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DAMPED FOURIER SPECTRUM AND RESPONSE SPECTRA

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ABSTRACT

This paper describes the physical relationships that exist between the Fourier transform and the response spectrum of a strong-motion accelerogram. By developing the new concept of the "Damped Fourier Spectrum" (D.F.S.), we show that the velocity and displacement of the damped oscillator can be represented by a linear combination of the real and imaginary parts of the D.F.S. and by the initial conditions. The D.F.S. represents a new way of "smoothing" the classical Fourier Transform by using a physically based filter.

Introduction

The computational economy afforded by the Fast Fourier Transform algorithm (Tukey, 1967) has made Fourier analysis the *condicio sine qua non* in the processing of strong-motion data (e.g. Udwadia and Trifunac, 1973b; Trifunac, 1972). To take full advantage of this economy, it is now necessary to develop new methods that are capable of extracting the maximum possible information from the complex Fourier transform for use in vibration analysis.

In this paper, we show that the Damped Fourier Spectrum bears the same relationship to the damped velocity spectrum, as the classical Fourier transform does to the undamped velocity spectrum (Kawasumi, 1956; Rubin, 1961; Hudson, 1962; Jennings, 1972).

The Damped Fourier Spectrum

The governing equation of relative response $x(t)$ of a damped linear oscillator subjected to an absolute ground acceleration $-\ddot{z}(t)$ is

$$\ddot{x}+2\omega_n\xi\dot{x}+\omega_n{}^2x = \ddot{z}(t), \tag{1}$$

where ξ is the percentage of critical damping and $\omega_n = (k/m)^{\frac{1}{2}}$ is the natural frequency (Figure 1).

Using the transformation

$$y = x \exp(\omega_n\xi t) \tag{2}$$

we get

$$\ddot{y}+\omega_d{}^2y = \ddot{z}(t)\exp(\omega_d\beta t), \tag{3}$$

where $\omega_d=\omega_n(1-\xi^2)^{\frac{1}{2}}$ is the damped natural frequency of the oscillator and $\beta=\xi/(1-\xi^2)^{\frac{1}{2}}$.

Defining

$$\eta_d{}^*(\omega_d, t) = \dot{y}(t)+i\omega_d y(t), \tag{4}$$

equation (3) becomes

$$(d\eta_d{}^*/dt)-i\omega_d\eta_d{}^* = \ddot{z}(t)\exp(\omega_d\beta t) \tag{5}$$

whose solution is

$$\eta_d{}^*(\omega_d, t) = \exp(i\omega_d t)\left[\int_0^t \ddot{z}(\tau)\exp(\omega_d\beta\tau-i\omega_d\tau)d\tau+\eta_{d0}^*\right]. \tag{6}$$

Here η_{d0}^* is the value of $\eta_d{}^*$ at $t = 0$.

To cast equation (6) into a form resembling the classical Fourier transform of $\ddot{z}(t)$, we take the integrand in equation (6) as a product of two functions $g(t)$ and $\ddot{z}^*(t)$, where

$$g(t) = \begin{cases} \exp\{[\omega_n\xi - i\omega_n(1-\xi^2)^{\frac{1}{2}}]t\}; & -\infty < t \leqq t_0 \\ 0, & \text{otherwise} \end{cases} \tag{7a}$$

for $\xi > 0$ and $\omega_n > 0$ and

$$\ddot{z}^*(t) = \begin{cases} \ddot{z}(t) & 0 \leqq t \leqq T \\ 0, & \text{otherwise.} \end{cases} \tag{7b}$$

Here we are assuming that the forcing function $\ddot{z}(t)$ is nonzero only between 0 and T while the response is evaluated at time t_0.

