VECTOR ANALYSIS

by

Harold Wayland

California Institute of Technology

September 1970

VECTORS

2.1 The Characterization of a Vector

Familiarity with such vector quantities as velocity and force gives us what is usually called an "intuitive" notion of vectors. We are familiar with the fact that such vector quantities possess both magnitude and direction, as contrasted with scalar quantities which possess only magnitude. In physics, a vector quantity in three dimensions is frequently represented by a directed line segment, the length of which is proportional to the magnitude of the vector quantity, and the direction of which corresponds to the

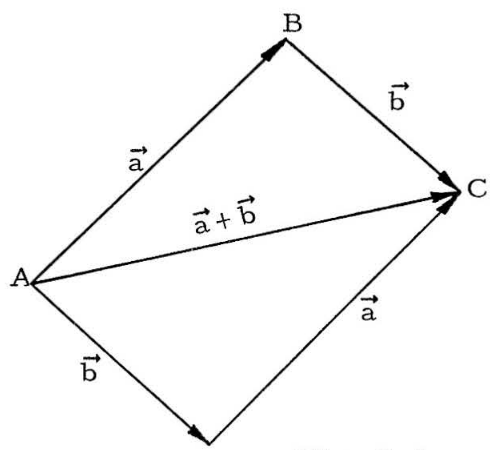


Fig. 2.1

direction of the vector. The simplest prototype vector is given by the displacement between two fixed points in space. Two successive displacements A to B then B to C will be represented by a vector drawn from the original starting point to the final point (AC in Fig. 2.1) and this vector is defined as the "sum" of the two displacement vectors AB and BC. Such a definition of addition insures the commutativity of vector addition, i.e.,

$$\vec{a}+\vec{b} = \vec{b}+\vec{a} \tag{2.1}$$

It is usual in vector analysis to permit vectors to be moved anywhere in space, provided their direction and length are preserved. Such vectors are called free vectors. In mechanics, the line of action of a force vector is important, and a vector constrained to act along a given line is called a bound vector or a sliding vector. We shall direct our attention primarily to free vectors. Multiplication by a positive scalar stretches or contracts the length of the vector without changing its direction or sense. Such multiplication by a scalar is distributive, i.e.,

$$N(\vec{a}+\vec{b}) = N\vec{a}+N\vec{b} \tag{2.2}$$

Multiplication by the scalar $N = 0$ produces a zero vector, a vector of length zero; whereas a multiplication by a negative scalar $N = -M$ stretches the length of the original vector by M and reverses its sense.

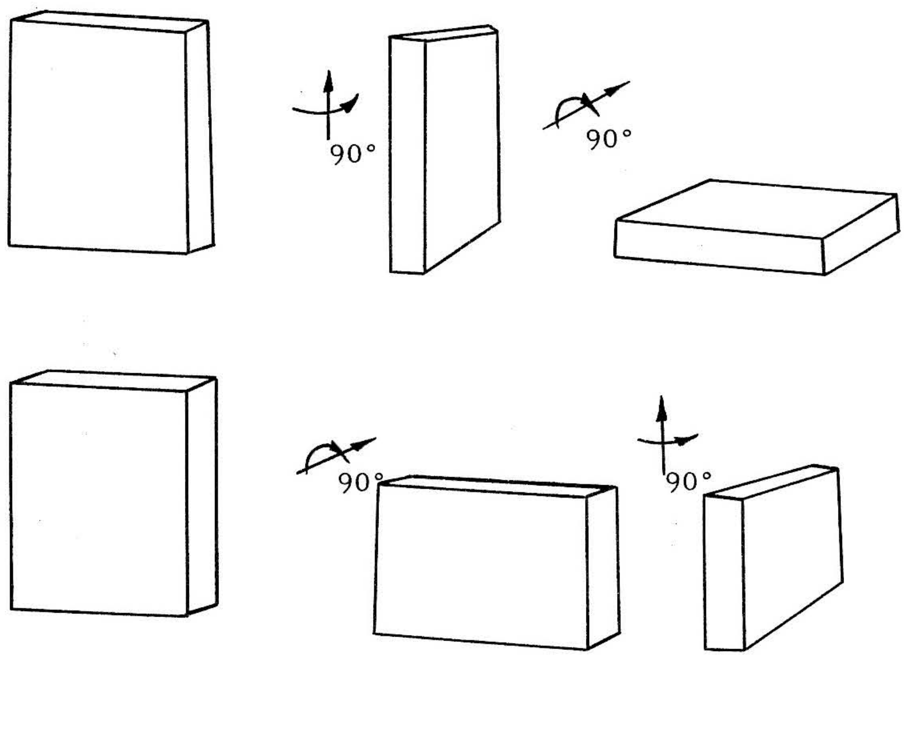


Fig. 2.2

Not all directed quantities which might be represented by directed line segments are vectors. For example, an angular displacement of a rigid body can be uniquely represented by a line parallel to the axis of rotation, of length proportional to the angle of rotation. The final orientation of a body subjected to two successive rotations about non-parallel axes will, in general, be dependent on the order in which the rotations are performed and will not be equal to the rotation obtained by vector addition of the two directed quantities representing each rotation as illustrated in Fig. 2.2. It is important, therefore, to be sure that a set of directed quantities obeys the laws of vector addition before being treated as vectors.

2.2 Vector Algebra

Addition. We have seen that vectors in three dimensions are added by the parallelogram or triangle method; i.e., if the tail of one vector is placed at the tip of the other, then a vector drawn from the tail of the first to the tip of the second is defined as the sum or resultant of the two original vectors (Fig. 2.1). It should be noted that two vectors are coplanar with

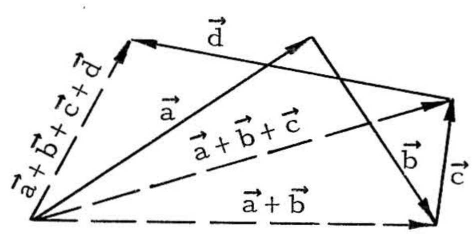


Fig. 2.3

their sum. More than two vectors can be added by first adding a pair, then adding a third to the resultant of the first two, and so on. The same result is obtained by constructing a space polygon as shown in Fig. 2.3.

Equality. Two vectors are defined as equal if they have the same magnitude, direction and sense, even if they do not lie in the same straight line.

Absolute Value. The absolute value of a vector in three dimensions is defined as a scalar numerically equal to the length of the vector.

Multiplication by a Scalar. Multiplication of a vector by a scalar yields a new vector along the same line as the original vector, but with the magnitude changed by the product of its length by the magnitude of the scalar multiplier. The sense remains the same, or is reversed, depending on whether the multiplier is positive or negative.

Scalar Product. The scalar product of two vectors is a number equal to the product of the absolute values of each of the vectors multiplied by the cosine of the angle between them. The most common notation in the U. S. is that of Gibbs (other notations are discussed at the end of this chapter), which represents the scalar product by a dot placed between the vectors. It should be noted that

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos(a, b) = \vec{b} \cdot \vec{a} \tag{2.3}$$

the result of "dotting" $\vec{a}$ with $\vec{b}$ is to form the product of the magnitude of the projection of $\vec{a}$ in the direction of $\vec{b}$ with the magnitude of $\vec{b}$. (Fig. 2.4).

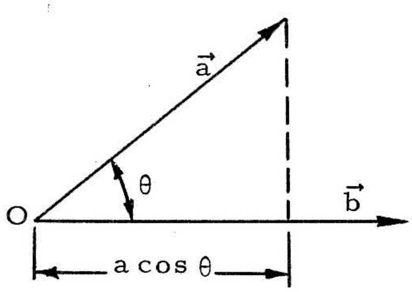


Fig. 2.4

Such scalar products are frequently met in mechanics: if $\vec{a}$ is a force $\vec{f}$ acting on a particle at O, and $\vec{b}$ a linear displacement of the particle, then $\vec{a} \cdot \vec{b} = \vec{f} \cdot \vec{b}$ is just the product of the component of $\vec{f}$ in the $\vec{b}$ direction by the displacement, hence the work done on the particle by the force $\vec{f}$ in moving through the distance b. If $\vec{a} = \vec{f}$ is a force and $\vec{b} = \vec{v}$ a velocity vector, $\vec{a} \cdot \vec{b} = \vec{f} \cdot \vec{v}$ represents the rate $\vec{f}$ is doing work in the $\vec{v}$ direction.

If two vectors are perpendicular, the scalar product vanishes. Conversely, the vanishing of the scalar product of two non-vanishing vectors insures their perpendicularity.

Vector Product. The vector product of two vectors is defined as a vector perpendicular to the plane defined by the two original vectors when translated to a common origin, and of magnitude equal to the product of the absolute values of the original vectors multiplied by the sine of the angle between them. The sense of the product vector is given by the right hand screw rule, i. e., the direction of progression of a right hand screw when turned from the first to the second term of the product (Fig. 2.5).

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin(a, b)\, \vec{\nu} \qquad (2.4)$$

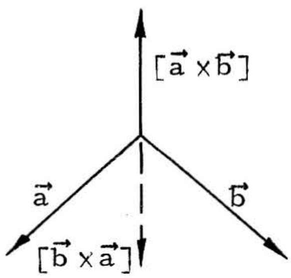


Fig. 2.5

where $\vec{\nu}$ is a unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$, the sense of which is given by the direction of progression of a right hand screw when turned from $\vec{a}$ to $\vec{b}$. From this definition it follows that

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \qquad (2.5)$$

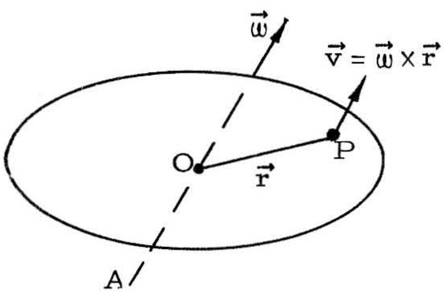


Fig. 2.6

A familiar example from mechanics arises in evaluating the linear velocity of a point in a rotating solid body. If the body is rotating about the axis A (Fig. 2.6) with angular velocity $\vec{\omega}$, and $\vec{r}$ represents the position vector of the point P with respect to any prescribed point O on the axis of rotation, then the linear velocity of P will be given by $\vec{v} = \vec{\omega} \times \vec{r}$.

