An H_∞-Optimal Alternative to the FxLMS Algorithm[1]

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Abstract

We study a general setting of active noise cancellation problems from the H_∞ point of view and present a solution that *optimally* limits the *worst case* energy gain from the interfering measurement errors, external disturbances, and initial condition uncertainty to the residual noise. The optimal bounding of this energy gain is the main characteristic of the proposed solution. To impose a finite impulse response (FIR) structure on the controller, we suggest an adaptation scheme for the weight vector of an FIR filter that approximates the H_∞-optimal solution. Our discussions in this paper explain; (i) why and how this new adaptive scheme generalizes previous results on the H_∞-optimality of the LMS algorithm, (ii) why it is an alternative for the widely used Filtered-X Least-Mean-Squares (FxLMS) algorithm, and (iii) how the formulation provides an appropriate framework to address the issues of modeling error and robustness. Simulations are used to compare the performance of the proposed (approximate) H_∞-optimal adaptive scheme with the FxLMS algorithm.

1 Introduction

The Least-Mean Squares (LMS) adaptive algorithm [1] has been used for over 35 years as the center piece of a wide variety of adaptive algorithms. Despite numerous successful applications, it was only recently that the H_∞ optimality of the LMS algorithm was established [2], and its important properties, such as bounds on adaptation rates, were rigorously derived.

Soon after its introduction, however, the direct implementation of the LMS algorithm was found inadequate for some applications. This sparked a variety of mostly heuristic implementation schemes in which the signals entering the LMS-based adaptation engine and/or its output were passed through appropriate filters (see [1] Chapter 11, and [5] Chapters 3 and 5 for instance). Investigating the H_∞-optimality and performance bounds for these, more general, classes of adaptive algorithms is the primary objective of this paper.

The general setup of the ANC problem (Figure 1) captures some common features among all these modified adaptive algorithms. Thus, our discussion here concentrates on an H_∞-optimal adaptive solution to the ANC problem. For a fair assessment of the performance of the H_∞-optimal adaptive scheme, a classical solution to the ANC problem (i.e. Filtered-X Least-Mean-Squares (FxLMS) algorithm) is used as a basis for comparison.

It is important to note that our approach is not a modification to the existing FxLMS algorithm. Instead it derives a new solution to the exact same problem for which FxLMS algorithm was devised. It presents an estimation interpretation of the adaptive ANC problem, for which an H_∞ solution is straight forward ([4,7,8] and references therein), and provides an appropriate framework in which main concerns associated with the FxLMS algorithm (namely stability, convergence and the guarantee of error bounds) are directly addressed without restrictive assumptions. This, in our view, distinguishes our approach from those in the literature that establish bounds and convergence properties for existing adaptive control algorithms ([3] for instance), or modify the existing algorithms for improved convergence and/or performance properties [6].

This paper is organized as follows. Section 2 provides a brief description of the active noise cancellation problem. Section 3 presents the estimation interpretation of the problem and the subsequent problem formulation. Section 4 discusses the H_∞-Optimal solution and its main features. Section 5 outlines the implementation scheme for the H_∞-optimal adaptive algorithm. Section 6 contains simulation results. Section 7 concludes the paper with a summary and final remarks.

2 Background

Figure 1 depicts the active noise cancellation (ANC) problem as it is addressed in this paper. Here, (i) $x(k)$ is the reference signal, (ii) $W(k) \triangleq [w_0(k)\ w_1(k)\ \ldots\ w_N(k)]^T$ is the *adaptable* weight vector in the FIR filter, (iii) $u(k) \triangleq [x(k)\ x(k-1)\ \cdots\ x(k-N)]\, W(k)$ is the control signal applied to the secondary path, (iv) $y(k)$ is the output of the secondary path (aimed at cancelling $d(k)$), and (v) $e(k)$ is the residual noise which is made available to the adaptation scheme that updates $W(k)$.