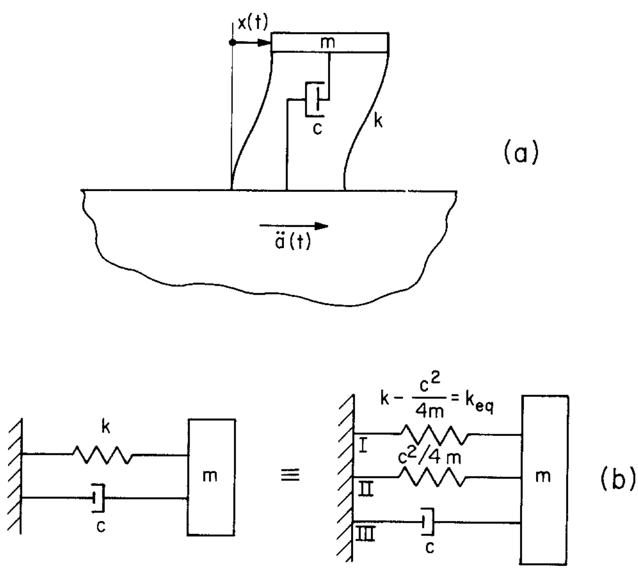

FIG. 1. (a) A single-degree-of-freedom oscillator subjected to ground acceleration $-\ddot{z}(t)$. (b) A mass spring-dashpot system and its equivalent as interpreted through the phase of the complex variable η_d.

Using Parseval's theorem and the above definitions we have,

$$I \equiv \int_{-\infty}^{\infty} \ddot{z}^*(t)g(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\lambda)G(-\lambda)d\lambda$$

where

$$Z(\lambda) = \int_{-\infty}^{\infty} \ddot{z}^*(t)\exp(-i\lambda t)dt$$

and

$$G(\lambda) = \int_{-\infty}^{\infty} g(t)\exp(-i\lambda t)dt.$$

Using the definition of $g(t)$ from equation (7a)

$$G(\lambda) = \int_{-\infty}^{t_0} \exp\{[\omega_n\xi - i\omega_n(1-\xi^2)^{\frac{1}{2}}]t\}\exp(-i\lambda t)dt$$

$$= \frac{\exp\{[\omega_n\xi - i\{\omega_n(1-\xi^2)^{\frac{1}{2}} + \lambda\}]t_0\}}{\omega_n\xi - i[\omega_n(1-\xi^2)^{\frac{1}{2}} + \lambda]}.$$

Hence,

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\lambda)\,\frac{\exp\{[\omega_n\xi - i\{\omega_n(1-\xi^2)^{\frac{1}{2}}-\lambda\}]t_0\}}{\omega_n\xi - i[\omega_n(1-\xi^2)^{\frac{1}{2}}-\lambda]}\,d\lambda$$

$$= \frac{1}{2\pi}\exp(\omega_n\xi t_0)\exp(-i\omega_d t_0)\int_{-\infty}^{\infty}\frac{Z(\lambda)\exp(i\lambda t_0)}{\omega_n\xi - i[\omega_n(1-\xi^2)^{\frac{1}{2}}-\lambda]}\,d\lambda. \tag{8}$$

Using equation (6)

$$\eta_d^*(\omega_d, t_0) = \frac{1}{2\pi}\exp(\omega_n\xi t_0)\int_{-\infty}^{\infty}\frac{Z(\lambda)\exp(i\lambda t_0)}{\omega_n\xi - i[\omega_n(1-\xi^2)^{\frac{1}{2}}-\lambda]}\,d\lambda + \eta_{d_0}^*\exp(i\omega_d t_0). \tag{9}$$

But

$$\eta_d^*(\omega_d, t) = \dot{y} + i\omega_d y = (\dot{x} + \omega_n\xi x + i\omega_d x)\exp(\omega_n\xi t) = \eta_d\exp(\omega_n\xi t)$$

where the damped complex response, η_d, is given by

$$\eta_d = \dot{x} + \omega_n\xi x + i\omega_d x. \tag{10}$$

Equation (9) then gives

$$\eta_d(\omega_d, t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{Z(\lambda)\exp(i\lambda t_0)}{\omega_n\xi - i[\omega_n(1-\xi^2)^{\frac{1}{2}}-\lambda]}\,d\lambda + \eta_{d_0}\exp[i\omega_n(1-\xi^2)^{\frac{1}{2}}t_0]\exp(-\omega_n\xi t_0) \tag{11}$$

$$= X^*(\omega_d, \xi, t_0) + iY^*(\omega_d, \xi, t_0) + \eta_{d_0}\exp[i\omega_n(1-\xi^2)^{\frac{1}{2}}t_0]\exp(-\omega_n\xi t_0)$$

$$\equiv F_d(\omega_d, \xi, t_0) + \eta_{d_0}\exp[i\omega_n(1-\xi^2)^{\frac{1}{2}}t_0]\exp(-\omega_n\xi t_0), \tag{12}$$

