PHYSICAL REVIEW E **88**, 057001 (2013)

Comment on "Hydrodynamics of fractal continuum flow" and "Map of fluid flow in fractal porous medium into fractal continuum flow"

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(Received 23 December 2012; published 21 November 2013)

In two recent papers [Phys. Rev. E **85**, 025302(R) (2012) and Phys. Rev. E **85**, 056314 (2012)], the authors proposed fractal continuum hydrodynamics and its application to model fluid flows in fractally permeable reservoirs. While in general providing a certain advancement of continuum mechanics modeling of fractal media to fluid flows, some results and statements to previous works need clarification. We first show that the nonlocal character those authors alleged in our paper [Proc. R. Soc. A **465**, 2521 (2009)] actually does not exist; instead, all those works are in the same general representation of derivative operators differing by specific forms of the line coefficient c_1. Next, the claimed generalization of the volumetric coefficient c_3 is, in fact, equivalent to previously proposed product measures when considering together the separate decomposition of c_3 on each coordinate. Furthermore, the modified Jacobian proposed in the two commented papers does not relate the volume element between the current and initial configurations, which henceforth leads to a correction of the Reynolds' transport theorem. Finally, we point out that the asymmetry of the Cauchy stress tensor resulting from the conservation of the angular momentum must not be ignored; this aspect motivates a more complete formulation of fractal continuum models within a micropolar framework.

DOI: 10.1103/PhysRevE.88.057001 PACS number(s): 47.56.+r, 47.53.+n, 47.10.ab

Background. Two recent papers [1,2] studied the fluid flow in the fractal pore space employing the local fractional calculus. In this Comment we discuss the relation of those operators to the product measure and fractal derivative employed in a previous formulation of continuum mechanics of fractal media [3–5]. For reference, the product measure is based on a dimensional regularization of fractal mass to a fractal continuum,

$$\begin{aligned} m(\mathcal{W}) &= \int_{\mathcal{W}} \rho dV_D = \int_{\mathcal{W}} \rho c_3(x_1,x_2,x_3) dV_3 \quad [6] \\ &= \int_{\mathcal{W}} \rho dl_{\alpha_1}(x_1) dl_{\alpha_2}(x_2) dl_{\alpha_3}(x_3) \\ &= \int_{\mathcal{W}} \rho c_1^{(1)} c_1^{(2)} c_1^{(3)} dx_1 dx_2 dx_3 \quad [3\text{–}5], \end{aligned} \tag{1}$$

with the length element along each coordinate axis given by a line coefficient $c_1^{(k)}$,

$$dl_{\alpha_k}(x_k) = c_1^{(k)}(\alpha_k, x_k) dx_k, \quad k = 1,2,3 \text{ (no sum)}, \tag{2}$$

where each α_k plays the role of a fractal dimension in the direction of the x_k axis.

Green-Gauss theorem, fractal derivative, and Hausdorff derivative. According to (1), the volumetric coefficient relating dV_D to dV_3 takes the form

$$c_3(x_i,x_j,x_k) = c_1(x_i,\alpha_i) c_1(x_j,\alpha_j) c_1(x_k,\alpha_k). \tag{3}$$

The product measure we previously proposed allows the treatment of fractal anisotropy by decoupling coordinate variables, and this further simplifies the Green-Gauss theorem and induces the definition of a fractal derivative [3–5]. To see it clearly, the Green-Gauss theorem is generalized as

$$\begin{aligned} \int_{\partial W} f_i n_k dS_d^{(k)} &= \int_{\partial W} f_i n_k c_2^{(k)} dS_2^{(k)} = \int_W \left(f_i c_2^{(k)}\right)_{,k} dV_3 \\ &= \int_W \left(f_i c_2^{(k)}\right)_{,k} c_3^{-1} dV_D \\ &= \int_W f_{i,k} c_2^{(k)} c_3^{-1} dV_D = \int_W \frac{f_{i,k}}{c_1^{(k)}} dV_D, \end{aligned} \tag{4}$$

where the surface coefficient relating $dS_d^{(k)}$ to $dS_2^{(k)}$ shows $c_2^{(k)} = c_1(x_i,\alpha_i) c_1(x_j,\alpha_j)$ $(k \neq i,j)$, which is independent of x_k. This leads to our definition of a *fractal derivative (gradient)*:

$$\nabla_k^D := \frac{1}{c_1^{(k)}} \frac{\partial}{\partial x_k} \quad \text{(no sum on } k\text{)}. \tag{5}$$