The FxLMS algorithm is adaptive control's classical solution to the ANC problem [5]. The objective of this adaptive scheme is to minimize the *instantaneous* squared error, $e^2(k)$. To achieve this, it intends to follow the LMS update criterion

$$W(k+1) = W(k) - (\mu/2) \times \nabla e^2(k) \qquad (1)$$

(i.e. to recursively adapt the weight vector in the negative gradient direction). Here μ is the adaptation rate, $e(k) = d(k) - s(k) * u(k)$ is the error signal, $s(k)$ is the impulse response of the secondary path, and "$*$" indicates convolution. The FxLMS algorithm, then *approximates*

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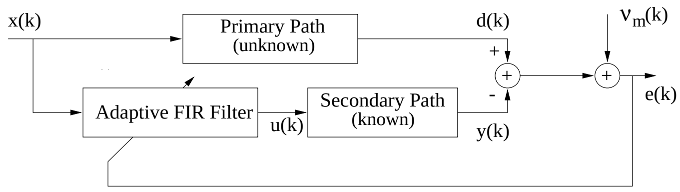

Fig. 1: A block diagram equivalent for the active noise cancellation problem

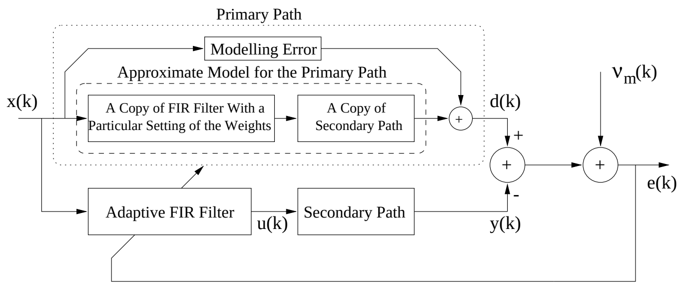

Fig. 2: Restatement of the ANC problem, where the primary path is replaced with an approximate model

the instantaneous gradient in the weight vector update with $\nabla e^2(n) \cong -2\left[x'(k)\ x'(k-1)\ \cdots\ x'(k-N)\right]^T e(k)$ where $x'(k) \triangleq s(k)*x(k)$ represents a filtered version of the reference signal (and hence the name Filtered-X LMS). In practice, however, only an approximate model of the secondary path (obtained via some identification scheme) is known, and it is this approximate model that is used to filter the reference signal.The length of the FIR filter is a design parameter. For broadband input signals $W(k)$ must essentially represent the impulse response of $P(z)S^{-1}(z)$. For narrow-band input signals, roughly speaking, $W(k)$ must cover a substantial fraction of input signal period. For further discussion on the derivation and analysis of the FxLMS algorithm please refer to [1,5] and the references therein.

3 Problem Formulation

The objective of ANC is to generate a control signal, $u(k)$, such that the output of the secondary path, $y(k)$, is in some measure (to be specified later) close enough to the output of the primary path, $d(k)$. For this to materialize, *the series connection of the FIR filter (for some particular setting of the weight vector) and the known secondary path must be an appropriate approximate model for the unknown primary path.* This observation, forms the basis for an estimation interpretation of the ANC problem.

Figure 2 depicts the suggested approximate model for the primary path. The series connection of a replica of the FIR filter (for a particular but unknown setting of the weight vector) and a replica of the secondary path are used to form this approximate model. Note that as long as the output of the "modeling error" block in Figure 2 is bounded, it can be treated as a component of the additive measurement disturbance signal, $\mathcal{V}_m(k)$.

In the estimation based approach to adaptive ANC our goal will be to construct $y(k)$ as the *estimate* of $d(k)$, based on the available measurements. To set the stage

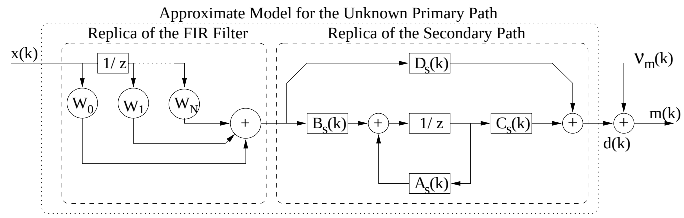

Fig. 3: Block diagram for an estimation interpretation of ANC problem

for the formulation of the estimation problem in Section 3.1, a closer look at the main signals in Figure 2 will be helpful. Note that $e(k) = d(k) - y(k) + \mathcal{V}_m(k)$, where (i) $e(k)$ is the available measurement, (ii) $\mathcal{V}_m(k)$ is the exogenous disturbance that captures measurement noise, modeling error and initial condition uncertainty, and (iii) $y(k)$ is the known output of the secondary path[1]. We can now introduce the *derived* measured quantity (to be constructed at each time step) that will be used in the estimation process to be described shortly

$$m(k) \triangleq e(k) + y(k) = d(k) + \mathcal{V}_m(k) \tag{2}$$

We should mention that, perhaps a more natural solution to the ANC problem is to treat it as a control problem in which a control signal $u(k)$ is directly generated such that the residual noise, $e(k)$, is (in some measure) minimized. This approach, however, has three main drawbacks:

1. The solution requires a backwards Riccati recursion (in addition to the forwards Riccati recursion for estimation), and therefore prohibits its real-time application. In particular, these recursions require *future* values of $x(k)$ which are not available in real-time.

