DISCUSSION OF

"ACCELERATED MOTION OF A SPHERICAL PARTICLE"
by M. R. CARSTENS

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Norman H. Brooks (Department of Civil Engineering, California Institute of Technology, Pasadena, Calif.)--In considering the accelerated motion of spheres in a viscous fluid, the author has restricted himself to consideration of only simple harmonic motions of the fluid field and the sphere. The author's equation of motion (Eq. 11) for a suspended sphere is true only when the solution yields a relative displacement $(x - a)$ which is sinusoidal in time. Otherwise, the apparent mass factor k, and the damping coefficient A have no meaning, because they are both based on the Stokes solution for an oscillating sphere given in LAMB [1945, see References at end of published paper, p. 721]. Both k and A are functions of the circular frequency ω as they are defined by (1), (2), and (3).

As an example of the inadequacy of (11) for general purposes, one may consider the effect of adding the gravity force $-(M_o - M)g$ to the right side of (11). Using $a = a_o \sin\omega t$, the solution will be just as given in (12) except for an added term; thus we get

$$x = x_o \sin(\omega t + \alpha) - [(M_o - M)g/A]\, t$$

The superposed settling velocity appears to be $(M_o - M)g/A$ where A depends on ω.

This is clearly incorrect, because the settling velocity has a constant value w which is independent of the frequency ω. This follows directly from the fact that the Navier-Stokes equations are linear in pressure and velocity when all velocity product terms (that is, convective acceleration terms) are neglected, according to the Stokes approximation. Thus the solution $x_1 = -wt$ can be superimposed on any other solution (which the author correctly did subsequently in deriving ϵ_s).

TCHEN [1947, chap. 4 and 5] has obtained a general equation of motion which is not based upon any presuppositions about the solution and might be given in place of (11). Tchen started with the integro-differential equation for the rectilinear accelerated motion of a sphere derived by BASSETT [1888, p. 291] and BOUSSINESQ [1903, p. 238]. Again all velocity product terms were neglected. In the author's notation this equation, including the gravity term, may be written as

$$(M_o + M/2)\,\ddot{x} = -3\pi\mu d\,[\dot{x} + (d/2\sqrt{\pi\nu})\int_{t_o}^{t} dt_1\, \ddot{x}(t_1)/\sqrt{t-t_1}\,] - (M_o - M)\,g \qquad (21)$$

where the entire system is at rest until the instant $t = t_o$. For convenience t_o may be taken as $-\infty$. From this, Tchen derived the equation of motion for a small sphere in a moving fluid

$$(M_o + M/2)\,\ddot{x} = (3/2)\,M\ddot{a} - 3\pi\mu d\left\{\dot{x} - \dot{a} + (d/2\sqrt{\pi\nu})\int_{-\infty}^{t} dt_1\,[\ddot{x}(t_1) - \ddot{a}(t_1)]/\sqrt{t-t_1}\right\} - (M_o - M)\,g \qquad (22)$$

He solved this equation for the case of simple harmonic fluid motion and obtained the same result as the author's (15).

By skillful manipulations which eliminated the integral term involving $\ddot{x}$, Tchen arrived at the following linear second-order differential equation for $\dot{x}$

$$\dddot{x} + 2K\ddot{x} + (K^2 + \sigma^2)\,\dot{x} = G(t) \qquad (23)$$

where

$$G(t) = -\alpha_o w + \alpha_o \dot{a}(t) + \alpha_1 \ddot{a}(t) + \alpha_2 \dddot{a}(t) - \alpha_3 \int_{o}^{\infty} dt_1\, \dddot{a}(t - t_1)/\sqrt{t_1} \qquad (24)$$

The various coefficients in (23) and (24) may be conveniently defined in terms of the two constants

$$\gamma = 12\,\nu/d^2, \quad s = 3\rho/(2\rho_s + \rho) \qquad (25)$$

and are as follows

$$\left.\begin{aligned} \alpha_o &= \gamma^2 s^2 \\ \alpha_1 &= \gamma s\,(1 - 2s) \\ \alpha_2 &= s \\ \alpha_3 &= (\sqrt{3\gamma/\pi})\,s\,(s - 1) \\ K &= \gamma s\,(1 - 3s/2) \\ \sigma^2 &= \gamma^2 s^2 - K^2 = 3\gamma^2 s^3\,(1 - 3s/4) \end{aligned}\right\} \qquad (26)$$

Also w, the Stokes settling velocity, is given by the familiar formula

$$w = g\,(\rho_s - \rho)\,d^2/18\mu$$

The general integral solution of (23) is

$$\dot{x}(t) = (1/\sigma)\int_{o}^{\infty} d\eta\, e^{-K\eta} \sin\sigma\eta\, G(t - \eta) \qquad (27)$$

Equation (27) will give the correct particle velocity $\dot{x}(t)$ for any arbitrary fluid velocity $\dot{a}(t)$, provided only that the convective acceleration terms are small, as was originally assumed.

