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SUTURED KHOVANOV HOMOLOGY DISTINGUISHES BRAIDS FROM OTHER TANGLES

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Abstract. We show that the sutured Khovanov homology of a balanced tangle in the product sutured manifold $D \times I$ has rank 1 if and only if the tangle is isotopic to a braid.

1. Introduction

In [11], Khovanov constructed a categorification of the Jones polynomial that assigns a bigraded abelian group to each link in S^3. Sutured Khovanov homology is a variant of Khovanov's construction that assigns

- to each link $\mathbb{L}$ in the product sutured manifold $A \times I$ (see Section 2.1) a triply-graded vector space $\mathrm{SKh}(\mathbb{L})$ over $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$ [1, 19], where $A = S^1 \times [0,1]$ and $I = [0,1]$, and
- to each balanced, admissible tangle T in the product sutured manifold $D \times I$ (see Section 2.2) a bigraded vector space $\mathrm{SKh}(T)$ over $\mathbb{F}$ [13, 6], where $D = D^2$.

Khovanov homology detects the unknot [14] and unlinks [8, 3], and the sutured annular Khovanov homology of braid closures detects the trivial braid [2]. In this note, we prove that the sutured Khovanov homology of balanced tangles distinguishes braids from other tangles.

Theorem 1.1. *Let $T \subset D \times I$ be a balanced, admissible tangle. (See Subsection 2.2 for the definition.) Then $SKh(T) \cong \mathbb{F}$ if and only if T is isotopic to a braid in $D \times I$.*

Theorem 1.1 is one of many results about the connection between Floer homology and Khovanov homology, starting with the work of Ozsváth and Szabó [18]. This theorem is an analogue of the fact that sutured Floer homology detects product sutured manifolds [17, 10], which is also an ingredient in our proof. Other ingredients include a spectral sequence relating sutured Khovanov homology and sutured Floer homology [6], Meeks–Scott's theorem on finite group actions on product manifolds [15], and Kronheimer–Mrowka's theorem that Khovanov homology is an unknot detector [14].

Given a link $\mathbb{L} \subset A \times I$, the *wrapping number* of $\mathbb{L}$ is the minimal geometric intersection number of all links isotopic to $\mathbb{L}$ with the meridional disk of $A \times I$. Theorem 1.1 combined with the observations in [5] (see Proposition 2.4) imply:

Corollary 1.2. *Let $\mathbb{L} \subset A \times I$ be a link with wrapping number ω, then the group*

$$SKh(\mathbb{L};\omega) = \bigoplus_{i,j} SKh^i(\mathbb{L}; j, \omega)$$

is isomorphic to $\mathbb{F}$ if and only if $\mathbb{L}$ is isotopic to a closed braid in $A \times I$.

This corollary is an analogue of the fact that knot Floer homology detects fibered knots.

2. Preliminaries

In this section, we will review the basics about sutured manifolds [4] and sutured Khovanov homology [1, 19, 6, 5].

Definition 2.1. A *sutured manifold* (M, γ) is a compact, oriented 3–manifold M, a set $\gamma \subset \partial M$, and a choice of orientation on each component of $R(\gamma) = \partial M \setminus \text{int}(\gamma)$ such that:
• γ consists of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$,
• if we define $R_+(\gamma)$ (resp., $R_-(\gamma)$) to be the union of those components of $R(\gamma)$ whose normal vectors point out of (resp., into) M, then each component of $A(\gamma)$ is adjacent to a component of $R_+(\gamma)$ and a component of $R_-(\gamma)$.

As an example, let S be a compact oriented surface, $M = S \times I$, $\gamma = (\partial S) \times I$, $R_-(\gamma) = S \times \{0\}, R_+(\gamma) = S \times \{1\}$, then (M, γ) is a sutured manifold. In this case we say that (M, γ) is a *product sutured manifold*.