2. In the case of an H_∞ control solution, expressions for γ_{opt} are not easily available, and positivity conditions must be checked at each step (hence increasing computational complexity of the scheme).

3. The controller is no longer of a finite impulse response (FIR) structure.

In view of the above, this paper will not consider this approach any further.

3.1 Estimation Problem

In Figure 3, we assume a state space model, $[A_s(k), B_s(k), C_s(k), D_s(k)]$, for the replica of the secondary path. We also treat the weight vector, $W(k) = [\ w_0(k)\ w_1(k)\ \cdots\ w_N(k)\ ]^T$, as the state vector that captures the dynamics of the replica of the FIR filter. $\xi_k^T = \left[W(k)^T\ \theta(k)^T\right]$ is then the state vector for the overall system. Note that $\theta(k)$ captures the dynamics of the replica

[1](a) $u(k)$ is exactly known (we directly set the weight vector in the FIR filter), and (b) we assume that θ_0 (the initial condition for the secondary path) is known. Note that as long as the effect of the initial condition in the output of the secondary path does not grow without bound, any error in $y(k)$ (due to an initial condition other than what we assumed) can be treated as a component of the measurement disturbance.

of the secondary path. The state space representation of the system is then

$$\overbrace{\begin{bmatrix} W(k+1) \\ \theta(k+1) \end{bmatrix}}^{\xi_{k+1}} = \overbrace{\begin{bmatrix} I_{(N+1)\times(N+1)} & 0 \\ B_s(k)h_k^* & A_s(k) \end{bmatrix}}^{F_k} \overbrace{\begin{bmatrix} W(k) \\ \theta(k) \end{bmatrix}}^{\xi_k} \qquad (3)$$

where $h_k = [x(k)\ x(k-1)\ \cdots\ x(k-N)]^T$ captures the effect of the reference input $x(\cdot)$. For this system, the *derived* measured output is

$$m(k) = \overbrace{\begin{bmatrix} D_s(k)h_k^* & C_s(k) \end{bmatrix}}^{H_k} \begin{bmatrix} W(k) \\ \theta(k) \end{bmatrix} + \mathcal{V}_m(k) \qquad (4)$$

where $m(k)$ is defined in Eq. (2). Now, define a generic linear combination of the states as the desired quantity to be estimated

$$s(k) \quad = \quad \overbrace{\begin{bmatrix} L_{1,k} & L_{2,k} \end{bmatrix}}^{L_k} \begin{bmatrix} W(k) \\ \theta(k) \end{bmatrix} \qquad (5)$$

Note that $m(\cdot) \in \mathcal{R}^{p\times 1}$, $s(\cdot) \in \mathcal{R}^{q\times 1}$, $\theta(\cdot) \in \mathcal{R}^{r\times 1}$, and $W(\cdot) \in \mathcal{R}^{(N+1)\times 1}$. All matrices are then of appropriate dimensions. To allow for a simplified solution (see Section 4), of all possible choices for L_K, $L_K = H_k$ is considered here. In principle, any estimation algorithm can now be used to generate $\hat{s}(k|k) \stackrel{\triangle}{=} \mathcal{F}(m(0), \cdots, m(k))$ (a causal estimate of the desired quantity, $s(k)$) such that some *closeness* criterion is met. This paper focuses on an H_∞ estimation criterion[2]. Here, the main objective is to limit the worst case energy gain from the measurement disturbance and the initial condition uncertainty to the error in a causal estimate of $s(k)$. In other words, it is desired to find an H_∞ suboptimal causal estimator $\hat{s}(k|k) = \mathcal{F}(m(0), \cdots, m(k))$ such that

$$\sup_{\mathcal{V}_m,\ \xi_0} \frac{\sum_{k=0}^{M} [s(k) - \hat{s}(k|k)]^* [s(k) - \hat{s}(k|k)]}{\xi_0^* \Pi_0^{-1} \xi_0 + \sum_{k=0}^{M} \mathcal{V}_m^*(k) \mathcal{V}_m(k)} \leq \gamma^2 \qquad (6)$$

for a given scalar $\gamma > 0$. The question of optimality of the solution is then answered by finding the *infimum* value among all feasible γs. Here Π_0 is a positive-definite matrix. Note that, in this case

1. There is no statistical assumption regarding the measurement disturbance. Therefor, the error in the modeling of the primary path can be easily treated as a component of the measurement disturbance. For large modeling error, however, the performance can be expected to deteriorate.