Multiplication is Distributive. All three types of multiplication are distributive, provided that the order of terms is retained for the vector product. The proof follows readily from the geometric interpretations of the various types of products.

Division. Division of a vector by a scalar is covered by the definition of multiplication by a scalar. Division of one vector by another is not defined.

Triple Products. Given three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, there are three types of triple products which have meaning in vector analysis.

1. The dot product can be formed for any pair and the resulting scalar multiplied into the third vector: $\vec{a}(\vec{b} \cdot \vec{c})$, a vector in the direction of $\vec{a}$.
2. The cross product can be formed for any pair and the resulting vector dotted into the third vector: $\vec{a} \cdot (\vec{b} \times \vec{c})$, a scalar. This is called the scalar triple product and is sometimes written $(\vec{a}\,\vec{b}\,\vec{c})$.

3. The cross product can be found for any pair and the resulting vector crossed into the third vector: $(\vec{a} \times \vec{b}) \times \vec{c}$, a vector. This is called the vector triple product.

EXERCISES[1]

2.1 Show by vector methods, that is, without using components, that the diagonals of a parallelogram bisect each other.

2.2 Show by vector methods that the line which joins one vertex of a parallelogram to the middle point of an opposite side trisects the diagonal.

2.3 The vectors $\vec{a}$ and $\vec{b}$ extend from the origin O to the points A and B. Determine the vector $\vec{c}$ which extends from O to the point C which divides the line segment from A to B in the ratio m:n. Do not use components.

2.4 Without using components, show that

$$(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) = (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2$$

for any vectors $\vec{a}$ and $\vec{b}$.

2.5 A natural way to attempt to define division by a vector would be to seek the vector $\vec{b}$ such that the equality $\vec{a} \times \vec{b} = \vec{c}$ holds when $\vec{a}$ and $\vec{c}$ are given nonparallel vectors. Show that this equation does not define $\vec{b}$ uniquely.

2.6 Without using components, show

a. Vector addition is commutative. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

b. Vector addition is associative. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

c. Multiplication by a scalar is distributive. $N(\vec{a} + \vec{b}) = N\vec{a} + N\vec{b}$

d. The scalar product is commutative. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

e. The vector product is not commutative, but $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

f. The scalar product is distributive. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

---

[1] Many important results are included only in the problems and the reader should familiarize himself with the results even when he does not work all of the problems.

2.7 Show that for two nonvanishing vectors:

$\vec{a}\cdot\vec{b}=0$ is the condition that $\vec{a}$ is perpendicular to $\vec{b}$

$\vec{a}\times\vec{b}=0$ is the condition that $\vec{a}$ is parallel to $\vec{b}$

2.8 Show that $\vec{a}\cdot(\vec{b}\times\vec{c})$ is the volume of the parallelopiped, the edges of which are the vectors a, b and c. From this geometrical fact establish the relation

$$\vec{a}\cdot\vec{b}\times\vec{c}=\vec{b}\cdot\vec{c}\times\vec{a}=\vec{c}\cdot\vec{a}\times\vec{b}$$

2.9 Show that the vector product is distributive.

$$\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

2.10 Show that

$$(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$$

and

$$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{a})\vec{c}$$

2.3 Differentiation of a Vector

If a vector is a function of a scalar variable such as time, then for each instant the magnitude and direction will be known. Between two successive instants the vector will change by an amount $\Delta\vec{a}$ (Fig. 2.7), while the time changes by an amount Δt. The vector

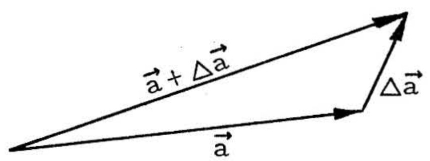


Fig. 2.7

$$\frac{d\vec{a}}{dt}=\lim_{\Delta t\to 0}\frac{\Delta\vec{a}}{\Delta t} \qquad (2.6)$$

is defined as the derivative of $\vec{a}$ with respect to t if the limit exists. The ordinary rule for differentiation of a product is valid, as can easily be demonstrated by applying the definition of differentiation coupled with the rules of multiplication to such a product, but care must be taken not to interchange the order of the factors if cross products are involved. For example

$$\frac{d}{dt}(\vec{u}\times\vec{v})=\vec{u}\times\frac{d\vec{v}}{dt}+\frac{d\vec{u}}{dt}\times\vec{v}=\vec{u}\times\frac{d\vec{v}}{dt}-\vec{v}\times\frac{d\vec{u}}{dt}$$

EXERCISES

2.11 A vector of $\vec{a}$ of constant length (but varying direction) is a function of time. Show that $d\vec{a}/dt$ is perpendicular to $\vec{a}$.

2.12 Show that if $\vec{F}$ is a force directed along $\vec{r}$ and if $\vec{F} \times d\vec{r}/dt = 0$ at all times, the vector $\vec{r}$ has a constant direction. $\vec{r}$ is the position vector from the origin to the point in question.

2.4 Space Curves

Each point of a space curve C (Fig. 2.8), whether plane or skew, can be described by means of the position vector $\vec{r}$ from a fixed origin O.

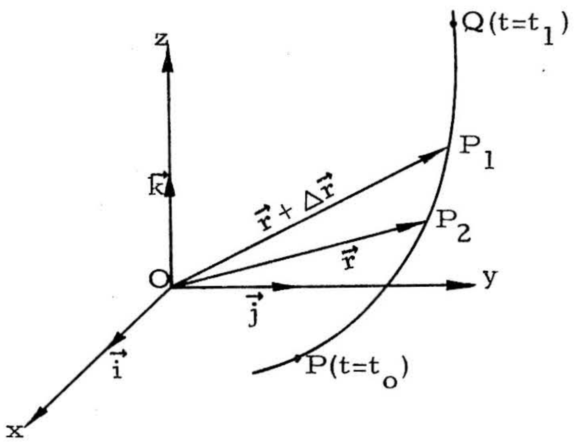


Fig. 2.8

In the cartesian coordinates of the figure we can write

$$\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z \tag{2.8}$$

If now,

$$x = f(t) \quad ; \quad y = g(t) \quad ; \quad z = h(t) \qquad (t_o \le t \le t_1) \tag{2.9}$$

where f(t), g(t) and h(t) are continuous functions of t for $t_o \le t \le t_1$, the curve can be expressed in terms of the parameter t as

$$r = r(t) = \vec{i}f(t) + \vec{j}g(t) + \vec{k}h(t) \tag{2.10}$$

The curves most frequently met in physical problems are continuous, rectifiable (i.e., their length can be measured) and made up of segments of finite length, each of which has a continuously turning tangent. For the moment we shall confine our attention to portions of such curves without singularities and with a continuously turning tangent.

The length s along the arc of the curve, measured from some fixed point P, can be used as the parameter for the analytic description of the curve

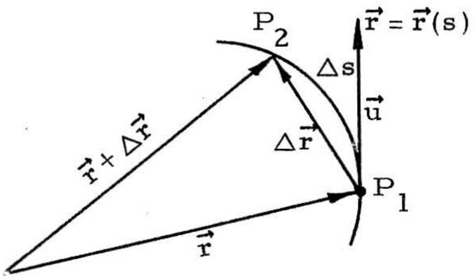


Fig. 2.9

$$\vec{r} = \vec{r}(s) \tag{2.11}$$

If we consider the points P_1 and P_2 (Fig. 2.9) where P_1 is given by the position vector $\vec{r}$ and P_2 by $(\vec{r}+\Delta\vec{r})$ we see that $\Delta\vec{r}$ will be a vector equal in length of the chord of the curve between P_1 and P_2 and for a smooth curve

$$\lim_{\Delta s \to 0} \left|\frac{\Delta\vec{r}}{\Delta s}\right| = 1 \tag{2.12}$$

and

$$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \left[\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2\right]^{1/2} \tag{2.13}$$

if $f'(t)$, $g'(t)$ and $h'(t)$ exist. We shall assume that these derivatives do not all vanish simultaneously on C; hence $|d\vec{r}/dt| \neq 0$ on C.

At any interior point on a space curve of the type we have been describing we can define a set of three orthogonal unit vectors: (a) the unit tangent vector $\vec{u}$; (b) the unit principal normal vector $\vec{n}$; and (c) the unit binormal vector $\vec{b}$, perpendicular to both $\vec{u}$ and $\vec{n}$. This triple of orthogonal unit vectors $(\vec{u}, \vec{n}, \vec{b})$ is called the principal triad of the curve, and will be chosen to form a right-handed system in the order given.

(a) The unit tangent vector $\vec{u}$. The vector $d\vec{r}/dt$ is tangent to the curve, hence we can define the unit tangent vector as

$$\vec{u} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{d\vec{r}}{dt}\frac{dt}{ds} = \frac{d\vec{r}}{ds} \tag{2.14}$$

(b) The unit prinicpal normal $\vec{n}$. If we consider the unit tangent vectors at the points P_1 and P_2 of Fig. 2.10, it appears as if, in the limit as $\Delta s \to 0$, $\Delta\vec{u}$ will be perpendicular to $\vec{u}$. This is readily shown analytically from the fact that $\vec{u} \cdot \vec{u} = 1$; hence $d\vec{u}/ds \cdot \vec{u} + \vec{u} \cdot d\vec{u}/ds = 0$. Except in the case in which $d\vec{u}/ds = 0$ (the curve is a straight line) this insures the perpendicularity of $\vec{u}$ and $d\vec{u}/ds$, and defines a unique normal direction to the curve.