Definition 2.2. [9, Definition 2.2] A *balanced sutured manifold* is a sutured manifold (M, γ) satisfying
(1) M has no closed components.
(2) $T(\gamma) = \emptyset$.
(3) Every component of ∂M intersects γ nontrivially.
(4) $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$.

If (M, γ) is a balanced, sutured manifold, then $SFH(M, \gamma)$ will denote its *sutured Floer homology*, as defined by Juhász in [9]. Whenever γ is implicit (e.g., when M is a product), we shall omit it from the notation.

We will be interested in Khovanov-type invariants for certain links and tangles in product sutured manifolds.

2.1. Sutured Khovanov homology of links in $A \times I$.

Sutured annular Khovanov homology, originally defined in [1], [19] (see also [5]) associates to an oriented link $\mathbb{L}$ in the product sutured manifold $A \times I$ a triply-graded vector space

$$\text{SKh}(\mathbb{L}) = \bigoplus_{i,j,k} \text{SKh}^i(\mathbb{L}; j, k),$$

which is an invariant of the oriented isotopy class of $\mathbb{L} \subset A \times I$.

To define it, one chooses a diagram $\mathcal{D}_{\mathbb{L}}$ of $\mathbb{L}$ on $A \times \{\frac{1}{2}\}$. By filling in one boundary component of $A \times \{\frac{1}{2}\}$ with a disk marked with a basepoint X at its center and the other boundary component with a disk marked with a basepoint at its center, one obtains a diagram on $S^2 - \{X, O\}$. Ignoring the X basepoint yields a diagram on $\mathbb{R}^2 = S^2 - \{O\}$ from which the ordinary bigraded Khovanov chain complex

$$\text{CKh}(\mathcal{D}_{\mathbb{L}}) := \bigoplus_{i,j} \text{CKh}^i(\mathcal{D}_{\mathbb{L}}; j)$$

can be constructed from a cube of resolutions. Here, i and j are the homological and quantum gradings, respectively. The basepoint X gives rise to a filtration on $\text{CKh}(\mathcal{D}_{\mathbb{L}})$, and $\text{SKh}(\mathbb{L})$ is the homology of the associated graded object.

To define this filtration, choose an oriented arc from X to O missing all crossings of $\mathcal{D}_{\mathbb{L}}$. As described in [7, Sec. 4.2], the generators of $\text{CKh}(\mathcal{D}_{\mathbb{L}})$ are in one-to-one correspondence with *oriented* resolutions, where the counterclockwise orientation on each circle corresponds to the generator v_+. The "k" grading of an oriented resolution is defined to be the algebraic intersection number of this resolution with our oriented arc. Roberts proves ([19, Lem. 1]) that the Khovanov differential does not increase this extra grading.

One therefore obtains a bounded filtration,

$$0 \subseteq \ldots \subseteq \mathcal{F}_{n-1}(\mathcal{D}_{\mathbb{L}}) \subseteq \mathcal{F}_n(\mathcal{D}_{\mathbb{L}}) \subseteq \mathcal{F}_{n+1}(\mathcal{D}_{\mathbb{L}}) \subseteq \ldots \subseteq \mathrm{CKh}(\mathcal{D}_{\mathbb{L}}),$$

where $\mathcal{F}_n(\mathcal{D}_{\mathbb{L}})$ is the subcomplex of $\mathrm{CKh}(\mathcal{D}_{\mathbb{L}})$ generated by oriented resolutions with k grading at most n. Let

$$\mathcal{F}_n(\mathcal{D}_{\mathbb{L}}; j) = \mathcal{F}_n(\mathcal{D}_{\mathbb{L}}) \cap \bigoplus_i \mathrm{CKh}^i(\mathcal{D}_{\mathbb{L}}; j).$$

The sutured annular Khovanov homology groups of $\mathbb{L}$ are defined to be

$$\mathrm{SKh}^i(\mathbb{L}; j, k) := H^i\left(\frac{\mathcal{F}_k(\mathcal{D}_{\mathbb{L}}; j)}{\mathcal{F}_{k-1}(\mathcal{D}_{\mathbb{L}}; j)}\right).$$

It is an immediate consequence of the definitions that if $\mathbb{L}$ has wrapping number ω, then $\mathrm{SKh}^i(\mathbb{L}; j, k) \cong 0$ for $k \notin \{-\omega, -(\omega-2), \ldots, \omega-2, \omega\}$.