2. For an H_∞ estimation solution, in general, some accompanying conditions must be verified at each step. The next section will address this requirement and explains why for a proper choice of L_k, the H_∞-optimal solution eliminates the need for such checks.

4 H_∞-Optimal Solution

To discuss the solution, first we quote (from [4]) the solution to the γ-suboptimal estimation problem of Eq. (6). Then, we find the optimal value of γ and show how $\gamma = \gamma_{opt}$ simplifies the solution.

4.1 γ-Suboptimal Finite Horizon Filtering Solution

Theorem [4]: Consider the system depicted by Figure 3 and described by Equations (3)-(5). A level γ H_∞ filter that achieves (6) exists if, and only if, the matrices R_k and $R_{e,k}$ defined by

$$R_{e,k} = \overbrace{\begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix}}^{R_k} + \begin{bmatrix} H_k \\ L_k \end{bmatrix} P_k \begin{bmatrix} H_k^* & L_k^* \end{bmatrix} \qquad (7)$$

have the same inertia for all $0 \leq k \leq M$, where $P_0 = \Pi_0$ and $P_k > 0$ satisfies the Riccati recursion

$$P_{k+1} = F_k P_k F_k^* - K_{p,k} R_{e,k} K_{p,k}^* \qquad (8)$$

where $K_{p,k} = \left(F_k P_k \begin{bmatrix} H_k^* & L_k^* \end{bmatrix} \right) R_{e,k}^{-1}$. If this is the case, then the central H_∞ estimator is given by

$$\hat{\xi}_{k+1} = F_k \hat{\xi}_k + K_{1,k} \left(m(k) - H_k \hat{\xi}_k \right), \quad \hat{\xi}_0 = 0 \qquad (9)$$

$$\hat{s}(k|k) = L_k \hat{\xi}_k + (L_k P_k H_k^*) R_{He,k}^{-1} \left(m(k) - H_k \hat{\xi}_k \right) \qquad (10)$$

with $K_{1,k} = (F_k P_k H_k^*) R_{He,k}^{-1}$ and $R_{He,k} = I_p + H_k P_k H_k^*$.

4.2 The Optimal Value of γ

First we show that $\gamma_{opt} \leq 1$. Going back to Eq. (6), we can always pick $\hat{s}(k|k)$ to be simply $m(k)$. With this choice, $\hat{s}(k|k) - s_k = \mathcal{V}_m(k)$ for all k, and Eq. (6) can never exceed 1 (i.e. $\gamma_{opt} \leq 1$).

To show that γ_{opt} is indeed 1, we need to construct an admissible sequence of disturbances and a valid initial condition for which γ could be made arbitrarily close to 1. Assume that $\hat{\xi}_{-1}^T = \left[\hat{W}_{-1}^T\ \hat{\theta}_{-1}^T \right]$ is the estimator's guess for the initial condition of the system in Figure 3. Moreover, assume that $\hat{\theta}_{-1}$ is indeed the actual initial condition for the replica of the secondary path. Then, in Figure 3, one may conceive of a disturbance such that $m(k)$ coincides with the output expected from $\hat{\xi}_{-1}$. For this to be the case, it is easy to see that $\mathcal{V}_m(k) = D_s(k)h_k^* \left(W - \hat{W}_0 \right)$ should hold for all k (W is the actual initial weight vector). Note that this renders $m(k) - H_k \hat{\xi}(k) = 0$, leaving the weight vector unchanged, and $s(k) - \hat{s}(k|k) = \mathcal{V}_m(k)$, reducing Equation (6) to

$$\sup_{\xi_0} \frac{(W - \hat{W}_0)^* \left[\sum_{k=0}^{M} h_k D_s(k) D_s^*(k) h_k^* \right] (W - \hat{W}_0)}{\xi_0^* \Pi_0^{-1} \xi_0 + (W - \hat{W}_0)^* \left[\sum_{k=0}^{M} h_k D_s(k) D_s^*(k) h_k^* \right] (W - \hat{W}_0)}$$

[2] We do not pursue an H_2-optimal filtering solution mainly for the following reasons: (i) An H_2 optimal solution is valid as long as the assumptions in problem formulation are valid. If the external disturbance is not Gaussian (for instance when there is a considerable modeling error that should be treated as a component of the measurement disturbance) then an H_2 filtering solution may yield undesirable performance. (ii) In general, regardless of the choice for L_k, the H_2 filtering solutions do not simplify as the H_∞ solutions do. This can be of practical importance when the real-time computational power is limited.