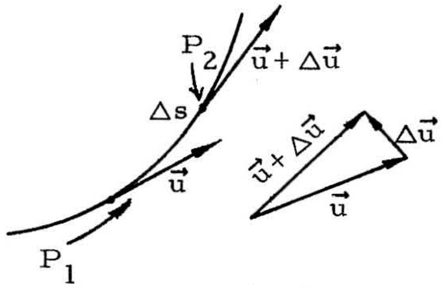


Fig. 2.10

(In the case of a straight line there is no way in which to define a unique normal from the intrinsic properties of the curve.) The unit principal normal is defined as

$$\vec{n} = \frac{\frac{d\vec{u}}{ds}}{\left|\frac{d\vec{u}}{ds}\right|} \tag{2.15}$$

This can be written in the form

$$\frac{d\vec{u}}{ds} = (\kappa\vec{n}) \tag{2.16}$$

where κ is the principal curvature of the curve at the point at which $d\vec{u}/ds$ is evaluated, and $\rho = 1/\kappa$ is called the principal radius of curvature. From the mode of definition of the unit principal normal, we see that the element of the curve adjacent to P_1 is contained in the plane defined by the vectors $\vec{u}$ and $\vec{n}$. This is called the osculating plane for the curve at that point

(c) The unit binormal vector $\vec{b}$. The unit binormal, the third vector of the principal triad, is defined as being perpendicular to both $\vec{u}$ and $\vec{n}$ and in such a sense as to form a right-handed system in the order $(\vec{u}, \vec{n}, \vec{b})$, hence we must have (Fig. 2.11)

$$\vec{b} = [\vec{u} \times \vec{n}] \tag{2.17}$$

The Frenet-Serret Formulas. The derivatives of the unit vectors $\vec{u}$, $\vec{n}$, and $\vec{b}$ with respect to s are related to the vectors themselves by the Frenet-Serret formulas

$$\left.\frac{d\vec{u}}{ds} = \vec{n} \quad \right\} \tag{2.18}$$

$$\left.\begin{aligned} \frac{d\vec{n}}{ds} &= \tau\vec{b} - \kappa\vec{u} \\ \frac{d\vec{b}}{ds} &= -\tau\vec{n} \end{aligned}\right\} \qquad \text{(2.18) cont.}$$

κ has already been defined as the principal curvature (Eq. 2.16). τ is called the torsion and is a measure of tendency of the curve to "twist" out of the osculating plane. For a plane curve, $\vec{b}$ at any point on the curve will be parallel to its value at any other point, hence $d\vec{b}/ds$, and consequently τ, will vanish. Its reciprocal $1/\tau$ is called the radius of torsion. The first of Eqs. 2.18 has already been established. To establish the other two

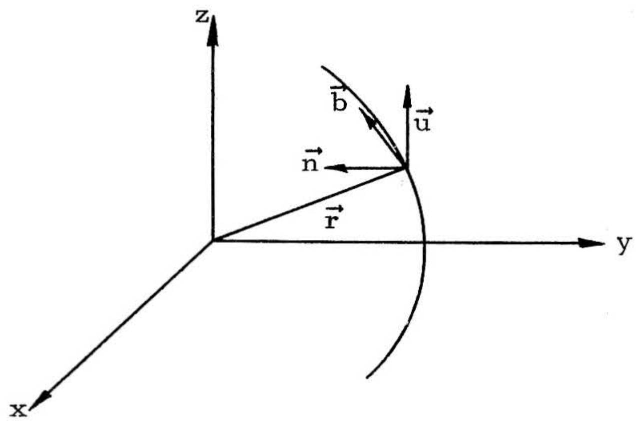


Fig. 2.11

equations we first differentiate the equation $\vec{b} = \vec{u} \times \vec{n}$ and substitute the known value for $d\vec{u}/ds$.

$$\frac{d\vec{b}}{ds} = \frac{d\vec{u}}{ds} \times \vec{n} + \vec{u} \times \frac{d\vec{n}}{ds} = \kappa\vec{n} \times \vec{n} + \vec{u} \times \frac{d\vec{n}}{ds} = \vec{u} \times \frac{d\vec{n}}{ds} \qquad (2.19)$$

Next we differentiate $\vec{n} = \vec{b} \times \vec{u}$ to obtain

$$\frac{d\vec{n}}{ds} = \frac{d\vec{b}}{ds} \times \vec{u} + \vec{b} \times \frac{d\vec{u}}{ds} = \frac{d\vec{b}}{ds} \times \vec{u} + (\vec{b} \times \vec{n})\kappa = \frac{d\vec{b}}{ds} \times \vec{u} - \kappa\vec{u} \qquad (2.20)$$

Now since $\vec{b}$ is a unit vector — i.e., it can change direction but not magnitude — $d\vec{b}/ds$ must lie in a plane perpendicular to $\vec{b}$; hence it can be expressed as a linear combination of $\vec{u}$ and $\vec{n}$. Hence

$$\frac{d\vec{b}}{ds} = \alpha\vec{u} + \beta\vec{n} \qquad (2.21)$$

where α and β are numbers which we wish to determine. Putting this value for $d\vec{b}/ds$ into Eq. 2.20 we obtain

$$\frac{d\vec{n}}{ds} = (\alpha\vec{u} + \beta\vec{n}) \times \vec{u} - \kappa\vec{u} = -\beta\vec{b} - \kappa\vec{u} \qquad (2.22)$$

Introducing Eq. 2.22 into Eq. 2.19 we obtain

$$\frac{d\vec{b}}{ds} = \vec{u} \times (-\beta\vec{b} - \kappa\vec{u}) = \beta\vec{n} \qquad (2.23)$$

This shows that $d\vec{b}/ds$ is, indeed, parallel to $\vec{n}$. We arbitrarily define $\beta = -\tau$, giving the third of the Frenet-Serret formulas. Inserting this value for β in Eq. 2.22 we obtain the second of the formulas

$$\frac{d\vec{n}}{ds} = \tau\vec{b} - \kappa\vec{u} \qquad (2.24)$$

Examples

1. For a straight line, $d\vec{u}/ds = 0$, the curvature is zero, the radius of curvature infinite and $\vec{b}$ and $\vec{n}$ are not defined.

2. For a circle of radius a, the curvature is 1/a and the torsion is zero.

3. Consider the curve given by the set of parametric equations

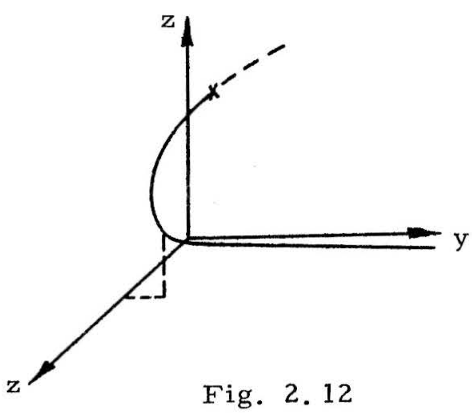


Fig. 2.12

$$\left.\begin{aligned} x &= 3t - t^3 \\ y &= 3t^2 \\ z &= 3t + t^3 \end{aligned}\right\} \qquad (2.25)$$

This curve starts from the origin at t=0, moves into the first octant, and then penetrates the y-z plane when $t = \sqrt{3}\ (0, 9, 6\sqrt{3})$, remaining in the octant in which y and z are positive and x negative for all subsequent positive t. We can use the parametric Equations 2.25 to calculate ds/dt

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 18(1+t^2)^2 \qquad (2.26)$$

Since $\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$, we can calculate $\vec{u}$ from Eq. 2.14

$$\vec{u} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds} = \frac{1-t^2}{\sqrt{2}\,(1+t^2)}\vec{i} + \frac{2t}{\sqrt{2}\,(1+t^2)}\vec{j} + \frac{1}{\sqrt{2}}\vec{k} \tag{2.27}$$

From Eq. 2.16

$$\frac{d\vec{u}}{ds} = \kappa\vec{n} = \frac{d\vec{u}}{dt}\frac{dt}{ds} = \frac{-2t}{3(1+t^2)^3}\vec{i} + \frac{(1-t^2)}{3(1+t^2)^3}\vec{j} \tag{2.28}$$

whence

$$\vec{n} = -\frac{2t}{(1+t^2)}\vec{i} + \frac{(1-t^2)}{(1+t^2)}\vec{j} \tag{2.29}$$

and

$$\kappa = \frac{1}{3(1+t^2)^3} \tag{2.30}$$

From Eqs. 2.27 and 2.29 we find

$$\vec{b} = \vec{u}\times\vec{n} = -\frac{(1-t^2)}{\sqrt{2}\,(1+t^2)}\vec{i} - \frac{\sqrt{2}t}{1+t^2}\vec{j} + \frac{\sqrt{2}}{2}\vec{k} \tag{2.31}$$

Comparing Eqs. 2.27 and 2.31 we see that the only components which vary along the curve have opposite signs, hence we can conclude that for this curve

$$\frac{d\vec{b}}{ds} = -\frac{d\vec{u}}{ds} \tag{2.32}$$

hence $\tau = \kappa$, so that the torsion and curvature are equal.

EXERCISES

2.13 (a) Describe the space curve whose parametric equations are

$$x = a\cos t \quad , \quad y = a\sin t \quad , \quad z = ct$$

where a and c are constants. Compute the unit tangent vector, the unit principal normal and the unit binormal.

(b) Find the radius of curvature, the radius of torsion and the angle between the unit tangent vector at any point and the positive z-axis.

2.14 (a) A particle of mass m moves along the curve C whose vector equation is $\vec{r} = \vec{r}(t)$, where t is time. Compute the velocity and acceleration vectors in terms of the unit tangent vector, the principal normal vector and the binormal vector for C.

(b) Suppose C is the helix of Problem 2.13(a). Compute the force vector $\vec{F}$ which must act on the particle in order to produce the observed motion.