We shall denote by $\Sigma(A \times I, \mathbb{L})$ the sutured manifold obtained as the double cover of $A \times I$ branched along $\mathbb{L}$ (cf. [5, Rmk. 2.6]), where γ is the cover of $(\partial A) \times I$, and R_+ (resp., R_-) is the cover of $A \times \{1\}$ (resp., $A \times \{0\}$).

2.2. Sutured Khovanov homology of balanced tangles in $D \times I$. A tangle T in the product sutured manifold $(D \times I, \gamma)$ is said to be *admissible* if $\partial T \cap \gamma = \emptyset$, and *balanced* if $|T \cap (D \times \{0\})| = |T \cap (D \times \{1\})|$. To make sense of tangle composition (stacking), we will fix an identification of D with the standard unit disk in $\mathbb{C}$ and assume that ∂T intersects both $D \times \{0\}$ and $D \times \{1\}$ along the real axis.

The sutured Khovanov homology of an admissible, balanced tangle in $D \times I$ was defined by Khovanov in [13, Sec. 5] in the course of constructing a categorification of the reduced n–colored Jones polynomial. An elaboration of Khovanov's construction is given in [6, Sec. 5], where it is also related to sutured Floer homology. We briefly recall the main points of the construction here.

Let $T \subset D \times I$ be a balanced, admissible tangle and choose a diagram $\mathcal{D}_T$ of T on $[-1, 1] \times I$. Then the sutured Khovanov homology of T, $\mathrm{SKh}(T) = \bigoplus_{i,j} \mathrm{SKh}^i(T; j)$, is obtained as the homology of the complex,

$$\mathrm{CKh}(\mathcal{D}_T) := \bigoplus_{i,j} \mathrm{CKh}^i(\mathcal{D}_T; j)$$

obtained as follows.

Number the c crossings, and construct a Khovanov-type *cube of resolutions* whose vertices are in one-to-one correspondence with elements of $\{0,1\}^c$. Associated to each such $\mathcal{I} \in \{0,1\}^c$ is a *complete resolution* $R_{\mathcal{I}}$ with $a_{\mathcal{I}}$ closed components (circles) $T_1, \ldots, T_{a_{\mathcal{I}}}$ and $b_{\mathcal{I}}$ non-closed components (arcs) $T_{a_{\mathcal{I}}+1}, \ldots, T_{a_{\mathcal{I}}+b_{\mathcal{I}}}$. We say that $R_{\mathcal{I}}$ *backtracks* if the boundary of at least one of its non-closed components is contained in $[-1, 1] \times \{1\}$. We now assign to the corresponding vertex in the cube of resolutions the vector space

$$V(R_{\mathcal{I}}) := \begin{cases} 0 & \text{if } R_{\mathcal{I}} \text{ backtracks} \\ \Lambda^*(Z(R_{\mathcal{I}})) & \text{otherwise,} \end{cases}$$

where

$$Z(R_{\mathcal{I}}) := \frac{\mathrm{Span}_{\mathbb{F}}\{[T_1], \ldots, [T_{a_{\mathcal{I}}+b_{\mathcal{I}}}]\}}{\mathrm{Span}_{\mathbb{F}}([T_{a_{\mathcal{I}}+1}], \ldots, [T_{a_{\mathcal{I}}+b_{\mathcal{I}}}])}$$

is the vector space formally generated by the closed components of $R_{\mathcal{I}}$, which for convenience we realize as a quotient space of the vector space formally generated by *all* components of $R_{\mathcal{I}}$.