Then, as long as $\sum_{k=0}^{M} h_k D_s(k) D_s^*(k) h^*(k) \to \infty$ as $M \to \infty$, for any given $\epsilon > 0$, there exist an $M > 0$ such that the ratio in (6) is $\geq (1-\epsilon)$ (i.e. arbitrarily close to 1). Such input vector is referred to as *exciting*.

From a computational point of view, this optimal value for γ leads to a significant simplification in the Riccati equation (8). Applying matrix inversion lemma to $R_{e,k}$ in (7) and noting the fact that $L_k = H_k$, it is straightforward to show that (8) reduces to the simple *Lyapanov* recursion $P_{k+1} = F_k P_k F_k^*$ with $P_0 = \Pi_0$. Before we specifically outline the adaptive algorithm that is based on the above mentioned *simplified* solution, we would like to highlight main features of the solution so far.

4.3 Important Remarks

1. The estimation-based approach to the design of the adaptive filter in the ANC problem yields a solution which only requires one Riccati recursion. The recursion propagates *forward* in time, and does not require any information about the future of the system and the reference signal (thus allowing a real-time implementable algorithm). This has come at the expense of restricting the controller to an FIR structure in advance.

2. With $K_{p,k} R_{e,k} K_{p,k}^* = 0$, $P_{k+1} = F_k P_k F_k^*$ is the simplified Riccati equation. Thus the Riccati update, always generates a positive definite P_k, as long as P_0 is selected to be positive definite. This eliminates the need for computationally expensive checks for positive definiteness of P_k at each step.

3. In general, the solution to an H_∞ filtering problem requires verification of the fact that R_k and $R_{e,k}$ are of the same inertia at each step. This can be another computationally expensive task. Moreover, it may lend to a breakdown in the solution if the condition is not met at some time k. Our formulation of the problem has eliminated the need for such checks, as well as the potential breakdown of the solution, by allowing a definitive answer to the feasibility of $\gamma = 1$. We have also avoided the potential conservatism incurred, by proving that $\gamma = 1$ is indeed H_∞-optimal.

4. When $A_s(k)$, $B_s(k)$, and $C_s(k)$ are all zero, and $D_s(k) = I$ (for all k), (i.e. the output of the FIR filter directly cancels $d(k)$), the results we have derived so far reduces to the simple LMS algorithm [2].

5. When the secondary path is causally invertible, our approach produces results similar to those known as *inverse secondary path modeling* [5]. It also explains why the performance of an inverse secondary path modeling approach could be poor (Appendix A).

6. With no need to verify the solutions at each step, the computational complexity of the estimation based approach is $O(n^3)$ (in calculating $F_k P_k F_K^*$), where n = length of the FIR filter $(N+1)$ + order of the secondary path $(n_{sec-path})$. The special structure of F_k however reduces the computational complexity involved to $O(n^3_{sec-path})$. This reduction is significant when $n_{sec-path} \ll N$.

7. Note that the physical setting of the problem, only allows for the adaptation of the weight vector in the FIR filter. In other words, only $u(k)$ in Figure 1 can be directly generated, and we can not directly access $y(k)$.[3]

5 The H_∞-Optimal Adaptive Algorithm

In this section, we suggest an implementation scheme in which the weight vector in the FIR filter follows the H_∞-optimal estimate of the weight vector in the approximate model of the primary path (Equation 9). It is instructive to point out that we use three sets of variables in what follows,

a) *Estimator's best estimate of a variable* which includes (i) $\hat{W}(k)$: H_∞-optimal estimator's estimate of *the* weight vector, and (ii) $\hat{\theta}(k)$: H_∞-optimal estimator's estimate of the state of the replica of the secondary path.

b) *The actual value of a variable* which includes (i) θ_{actual} : the actual state of the secondary path (not directly available to the adaptation algorithm unless $\theta_{actual,0}$ is known), (ii) $u(k) \triangleq h_k^* \hat{W}(k)$: the actual input to the secondary path (note that at each iteration, the weight vector in the adaptive FIR filter is set to $\hat{W}(k)$), (iii) $y(k)$: the actual output of the secondary path, and (iv) $d(k)$: the actual output of the primary path. Note that $d(k)$ and $y(k)$ are not directly measurable.