2.5 Surfaces

A surface is a two-parameter system, which can be defined vectorially by

$$\vec{r} = \vec{r}(u, v) \tag{2.33}$$

For the sake of this discussion, we shall confine our attention to intervals on u and v throughout which $\vec{r}(u, v)$ is single-valued. Let $(u_o \le u \le u_1;\ v_o \le v \le v_1)$ be such an interval. If v is held constant and u permitted to range from v_o to v_1, $\vec{r}$ will sweep out a space curve (Fig. 2.13) lying in the surface. Similarly if u is held constant and v permitted to vary. Since the curves u = const. and v = const. lie in the surface, we can construct two tangent vectors to the surface $\partial\vec{r}/\partial u$ and $\partial\vec{r}/\partial v$. These vectors will not, in general, be perpendicular to one another nor will they be unit vectors,

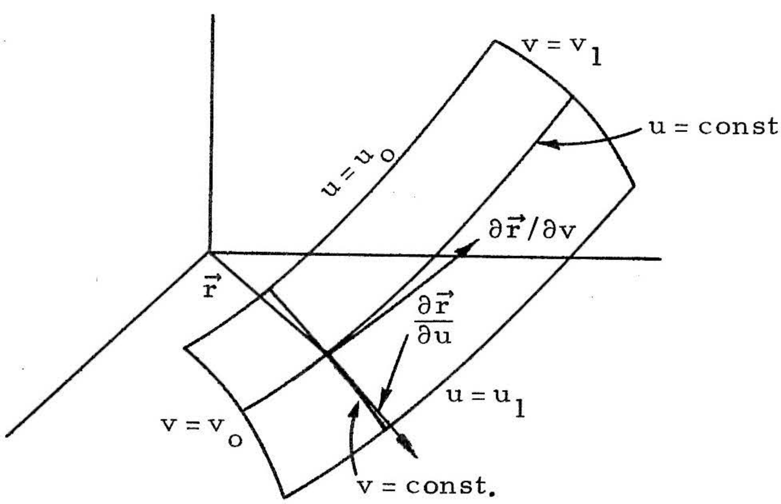


Fig. 2.13

although normalization is readily accomplished by dividing by the absolute value. There is an infinite number of tangent vectors to a smooth surface at any point, but the direction of the normal is uniquely defined, although some convention must be adopted to define the sense. A vector normal to S can readily be constructed by taking the cross product of the two tangent vectors already obtained, normalizing it to obtain the unit normal vector $\vec{\nu}$ where

$$\vec{\nu} = \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right|} \tag{2.34}$$

<u>Example.</u> Consider the paraboloid of revolution

$$x^2 + y^2 = 2z - 2 \tag{2.35}$$

In vector form this can be written as

$$\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z = \vec{i}x + \vec{j}y + \vec{k}\left(\frac{x^2+y^2}{2} + 1\right) \tag{2.36}$$

Tangents are obtained by taking the partial derivatives

$$\frac{\partial \vec{r}}{\partial x} = \vec{i} + \vec{k}x \quad ; \quad \frac{\partial \vec{r}}{\partial y} = \vec{j} + \vec{k}y \tag{2.37}$$

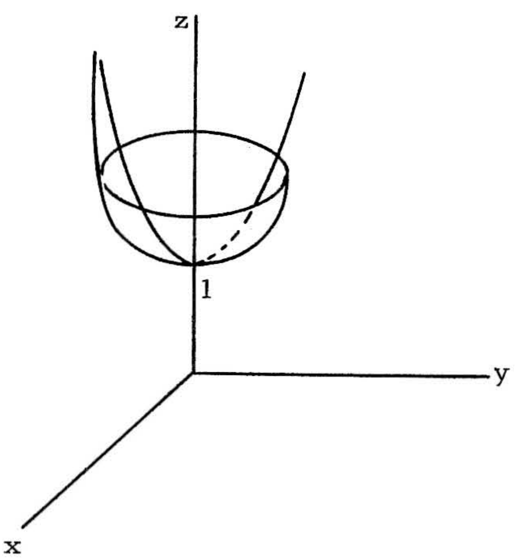


Fig. 2.14

The normal is

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \vec{i}x - \vec{j}y + \vec{k} \tag{2.38}$$

and the unit normal

$$\vec{\nu} = \frac{-\vec{i}x - \vec{j}y + \vec{k}}{\sqrt{x^2+y^2+1}} \tag{2.39}$$

In this case the normal points toward the z-axis: to the interior of the surface if we think of it as a cup.

2.6 Coordinate Systems

Any pair of non-parallel intersecting surfaces will in general intersect in a space curve. If a third surface intersects the curve in a single point, then these three surfaces can be used to define that point. A family of such surfaces can be used as a curvilinear coordinate system: the term "curvilinear" arising from the fact that the three curves formed by the intersections of the surfaces in pairs will pass through the point. The reader should already be familiar with the three sets of coordinates shown in Fig. 2.15. In Fig. 2.15(a) we have rectangular coordinates in which the coordinate surfaces are three planes, parallel respectively to the y-z, x-z, and x-y planes. Their curves of intersection are lines parallel to the coordinate axes. The coordinate surfaces for cylindrical coordinates, Fig. 2.15(b), are cylinders ($r = \text{const.}$), half planes ($\varphi = \text{const.}$, $0 < \varphi \leq 2\pi$), and planes ($z = \text{const.}$). The curves of intersection are readily seen in the figure. For spherical coordinates (Fig. 2.15(c)) the surfaces are spheres ($r = \text{const.}$), half-cones ($\theta = \text{const.}$, $0 \leq \theta \leq \pi$) and half planes ($\varphi = \text{const.}$, $0 < \varphi \leq 2\pi$). Again the curves of intersection can be seen in the figure. At the point of intersection of three surfaces a triad of unit normal vectors can be defined uniquely except for sense. Such triads are shown in Fig. 2.15(a), (b), (c), with a standard convention regarding sense. As long as these unit vectors are not coplanar, any vector quantity can be described in terms of its components along these three normal vectors. They do not have to be orthogonal.

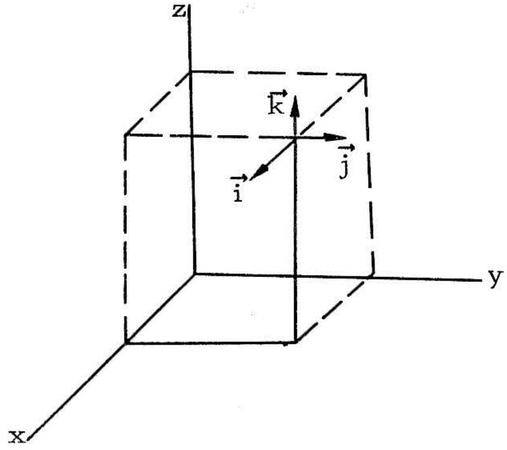


Fig. 2.15(a)

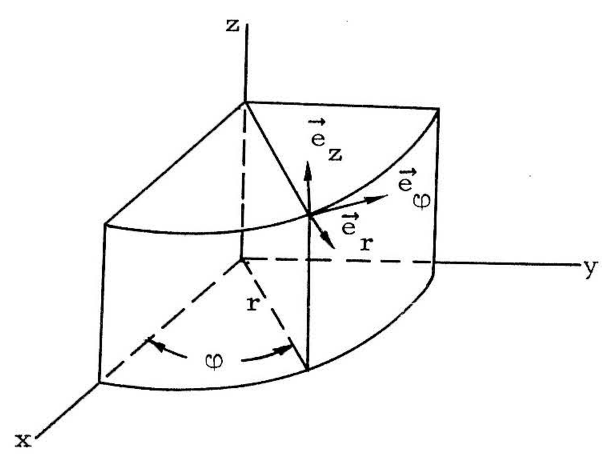


Fig. 2.15(b)

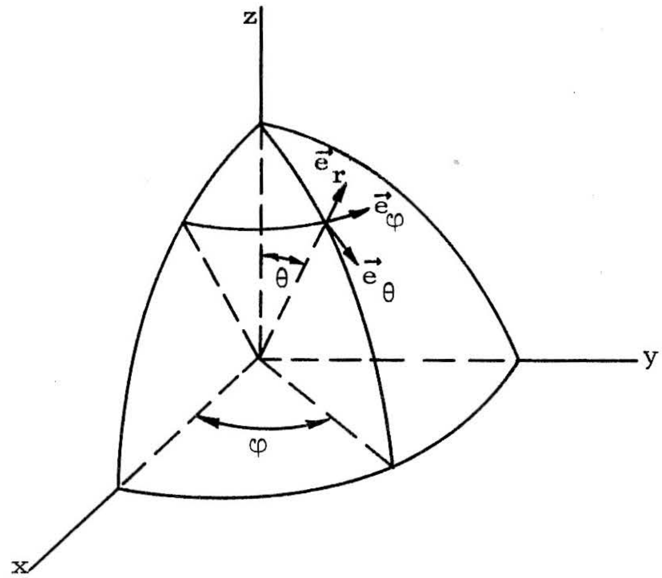


Fig. 2. 15(c)

Suppose we are given the three sets of surfaces

$$\begin{aligned} f(x, y, z) &= u \\ g(x, y, z) &= v \\ h(x, y, z) &= w \end{aligned} \qquad (2.40)$$

If these are non-parallel surfaces, each pair of which intersects in a space curve for some range of values of u, v, and w, then a point will be defined for each allowable triple of values of (u, v, w). Since any point in space can be uniquely described in terms of its cartesian coordinates (x, y, z), then if the three numbers (u, v, w) represent a point we would expect it to be possible to invert the Equations 2. 40 and solve them for x, y and z as functions of u, v, w. This is not always possible to do explicitly even when such a relationship theoretically exists. We can, however, establish criteria which tell us when such an inversion is theoretically possible. To explore this in the neighborhood of a given point we shall take a linear approximation, using the first terms in the Taylor expansion, assuming that the various functions are continuous and possess the required derivatives. To do this we must calculate $\partial x/\partial u$, $\partial y/\partial u$, etc., from Eqs. 2. 40. Differentiating these equations with respect to u we obtain

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} = 1$$

$$\frac{\partial g}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial u} = 0 \qquad (2.41)$$

$$\frac{\partial h}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial h}{\partial z}\frac{\partial z}{\partial u} = 0$$

Solving for $\partial x/\partial u$ we obtain

$$\frac{\partial z}{\partial u} = \frac{\begin{vmatrix} 1 & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ 0 & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ 0 & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{vmatrix}} = \frac{\frac{\partial (g, h)}{\partial (y, z)}}{\frac{\partial (f, g, h)}{\partial (x, y, z)}} = \frac{\frac{\partial (v, w)}{\partial (y, z)}}{\frac{\partial (u, v, w)}{\partial (x, y, z)}}$$

where the notation on the right of Eq. 2.41 is a short-hand notation for the determinants shown. Such determinants are called Jacobians. The other partials can be similarly obtained. In all of them the denominator will be the same: the Jacobian of (u, v, w) with respect to (x, y, z). This Jacobian must not vanish for the inversion to exist.

