{\mu \sigma E_z}{G} B$$

To obtain the complete solution integrals of the following form must be evaluated.

$$I = \int_0^x s\, J_1(s)\, I_1(\alpha_1 s)\, ds$$

where

$$\alpha_1 = \lambda_2/\lambda_1$$

This indefinite integral involving a product of Bessel functions with complex arguments is difficult to evaluate. In fact, the best that may conveniently be done is to expand one of the Bessel functions in an appropriate asymptotic expansion and integrate term by term. If this task is accomplished the boundary condition must be applied to obtain the numerical values of λ_1 and λ_2. Recalling that the task of evaluating λ_1, for the homogeneous case, which corresponds to case III, required machine computation the full difficulty of this problem will be recognized. It is therefore deemed an unwise course of action to pursue this problem in this manner any further. An alternative method to this general type of problem is suggested by Miles. [7] He studied the radial mode of vibration of a hollow rod as a shallow shell. Applying membrane theory he obtained a solution; however, even with this approach he was forced to restrict the problem to one in which the hollow rod had zero hoop stress.

SECTION D: CONCLUSIONS

The results of this investigation may be summarized as follows:

Case I. Solid rod with axial electric and magnetic fields.

It was found that a pure torsional wave that is coupled to the electromagnetic fields will not propagate under these conditions.

Case II. Solid rod with axial magnetic field only.

A pure torsional wave was found that will propagate over a restricted range of frequencies. Below this range the waves uncouple and the pure elastic torsional wave results. Above this range higher modes are introduced which are highly damped.

Case III. Hollow rod with tangential magnetic field only.

The elastic wave and magnetic effects are completely uncoupled.

Case IV. Hollow rod with tangential magnetic field and axial electric field.

The equations of motion are obtained and the secondary coupling due to the magnetic field induced by the axial electric field is noted. The coupling in this case is not thoroughly investigated due to mathematical difficulties which arise.

In general the effect of the magnetic field on a torsional wave is found to be quite small. Case II was found to be the most interesting from the viewpoint of a possible configuration for experiment investigation. However, if an axial displacement were permitted as well as the pure torsional displacement, Case I would become of equal importance.

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