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RELATIONS BETWEEN NEUTRON-STAR PARAMETERS IN THE HARTLE-THORNE APPROXIMATION

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ABSTRACT

Using stellar structure calculations in the Hartle-Thorne approximation, we derive analytic expressions connecting the ellipticity of the stellar surface to the compactness, the spin angular momentum, and the quadrupole moment of the spacetime. We also obtain empirical relations between the compactness, the spin angular momentum, and the spacetime quadrupole. Our formulae reproduce the results of numerical calculations to within a few percent and help reduce the number of parameters necessary to model the observational appearance of moderately spinning neutron stars. This is sufficient for comparing theoretical spectroscopic and timing models to observations that aim to measure the masses and radii of neutron stars and to determine the equation of state prevailing in their interiors.

Subject headings: graviation — relativistic processes — stars: neutron

1. INTRODUCTION

Measuring the masses and radii of neutron stars provides one of the most stringent tests of our understanding of the properties of matter under extreme conditions. Several methods of measuring these properties involve analyzing the emission originating from the stellar surface. In order to correctly interpret this surface emission, it is necessary to understand the strong-field gravitational lensing experienced by the photons when traveling through the curved spacetime in the vicinity of a neutron star.

To date, considerable effort has been expended to accurately measure neutron-star radii, primarily through the spectroscopic observations of quiescent neutron stars (e.g., Heinke et al. 2006; Webb & Barret 2007; Guillot et al. 2011) and X-ray bursters (e.g., Özel et al. 2009: see Özel 2013 for a recent review). In the near future, X-ray missions such as NICER and LOFT as well as gravitational-wave detectors such as Advanced LIGO will allow even more precise measurements of various neutron star properties.

Many of the primary targets for future measurements have moderate spins (∼300–800 Hz; e.g., Galloway et al. 2008; Bogdanov et al. 2008). At these frequencies, gravitational effects depend not only on the mass and radius, but also on other parameters, such as the quadrupole moment of the neutron star and the oblateness of its surface (Morsink et al. 2007; Bauböck et al. 2013). Exploiting the upcoming high-quality observations and measuring the masses and radii of neutron stars at the accuracy necessary to constrain their equation of state requires taking these non-negligible effects into account.

For the moderate spin frequencies of weakly magnetic neutron stars, the Hartle-Thorne metric provides an accurate approximation to their spacetime (Hartle & Thorne 1968). In this regime, the appearance of a neutron star as measured by an observer at spatial infinity depends on seven macroscopic parameters: the mass M, the equatorial radius $R_{\rm eq}$, the spin frequency f, the inclination of the rotational pole with respect to the observer θ_0, the angular momentum J, the quadrupole moment Q, and the eccentricity of the surface e_s. Three of these parameters (f, θ_0, and, e.g., M) are independent of the equation of state for any given observed source. The remaining four parameters ($R_{\rm eq}$, J, Q, and e_s) are uniquely determined by the equation of state, given a neutron star mass and spin frequency. Therefore, it is any of these four parameters that need to be measured observationally, in addition to the mass and spin frequency, in order for the underlying equation of state to be constrained.

Even though measuring only one of the four dependent parameters with high precision would be sufficient, typical observables depend on all seven parameters in a complex manner. It is unlikely that spectroscopic or timing observations in the near future will be accurate enough to allow for independent measurements of all of these parameters in individual neutron stars. In order to make progress, we can reduce the dimensionality of the problem by using approximate relations that connect quantities that are higher order in spin frequency (such as the spin angular momentum J, the spacetime quadrupole Q, and the ellipticity of the stellar surface e_s) to ones that are of lower order (such as the mass M and equatorial radius $R_{\rm eq}$). In this way, observable phenomena from a moderately spinning neutron star can be calculated based only on its mass and equatorial radius, given a spin frequency and an observer's inclination.