As in ordinary Khovanov homology, if $\mathcal{I}'$ is an *immediate successor* of $\mathcal{I}$ in the language of [18, Sec. 4] and [6, Sec. 4], then one obtains $R_{\mathcal{I}'}$ from $R_{\mathcal{I}}$ by either merging two components T_i and T_j of $R_{\mathcal{I}}$ to form a component T' of $R'_{\mathcal{I}}$ or splitting a single component T of $R_{\mathcal{I}}$ into two components T'_i and T'_j of $R_{\mathcal{I}'}$, and in both cases leaving all other components unchanged.

With the above understood, we now associate a map

$$F_{R_{\mathcal{I}} \to R_{\mathcal{I}'}} : V(R_{\mathcal{I}}) \to V(R_{\mathcal{I}'})$$

to every pair of immediate successors as follows.

If at least one of $R_{\mathcal{I}}$, $R_{\mathcal{I}'}$ backtracks, we define $F_{R_{\mathcal{I}} \to R_{\mathcal{I}'}} := 0$.

Otherwise, $R_{\mathcal{I}} \to R_{\mathcal{I}'}$ is either a merge or split cobordism involving either two closed components or one closed component and one non-backtracking arc.

If $R_{\mathcal{I}} \to R_{\mathcal{I}'}$ is a merge, we define $F_{R_{\mathcal{I}} \to R_{\mathcal{I}'}}$ to be the composition

$$V(R_{\mathcal{I}}) \xrightarrow{\pi} \frac{V(R_{\mathcal{I}})}{[T_i] \sim [T_j]} \xrightarrow{\alpha} V(R'_{\mathcal{I}}) \ ,$$

where α is the isomorphism on exterior algebras induced by the isomorphism

$$\frac{Z(R_{\mathcal{I}})}{[T_i] \sim [T_j]} \cong Z(R_{\mathcal{I}'})$$

identifying $[T_i] = [T_j]$ with $[T']$.

If $R_{\mathcal{I}} \to R_{\mathcal{I}'}$ is a split, we define $F_{R_{\mathcal{I}} \to R_{\mathcal{I}'}}$ to be the composition

$$V(R_{\mathcal{I}}) \xrightarrow{\alpha^{-1}} \frac{V(R_{\mathcal{I}})}{[T'_i] \sim [T'_j]} \xrightarrow{\varphi} V(R_{\mathcal{I}'}) \ ,$$

where $\varphi(a) := ([T'_i] + [T'_j]) \wedge \widetilde{a}$, and $\widetilde{a}$ is any lift of a in $\pi^{-1}(a)$.

The image of $\theta \in V(R_{\mathcal{I}})$ under the boundary map ∂ on the complex is now defined to be

$$\partial(\theta) := \sum_{R_{\mathcal{I}'}} F_{R_{\mathcal{I}} \to R_{\mathcal{I}'}}(\theta),$$

where the sum is taken over all immediate successors $\mathcal{I}'$ to $\mathcal{I}$. Extend linearly.

Remark 2.3. If T is an admissible (n, n) tangle in $D \times I$ and $\mathcal{D}_T$ is a diagram of T, then we can alternatively associate to T a left H^n–module, $\mathcal{F}(\mathcal{D}_T)$, as in [12], by viewing T as a tangle with $2n$ upper endpoints (cf. [6, Rmk. 5.9]). The chain complex $\mathrm{CKh}(\mathcal{D}_T)$ may then be identified with $\vec{\mathbf{v}}_- \otimes_{H^n} \mathcal{F}(\mathcal{D}_T)$, where $\vec{\mathbf{v}}_-$ is the right H^n module constructed as follows. Let b denote the fully-nested crossingless match on $2n$ points; then $\vec{\mathbf{v}}_-$ is the two-sided ideal of the H^n module $\mathcal{F}(W(b)b)$ corresponding to the generator whose strands are all labeled with a v_-. Via the correspondence between *oriented resolutions* and Khovanov generators described in the previous section (cf. [7, Sec. 4.2]), we may then identify $\mathrm{CKh}(\mathcal{D}_T)$ as the quotient complex obtained from the ordinary Khovanov complex of the closure, $\widehat{\mathcal{D}}_T$, of $\mathcal{D}_T$ by the subcomplex generated by all generators with Roberts' "k"–grading less than n. This has the effect of setting to 0 any vertex associated to a backtracking resolution and treating the non-backtracking non-closed components of a resolution just as basepointed strands are treated in Khovanov's *reduced* theory.