The product measure was restricted to fractals that can be decomposed into products. In an attempt to model a wider class, the authors of Refs. [1,2] generalized the form of c_3 as follows:

$$c_3(x_i,x_j,x_k) = c_1(x_k,\zeta_k) c_2^{(k)}(x_i,x_j,d_k), \tag{6}$$

with $c_2^{(k)}$ not necessarily a product form, and ζ_k the codimension of the intersection between the fractal and the Cartesian plane (x_i,x_j): $\zeta_k = D - d_k$. Note, however, that our formulation of the Green-Gauss theorem (4) does not require an explicit expression of $c_2^{(k)}$ but only that it be independent of x_k. Therefore, in Refs. [1,2] the authors arrived at the same form of the fractal derivative (5), albeit a different form of $c_1(x_k,\alpha_k)$ was employed. The expression of

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$c_1(x_k,\alpha_k)$ should reflect the physical scaling law of fractal mass $m \sim x_k^{\alpha_k}$. As such, various forms could be proposed [1–6] and, for example, Tarasov [6] wrote $c_1 \sim \alpha_k x_k^{\alpha_k - 1}$ in relation to the left-sided fractional Riemann-Liouville integral. Regarding the singularity of the point mass distribution at the origin [3], we used $c_1 \sim [(l_k - x_k)/l_0]^{\alpha_k-1} \sim (1 - x_k/l_k)^{\alpha_k-1}$ based on Jumarie's modified fractional Riemann-Liouville integral [7,8], where l_k refers to the boundary position and l_0 is the characteristic size. The authors of Refs. [1,2] directly proposed $c_1 \sim (1 + x_k/l_0)^{\alpha_k-1}$. However, within the same general representation of the proposed fractal derivative (5), the authors of Ref. [1] incorrectly commented on our derivatives in Refs. [3–5] as being nonlocal and unrecoverable from Jumarie's fractional derivative. To clarify it, observe the following:

(1) The apparent dependence of the distance to the boundary position is not a nonlocal property: The nonlocal property usually involves an integral of some field quantities (e.g., force or mass distributions) outside the neighborhood of a particle, whereas the fractal derivative (5) does not involve any such relation. It is also evident that the fractal derivative (5) has no specific relation to the usual fractional derivative involving an integral convolution being nonlocal.

(2) The form $(1 - x_k/l_k)^{\alpha_k-1}$ [3–5] removes the singularity at the origin and the distance to the boundary position satisfies the frame-independence requirement (i.e., objectivity), while it introduces singularities on the boundaries. On the other hand, the form of $(x_k/l_0 + 1)^{\alpha_k-1}$ [1,2] removes singularities both at the origin and the boundaries, while the objectivity (frame independence) is violated since a different choice of the origin results in the change of the coordinate value x_k for a fixed point and, further, a different $c_1(x_k,\alpha_k)$ and fractal derivative associated with that point. Besides, the scaling $m \sim x_k^{\alpha_k}$ is precisely recovered in $c_1 \sim \alpha_k x_k^{\alpha_k-1}$ or $c_1 \sim (1 - x_k/l_k)^{\alpha_k-1}$ based on established fractional integrals, while the form $c_1 \sim (1 + x_k/l_0)^{\alpha_k-1}$ does not produce that scaling when x_k is close to the order of l_0. Nonetheless, appropriate forms of c_1 leave open questions in this framework and would best be chosen in applications.

Furthermore, we would like to point out that the product form (3) is actually equivalent to the generalized form (6), given that the volumetric coefficient c_3 can be decomposed just as (6) along each coordinate axis separately:

$$\begin{aligned} c_3 &= c_1(x_k,\alpha_k)c_2^{(k)}(x_i,x_j,d_k) = c_1(x_i,\alpha_i)c_2^{(i)}(x_j,x_k,d_i) \\ &= c_1(x_j,\alpha_j)c_2^{(j)}(x_k,x_i,d_j). \end{aligned} \tag{7}$$

The above condition is implied by the Green-Gauss theorem (4), which involves the summation of vector components of (4) and, therefore, the decomposition of c_3 (6) along each coordinate axis. The first equality assumes a factor of $c_1(x_k,\alpha_k)$ for all expressions including x_k in c_3. Similar arguments relating to the other two equalities jointly lead to a product form of c_3 in (3). It turns out that the generalized Green-Gauss theorem and the fractal derivative (5) result in the proposed product measure, although not explicitly shown in Refs. [1,2]. As a side note, the product measure only relies on separated scaling laws along each coordinate axis and, in general, it is not necessarily related to the summation relation of fractal dimensions $D = \alpha_1 + \alpha_2 + \alpha_3$.