This reduction of the parameter space by means of approximate relations allows for the properties of dense matter to be constrained only if the relations themselves do not depend strongly on the details of the equation of state. Andersson & Kokkotas (1998) modeled pulsation modes of neutron stars and showed that the relations between several parameters of interest have a significant

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dependence on the equation of state. However, given the constraints imposed on the equation of state of dense matter by recent observations (e.g., Demorest et al. 2010; Antoniadis et al. 2013), we show that it is possible to find relations between the parameters described above that are valid over the astrophysically relevant parameter range and for a variety of equations of state.

Several authors to date have explored such approximate relations in different contexts. Ravenhall & Pethick (1994) and Lattimer & Prakash (2001) provide empirical formulae for the moments of inertia and binding energies of slowly spinning neutron stars as a function of their masses and radii. Morsink et al. (2007) obtained empirical formulae that connect the ellipticity of the surfaces of spinning neutron stars to their masses, equatorial radii, and spin frequencies. More recently Urbanec et al. (2013) modeled the angular momenta and quadrupole moments of both neutron stars and strange stars, showing that different relations exist for these two classes of objects. Finally, Yagi & Yunes (2013a) found relations between the moment of inertia, the quadrupole moment, and the tidal Love number that are highly accurate for several equations of state.

In this paper, we model the properties of moderately spinning neutron stars in the Hartle-Thorne approximation, which is adequate for spin frequencies $\lesssim 800$ Hz. We derive an analytic expression connecting the ellipticity of the stellar surface to the compactness, the spin angular momentum, and the spacetime quadrupole. We also obtain empirical relations between the compactness, the spin angular momentum, and the spacetime quadrupole similar to those found in Lattimer & Prakash (2001) and Yagi & Yunes (2013a). These relations allow us to fully determine the parameters of a neutron star given a measurement of its mass, radius, and spin frequency. We demonstrate that our formulae reproduce the results of numerical calculations of neutron-star spacetimes to within a few percent. This is sufficient for comparing theoretical spectroscopic and timing models to observations that aim to measure the masses and radii of neutron stars and to determine the high-density equation of state prevailing in their interiors.

2. NUMERICAL MODELS IN THE HARTLE-THORNE APPROXIMATION

The Hartle-Thorne metric is based on a slow-rotation expansion. If the expansion is truncated at second order in the spin frequency, the spacetime exterior to a rotating object can be described by three parameters: the total mass, the angular momentum, and the quadrupole moment of a neutron star. Observationally, the appearance also depends on the geometry of its surface, i.e., on its equatorial radius and ellipticity.

Out of these parameters, we choose the mass M and the equatorial radius $R_{\rm eq}$ to characterize each neutron star. The other three parameters depend on the equation of state but are of higher order in spin frequency and introduce small corrections to most observables (e.g., Poisson 1998; Morsink et al. 2007; Racine 2008; Bauböck et al. 2013; Psaltis & Özel 2013). Therefore, we aim to find relations that allow for these parameters to be approximated given a neutron-star mass and radius, independent of the equation of state.

The angular momentum J of a neutron star is often represented by the dimensionless spin parameter

$$a \equiv \frac{cJ}{GM^2}, \tag{1}$$

which is zero for a non spinning object. Neutron stars typically have a spin $a \leq 0.7$ for uniform rotation and physically motivated equations of state (Cook, Shapiro & Teukolsky 1994; Berti & Stergioulas 2004; Lo & Lin 2011), but the spin magnitudes of neutron stars in binaries observable by Advanced LIGO are likely to be much smaller than this theoretical upper bound (Mandel & O'Shaughnessy 2010; Brown et al. 2012). The spin periods of isolated neutron stars at birth should be in the range 10-140 ms (Lorimer 2001), or $a \lesssim 0.04$. Accretion from a binary companion can spin up neutron stars but is unlikely to produce periods less than 1 ms, i.e., $a \lesssim 0.4$ (Ferdman et al. 2008). The fastest-spinning observed pulsar has a period of 1.4 ms, ($a \sim 0.3$) (Hessels et al. 2006); the fastest known pulsar in a neutron star-neutron star system, J0737-3039A, has a period of 22.70 ms ($a \sim 0.02$; Burgay et al. 2003). The spin parameter depends both on the rotational period and on the moment of inertia of the neutron star, which is determined by the equation of state.