Unit Vectors. If we have a set of coordinate surfaces u_1 = const., u_2 = const., u_3 = const. which are non-parallel, then at any point of intersection we can set up the triad of non-coplanar unit normal vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}$. Another logical triad of unit vectors which can be associated with each point will be tangent to the coordinate curves $\vec{i}_1$, $\vec{i}_2$, $\vec{i}_3$ in Fig. 2.16. These will coincide with $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ only if the coordinates are orthogonal.

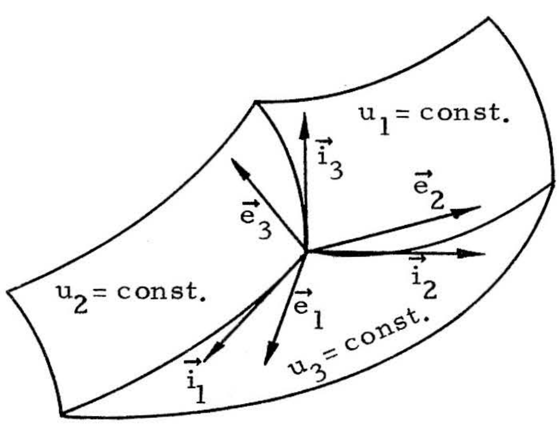


Fig. 2.16

Element of Length. If the unit normals $\vec{i}_1$, $\vec{i}_2$, $\vec{i}_3$ are non-coplanar, any vector can be expressed in terms of its components in these three directions. In particular, if a curve passes through the point P associated with this particular triad of unit vectors, we can express the element of length along the curve, $d\vec{s}$, in terms of such components:

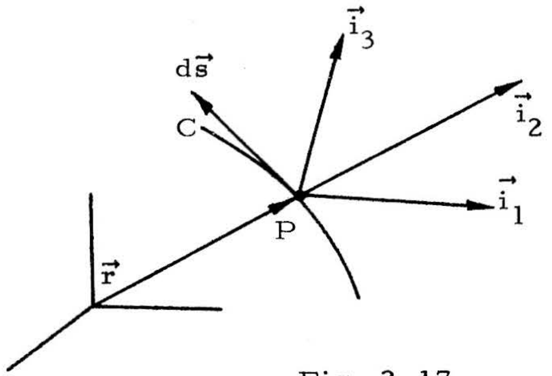


Fig. 2.17

$$d\vec{s} = h_1 du_1 \vec{i}_1 + h_2 du_2 \vec{i}_2 + h_3 du_3 \vec{i}_3 \tag{2.42}$$

(It should be noted that it is possible to express a vector in terms of its components with respect to any non-coplanar set of directions. Since the unit vectors of a curvilinear coodrinate system will, in general, change direction from point to point we will have to specify the point at which the basic vector triad is defined. In the case of a space curve, it is most convenient to use the triad associated with the point being examined.)

If $\vec{r} = \vec{r}(u_1, u_2, u_3)$ and we allow $\vec{r}$ to travel along the curve C of Fig. 2.17, we can write

$$\begin{aligned} d\vec{s} &= \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3 \\ &= h_1 \vec{i}_1 du_1 + h_2 \vec{i}_2 du_2 + h_3 \vec{i}_3 du_3 \end{aligned} \tag{2.43}$$

Now $\partial \vec{r}/\partial u_i$ will be a vector in the $\vec{i}_i$ direction, hence we can write

$$\frac{\partial \vec{r}}{\partial u_i} = h_i \vec{i}_i \tag{2.44}$$

If we now put $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and think of x, y, z as functions of u_1, u_2, u_3, we have

$$\frac{\partial x}{\partial u_i}\vec{i} + \frac{\partial y}{\partial u_i}\vec{j} + \frac{\partial z}{\partial u_i}\vec{k} = h_i \vec{i}_i \tag{2.45}$$

Dotting this vector with itself we obtain

$$h_i^2 = \left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2 \qquad (i = 1, 2, 3) \tag{2.46}$$

which is valid whether the coordinate system is orthogonal or not.

Element of Volume. With the same notation as used in the previous section, an element of volume associated with a curvilinear parallelopiped bounded by the coordinate surfaces u_1, $u_1 + du_1$; u_2, $u_2 + du_2$; u_3, $u_3 + du_3$ will be

$$dV = h_1 \vec{i}_1 du_1 \cdot \left[h_2 \vec{i}_2 du_2 \times h_3 \vec{i}_3 du_3\right] = h_1 h_2 h_3 du_1 du_2 du_3 \vec{i}_1 \cdot \left[\vec{i}_2 \times \vec{i}_3\right] \tag{2.47}$$

If the system is orthogonal this reduces to

$$dV = h_1 h_2 h_3 du_1 du_2 du_3 \tag{2.48}$$

If not, we can obtain an analytic expression by considering

$$dV = \frac{\partial \vec{r}}{\partial u_1} du_1 \cdot \left[\frac{\partial \vec{r}}{\partial u_2} du_2 \times \frac{\partial \vec{r}}{\partial u_3} du_3\right]$$

$$= \left[\frac{\partial x}{\partial u_1}\vec{i} + \frac{\partial y}{\partial u_1}\vec{j} + \frac{\partial z}{\partial u_1}\vec{k}\right] \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix} du_1 du_2 du_3 \tag{2.49}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix} du_1 du_2 du_3$$

Since rows and columns can be interchanged in a determinant without changing its value, we see that the determinant is just the Jacobian of the transformation or

$$dV = \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} du_1 du_2 du_3 \tag{2.50}$$

<u>Variation of Unit Vectors.</u> In a curvilinear coordinate system the direction of the unit vectors will depend on their position, and we need to know just how they vary from point to point. Since

$$d\vec{s} = h_1 du_1 \vec{i}_1 + h_2 du_2 \vec{i}_2 + h_3 du_3 \vec{i}_3 \tag{2.51}$$

then

$$\frac{\partial \vec{s}}{\partial u_1} = h_1 \vec{i}_1 ; \quad \frac{\partial \vec{s}}{\partial u_2} = h_2 \vec{i}_2 ; \quad \frac{\partial \vec{s}}{\partial u_3} = h_3 \vec{i}_3 \tag{2.52}$$

Differentiating the first and second of these equations with respect to u_2 and u_1, respectively:

$$\begin{aligned} \frac{\partial^2 \vec{s}}{\partial u_2 \partial u_1} &= \frac{\partial}{\partial u_2}(h_1 \vec{i}_1) = \frac{\partial h_1}{\partial u_2}\vec{i}_1 + h_1 \frac{\partial \vec{i}_1}{\partial u_2} \\ \frac{\partial^2 \vec{s}}{\partial u_1 \partial u_2} &= \frac{\partial}{\partial u_1}(h_2 \vec{i}_2) = \frac{\partial h_2}{\partial u_1}\vec{i}_2 + h_2 \frac{\partial \vec{i}_2}{\partial u_1} \end{aligned} \tag{2.53}$$

Equating the two mixed partials

$$\frac{\partial h_1}{\partial u_2}\vec{i}_1 + h_1 \frac{\partial \vec{i}_1}{\partial u_2} = \frac{\partial h_2}{\partial u_2}\vec{i}_2 + h_2 \frac{\partial \vec{i}_2}{\partial u_1} \tag{2.54}$$

This equation is valid whether the coordinates are orthogonal or not. <u>In case they are orthogonal,</u> $(\partial \vec{i}_1 / \partial u_2)$ will be parallel to $\vec{i}_2$ and $(\partial \vec{i}_2 / \partial u_1)$ will be parallel to $\vec{i}_1$. For orthogonal coordinates, $\vec{i}_i = \vec{e}_i$, so we can write

$$\left.\begin{aligned} \frac{\partial \vec{e}_2}{\partial u_1} &= \frac{1}{h_2}\frac{\partial h_1}{\partial u_2}\vec{e}_1 \\ \frac{\partial \vec{e}_1}{\partial u_2} &= \frac{1}{h_1}\frac{\partial h_2}{\partial u_1}\vec{e}_2 \end{aligned}\right\} \tag{2.55}$$

By cyclic permutation we can fill out the table to obtain

$$\frac{\partial \vec{e}_1}{\partial u_3} = \frac{1}{h_1}\frac{\partial h_3}{\partial u_1}\vec{e}_3 ; \quad \frac{\partial \vec{e}_3}{\partial u_1} = \frac{1}{h_3}\frac{\partial h_1}{\partial u_3}\vec{e}_1 \tag{2.56}$$

$$\frac{\partial \vec{e}_2}{\partial u_3} = \frac{1}{h_2}\frac{\partial h_3}{\partial u_2}\vec{e}_2 \; ; \quad \frac{\partial \vec{e}_3}{\partial u_2} = \frac{1}{h_3}\frac{\partial h_2}{\partial u_3}\vec{e}_2 \qquad (2.56) \text{ cont.}$$

For the terms with the same subscript on the vector and coordinate we take advantage of the relations of the type $\vec{e}_3 = \vec{e}_1 \times \vec{e}_2$. For example,

$$\frac{\partial \vec{e}_3}{\partial u_3} = \frac{\partial \vec{e}_1}{\partial u_3} \times \vec{e}_2 + \vec{e}_1 \times \frac{\partial \vec{e}_2}{\partial u_3} = \frac{1}{h_1}\frac{\partial h_3}{\partial u_1}[\vec{e}_1 \times \vec{e}_2] + \vec{e}_1 \times \frac{1}{h_2}\frac{\partial h_3}{\partial u_2}\vec{e}_3$$

$$= -\frac{1}{h_1}\frac{\partial h_3}{\partial u_1}\vec{e}_1 - \frac{1}{h_2}\frac{\partial h_3}{\partial u_2}\vec{e}_2 \qquad (2.57)$$

$$\left.\begin{aligned} \frac{\partial \vec{e}_1}{\partial u_1} &= -\frac{1}{h_2}\frac{\partial h_1}{\partial u_2}\vec{e}_2 - \frac{1}{h_3}\frac{\partial h_1}{\partial u_3}\vec{e}_3 \\ \frac{\partial \vec{e}_2}{\partial u_2} &= -\frac{1}{h_1}\frac{\partial h_2}{\partial u_1}\vec{e}_1 - \frac{1}{h_3}\frac{\partial h_2}{\partial u_3}\vec{e}_3 \end{aligned}\right\} \qquad (2.58)$$

It should be kept in mind that Eqs. 2.55 - 2.58 are valid only for orthogonal sets of coordinates.