It is pertinent to point out the differences between the fractal derivative and Hausdorff derivative, defined in Ref. [9] as

$$\frac{d^H f}{dx^\zeta} = \lim_{x \to x'} \frac{f(x') - f(x)}{x'^\zeta - x^\zeta} = \frac{df}{d^\zeta x}. \tag{8}$$

Although they lead to the same expressions shown in Refs. [1,2] when $c_1 \sim (1 + x_k/l_0)^{\alpha_k-1}$, the underlying assumptions are intrinsically different. The Hausdorff derivative is based on a specific interpretation of $(dx)^\zeta = \lim_{x\to x'} x'^\zeta - x^\zeta = d(x^\zeta)$, or moreover, on the fractal metric proposed in Refs. [10,11]:

$$\triangle_i(a_i,b_i) = \left| \int_{a_i}^{b_i} d^{\zeta_i} x_i \right| = l_0 |(1 + a_i/l_0)^{\zeta_i} - (1 + b_i/l_0)^{\zeta_i}|. \tag{9}$$

The fractal derivative stems from the Green-Gauss theorem to transform the fractal surface integral to the volume integral. It is noted that the product measure is essential to give the form (5), otherwise $\nabla_k^D = c_3^{-1}(c_2 \cdot)_{,k}$ [6]. The fractal derivative (5) provides a more general expression involving the line coefficient c_1, and the Hausdorff derivative is consistent with a specific c_1. We note that an alternative way to interpret the form of the fractal derivative (5) could be possibly from a fractal metric [3]. Very recently Balankin [11] formally proposed a fractal metric in terms of the fractional integral (9) to define the fractal derivative (5). However, we want to clarify two issues:

(1) The relation between the fractional integral and the fractal metric is moot. Back in 1992 Nigmatullin [12] proposed the connection, which in turn was criticized by Rutman [13]. Podlubny [14] then summed up these efforts by saying "in principle, fractals themselves are not directly related to fractional integrals or fractional derivatives; only description of dynamical processes in fractal structures can lead to models involving fractional order operators." The promising applications of fractional calculus in various physical systems and dynamics of fractal media were recently reviewed in Ref. [15]. Note that contemporary mathematics offers no general "formula" to represent a fractal metric [16]. It is therefore questionable to formally define a fractal metric on which the fractal derivative is to be built, and then the fractal strain [1,2]. Although the explicit expression of the fractal metric is not available, the covariance functions on fractals can be obtained and studied [17]. Thus, it is suggested in the fractal continuum (i) to use the Green-Gauss theorem (4) to introduce the fractal derivative, and (ii) to use the variational principles to find the fractal stress and fractal strain as energy conjugate pairs of kinetic and kinematic quantities [3–5], where the fractal strain was found related to the fractal derivative.

(2) The definition of the fractal metric (9) violates the frame-independence (objectivity, or invariance) principle of continuum mechanics. Specifically, a simple translation of the coordinate frame changes the coordinate value (a_i,b_i) by $(a_i + c, b_i + c)$, which alters the fractal metric $\triangle_i(a_i,b_i) \neq \triangle_i(a_i + c, b_i + c)$. The underlying reason is possibly that the fractional integral does not satisfy the combination property on fractals. To resolve this issue, we have recently proposed an alternative representation of fractal mass, $m = [\int (\rho(x))^{1/\alpha} dx]^\alpha$ [18], but then the generalized Green-Gauss theorem becomes much more complicated due to the intrinsic nonlinearity.

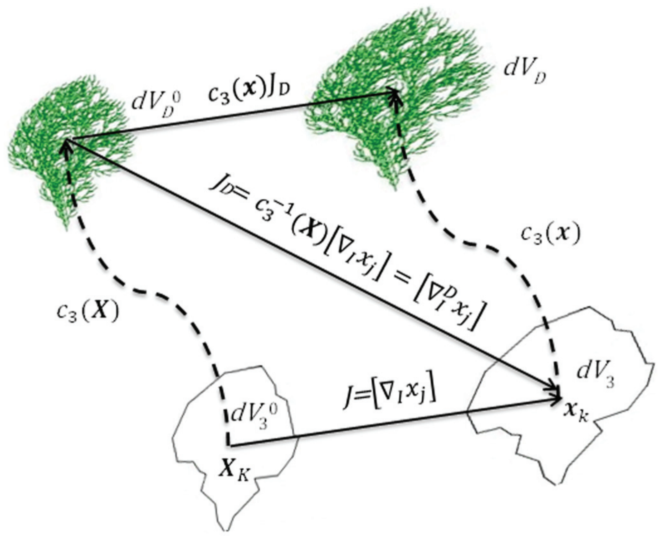

FIG. 1. (Color online) Mapping (top: continuous lines) of a fractal from the original (X_K) to deformed (x_k) configuration, also showing the relations (dashed lines) to respective continuum configurations (bottom: continuous lines) via dimensional regularization.