A spinning neutron star also acquires a nonzero quadrupole moment Q. We characterize the quadrupole moment by the dimensionless quantity

$$q \equiv -\frac{c^4 Q}{G^2 M^3}. \tag{2}$$

Laarakkers & Poisson (1999), Berti & Stergioulas (2004), and Pappas & Apostolatos (2012) computed the quadrupole moment of rapidly spinning neutron stars for a range of equations of state. They found values of q ranging between 1 and 11 (see also Bauböck et al. 2012).

Lastly, a spinning neutron star also becomes oblate in shape. In the Hartle-Thorne approximation, this oblateness is described by

$$R(\theta) = R_0 + \xi_2 P_2(\cos\theta), \tag{3}$$

where P_2 is the second-order Legendre polynomial and ξ_2 is a coefficient depending on the equation of state and spin frequency of the neutron star. In the non-spinning limit, $\xi_2 = 0$ and $R(\theta) = R_0$. For moderately spinning neutron stars, there are two frequently used parameters to characterize the oblate shape: the eccentricity of the surface,

$$e_s \equiv \sqrt{\left(\frac{R_{\rm eq}}{R_{\rm p}}\right)^2 - 1}, \tag{4}$$

and its ellipticity,

$$\varepsilon_s \equiv 1 - \frac{R_{\rm p}}{R_{\rm eq}}, \tag{5}$$

where $R_{\rm eq}$ is the equatorial radius and $R_{\rm p}$ is the radius at the pole.

As in Berti et al. (2005), we expand the parameters a, q, e_s, and ε_s to second order in the spin frequency of the neutron star. Specifically, we define the parameter

$$\epsilon_0 \equiv \frac{f}{f_0} \tag{6}$$

in terms of the characteristic frequency

$$f_0 \equiv \sqrt{\frac{GM_0}{R_0^3}}. \quad (7)$$

In this equation, M_0 and R_0 are the non-spinning mass and radius of the neutron star. The characteristic frequency f_0 corresponds to the Keplerian orbital period of a test particle at a radius R_0 around a mass M_0 and thus corresponds roughly to the maximum frequency a neutron star can be spun up to before breakup. For spin frequencies much smaller than this characteristic frequency ($f < f_0$), ϵ_0 serves as a suitable small parameter about which we can expand the metric. When f approaches f_0, the parameter ϵ_0 approaches unity, and the Hartle-Thorne approximation is no longer valid. The spin frequency at which this occurs depends on M_0 and R_0 and, therefore, on the equation of state. However, for most proposed equations of state, this approximation is valid for even the most rapidly spinning neutron stars observed to date (Berti et al. 2005).

For a non-spinning neutron star, the parameter ϵ_0 is equal to zero, and thus the spin a, the quadrupole moment q, and the eccentricity of the surface e_s are all zero, as well. As the spin frequency increases, corrections to the metric enter at different orders in ϵ_0. To first order in the spin frequency, the star acquires a non-zero angular momentum, characterized by the spin parameter a. To the lowest order, we can approximate the spin parameter as a linear function of spin frequency, i.e.,

$$a = \epsilon_0 a^*, \quad (8)$$

where a^* is a constant that depends on the equation of state. To second order in the spin frequency, the star acquires a quadrupole moment and an elliptical shape, i.e.,

$$q = \epsilon_0^2 q^* \quad (9)$$

and

$$\frac{R_{\rm eq}}{R_{\rm p}} = 1 + \epsilon_0^2 R^*, \quad (10)$$

where q^* and R^* are again constants depending on the equation of state. Substituting Equation (10) into Equations (4) and (5) shows that the eccentricity of the surface of the neutron star has a first order dependence on the spin frequency

$$e_s = \epsilon_0 e_s^*, \quad (11)$$

while the ellipticity has a second-order dependence on the spin frequency, i.e.,

$$\varepsilon_s = \epsilon_0^2 \varepsilon_s^*. \quad (12)$$

The above relations all depend on ϵ_0, which in turn depends on the non-spinning values M_0 and R_0. For a spinning neutron star, however, these quantities are not readily measurable. Instead, observations can only constrain the spinning mass and equatorial radius, M and $R_{\rm eq}$, respectively. These parameters differ from their non-spinning values at second order in ϵ_0:

$$M = M_0 + \epsilon_0^2 \delta M^*, \quad (13)$$

$$R_{\rm eq} = R_0 + \epsilon_0^2 \delta R^*, \quad (14)$$

where δM^* and δR^* are again constants depending on the equation of state. Since M and $R_{\rm eq}$ differ from M_0 and R_0 at second order in ϵ_0, the corrections introduced to Equations (8)–(11) by the altered mass and radius will necessarily enter at third or fourth order in ϵ_0. Therefore, the lowest-order effects will be unchanged. For the remainder of this work, we will use the spinning mass and radius interchangeably with the nonspinning values.

We use the procedure described in Berti et al. (2005) to calculate the values of the parameters described above for several neutron-star equations of state. For a given equation of state, a central density and spin frequency uniquely determine the properties of a neutron star in the Hartle-Thorne approximation. First, we solve the Tolman-Oppenheimer-Volkoff equations to find the parameters of a non-spinning star with the same equation of state and central density. In this non-spinning case, a, q, and e_s^* are equal to 0. Next, we solve the full Hartle-Thorne equations for the perturbative quantities, i.e. q^*, a^*, R^*, e_s^*, and ε_s^*. Once we have found the values of these starred parameters, we can then use Equations (8)–(12) to determine the parameters of a neutron star spinning at any intermediate rate characterized by $\epsilon < \epsilon_0$.

3. RELATIONS BETWEEN SPIN, QUADRUPOLE, AND COMPACTNESS

As in Lattimer & Prakash (2001) and Yagi & Yunes (2013a), we find that tight empirical relations exist between the spin parameter a^*, the dimensionless quadrupole moment q^*, and the compactness $\zeta = GM_0/R_0c^2$ of neutron stars that depend very weakly on the assumed equation of state. In addition, we derive an analytic formula relating these four quantities to the eccentricity parameter e_s^* and the ellipticity parameter ε_s^* of the neutron star surface.

In order to generate our fits, we selected several modern equations of state. Observations by Demorest et al. (2010) of a 1.97 $M_\odot$ neutron star and by Antoniadis et al. (2013) of a 2.01 $M_\odot$ neutron star place significant constraints on the properties of dense matter and strongly disfavor several equations of state. We selected only equations of state that allow a maximum mass of at least 2.0 $M_\odot$. Following the naming convention of Lattimer & Prakash (2001), we chose for our fits the equations of state AP4, ENG, MPA1, and MS0, which cover a wide range of microphysics assumptions and calculational procedures.

For each equation of state, we use a large number of numerical models covering the astrophysically relevant range of masses $M > 1.0\ M_\odot$ (Özel et al 2012). A least-squares polynomial fit of the spin parameter a^* as a function the compactness yields

$$a^* = 1.1035 - 2.146\zeta + 4.5756\zeta^2\,. \quad (15)$$

Figure 1 shows this fit as a solid line, along with the results of numerical calculations for four different equations of state. The lower panel shows the residuals: for all equations of state, the residuals over the range of masses considered here are less than 4%.

Both Lattimer & Prakash (2001) and Yagi & Yunes (2013b) found similar empirical relations between the moment of inertia and the compactness. These authors consider a wider range of equations of state and neutron star parameters than those included in Figure 1. For less