where $X^*(\omega_d, \xi, t_0)$ and $Y^*(\omega_d, \xi, t_0)$ are the real and imaginary parts of the "Damped Fourier Transform", $F_d(\omega_d, \xi, t_0)$, defined by the integral in equation (11). At the frequencies $\omega_d^m = (2\pi m/t_0)$, we get

$$\left.\begin{aligned} x(\omega_d^m, \xi, t_0) &= (2\pi m/t_0)^{-1}Y^*(\omega_d^m, \xi, t_0) + x_0\exp(-\omega_n\xi t_0) \\ \dot{x}(\omega_d^m, \xi, t_0) &= X^*(\omega_d^m, \xi, t_0) - \beta Y^*(\omega_d^m, \xi, t_0) + \dot{x}_0\exp(-\omega_n\xi t_0) \end{aligned}\right\}. \tag{13}$$

Equation (13) states that the pseudo-velocity of a damped oscillator with zero initial conditions having a damped natural frequency of ω_d^m and the fraction of critical damping, ξ, is given by the imaginary part of the Damped Fourier Transform evaluated at the frequency ω_d^m. Similar interpretations on the basis of the real part of the Damped Fourier Spectrum are possible. It may be noted here that the above formulation is valid for any ξ, however small, although not exactly equal to zero. When $\xi = 0$, $G(\lambda)$ as defined above does not exist, and the domain in which $g(t)$ takes nonzero values needs to be redefined as the interval $(-\infty, \infty)$. At the same time, $\ddot{z}^*(t)$ given by (7b) has to be redefined into the interval between 0 and t_0. With these new definitions $G(-\lambda)$ becomes $2\pi\delta(\omega_n - \lambda)$, and I reduces to $Z(\omega_n)$, so that equation (13) still holds. However, $X^*(\omega_d^m = \omega_n^m, 0, t_0)$ and $Y^*(\omega_d^m = \omega_n^m, 0, t_0)$ now become the real and imaginary parts of the Fourier transform of $\ddot{z}^*(t)$, which is nonzero between 0 and t_0. When $X^* \equiv Y^* \equiv 0$, we have the free vibration problem of the damped oscillator indicating that the velocity and displacement of an oscillator at time t_0 (a complete multiple of $2\pi/\omega_d^m$) are, respectively, equal to the initial velocity and displacement multiplied by the factor $\exp(-\omega_n\xi t_0)$. Next we show that the phase ϕ_d of the damped complex response, η_d, is related to the partition of the oscillator's energy and to the phase of the Damped Fourier Transform. The phase of the complex variable η_d is given by

$$\tan \phi_d(t) = \frac{\omega_n(1-\xi^2)^{\frac{1}{2}}x}{\dot{x}+\omega_n\xi x}. \tag{14}$$

When the oscillator starts from rest, for frequencies $\omega_d{}^m$,

$$\tan \phi_d{}^m(t_0) = \frac{Y^*(\omega_d{}^m, \xi, t_0)}{X^*(\omega_d{}^m, \xi, t_0)} = \tan \psi_d(\omega_d{}^m, t_0) \tag{15}$$

where ψ is the phase of the Damped Fourier Transform of $\ddot{z}^*(t)$. From equation (14) we get

$$\tan^2 \phi_d = \frac{kx^2 - k(\xi x)^2}{m\dot{x}^2 + cx\dot{x} + k(\xi x)^2}. \tag{16}$$

Here k is the force per unit displacement of the spring and c is the viscous damping of the dashpot. The numerator of equation (16) is related to the potential energy of an equivalent spring while the denominator is simply the total energy less the potential energy of the equivalent spring. The potential energy is reduced from the undamped case by a term $k(\xi x)^2$. This reduction may then be looked upon as being the cause of a reduced natural frequency in the damped system. The damping thus has the effect of reducing the apparent spring stiffness. Noting that $k(\xi x)^2 = (c^2/4m)x^2 = k_1x^2$, we observe that the dashpot can be interpreted as acting as a negative spring of stiffness $c^2/4m$ (Figure 1). We then propose that the mass-spring-dashpot system can be looked upon as being composed of three different elements: an equivalent spring (I), a spring related to the damping characteristics of the dashpot (II), and a velocity dependent dissipative element (III). Element I, which represents the equivalent spring, yields the frequency characteristics of the system while element III yields the dissipative qualities associated with any oscillation of the mass m.