EXERCISES

2.15 (a) Find the scalar products of the unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ with each other.

(b) Find the vector products of these unit vectors with each other.

2.16 Show that, in cartesian coordinates,

$$(\vec{a}\vec{b}\vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

2.17 (a) Find the scalar product of the vector $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$ with each of the unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$.

(b) Find the vector product of $\vec{a}$ with each of these unit vectors.

2.18 If $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\varphi$ are mutually perpendicular unit vectors in the r, θ, φ directions for a set of spherical polar coordinates,

(a) Find the scalar products with each other

(b) Find the vector products with each other.

2.19 Using the notation of Prob. 2.18 find the vector product of two vectors when expressed in terms of components in spherical polar coordinates

$$\vec{a} \times \vec{b} = (a_r \vec{e}_r + a_\theta \vec{e}_\theta + a_\varphi \vec{e}_\varphi) \times (b_r \vec{e}_r + b_\theta \vec{e}_\theta + b_\varphi \vec{e}_\varphi)$$

2.20 Cylindrical coordinates r, φ, z are defined as shown in Fig. 2.15(c).

(a) Show that the time derivatives of the unit vectors are

$$\frac{d\vec{e}_r}{dt} = \dot{\vec{e}}_r = \dot{\varphi}\vec{e}_\varphi$$

$$\dot{\vec{e}}_\varphi = -\dot{\varphi}\vec{e}_r$$

$$\dot{\vec{e}}_r = 0$$

(b) Show that the velocity of the displacement vector

$$\vec{r} = r\vec{e}_r + x\vec{e}_x \quad \text{is} \quad \frac{d\vec{r}}{dt} = \vec{v} = \dot{r}\vec{e}_r + r\dot{\varphi}\vec{e}_\varphi + \dot{z}\vec{e}_z$$

(c) Show that the acceleration is given by

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2} = (\ddot{r} - r\dot{\varphi}^2)e_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\vec{e}_\varphi + \ddot{z}\vec{e}_z$$

2.21 A particle is moving in the x-y plane, and r is the vector from the origin to the particle. Show that the components of the velocity dr/dt along, and perpendicular to, the radius vector are $e_r dr/dt$ and $e_\varphi r d\varphi/dt$.

2.22 Show that in spherical coordinates

$$\dot{\vec{e}}_r = \dot{\theta}\vec{e}_\theta + \dot{\varphi}\sin\theta\vec{e}_\varphi$$

$$\dot{\vec{e}}_\theta = -\dot{\theta}\vec{e}_r + \dot{\varphi}\cos\theta\vec{e}_\varphi$$

$$\dot{\vec{e}}_\varphi = -\dot{\varphi}\cos\theta\vec{e}_\theta - \dot{\varphi}\sin\theta\vec{e}_r$$

2.22 (continued)

$$\vec{V} = \dot{\vec{r}} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\dot{\varphi}\sin\theta\,\vec{e}_\varphi$$

$$\vec{a} = \ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2 - 2\dot{\varphi}^2 \sin^2\theta)\vec{e}_r$$

$$+ (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2 \sin\theta\cos\theta)\vec{e}_\theta$$

$$+ (2\dot{r}\dot{\varphi}\sin\theta + r\ddot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\vec{e}_\varphi$$

2.23 Calculate the h's for cylindrical coordinates from Eq. 2.46.

2.24 For spherical coordinates

$$x = r\sin\theta\cos\varphi \quad , \quad y = r\sin\theta\sin\varphi \quad , \quad z = r\cos\theta$$

Find h_r, h_θ, h_φ.

2.25 One can define an elliptic cylindrical coordinate system (σ, τ, z) in which $x = 2A\cosh\sigma\cos\tau$, $y = 2A\sinh\sigma\sin\tau$, $z = z$.

(a) Show that the system of coordinates is orthogonal, and that

$$h_\sigma^2 = h_\tau^2 = 4A^2(\sinh^2\sigma + \sin^2\tau) \,, \quad h_z^2 = 1$$

(b) Sketch the surfaces $\sigma = \text{const.}$, $\tau = \text{const.}$, $z = \text{const.}$, and give the permissible range of variation of each coordinate to define a unique coordinate system.

2.7 Line and Surface Integrals

Line Integrals. In discussing the scalar product in Section 2.2 we saw that it is useful in giving an analytic expression for many quantities met in mechanics; e.g., the scalar product of a force vector with a displacement vector gives work, and the scalar product of force and velocity vectors gives rate of doing work. If the magnitude and directions of the vectors in such a product change, however, we must introduce the concept of the line integral in order to obtain physically meaningful quantities.

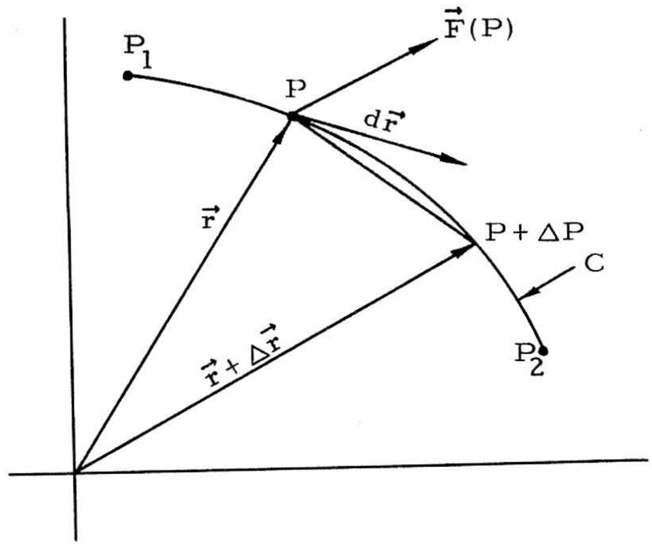


Fig. 2.18

If a particle is constrained to move along a curve C (Fig. 2.18) and is acted upon by a force $\vec{F}(P)$ which depends on the point P, then the work required to move the particle from P to P+ΔP will be approximately

$$\Delta W = \vec{F}(P') \cdot \Delta\vec{r} \tag{2.59}$$

where P′ is a point on C between P and P+ΔP. The work required to move the particle from P_1 to P_2 will be approximately

$$W = \sum_i \vec{F}(P_i) \cdot \Delta\vec{r}_i \tag{2.60}$$

where the summation covers the length of the curve from P_1 to P_2. We define the line integral as the limit of this sum as the largest of the increments $\Delta\vec{r}_i$ approaches zero, and write

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} \tag{2.61}$$

This definition of the line integral can be applied to any vector point function $\vec{F}(P)$. The value of the integral will, in general, depend on the path chosen between P_1 and P_2. The actual evaluation of such an integral will require

$\vec{F}$ and $\vec{r}$ in terms of some convenient parameter. As an example, let us consider the line integral of the vector $\vec{F} = F_o \vec{e}_r$ (where $\vec{e}_r$ is the radial unit vector in spherical polar coordinates) along two different paths in Fig. 2.19.

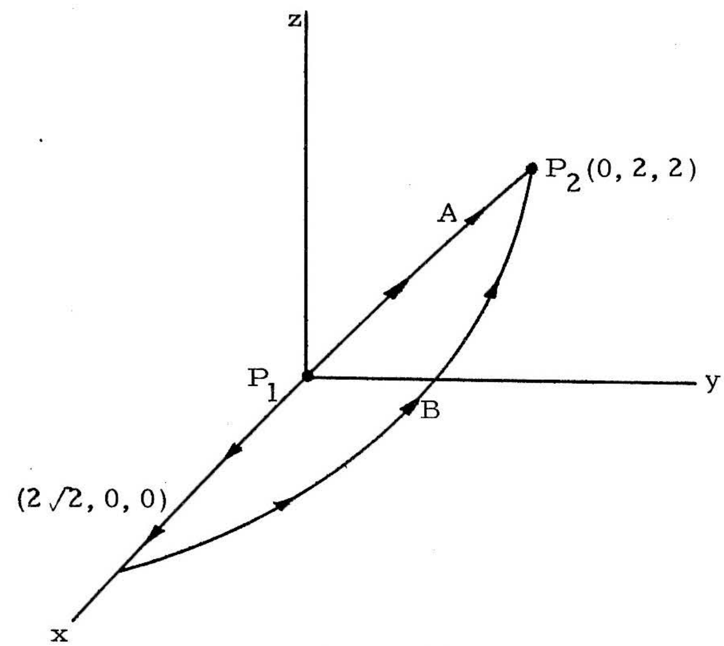


Fig. 2.19

Path A is a line from the origin to the point (0, 2, 2). On A we have $d\vec{r} = \vec{e}_r dr$ and

$$\int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_0^{2\sqrt{2}} F_o dr = 2\sqrt{2}\, F_o \tag{2.62}$$

Path B will be taken along the x-axis to the point $(2\sqrt{2}, 0, 0)$ and then along a circular arc from this point to P_2. We see that the result will be the same as we already obtained, since the integral along the x-axis is identical with that of Eq. 2.62, and along the circular arc $\vec{F}$ will always be perpendicular to $d\vec{r}$, hence the scalar product $\vec{F} \cdot d\vec{r}$ will vanish.

EXERCISES

2.26 Given the force field

$$\vec{F} = (y^2+1)z\vec{i} + 2xyz\vec{j} + xy^2\vec{k}$$

2.26 (continued)

(a) Calculate the work done in moving a particle from the point (0, 0, 2) to the point (0, 0, -2) along a semicircle lying entirely in the positive x half of the xz-plane.