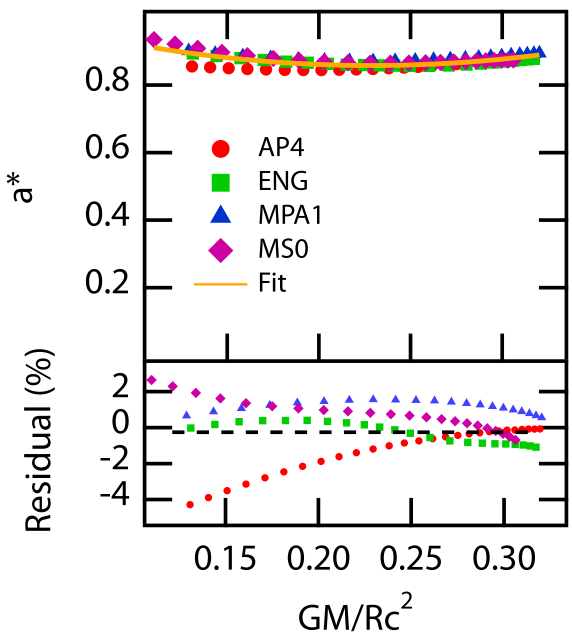

FIG. 1.— Empirical fit to the correlation between spin and compactness of a neutron star for four equations of state, corresponding to equation (15). The lower panel shows the residual in percent.

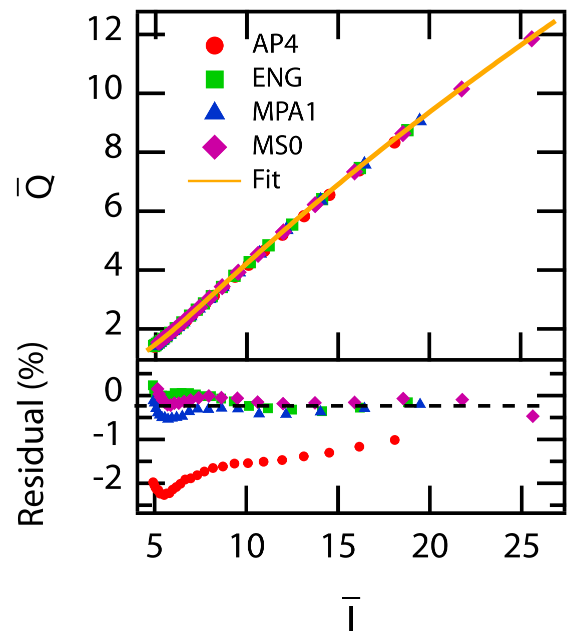

FIG. 2.— Empirical fit to the correlation between the dimensionless quadrupole moment $\bar{Q}$ and angular momentum $\bar{I}$. The fit corresponds to Equation (18) and is equivalent to that proposed in Yagi & Yunes (2013b). The lower panel shows the residual to the fit in percent.

compact neutron stars than those shown in Figure 1, i.e., typically those with masses $<1\ M_\odot$, different equations of state predict more divergent values for the moment of inertia (or equivalently the spin parameter a^*). For the purpose of modeling observations of astrophysical neutron stars, however, the relation given in Equation (15) is adequate.

In order to determine the quadrupole moment, we adopt the relation proposed by Yagi & Yunes (2013b). These authors present a relation between the quadrupole moment and moment of inertia of spinning neutron stars with a variety of equations of state. They define a dimensionless quadrupole moment $\bar{Q}$ and moment of inertia $\bar{I}$ that relate to our q^* and a^* via

$$\overline{Q} \equiv \frac{q^*}{a^{*2}}, \tag{16}$$

and

$$\overline{I} \equiv a^* \zeta^{-3/2}. \tag{17}$$

They then find an empirical expression for $\bar{I}$ as a function of $\bar{Q}$. Since the inverse of this relation is required for use with our fit for the spin parameter a^* in Equation (15), we find instead an analogous fit for $\bar{Q}$ as a function of $\bar{I}$,

$$\begin{aligned} \ln \overline{Q} = & -2.014 + 0.601 \ln \overline{I} + 1.10 (\ln \overline{I})^2 \\ & -0.412 (\ln \overline{I})^3 + 0.0459 (\ln \overline{I})^4. \end{aligned} \tag{18}$$

Alternatively, the fit of Yagi & Yunes (2013b) can be inverted numerically to obtain an equivalent relation. The above fit is shown as the solid line in Figure 2 along with numerical calculations for our chosen equations of state. Again, the residuals shown in the lower panel are less than 2% for each considered equation of state.