The response η_d then brings about a split-up of the energy which can be expressed through its phase angle as

$$\tan^2 \phi_d = \frac{(\text{P.E.})_{\text{spring}} - (\text{P.E.})_{\text{“damper spring”}}}{(\text{K.E.})_{\text{mass}} + (\text{D.E.})_{\text{dashpot}} + (\text{P.E.})_{\text{“damper spring”}}}.$$

When $c^2/4m \to k$, the equivalent spring in the system has zero stiffness ($k_{\text{eq}} \equiv 0$), and an exponential decay sets in, thus leading to the concept of critical damping. For such an oscillator, the phase of the damped transform tends to zero.

The complex number η_d can be looked upon as a vector whose magnitude equals $\sqrt{E_d}$ while its phase angle is given by equation (14). Thus

$$\eta_d = \sqrt{(E_d)} \exp(i\phi_d). \tag{17}$$

E_d is a positive definite quantity and is given by

$$E_d = \dot{x}^2 + \omega_n{}^2x^2 + 2\omega_n\xi x\dot{x} = 2(\text{K.E.} + \text{P.E.} + \text{D.E.})/m,$$

where K.E. represents the kinetic energy, P.E. the potential energy, and D.E. the damping energy. The rate of rotation of this vector is given by

$$\frac{d\phi_d}{dt} = \omega_d\left[1 + \frac{x\ddot{a}}{E_d}\right]. \tag{18}$$

For the free vibration case, $d\phi_d/dt = \omega_d$, the damped natural frequency of vibration.

Just as the Fourier transform gives the response (η) of an undamped oscillator starting from rest at the end of the excitation, t_0 (Udwadia and Trifunac, 1973a), so also the Damped Fourier Transform (X^*+iY^*) yields the response of a damped oscillator (η_d) at time t_0. The Damped Fourier Spectrum can then be defined as $|\eta_d|_{\max} = |\dot{x}+\beta\omega_d x+ i\omega_d x|_{\max} \geqq |\eta_d|_{t_0}$. From equation (17) we have that

$$\dot{x} = \sqrt{(E_d)}(\cos\phi_d - \beta \sin\phi_d).$$

Remembering that the damped velocity spectrum $S_v(\omega_n, \xi)$ is $|\dot{x}|_{\max}$ we get,

$$S_v(\omega_n, \xi) \equiv |\dot{x}|_{\max} \lesssim |\eta_d|_{\max}.$$

Equation (11) indicates that the damped Fourier spectral amplitude cannot be directly obtained from the Fourier spectral amplitudes by the use of a simple linear filtering operation performed on the Fourier spectrum. This damped spectral amplitude for any particular damping $\xi = \xi_0$ computed at the end of the excitation will serve as a lower bound on the damped ($\xi = \xi_0$) velocity spectrum for an oscillator with natural frequency ω_n and percentage of critical damping, ξ_0.

Calculation of the Damped Fourier Spectrum (D.F.S.)

The Damped Fourier Spectrum $F_d(\omega_d, \xi, t_0)$ is defined for $\xi > 0$ [refer to equation (11)] as

$$F_d(\omega_d, \xi, t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{Z(\lambda)\exp(i\lambda t_0)}{\omega_n\xi - i[\omega_n(1-\xi^2)^{\frac{1}{2}}-\lambda]}\, d\lambda. \tag{19}$$

In what follows it shall be assumed that $t_0 = T$. Physically, equation (19) implies that the response of a damped oscillator at any time t_0 to a given excitation can be obtained if a knowledge of the response at time t_0 (to that excitation) of undamped oscillators of all possible frequencies is known. Since the calculation of $Z(\lambda)$ is generally done using the Fast Fourier Transform (F.F.T.) its values are known only at $\lambda = (2\pi n/t_0)$; $n = 0, 1, \ldots$. Hence $Z(\lambda)$ needs to be reconstructed for intermediate frequencies between these discrete values using the sampling theorem.

$$Z(\omega) = \sum_{n=-\infty}^{\infty} Z(2\pi m/t_0)\exp(-i\omega t_0/2)\exp(in\pi)\frac{\sin[(\omega t_0/2)-n\pi]}{[(\omega t_0/2)-n\pi]}.$$