c) *Adaptive algorithm's internal copy of a variable:* From Equation (2), recall that to construct the derived measurement $m(k)$ (to be used by the estimator), $y(k)$ is needed. Since $y(k)$ is not directly available, the adaptive algorithm needs to generate an internal copy of this variable. With the dynamics of the secondary path and its input $u(k)$ known, the adaptive algorithm's copy of $y(k)$ will be exact if the actual initial condition of the secondary path is exactly known. Obviously, one can not expect to have the exact knowledge of the actual initial condition of the secondary path. However, any error in the output of the secondary path (due to an initial condition other than that assumed by the estimation-based adaptive algorithm) can be treated as a portion of the exogenous disturbance, $\mathcal{V}_m(k)$, as long as it does not grow without bound. We use the subscript "$_{copy}$" to refer to the adaptive algorithm's internal copy of a variable.

Now, we can propose the following implementation scheme for the adaptive algorithm

1. Start with $\hat{W}(0) = \hat{W}_0$, $\hat{\theta}(0) = \hat{\theta}_0$ as estimator's best initial guess for the state vector in the approximate model of the primary path. Also assume that $\theta_{actual}(0) = \theta_{actual,0}$, while the adaptive algorithm assumes that $\theta_{copy}(0) = \theta_{copy,0}$. Furthermore, assume that $d(0)$ is the initial output of the primary path. Now, for $0 \leq k \leq M(\textit{finite horizon})$:

2. Form the control signal $u(k) = h_k^* \hat{W}(k)$,

3. Applying the control signal to the secondary path, the actual output and the new state vector obey the following dynamics

$$\begin{aligned} \theta_{actual}(k+1) &= A_s(k)\theta_{actual}(k) + B_s(k)u(k) \\ y(k) &= C_s(k)\theta_{actual}(k) + D_s(k)u(k) \end{aligned} \qquad (11)$$

[3] When the secondary path $S(z)$ is minimum phase, we can reconstruct $u(k)$ from $\hat{s}(k|k)$ by using the causal and stable inverse of $S(z)$. Even though this solution is no longer of an FIR structure, it is certainly an implementable solution. We shall not discuss this solution any further.

4. Propagate the internal copy of the state vector and the output of the secondary path as

$$\theta_{copy}(k+1) = A_s(k)\theta_{copy}(k) + B_s(k)u(k)$$
$$y_{copy}(k) = C_s(k)\theta_{copy}(k) + D_s(k)u(k) \quad (12)$$

5. Form the *derived* measurement, $m(k)$, using the direct measurement $e(k)$ and the controller's copy of the output of the secondary path $m(k) = e(k) + y_{copy}(k)$,

6. Use the H_∞-optimal estimator's state update, Equation (9), to find the H_∞-optimal estimate of the weight vector in the replica of the FIR filter (i.e. $\hat{W}(k+1)$). Note that $\hat{\theta}(k+1)$ should also be stored for the next estimation update.

7. If $k \leq M$ go to **2.**

6 Simulation Results

In this section we compare the performance of the estimation-based adaptive algorithm to the performance of the FxLMS algorithm. For the simulations included here, **(i)** the primary path is $P(z) = \frac{z-0.3}{(z+0.4-j0.8)(z+0.4+j0.8)}$, **(ii)** the actual model of the secondary path is $S(z) = \frac{z-0.3}{(z+0.66-j0.75)(z+0.66+j0.75)}$, **(iii)** the adaptive algorithm has access to an approximate model of the secondary path described by $\tilde{S}(z) = \frac{z-0.3}{(z+0.2-j0.8)(z+0.2+j0.8)}$, **(iv)** the length of the FIR filter is 4, and **(v)** $\triangle t = 0.05$ (sec). We consider a multi-tone reference signal ($x(k) = \sum_{i=1}^{3} 4\sin(2\pi f_i k \triangle t)$, with $f_1 = 0.1$, $f_2 = 0.15$, and $f_3 = 0.25$ Hz for our simulations. As measurement noise, $\mathcal{V}_m(k)$, we use a zero mean, normally distributed random variable with variance 0.01.