Deformation gradient, Jacobian, and Reynolds' transport theorem. Working in the context of finite deformation, which involves the current (Eulerian) configuration x_k and the initial (Lagrangian) configuration X_K, Refs. [1,2] postulated the deformation gradient $\mathbf{F}_D = [\nabla_I^D x_j]$ and the transformation Jacobian $J_D = \det \mathbf{F}_D$. Here we follow the usual convention of continuum mechanics for the usage of x_k and X_K that was interchanged in Refs. [1,2]. It is pertinent to note that the deformation gradient $\mathbf{F}_D$ is consistent with the fractal strain $\epsilon_{ij} = 1/2(\nabla_i^D u_j + \nabla_j^D u_i)$ introduced earlier for a small deformation from a conjugate pair of stress and strain [3–5]. The material derivative was postulated in Refs. [1,2] as $(\frac{d}{dt})^D \psi := \frac{\partial}{\partial t}\psi + v_k \nabla_k^D \psi$. However, the Jacobian J_D does not relate the volume element between the two configurations in $dV_D = J_D dV_D^0$. This is seen by first recalling the conventional Jacobian in the Euclidean space $\mathbb{E}^3$, $dV_3 = J dV_3^0 = \det[\nabla_I x_j] dV_3^0$, so that we have

$$\begin{aligned} dV_D &= c_3(x) dV_3 = c_3(x) J dV_3^0 = c_3(x) c_3^{-1}(X) J dV_D^0 \\ &= c_3(x) J_D dV_D^0. \end{aligned} \tag{10}$$

Figure 1 shows all the relations between current and initial configurations. Accordingly, the Reynolds' transport theorem for a fractal continuum should be modified as follows:

$$\begin{aligned} &\left(\frac{d}{dt}\right)^D \int_{W_t} \psi dV_D \\ &= \left(\frac{d}{dt}\right)^D \int_{W_0} \psi c_3(x) J_D dV_D^0 \\ &= \int_{W_0} \left[c_3(x) J_D \left(\frac{d}{dt}\right)^D \psi + \psi c_3(x) \left(\frac{d}{dt}\right)^D J_D \right. \\ &\quad \left. + \psi J_D \left(\frac{d}{dt}\right)^D c_3(x) \right] dV_D^0 \\ &= \int_{W_0} \left[\left(\frac{d}{dt}\right)^D \psi + \psi \nabla_k^D v_k + \psi v_k c_3^{-1}(x) \nabla_k^D c_3(x) \right] \\ &\quad \times c_3(x) J_D dV_D^0 \\ &= \int_{W t} \left[\frac{\partial}{\partial t}\psi + \nabla_k^D (\psi v_k) + \psi v_k c_1^{-1}(x_k) \nabla_k^D c_1(x_k) \right] dV_D \\ &= \int_{W t} \left[\frac{\partial}{\partial t}\psi + \psi v_k x_k^{-\alpha_k} \right] dV_D + \int_A \psi v_k n_k dA_d^{(k)}. \end{aligned} \tag{11}$$

Note that an extra term $v_k x_k^{-\alpha_k}$ arises due to the fact that $\nabla_k^D c_1(x_k) \neq 0$. To resolve this issue, the fractal Jacobian should be modified as $c_3(x) c_3^{-1}(X) J$ (already included in deriving the Reynolds' transport theorem), and the material derivative would then be corrected involving $c_1(x)$ in addition to the previous $c_1(X)$. Moreover, the deformation gradient and the fractal strain all need the addition of $c_1(x)$. It is necessary to point out that the definition of fractal stress and fractal strain postulated in Refs. [1,2] needs careful consideration from the standpoint of conjugate quantities dictated by the energy density. While for small deformation the conjugate relation was thoroughly investigated in Ref. [3], the finite deformation case is outside the scope of this Comment.