Comparing the above description with the description of the sutured annular Khovanov invariant in the previous section, we have:

Proposition 2.4. [5, Thm. 3.1] *If $\mathbb{L} \subset A \times I$ is an oriented annular link with wrapping number ω, and T_θ is the oriented, admissible balanced tangle obtained by decomposing $A \times I$ along a meridional disk D_θ for which $|\mathbb{L} \cap D_\theta| = \omega$,*

$$SKh^i(\mathbb{L}; j, \omega) \cong SKh^i(T_\theta; j).$$

Since all but one resolution of a braid backtracks, we have:

Proposition 2.5. *If $T \subset D \times I$ is isotopic to a braid, then $SKh(T) \cong \mathbb{F}$.*

3. Proof of the main theorem

Definition 3.1. A tangle $T \subset D \times I$ is a *string link* if it consists of proper arcs, each of which has one end on $D \times \{0\}$ and the other end on $D \times \{1\}$.

As a consequence, a string link T contains no closed components, and T does not backtrack.

Lemma 3.2. *Let $T \subset D \times I$ be a balanced, admissible tangle, then $\dim_{\mathbb{F}} SKh(T)$ is odd if and only if T is a string link.*

Proof. We observe that if two tangles T_+, T_- differ by a crossing change, then the corresponding chain complexes $\mathrm{CKh}(T_+)$ and $\mathrm{CKh}(T_-)$ have the same set of generators, thus the parities of the total dimensions of their homology are the same.

If a tangle T has closed components, after crossing changes we can transform T to a tangle T' with a diagram $\mathcal{D}'$ containing a trivial loop. This loop persists in any complete resolution of $\mathcal{D}'$, so it follows from the construction that the dimension of $\mathrm{CKh}(\mathcal{D}')$ is even, hence $\dim_{\mathbb{F}} \mathrm{SKh}(T)$ is even.

If T backtracks, after crossing changes we can transform T to a tangle T' with an arc which can be isotoped rel boundary into $D \times \{0\}$ or $D \times \{1\}$ without crossing other components. We can find a diagram $\mathcal{D}'$ of T' such that any complete resolution of $\mathcal{D}'$ backtracks. So $\mathrm{CKh}(\mathcal{D}') = 0$, and $\dim_{\mathbb{F}} \mathrm{SKh}(T)$ is even.

If T is a string link, after crossing changes we can transform T to a braid B. By Proposition 2.5, $\mathrm{SKh}(B) \cong \mathbb{F}$, so $\dim_{\mathbb{F}} \mathrm{SKh}(T)$ is odd. □

Definition 3.3. A tangle $T \subset D \times I$ is *split*, if there exists a 3–ball $B \subset D \times I$, such that $L_2 = T \cap B$ is a link and $L_2 \neq T$. In this case, let $T_1 = T - L_2$, then we write $T = T_1 \sqcup L_2$. We say T is *nonsplit* if it is not split.