Then

$$F_d = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} Z(2n\pi/t_0)\exp(-i\omega t_0/2)\exp(in\pi)\frac{\sin[(\omega t_0/2)-n\pi]}{[(\omega t_0/2-n\pi)]}$$

$$\times\frac{\exp(i\omega t_0)}{[\omega_n\xi - i\{\omega_n(1-\xi^2)^{\frac{1}{2}}-\omega\}]}\, d\omega$$

$$= \frac{1}{t_0}\sum_{n=-N}^{n=N} Z(2\pi n/t_0)\left[\frac{1-\exp\{-[\omega_n\xi - i\omega_n(1-\xi^2)^{\frac{1}{2}}]t_0\}}{\omega_n\xi - i\{\omega_n(1-\xi^2)^{\frac{1}{2}}-2\pi n/t_0\}}\right].$$

If further $\omega_n = (2\pi m/t_0)$,

$$F_d(\omega_d{}^m, \xi, t_0) = \sum_{n=-N}^{n=N} Z(2\pi n/t_0)\left[\frac{1-\exp\{-2\pi m(\xi - i(1-\xi^2)^{\frac{1}{2}})\}}{2\pi m\xi - i2\pi\{m(1-\xi^2)^{\frac{1}{2}}-n\}}\right]. \tag{20}$$

The interchange of summation with integration can be justified on the physical grounds that the signal is almost frequency-band limited.

Although the summation in equation (20) does not represent a simple convolution, it is done on a product of $Z(\lambda)$ and a sharply peaked function so that the actual summation may be truncated to a smaller number of frequency estimates around the frequency of interest. This is what one would actually expect, for at a given frequency the damped

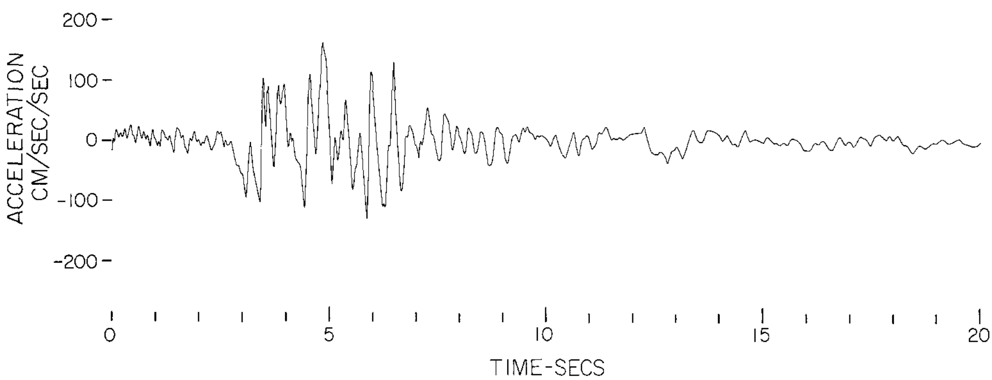

FIG. 2. Acceleration-time history from the record of the Eureka Earthquake of December 21, 1954, N11W component.