(b) Calculate the work done in moving a particle from the point (0, 0, 2) to the point (0, 0, -2) along the z-axis.

2.27 If we form the scalar product of both sides of Eq. 2.14 with the unit tangent vector $\vec{u}$ we find

$$\vec{u}\cdot\vec{u} = \vec{u}\cdot\frac{d\vec{r}}{ds} = 1. \qquad \text{Whence } \vec{u}\cdot d\vec{r} = ds.$$

Using this expression, find the length of the curve expressed parametrically in Eq. 2.25 between the origin and the point at which it penetrates the xz plane.

2.28 Find the line integral of $\vec{F}\cdot d\vec{r}$ from (1, 0, 0) to (1, 0, 4) if $\vec{F} = x\vec{i} - y\vec{j} + z\vec{k}$

(a) along a line segment joining the end points;

(b) along the helix $x = \cos 2\pi t$, $y = \sin 2\pi t$, $z = 4t$.

Surface Integrals. The flux of a vector point function (such as mass flow) through a surface can be obtained by a surface integral of the form

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{\nu}\, dS \qquad (2.63)$$

where $\vec{\nu}$ is the unit normal to the surface. Since the integrand is a scalar quantity, such an integral can be reduced to a double integral over an appropriately chosen pair of parameters. For example, if we have an incompressible fluid of density ρ, flowing with a velocity

$$\vec{v} = v_o\left[\frac{a}{r}\vec{e}_r + \frac{r}{a}\vec{e}_\theta\right],$$

what is the flux through the curved surface of the cylinder of Fig. 2.20? Here we must evaluate the surface integral

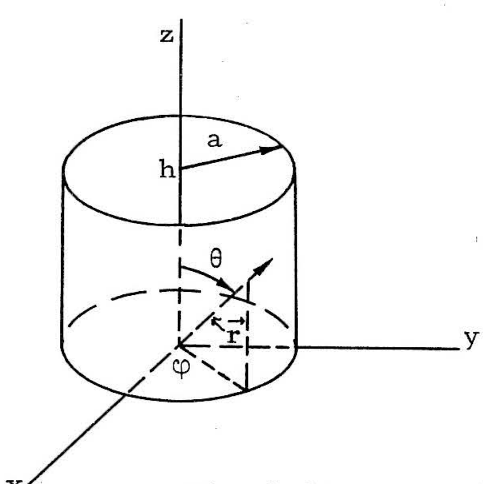


Fig. 2.20

$$\iint_S \rho\vec{v}\cdot d\vec{S}$$

We can readily obtain $d\vec{S}$ from the cross product of two line elements:

$$d\vec{S} = \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \varphi}\, d\theta\, d\varphi \tag{2.64}$$

The surface in question has the vector equation

$$\vec{r} = a \csc \theta \vec{e}_r \tag{2.65}$$

Hence

$$\begin{aligned} d\vec{S} &= \frac{\partial}{\partial \theta}(a \csc \theta \vec{e}_r) \times \frac{\partial}{\partial \theta}(a \csc \theta \vec{e}_r)\, d\theta\, d\varphi \\ &= (-a \operatorname{ctn} \theta \csc \theta \vec{e}_r + a \csc \theta \vec{e}_\theta) \times (a\vec{e}_\varphi)\, d\theta\, d\varphi \\ &= a^2 \csc \theta \operatorname{ctn} \theta \vec{e}_\theta + a^2 \csc \theta \vec{e}_r \end{aligned} \tag{2.66}$$

The integral becomes

$$\iint_S \rho\vec{v}\cdot d\vec{S} = \rho \int_{\pi/2}^{\tan^{-1} a/h} \int_0^{2\pi} \left(\frac{r}{a} a^2 \csc \theta \operatorname{ctn} \theta + \frac{a}{r} a^2 \csc \theta\right) d\varphi\, d\theta \tag{2.67}$$

Since $r \sin \theta = a$ this reduces to

$$\begin{aligned} a^2 \rho \int_{\pi/2}^{\tan^{-1} a/h} \int_0^{2\pi} \left(\frac{\cos \theta}{\sin^3 \theta} + 1\right) d\varphi\, d\theta &= 2\pi a^2 \rho \left[\theta - \frac{1}{2 \sin^2 \theta}\right]_{\pi/2}^{\tan^{-1} a/h} \\ &= 2\pi a^2 \rho \left[\tan^{-1} \frac{a}{h} - \frac{\pi}{2} - \frac{a^2+h^2}{2a^2} + \frac{1}{2}\right] \end{aligned} \tag{2.68}$$

EXERCISES

2.29 Find the flux of the vector field defined by the expression

$$\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$$

through the closed surface consisting of the coordinate planes and the first octant of the sphere $x^2+y^2+z^2 = a^2$, first by direct calculation using cartesian coordinates and then using spherical polar coordinates.

2.8 The Directional Derivative and the Gradient.

In many physical problems we shall be interested in the rate of change of some scalar point function in a particular direction. For example, the rate of flow of heat across an element of surface is proportional to the rate of change of temperature normal to that surface. If the element of surface in question lies in one of the coordinate surfaces, the required rate of change will be related to the appropriate partial derivative. Since this will not be generally the case, we must extend the notion of partial derivatives.

Consider a scalar point function $\Phi(P)$ which is continuous and varies smoothly in every direction from any point interior to some region R. Let us consider the variation of $\Phi(P)$ in the direction of an arbitrary unit vector $\vec{e}$ (Fig. 2.21). If we start from the point P, let ΔP_e be the distance along $\vec{e}$ to a neighboring point $P + \Delta P_e$. Then we define

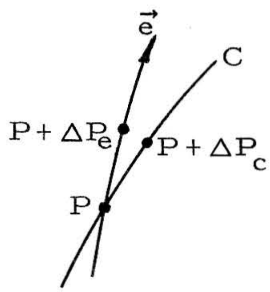


Fig. 2.21

$$\lim_{\Delta P_e \to 0} \frac{\Phi(P + \Delta P_{\vec{e}}) - \Phi(P)}{\Delta P_{\vec{e}}} \tag{2.69}$$

as the directional derivative of $\Phi(P)$ in the $\vec{e}$ direction. We see that this is a direct extension of the usual definition of a partial derivative.

If we had taken ΔP along a smooth curve C passing through P (Fig. 2.21) where $\vec{e}$ is tangent to the curve at P, then $\Delta P_e \approx |\Delta \vec{s}|$

$$\lim_{\Delta P_c \to 0} \frac{\Delta P_c}{\Delta P_{\vec{e}}} = \lim_{\Delta P_{\vec{e}} \to 0} \frac{|\Delta \vec{s}|}{\Delta P_{\vec{e}}} = 1 \tag{2.70}$$

we can consider Eq. 2.69 as giving us the directional derivatives along the curve C. If we now consider a set of coordinate curves in an orthogonal curvilinear coordinate system, we will have $\Delta \vec{s} = h_i \Delta u_i \vec{e}_i$, and the directional derivative in the $\vec{e}_i$ direction will be

$$\lim_{\Delta s_i \to 0} \frac{\Phi(P + h_i \Delta u_i) - \Phi(P)}{h_i \Delta u_i} = \frac{1}{h_1} \frac{\partial \Phi}{\partial u_i} \tag{2.71}$$

It is possible to construct an infinite number of directional derivatives of $\Phi(P)$ at any point, but these are by no means independent of each other. In fact, we can construct a unique directional derivative, called the gradient, which, when treated as a vector, has the property that its component in any direction is just the directional derivative in that direction. Consider two neighboring surfaces $\Phi(P) = C$ and $\Phi(P) = C + \Delta C$ (Fig. 2.22). Such surfaces are frequently called <u>level surfaces</u> of the function $\Phi(P)$. The directional derivatives of $\Phi(P)$ in the direction RQ', evaluated at the point R, will be

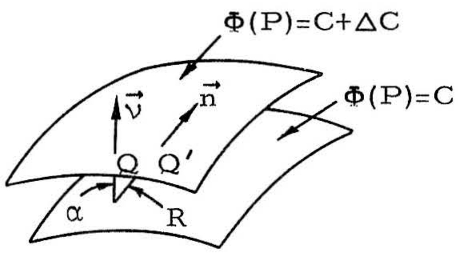


Fig. 2.22

$$\lim_{RQ'\to 0} \frac{\Phi(Q') - \Phi(R)}{RQ'} = \lim_{RQ'\to 0} \frac{(C+\Delta C) - C}{RQ'} = \lim_{RQ'\to 0} \frac{\Delta C}{RQ'} \qquad (2.72)$$

Let Δr be the distance between the two surfaces along the normal to $\Phi(P) = C$ erected at R, and let $\vec{\nu}$ be a unit vector in the direction of that normal. Then $RQ' = \Delta\nu \sec\alpha$ where α is the angle between RQ and RQ', except for terms of higher order than $\Delta\nu$, and $\Delta\nu$ will represent the minimal distance between the two surfaces. This means that the directional derivative normal to $\Phi(P) = C$ at R will be the maximal directional derivative at R. Furthermore, the normal direction can be defined uniquely relative to a surface at a point on the surface. This gives us the possibility of defining a unique directional derivative. We shall define the gradient of the function $\Phi(P)$ as a vector in the direction of the normal to $\Phi(P) = C$, equal in magnitude to the directional derivative in this normal direction

$$\text{gradient}\,\Phi = \text{grad}\,\Phi = \nabla\Phi = \lim_{\Delta n\to 0} \frac{\Delta\Phi}{\Delta n}\vec{\nu} \qquad (2.73)$$

Since $RQ' = \Delta\nu \sec\alpha$, we find that

$$\lim_{RQ'\to 0} \frac{\Phi(Q') - \Phi(R)}{RQ'} = \lim_{\Delta n\to 0} \frac{\Delta C}{\Delta\nu \sec\alpha} = |\text{grad}\,\Phi| \cos\alpha$$

If we let $\vec{n}$ be a unit vector in the RQ' direction, $\vec{n}\cdot\vec{\nu} = \cos\alpha$, and we have the result that

$$D_n\Phi = \text{grad}\,\Phi \cdot \vec{n} = \nabla\Phi \cdot \vec{n} \tag{2.74}$$

In fact, the directional derivative of a function in any direction will be given by the scalar product of a unit vector in that direction with the gradient of the function. We can use this property to construct the gradient vector in any coordinate system, whether orthogonal or not.