A tangle $T \subset D \times I$ is *nonprime*, if there exists a 3–ball $B \subset D \times I$, such that $T_2 = T \cap B$ is a $(1, 1)$–tangle in B, and T_2 does not cobound a disk with any arc in ∂B. In this case, Let $T_1 \subset D \times I$ be the tangle obtained by replacing T_2 with a trivial

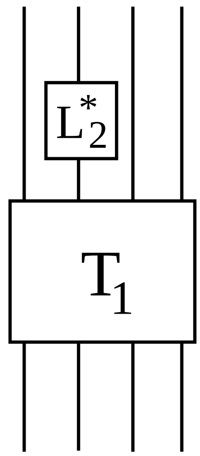

FIGURE 1. The tangle $T = T_1 \# L_2$, realized as a composition of T_1 and $L_2^{*,n}$.

arc in B, and let L_2 be the link obtained from T_2 by connecting the two ends of T_2 by an arc in ∂B. We denote $T = T_1 \# L_2$. We say T is *prime* if there does not exist such a B.

Lemma 3.4. *Let (M, γ) be the sutured manifold which is the double branched cover of $D^2 \times I$ branched along T. Then M is irreducible if and only if T is nonsplit and prime.*

Proof. The conclusion follows from the Equivariant Sphere Theorem [16] by the same argument as in [8, Proposition 5.1]. □

Lemma 3.5. *If $T = T_1 \# L_2$ is a nonprime string link, then*

$$SKh(T) \cong SKh(T_1) \otimes Kh_r(L_2).$$

In the above, $Kh_r(L_2)$ denotes the reduced Khovanov homology of L_2.

Proof. We choose a diagram $\mathcal{D}_T$ of T realized as the composition of diagrams $\mathcal{D}_{T_1}$ of T_1 and $\mathcal{D}_{L_2^{*,n}}$ $L_2^{*,n}$, where $L_2^{*,n}$ is an (n, n) tangle obtained from L_2 by removing a neighborhood of a point near the connected sum region and adjoining $n - 1$ trivial strands as pictured in Figure 3.

Now we claim that

$$\mathrm{CKh}(\mathcal{D}_T) \cong \mathrm{CKh}(\mathcal{D}_{T_1}) \otimes_{\mathbb{F}} \mathrm{CKh}(\mathcal{D}_{L_2^{*,n}}).$$

Since $\mathrm{CKh}(\mathcal{D}_{L_2^{*,n}})$ is canonically chain isomorphic to $\mathrm{CKh}(\mathcal{D}_{L_2^{*,1}})$, and the homology of the latter complex is the reduced Khovanov homology of L_2 with $\mathbb{F}$ coefficients, the lemma will then follow from the Künneth theorem.

To see the claim, note first that each resolution R of $\mathcal{D}_T$ is obtained by stacking a resolution R_1 of $\mathcal{D}_{T_1}$ and R_2 of $\mathcal{D}_{L_2^{*,n}}$.

Moreover:

- R backtracks iff at least one of R_1, R_2 backtracks, and
- If R does not backtrack, then the number of closed components of R is the sum of the number of closed components of R_1 and R_2.

Hence, the $\mathbb{F}$–vector space underlying the chain complex $\mathrm{CKh}(\mathcal{D}_T)$ is canonically isomorphic to $\mathrm{CKh}(\mathcal{D}_{T_1}) \otimes_{\mathbb{F}} \mathrm{CKh}(\mathcal{D}_{L_2^{*,n}})$.

To verify that the boundary map ∂_T on $\mathrm{CKh}(\mathcal{D}_T)$ agrees with the induced boundary map on the tensor product, i.e.:

$$\partial_T = \partial_{T_1} \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial_{L_2^{*,n}},$$

it is sufficient to verify that the two maps agree on any decomposable generator $\theta = \theta_1 \otimes \theta_2$ of $\mathrm{CKh}(\mathcal{D}_T)$ associated to a resolution $R = (R_1, R_2)$. We may further assume, without loss of generality, that R does not backtrack.

By definition

$$\partial_T(\theta) = \sum_{R'=(R_1',R_2')} F_{R\to R'}(\theta)$$

where the sum above is taken over all *immediate successors* R' to R.