Figure 5 compares the performance of the FxLMS algorithm with that of the estimation-based adaptive algorithm when the secondary path is exactly known. While estimation based algorithm effectively cancels the output of the primary path, $d(k)$, in about 2.0 seconds, the FxLMS needs about 7.0 seconds to do so. Figure 6 compares their performance when only an estimate of the secondary path is known to the adaptive algorithms. In this case, the estimation-based adaptive algorithm provides an acceptable performance, while the FxLMS algorithm (with the same $\mu = 0.005$ as before) goes unstable. Thus, the estimation-based adaptive algorithm exhibits less sensitivity to the error in the modeling of the secondary path. For a stable adaptive controller, the adaptation rate for the FxLMS algorithm must be reduced (yielding an even slower response). As Figure 7 indicates, the estimation-based adaptive algorithm results in "smooth" time variations in the weight vector of the FIR filter (as does the FxLMS algorithm).

7 Conclusion

We have given an estimation interpretation of the active noise cancellation (ANC) problem and have shown that it admits an H_∞-optimal filtering solution with interesting features. We have shown that γ_{opt}, the min-max energy gain, is unity. We have also simplified the filtering solution for this optimal value of γ. The most important simplification is that there is no need to validate the feasibility condition required of a general H_∞ filtering solution. Without this guaranteed existence, the proposed algorithm would not have been real-time implementable. We have then suggested an implementation scheme (based on the H_∞-optimal filtering solution) and compared its performance to the FxLMS adaptive algorithm. The proposed estimation-based adaptive algorithm meets a disturbance attenuation criterion and provides an appropriate framework to address the performance and robustness of the adaptive algorithm in the face of modeling uncertainty. It also allows for a systematic optimization of the FIR filter parameters (such as filter length).

Appendix A: Alternative H_∞ Optimal Adaptive Algorithm for Causally Invertible $S(z)$

To present the solution for the case when $S(z)$ is causally invertible, we go back to Figure 3. Note that since $S^{-1}(z)$ exists, knowing $d(k)$ at the output of the replica of the secondary path is equivalent to knowing $S^{-1}(k) \star d(k)$ at its input ($\star$ indicates convolution operator). Thus, for an estimation interpretation of the adaptive control problem one can simplify the block diagram of Figure 3 into the block diagram of Figure 4. Note that the exogenous disturbance, $\mathcal{V}_a(k)$, is included to reflect the fact that $d(k)$ is only known within the accuracy of the measurement error. The following is the state space representation for this simplified model

$$\begin{aligned} W(k+1) &= W(k) \\ v(k) \triangleq S^{-1}(k) \star d(k) &= h_k^* W(k) + \mathcal{V}_a(k) \\ u(k) &= h_k^* W(k) \end{aligned}$$

Here $v(k)$ is the available measurement to the estimation problem and $u(k)$ is the quantity we would like to estimate. The estimation objective is to find an H_∞-Optimal causal estimator $\hat{u}(k|k) = \mathcal{F}(v_0, v_1, ..., v_k)$ such that

$$\sup_{\mathcal{V}_a \in \mathcal{L}_2, W_0} \frac{\sum_{k=0}^{M} (u(k) - \hat{u}(k|k))^* (u(k) - \hat{u}(k|k))}{W_0^* \Pi_0^{-1} W_0 + \sum_{k=0}^{M} \mathcal{V}_a(k)^* \mathcal{V}_a(k)} \leq \gamma_{opt}^2$$

where we should find the optimal value of γ as well. To use the general solution presented in Section 4, note that here $F_k = I_{(N+1)\times(N+1)}$, and $H_k = L_k = h_k^*$. This leads to the following simple solution

$$P_{k+1} = P_k \quad (13)$$

$$\hat{W}(k+1) = \hat{W}(k) + P_k h_k (I + h_k^* P_k h_k)^{-1} \left(\overbrace{S^{-1}(k) * d(k)}^{v(k)} - h_k^* \hat{W}(k) \right) \quad (14)$$

$$\hat{u}(k|k) = h_k^* \hat{W}(k) + h_k^* P_k h_k (I + h_k^* P_k h_k)^{-1} \left(S^{-1}(k) * d(k) - h_k^* \hat{W}(k) \right) \quad (15)$$

Factoring $S^{-1}(k)$ outside the parentheses in Eq. (14), the estimate of the weight vector in the replica of the FIR