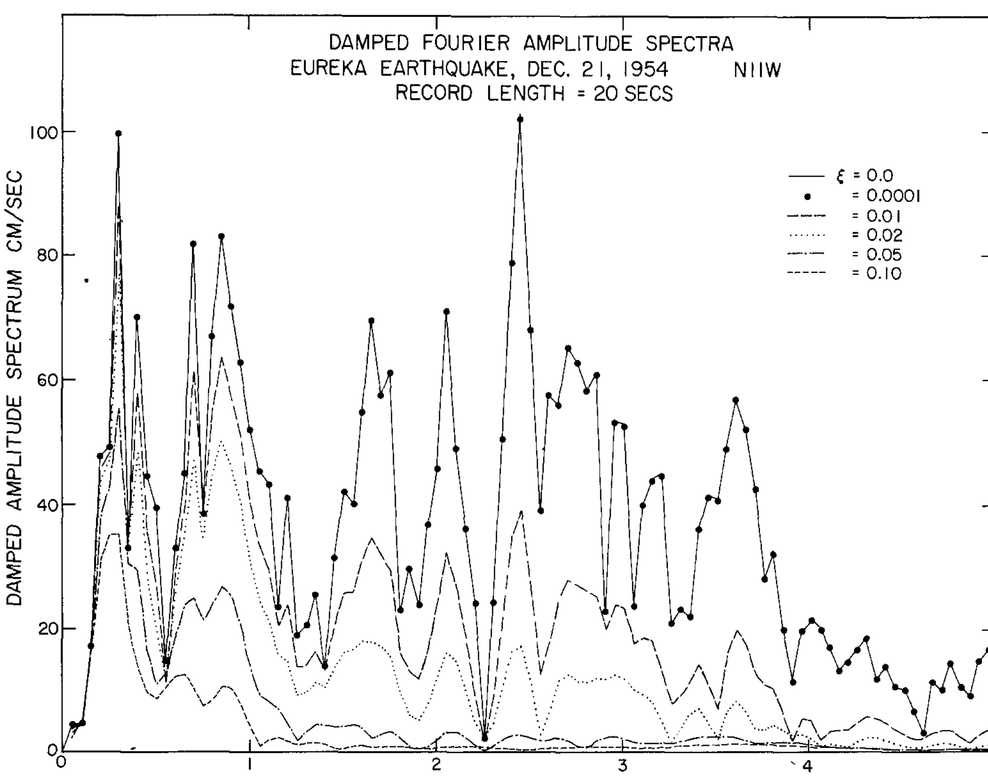

FIG. 3. Damped Fourier amplitude spectrum curves.

Fourier amplitude spectrum ought to depend more closely on the Fourier spectral amplitude at that and neighboring frequencies.

To illustrate the concept of the damped spectrum, a typical accelerogram was analyzed. It is shown in Figure 2. The spectrum curves corresponding to this acceleration time history for various values of ξ are shown in Figure 3. The spectra have been obtained by taking 500 frequency estimates of the Fourier spectrum around the frequency of interest.

The exponential decay of the damped oscillator leads to low responses at higher frequencies. The upper solid curve in Figure 3 is the Fourier amplitude spectrum, while the full circles indicate the damped spectrum estimates for a damping value of 0.01 per cent. We observe that these points lie below the spectrum curve, although they follow it closely. As seen from the figures, the effect of damping on the response as indicated by the damped spectral amplitudes is quite intense. For damping values as low as 1 per cent, the spectral amplitudes are reduced by about half to a third of those obtained from the undamped spectra. It may be noted that the spectral curves for various dampings occasionally cross each other and that the curves for higher ξ values show lesser undulations so that the introduction of damping in this way effects a physically meaningful smoothing.

On The Smoothing of Fourier Amplitude Spectra

Various investigators (Holloway, 1958; Robinson and Treitel, 1964; Tukey, 1967) have looked at the problem of smoothing of spectra from the point of view of time series analyses. Most smoothing operators suggested are linear. They show no preference for any range of frequencies and are so manipulated as to keep the area under the smoothed curves identical to the area under the unsmoothed curves. As observed from the damped spectra, the operation represented by the integral in equation (19) could be referred to as a smoothing operation that yields smoother spectral curves (Figure 3). However, there are some marked differences between this operation and the smoothing operators that have been suggested by workers in time series analysis. First, it is impossible to convert equation (19) into a classical convolution integral. The higher frequencies are modulated to a greater extent than lower frequencies. Second, the areas under the smoothed curves are not identical to those under the unsmoothed curves. These two results fall out naturally when we consider the fact that every point on the smoothed Fourier spectrum curve corresponds to the response of a damped oscillator which decays as $\exp(-\omega_n \xi t)$, so that the higher frequencies become attenuated to a greater extent. The dissipation of energy through the dashpot in the damped system causes the area under the smoothed and unsmoothed curves to be different.

The physical nature of this effect can be best illustrated through a simple example as follows. Let $\ddot{z}(t) = \delta(t)$, so that $Z(\lambda) = 1$. Then,

$$F_d = \dot{x} + \omega_n \xi x + i\omega_d x\big|_{t=t_0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{Z(\lambda)\exp(i\lambda t_0)}{\omega_n \xi - i[\omega_n(1-\xi^2)^{\frac{1}{2}} - \lambda]}\, d\lambda \tag{21}$$