But if $R' = (R_1', R_2')$ is an immediate successor of R, then either R_1' is an immediate successor of R_1 and $R_2' = R_2$, or vice versa. Assume for definiteness that it is the former, the latter case being analogous.

If R' backtracks, then so does R_1', so:

$$F_{R\to R'}(\theta) = \left(F_{R_1\to R_1'} \otimes \mathrm{Id}\right)(\theta_1 \otimes \theta_2) = 0.$$

If R' does not backtrack, then the saddle cobordism connecting R_1 to R_1' is a merge (resp., split) connecting either

- two closed components of R_1 (resp., of R_1'), or
- one closed and one vertical component of R_1 (resp., of R_1').

In either case, we see that

$$F_{R\to R'}(\theta) = \left[F_{R_1\to R_1'} \otimes \mathrm{Id}\right](\theta_1 \otimes \theta_2).$$

We conclude that

$$\begin{aligned}
\partial_T(\theta) &= \left[\sum_{R'=(R_1',R_2')} F_{R\to R'}\right](\theta) \\
&= \left[\left(\sum_{R_1'} F_{R_1\to R_1'}\right) \otimes \mathrm{Id} + \mathrm{Id} \otimes \left(\sum_{R_2'} F_{R_2\to R_2'}\right)\right](\theta_1 \otimes \theta_2) \\
&= \left[\partial_{T_1} \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial_{L_2^{*,n}}\right](\theta_1 \otimes \theta_2),
\end{aligned}$$

as desired. □

Proposition 3.6. *Suppose that $T \subset D^2 \times I$ is a balanced, admissible tangle. If the double branched cover of $D^2 \times I$ branched along T is a product sutured manifold, then T is isotopic to a braid.*

Proof. Let $\pi\colon F \times I \to D^2 \times I$ be the double branched covering map, then the nontrivial deck transformation ρ is an involution on $F \times I$ that preserves $F \times \partial I$ setwise. By Meeks–Scott [15, Theorem 8.1], ρ is conjugate to a map preserving the product structure.[1] In particular, $\pi^{-1}(T)$, being the set of fixed points of ρ, is homeomorphic to $P \times I \subset F \times I$ for some finite set $P \subset F$, via a homeomorphism of $F \times I$ which preserves $F \times \partial I$. It follows that T is isotopic to a braid. □

Proposition 3.7. *A knot $K \subset S^3$ is the unknot if and only if $Kh_r(K) \cong \mathbb{F}$.*

Proof. This result is essentially a theorem of Kronheimer and Mrowka [14]. The original theorem of Kronheimer and Mrowka states that K is the unknot if and only if $Kh_r(K;\mathbb{Z}) \cong \mathbb{Z}$, where the coefficients ring is $\mathbb{Z}$ while ours is $\mathbb{F}$. However, the version with $\mathbb{F}$ coefficients easily follows from Kronheimer and Mrowka's argument. As shown in [14, Corollary 1.3],

$$\operatorname{rank} Kh_r(K;\mathbb{Z}) \geq \operatorname{rank} I^\natural(K).$$

Kronheimer and Mrowka proved that $\operatorname{rank} I^\natural(K) > 1$ when K is nontrivial. (See the paragraph after [14, Corollary 1.3].) So $\operatorname{rank} Kh_r(K;\mathbb{Z}) > 1$ when K is nontrivial. It follows from the universal coefficients theorem that $\dim_{\mathbb{F}} Kh_r(K;\mathbb{F}) > 1$ when K is nontrivial. □

Proof of Theorem 1.1. By Lemma 3.2, if $\mathrm{SKh}(T) \cong \mathbb{F}$, then T is a string link. In particular, T has no closed components, hence T must be nonsplit.

Since T has no closed components, if T is nonprime it must be the connected sum of a tangle with a knot (rather than a link). Suppose that $T = T_1 \# K_2$, where K_2 is a knot. Then it follows from Lemma 3.5 that $Kh_r(K_2) \cong \mathbb{F}$. Using Proposition 3.7, we conclude that K_2 is the unknot. Hence T is prime.