In the derivation of (19) for ϵ_m, the author implies that ϵ_m is identical with the coefficient of diffusion of fluid particles (let this be called ϵ_o). Fluid particles or molecules in solution are transported bodily in a fluid element, and, except for a relatively insignificant amount of molecular diffusion, they are carried with the fluid element wherever it goes. But, on the other hand, momentum is continually being transferred from one fluid element to its neighbors by pressure differentials and viscous shear. Thus the momentum carried by a parcel of fluid may be expected to vary continuously during its excursion from one point to another. TAYLOR [1932, p. 685] pointed out the variability of momentum due to pressure fluctuations as a weakness of Prandtl's momentum transfer theory when he advocated his vorticity transfer theory to obviate this difficulty. BURGERS [1951, chap. 5] considers the viscous transfer of momentum between adjacent elements more important than the pressure forces in his derivation of the diffusion coefficient for momentum.

Because of these two special factors impeding momentum transfer, this writer believes that it is quite reasonable to presume that the diffusion coefficient for momentum (ϵ_m) is less than the diffusion coefficient for particles of the fluid itself (ϵ_o).

Consequently, the author would be nearer to the truth to give (20) as

$$\epsilon_s/\epsilon_o = (x_o/a)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (28)$$

Thus, for sediment in water, $\epsilon_s/\epsilon_o < 1$ (since $x_o < a_o$ for $\rho_s > \rho$), but since $\epsilon_m/\epsilon_o < 1$ also, it is not all clear when $\beta = \epsilon_s/\epsilon_m$ is greater than or less than one.

But, whether or not (19) should give the diffusion coefficient for momentum (ϵ_m) or fluid particles (ϵ_o), there is some question about the validity of the way in which the author has tried to derive diffusion coefficients from the equation of motion of a suspended particle. The argument the author follows on page 719 is based on the concept of a discontinuous mixing process with the use of some sort of mixing length, and with the assumption that the fluid elements carry the same concentration of sediment as the mean concentration at the starting point. This Prandtl type of approach is often satisfactory for engineering purposes where the interest is in some average bulk property like concentration of sediment. However, it is hardly a good framework in which to utilize detailed information about the behavior of individual particles. At best, one can talk only of a loosely defined "average behavior" at some representative frequency of a simplified type of motion; and, after all, if a body of fluid simply oscillates up and down with $a = a_o \sin \omega t$, so that the motion of particles in the fluid is $x = x_o \sin(\omega t + \alpha) - wt$, there is certainly no turbulent diffusion. Diffusion occurs only because the turbulent velocities of a fluid element are not periodic in any simple sense.

To make use of the equations of motions of a small particle in suspension (23) to (26), it seems more reasonable to consider diffusion as a continuous process, expressing the diffusion coefficients in terms of integrals of Lagrangian correlations. TCHEN [1947, chap. 4 and 5] used this approach and with (27) calculated the integral of the Lagrangian correlation for the sphere velocity in terms of the same integral for the velocity of the fluid surrounding the particle. Neglecting the effect of gravity, the integrals are equal for any general type of turbulent motion.

Tchen's results are reported in detail by BURGERS [1951, chap. 5] who carries the development still further. Results of theirs which differ from the author's are summarized very briefly here. (1) $\epsilon_s \doteq \epsilon_o$ if the particles generally stay well within their respective fluid elements so that the relative motion can be adequately described by (27). This is true regardless of the type or frequency of the fluid oscillations. This assumption is probably quite accurate for silt and clay in suspension. (2) Actually $\epsilon_s < \epsilon_o$ only by virtue of the fact that sediment particles slip from one fluid element into another in the course of their motion. The amount of this slippage and the ratio ϵ_s/ϵ_o depends on the nature of the motion and size of the fluid elements carrying the sediment; this information cannot be deduced from the equation of motion which simply relates x to a. (3) $\epsilon_m < \epsilon_o$, because of the role of viscous forces transferring momentum between fluid elements. (4) $\epsilon_s > \epsilon_m$ ($\beta > 1$) and $\epsilon_s < \epsilon_m$ ($\beta < 1$) are both physically plausible cases. For very fine material it is reasonable to expect $\epsilon_s > \epsilon_m$, because $\epsilon_s \doteq \epsilon_o$.