Integrating we get

$$F_d = \exp[i\omega_n(1-\xi^2)^{\frac{1}{2}} t_0]\exp(-\omega_n \xi t_0) \tag{22}$$

$$|F_d| = \exp(-\omega_n \xi t_0). \tag{23}$$

But, $|Z(\omega_n, t_0)|$ is proportional to the energy of an undamped oscillator caused to oscillate by the delta function pulse. Also, $|F(\omega_d, t_0)|$ is proportional to the energy of a damped oscillator. We observe that this energy, unlike in the undamped case, is a function of the frequency and the time duration t_0. In this case, the unsmoothed and the smoothed spectra would be represented as in Figure 4. The damped curves represent the response at time $t = t_0$ of an oscillator of natural frequency ω_n and damping ξ to a delta function applied at time $t = 0$. The nondimensional frequency clearly indicates that for a given damping value, the response at larger times (t_0) for smaller natural frequencies (ω_n) will be the same as the response at shorter times at higher frequencies provided the product $\omega_n t_0$ is the same. As observed from Figure 4, at higher frequencies the damped oscillator shows a very short memory of past excitation.

Conclusions

The development of the new functional called the "Damped Fourier Transform" has enabled us to formulate the meaningful physical link between the Fourier transform of an accelerogram, recorded during the strong ground motion and the response of a viscously damped single-degree-of-freedom system excited by the same motion. This new result represents the logical extension of the relation previously known to exist only in the case of an undamped oscillator.

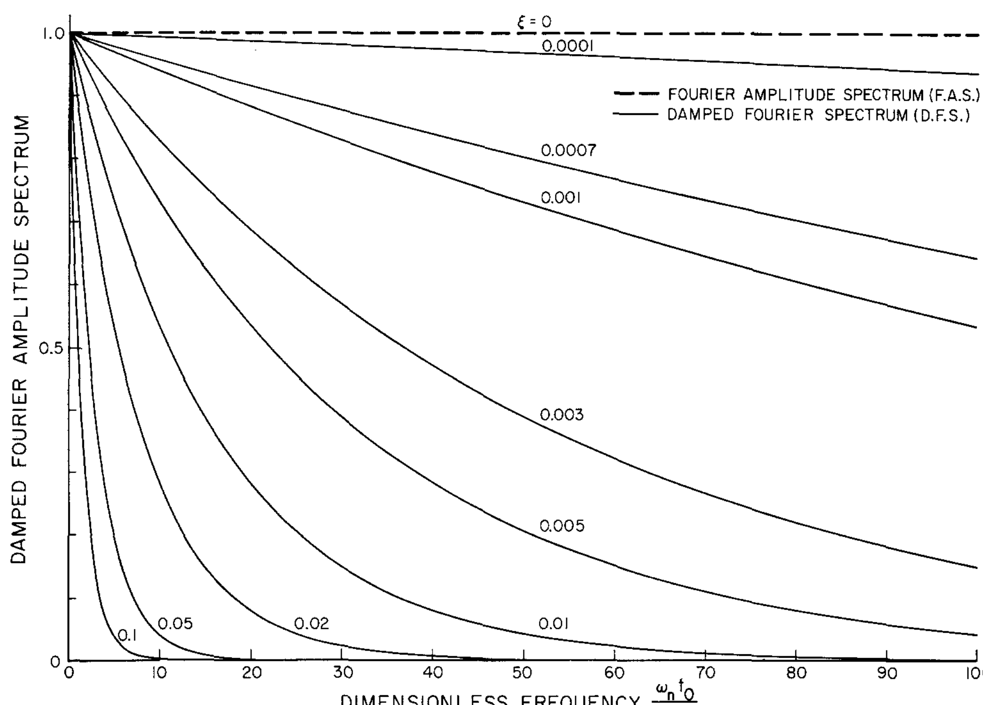

FIG. 4. Damped Fourier amplitude spectrum curves indicating the response $|F_d|$ of an oscillator of natural frequency ω_n and percentage of critical damping, ξ, to a delta function input at time $t = 0$.

The application of this functional (equation 19) to several accelerograms shows that it can be used as a good estimate of the damped relative velocity spectrum up to about 2 to 3 Hz. The estimate can no doubt be effectively improved by a proper choice of the time length of record chosen for analysis. At higher frequencies, however, the D.F.S. serves only as a lower bound to the damped relative velocity spectrum. The difference between the two increases with increasing frequency due to the frequency-dependent exponential decay of the damped oscillator response. The functional also serves as a physically meaningful spectral smoothing operator.

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