Since T is nonsplit and prime, Lemma 3.4 implies that $\Sigma(D \times I, T)$ is irreducible. Suppose that $\mathrm{SKh}(T) \cong \mathbb{F}$. By [6, Proposition 5.20], there is a spectral sequence whose E^2 term is $SKh(T)$ and whose E^∞ term is the sutured Floer homology group $SFH(\Sigma(D \times I, T))$. Hence $SFH(\Sigma(D \times I, T)) \cong \mathbb{F}$. In [17, 10], it is shown that an irreducible balanced sutured manifold (M, γ) is a product sutured manifold if and only if $SFH(M, \gamma) \cong \mathbb{F}$. Hence $\Sigma(D \times I, T)$ is a product sutured manifold. Proposition 3.6 then implies that T is isotopic to a braid. □

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References

[1] M. Asaeda, J. Przytycki, and A. Sikora, *Categorification of the Kauffman bracket skein module of I-bundles over surfaces*, Algebr. Geom. Topol. **4** (2004) 1177–1210.

[2] J. Baldwin and J. Grigsby, *Categorified invariants and the braid group* (2012). To appear in Proc. Amer. Math. Soc., available at arXiv:math.GT/1212.2222.

[1] A homeomorphism of $X \times Y$ *preserves the product structure* if it is the product of homeomorphisms of X and Y.

[3] J. Batson and C. Seed, *A link splitting spectral sequence in Khovanov homology* (2013). Available at arXiv:math.GT/1303.6240.

[4] D. Gabai, *Foliations and the topology of* 3*–manifolds*, J. Differential Geom. **18** (1983), no. 3, 445–503.

[5] J. Grigsby and S. Wehrli, *Khovanov homology, sutured Floer homology, and annular links*, Algebr. Geom. Topol. **10** (2010), no. 4, 2009–2039.

[6] ———, *On the colored Jones polynomial, sutured Floer homology, and knot Floer homology*, Adv. Math. **223** (2010), no. 6, 2114–2165.

[7] ———, *On gradings in Khovanov homology and sutured Floer homology*, in Topology and geometry in dimension three, Vol. 560 of *Contemporary Mathematics*, 111–128, Amer. Math. Soc., Providence, RI (2011).

[8] M. Hedden and Y. Ni, *Manifolds with small Heegaard Floer ranks*, Geom. Topol. **14** (2010), no. 3, 1479–1501.

[9] A. Juhász, *Holomorphic discs and sutured manifolds*, Algebr. Geom. Topol. **6** (2006) 1429–1457.

[10] ———, *Floer homology and surface decompositions*, Geom. Topol. **12** (2008), no. 1, 299–350.

[11] M. Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), no. 3, 359–426.

[12] ———, *A functor-valued invariant of tangles*, Algebr. Geom. Topol. **2** (2002) 665–741.

[13] ———, *Categorifications of the colored Jones polynomial*, J. Knot Theory Ramifications **14** (2005), no. 1, 111–130.

[14] P. Kronheimer and T. Mrowka, *Khovanov homology is an unknot-detector*, Publ. Math. Inst. Hautes Études Sci. **113** (2011), no. 1, 97–208.

[15] W. Meeks and P. Scott, *Finite group actions on* 3*-manifolds*, Invent. Math. **86** (1986), no. 2, 287–346.

[16] W. Meeks, L. Simon, and S. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Ann. of Math. (2) **116** (1982), no. 3, 621–659.

[17] Y. Ni, *Knot Floer homology detects fibred knots*, Invent. Math. **170** (2007), no. 3, 577–608.

[18] P. Ozsváth and Z. Szabó, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. **194** (2005), no. 1, 1–33.

[19] L. Roberts, *On knot Floer homology in double branched covers* (2007). Available at arXiv:math.GT/0706.0741.

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