For any curvilinear coordinate system with the line element

$$d\vec{s} = h_1 du_1 \vec{e}_1 + h_2 du_2 \vec{e}_2 + h_3 du_3 \vec{e}_3 \tag{2.75}$$

the directional derivatives in the three directions normal to the coordinate surfaces will be

$$\frac{1}{h_1}\frac{\partial\Phi}{\partial u_1} \quad , \quad \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2} \quad , \quad \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3} \tag{2.76}$$

and the gradient will be

$$\text{grad}\,\Phi = \nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\vec{e}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\vec{e}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\vec{e}_3 \tag{2.77}$$

Since the vector $\nabla\Phi$ is normal to the surface $\Phi = \text{const.}$, we can obtain the unit normal from the gradient

$$\vec{\nu} = \frac{\nabla\Phi}{|\nabla\Phi|} \tag{2.78}$$

By the operation of finding the gradient of a scalar field we have derived a related vector field. We can hardly expect all vector fields to be derivable as gradients of scalar point functions, so we might expect that such vector fields will possess certain special characteristics. For example, consider the line integral of grad F between two points P and Q

$$\int_P^Q \text{grad}\,F \cdot d\vec{s} = \int_P^Q \frac{\partial F}{\partial s} ds = F(Q) - F(P) \tag{2.79}$$

This result depends only on the value of F at the end points, and is independent of the path of integration. A further consequence of this fact is that the line integral of such a vector field around a closed path will vanish.

EXERCISES

2.30 Using the general definition of the directional derivative, show that the directional derivative of the radius vector r is unity in the direction $\vec{r}$. Check by using the expression for the directional derivative in cartesian coordinates and the fact that $r = (x^2+y^2+z^2)^{1/2}$.

2.31 Show that

(a) In cartesian coordinates

$$\nabla\Phi = \vec{i}\,\frac{\partial\Phi}{\partial x} + \vec{j}\,\frac{\partial\Phi}{\partial y} + \vec{k}\,\frac{\partial\Phi}{\partial z}$$

(b) In cylindrical coordinates

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\vec{e}_r + \frac{1}{r}\,\frac{\partial\Phi}{\partial\varphi}\vec{e}_\varphi + \frac{\partial\Phi}{\partial z}\vec{e}_z$$

(c) In spherical polar coordinates

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\vec{e}_r + \frac{1}{r}\,\frac{\partial\Phi}{\partial\theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\,\frac{\partial\Phi}{\partial\varphi}\vec{e}_\varphi$$

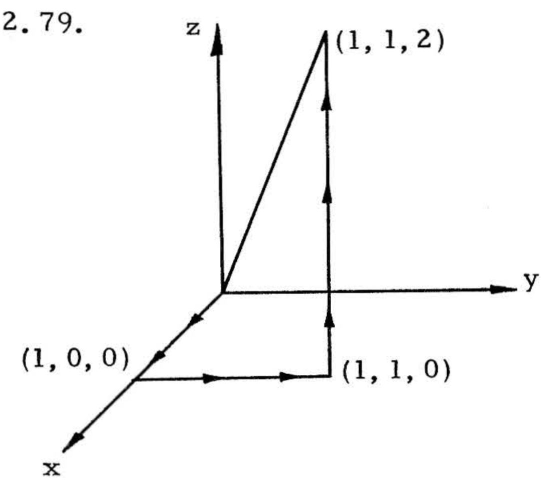
2.32 Given $F(x, y, z) = x^2+y^2+z^2 = r^2$

$$\text{grad}\, F = 2x\vec{i} + 2y\vec{j} + 2z\vec{k} = 2r\vec{e}_r$$

(a) Evaluate the line integral of grad F along the path indicated in the sketch.

(b) Evaluate the line integral of grad F between the same limits along the radius vector.

(c) Evaluate the line integral of grad F between the same limits using Eq. 2.79.



2.9 Divergence

The divergence of a vector field can probably be most easily illustrated by considering the example of fluid flow. Suppose that we have a fluid flowing in a region R such that the velocity at any point P and at time t is given by the vector $\vec{v}(P, t)$. Let us consider a small closed volume V (Fig. 2.23) and write the expression for the net inflow or outflow of fluid from that volume. If we represent an element of the surface of V by the vector $d\vec{S}$ directed normally outward from the enclosed volume, the net flow of fluid through that surface element per unit time will be $d\vec{S} \cdot (\rho\vec{v})$

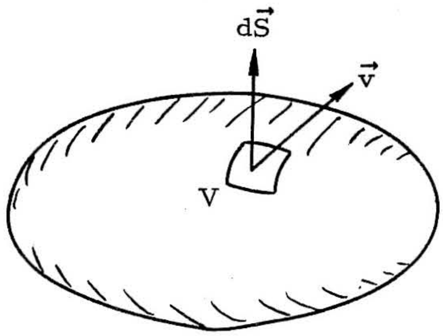


Fig. 2.23

where ρ is the mass density of the fluid. The net outflow or inflow from the volume V will be given by the integral over the entire bounding surface of this scalar expression

$$\text{Flux} = \iint_S d\vec{S} \cdot (\rho\vec{v}) \tag{2.80}$$

and the average flux per unit volume throughout the volume V will be

$$\frac{\iint_S d\vec{S} \cdot (\rho\vec{v})}{V} \tag{2.81}$$

We define the <u>divergence</u> of the mass flow at the point P by the expression

$$\text{divergence}\,(\rho\vec{v}) = \text{div}\,(\rho\vec{v}) = \nabla \cdot (\rho\vec{v}) = \lim_{\Delta V \to 0} \frac{\iint_S d\vec{S} \cdot (\rho\vec{v})}{\Delta V} \tag{2.82}$$

where, in the limiting process $\Delta V \to 0$, the point P remains interior to ΔV, and the greatest distance from P to any point on the surface of ΔV approaches zero with ΔV. The expression represents the net outflow (or inflow) of mass per unit volume at the point P. If the density ρ is constant, this flux must come from sources and/or sinks located at P. If ρ is not constant, such a flux could arise from a local density change. If no sources or sinks are present we can write

$$\iint_S d\vec{S} \cdot \rho\vec{v} = -\overline{\frac{\partial\rho}{\partial t}}\Delta V \qquad (2.83)$$

where $(\overline{\partial\rho/\partial t})$ represents the average value of $(\partial\rho/\partial t)$ over the small volume ΔV. In the limit this becomes

$$\nabla \cdot (\rho\vec{v}) = -\frac{\partial\rho}{\partial t} \qquad (2.84)$$

If, of course, the density is constant with time we have

$$\nabla \cdot (\rho\vec{v}) = 0 \qquad (2.85)$$

The concept of the divergence of a vector field is readily generalized to define the divergence of a vector point function $\vec{F}$ by the equation

$$\nabla \cdot \vec{F} = \lim_{\Delta V \to 0} \frac{\iint d\vec{S} \cdot F}{\Delta V} \qquad (2.86)$$

if this limit exists when, in the limiting process, the shape of ΔV is not restricted except that P be interior to ΔV and the greatest distance from P to any point on the surface of ΔV must approach zero as $\Delta V \to 0$.

Gauss's Theorem or the Divergence Theorem. Equation 2.86 can be rewritten in the form

$$\iint d\vec{S} \cdot \vec{F} = \nabla \cdot \vec{F}\Delta V + \zeta\Delta V$$
$$\lim_{\Delta V \to 0} \zeta = 0 \qquad (2.87)$$

For a finite volume, which can be broken up into n cells ΔV_i, we have

$$\sum_{i=1}^{n} \iint d\vec{S}_i \cdot \vec{F} = \sum_{i=1}^{n} \nabla \cdot \vec{F}_i \Delta V_i + \sum_{i=1}^{n} \zeta_i \Delta V_i \qquad (2.88)$$

If $\vec{F}$ is continuous and possesses continuous first derivatives throughout V and if the bounding surface S of V is continuous and piecewise smooth, we get as a limit

$$\iint_S d\vec{S}\cdot\vec{F} = \iiint_V \nabla\cdot\vec{F}\,dV \tag{2.89}$$

This is known as Gauss's Theorem or the divergence theorem. The conditions on $\vec{F}$ and S can be somewhat relaxed, but no simple catalogue will suffice, and the conditions enumerated will be satisfied in most physical situations arising in classical field theory.

The definition of the divergence given by Eq. 2.86 should make clear the fact that the divergence is a property of the original vector field, and does not depend on the coordinate system in which the vector field is described. Since many physical laws relate the value of a field quantity at a point to the values at neighboring points, we might expect to take advantage of expressions such as Eq. 2.86 to permit us to obtain rather general mathematical formulations of such laws. We shall illustrate this with a formulation of the laws of heat conduction in which we shall be concerned with the temperature as a scalar point function.

Heat Conduction. The formal laws of heat conduction and their mathematical formulation can be stated as follows:

(1) If the temperature of a body is changed by an increment of temperature ΔT, then the change in the heat content of an element of volume of the body is given by

$$\Delta q = \overline{c_v \rho}\,\Delta V \Delta T \tag{2.90}$$

where c_v is the specific heat at constant volume, ρ the density, and the bar represents the average value over ΔV. Both c_v and ρ will usually depend on T. If the temperature changes by ΔT in time ΔT, then

$$\lim_{\Delta t\to 0} \frac{\Delta q}{\Delta t} = \frac{\partial q}{\partial t} = \overline{c_v \rho}\,\Delta V \frac{\partial T}{\partial t} \tag{2.91}$$

Since q is the amount of heat in the volume element ΔV (ΔV does not vary with time) we can put $q = \Delta Q$ and sum up over a large body. In incremental form

$$\frac{\partial \Delta Q}{\partial t} = \overline{c_v \rho}\,\Delta V \frac{\partial T}{\partial t} \tag{2.92}$$