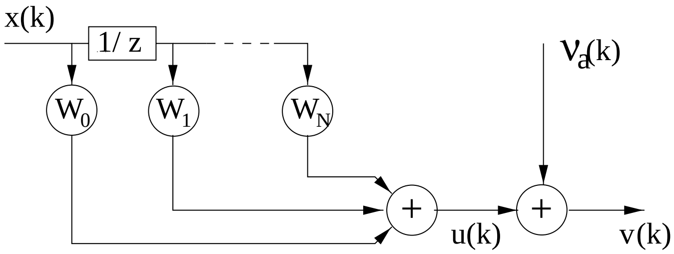

Fig. 4: Simplified block diagram of the estimation interpretation when the secondary path is invertible

filter obeys the following simple update

$$\hat{W}(k+1) = \hat{W}(k) + P_k h_k \left(I + h_k^* P_k h_k\right)^{-1} \overbrace{S^{-1}(k) \star \Big(\overbrace{d(k) - S(k) \star h_k^* \hat{W}(k)}^{Error} \Big)}^{Filtered\ Error} \quad (16)$$

which shares the error-filtering feature of secondary-path equalization techniques. Furthermore, according to the results in the general case of Section 4, the optimal value for γ is 1. Note that adjusting the weight vector in the FIR filter according to (14) will limit the energy gain from the exogenous disturbance to the error in noise cancellation, i.e. $d(k) - y(k)$, by the H_∞ gain of the secondary path, and hence it can be large. Thus the performance of the adaptive controller that is based on (14)-like update of the weight vector can be poor.

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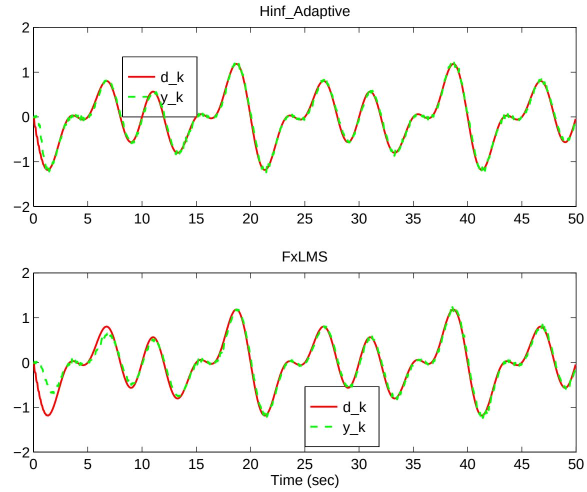

Fig. 5: Performance comparison when the secondary path is exactly known

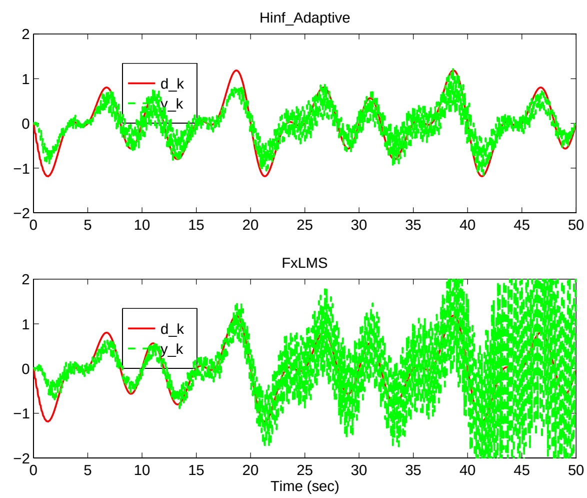

Fig. 6: Performance comparison when the secondary path is not fully known

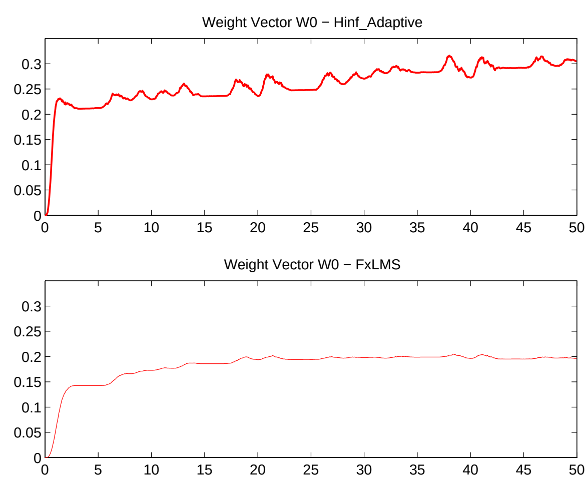

Fig. 7: First element of the weight vector in the FIR Filter for the estimation-based and FxLMS adaptive algorithms