Hence this writer is unable to accept (20) which implies that β is always less than unity for particles heavier than the fluid.

M. R. Carstens (School of Civil Engineering, Georgia Institute of Technology, Atlanta, Ga., Author's closure)--The experimental program reported concerned a single oscillating sphere in a fluid at rest. The experimental results are presented graphically in Figures 2 and 3. In an elementary manner these results were applied to the case of a spherical particle in an oscillating fluid. Brooks' remarks are confined to this subsequent analysis and not to the experimental program.

The mechanism of suspended-sediment transportation is extremely complex. The problems can be classified in three categories: (1) the description of the fluid motion; (2) the relationship between the fluid motion and the particle motion; and (3) the description of the particles. The description of the fluid motion is very incomplete despite the great amount of analytical and experimental work of the past 30 years. The result is that the present statistical theory of turbulence is mainly a qualitative tool. There have been practically no studies correlating the particle motion with the fluid motion. A considerable amount of work has been done with the particles in the determination of the shape, size, surface condition, and other factors. In view of the complexity of the suspended-sediment transportation process, it is understandable that simplifications, such as the use of the settling velocity to describe the sediment particles and the use of the Fickian diffusion concept (16), are extensively employed.

The very complexity of the suspended-sediment transportation process is an incentive for a simultaneous analysis of the three categories. Ultimately, a description of the transporting fluid motion will be necessary before a general solution of the mechanism of sediment transportation can be considered to exist. Lacking even this knowledge, the research worker is forced to make drastic simplifying assumptions. In this case the assumptions of spherical particles, of uniformly sized particles, of the absence of adjacent-particle interference, and of simple harmonic motion in the vertical direction would appear to preclude a solution of any significance. Although such a solution may fail to indicate correct numerical magnitudes, it may correctly indicate the importance of the individual variables and the order of magnitude of the result. For instance, in the hypothesized situation the value of the amplitude ratio x_o/a_o and the value of β was found to be independent of the absolute magnitude of the fluid amplitude. Experimental studies will be needed in order to test the validity of this conclusion in more realistic diffusion motions.

Thus, from this study the conclusion is that the frequency of motion is a prime variable, whereas the absolute value of the amplitude is of no significance in the determination of the value of β. In the discusser's conclusion (1) the exact opposite statement is presented. A portion of the statement is $\epsilon_s = \epsilon_o$ if the particles generally stay well within their respective fluid elements . . . Such a condition is impossible since a diffusion process is one in which both fluid particles and the foreign particles must be transported to other fluid elements or eddies. It is further implied that (27) will be indicative that $\epsilon_s \doteq \epsilon_o$, whereas it will be by means of (27) that the magnitude of ϵ_s/ϵ_o can be predicted.

In regard to the discusser's conclusion (2), it is correctly stated that $\epsilon_s \neq \epsilon_o$ only by virtue of the relative motion of the particle and the surrounding fluid. Thus ϵ_s/ϵ_o is a direct function of the slippage between the particle and the fluid. The ratio of the particle amplitude to the fluid amplitude x_o/a_o is a measure of this slippage. It can only follow that ϵ_s/ϵ_o must be a function of the amplitude ratio x_o/a_o.

The discusser's conclusions (3) and (4) are valid. A frequent assumption is that $\epsilon_m = \epsilon_o$. However, the experimental studies of CORRSIN [1943] with the diffusion of heat in an axially symmetric jet and the studies of HINZE and VAN DER HEGGE ZIJNEN [1949] with the diffusion of a gas and heat in an axially symmetric jet have indicated that ϵ_o is slightly greater than ϵ_m. Consequently, the ratio as used in this paper must be considered to be the ratio of the sediment diffusion coefficient and the fluid diffusion coefficient ϵ_s/ϵ_o.

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