Special note on fractal stress asymmetry. It is pertinent to note that the principle of angular momentum conservation (not considered in Refs. [1,2]) leads to the asymmetry of the Cauchy stress [5]

$$e_{ijk} \frac{\sigma_{jk}}{c_1^{(j)}} = 0, \tag{12}$$

since $c_1^{(j)} \neq c_1^{(k)}$ for $j \neq k$ in general. Thus, the generally asymmetric stress ($\sigma_{jk} \neq \sigma_{kj}$) physically means that the fractal structure carries a couple-stress tensor accompanied by its conjugate curvature-torsion tensor at any homogenized continuum point [19]; these effects are ignored in classical continua. As a result, the balance laws have to be augmented by the presence of these field quantities and, additionally, the microinertia balance has to be added. This influences the derivation of permeability and poromechanics of fractal models, which will be discussed in a separate study. As an example, the stress asymmetry and micropolar effects indeed have been experimentally observed in bone mechanics [20]. Given the fact that bone is fractally structured [21], a fractal micropolar continuum model could be developed. Overall, the field equations resulting from the fractal micropolar continuum can be solved quantitatively, as demonstrated on several wave propagation problems, e.g., Ref. [22].

Conclusion. This Comment clarifies the differences between Refs. [1,2] and the previous works [3–6] and discusses the challenges to current fractal continuum models. The nonlocal character alleged and criticized by the authors of Refs. [1,2] relative to our works [3–5] is actually not true but, rather, both derivatives fall within the same general representation while differing by specific expressions of the fractal line coefficient c_1. Our c_1 has a singularity at the boundary, while c_1 of Refs. [1,2] does not satisfy the objectivity (frame independence), raising open questions on appropriate forms of c_1. The proposed generalization of the fractal volumetric coefficient c_3 is indeed equivalent to previous product measures [3–5], given the decomposition in each axis when formulating the Green-Gauss theorem. The relation of the fractal metric [1,2] to the fractal derivative (5) is discouraged due to a debatable issue of representation of the fractal measure, and, instead, the use of the generalized Green-Gauss theorem is suggested. The definition of the fractal Jacobian in Refs. [1,2] does not relate the volume element between current and initial configurations and accordingly the Reynolds' transport theorem is modified. Finally, we point out the asymmetry of the Cauchy stress tensor resulting from the conservation of angular momentum (ignored in Refs. [1,2]), which, in turn, motivates a more complete formulation of fractal micropolar continuum models.

Acknowledgments. This work was made possible by support from NSF (CMMI-1030940) and Sandia-DTRA (HDTRA1-08-10-BRCWMD).

[1] A. S. Balankin and B. E. Elizarraraz, Phys. Rev. E **85**, 025302(R) (2012).
[2] A. S. Balankin and B. E. Elizarraraz, Phys. Rev. E **85**, 056314 (2012).
[3] J. Li and M. Ostoja-Starzewski, Proc. R. Soc. A **465**, 2521 (2009); **467**, 1214 (2010).
[4] P. N. Demmie and M. Ostoja-Starzewski, J. Elasticity **104**, 187 (2011).
[5] J. Li and M. Ostoja-Starzewski, Int. J. Eng. Sci. **49**, 1302 (2011).
[6] V. E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media* (Springer, Berlin, 2010).
[7] G. Jumarie, Appl. Math. Lett. **18**, 739 (2005).
[8] G. Jumarie, Appl. Math. Lett. **22**, 378 (2008).
[9] W. Chen, Chaos, Solitons Fractals **28**, 923 (2006).
[10] A. S. Balankin, B. Mena, J. Patiño, and D. Morales, Phys. Lett. A **377**, 783 (2013).
[11] A. S. Balankin, Phys. Lett. A **377**, 1606 (2013).
[12] R. R. Nigmatullin, Theoret. Math. Phys. **90**, 242 (1992).
[13] R. S. Rutman, Theoret. Math. Phys. **100**, 1154 (1994).
[14] I. Podlubny, V. Despotovic, T. Skovranek, and B. H. McNaughton, J. Online Math. Its Appl. **7**, 1664 (2007).
[15] V. E. Tarasov, Int. J. Mod. Phys. B **27**, 1330005 (2013).
[16] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications* (Wiley, Chichester, 2003).
[17] G. Christakos, D. T. Hristopulos, and P. Bogaert, Adv. Water Resour. **23**, 799 (2000).
[18] M. Ostoja-Starzewski, J. Li, H. Joumaa, and P. N. Demmie, Z. Angew. Math. Mech. **93**, 1 (2013).
[19] A. C. Eringen, *Microcontinuum Field Theories I: Foundations and Solids* (Springer, Berlin, 1999).
[20] P. M. Buechner and R. S. Lakes, Biomech. Model. Mechanobiol. **1**, 295 (2003).
[21] L. Pothuaud, C. L. Benhamou, P. Porion, E. Lespessailles, R. Harba and P. Levitz, J. Bone Miner. Res. **15**, 691 (2000).
[22] H. Joumaa, M. Ostoja-Starzewski and P. N. Demmie, Math. Mech. Solids, doi: 10.1177/1081286512454557 (2012).