4. RELATIONS FOR THE ELLIPTICAL SHAPE OF THE NEUTRON STAR SURFACE

Given these two empirical fits for the spin parameter and quadrupole moment, we now find an analytic expression for the eccentricity of the neutron star surface. Hartle & Thorne (1968) solved the equations of stellar structure at second order in spin frequency and showed that the eccentricity of the neutron star surface measured in flat space is given by

$$e_s = \sqrt{-3(v_2 - h_2 + \xi_2/R_0)}, \tag{19}$$

where v_2 and h_2 are functions of R_0 that are second order in spin, and ξ_2 is the parameter defined in Equation (3). Hartle & Thorne (1968) provide the exact forms of the quantities v_2 and h_2 as functions of the mass, radius, angular momentum, and quadrupole moment. Using Equations (1) and (2), we can reduce Equation (19) to depend only on the dimensionless parameters ζ, a, q, and ϵ_0 defined above. Substituting a^* and q^* for a and q via Equations (8) and (9) and using the definition of e_s^* in Equation (11), we can eliminate the dependence on spin and find an analytic expression for the eccentricity e_s^*,

$$\begin{aligned} e_s^*(\zeta, a^*, q^*) = & \Bigg[1 - 4a^*\zeta^{3/2} \\ & + \frac{15(a^{*2} - q^*)(3 - 6\zeta + 7\zeta^2)}{8\zeta^2} + \zeta^2 a^{*2}(3 + 4\zeta) \\ & + \frac{45}{16\zeta^2}(q^* - a^{*2})(\zeta - 1)(1 - 2\zeta + 2\zeta^2)\ln(1 - 2\zeta) \Bigg]^{1/2}. \end{aligned} \tag{20}$$

Figure 3 shows this expression for the eccentricity as a function of the compactness, along with numerical calculations from several equations of state. We have substituted Equations (15), (17) and (18) into Equation (20)

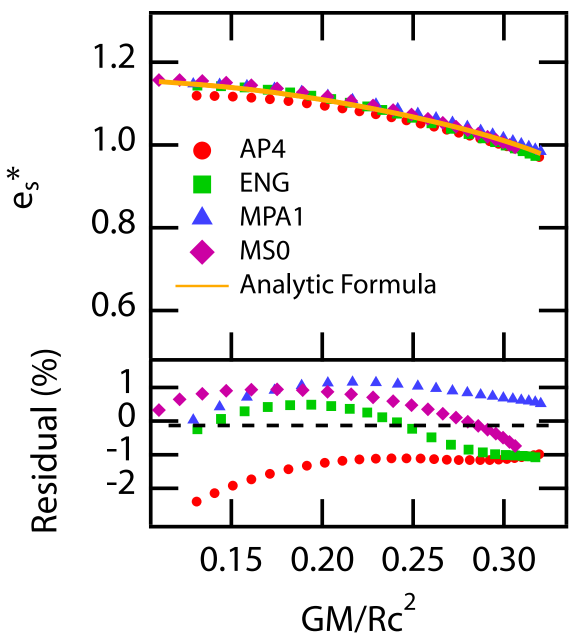

FIG. 3.— Analytic expression for the eccentricity of the neutron star surface, corresponding to Equation (20). In order to express this relation as a function of a single parameter, we have combined Equation (20) with our empirical fits for the quadrupole moment and spin parameter from Equations (15) and (18). The lower panel shows the residual to the relation in percent. Although the analytic relation is exact, the empirical fits between a^*, q^*, R, and M introduce some scatter about this relation.

in order to present the relation as a function of the single parameter ζ. The residuals to this relation are shown in the lower panel. The residuals are nonzero due to the empirical nature of the fits between a^*, q^*, and ζ.

Alternatively, Hartle (1967) gives an expression for the ellipticity of the neutron star surface in Hartle-Thorne coordinates as

$$\varepsilon_s = -\frac{3}{2R}\xi_2. \tag{21}$$

Again, we can reduce this equation to depend only on our dimensionless parameters and eliminate the spin dependency to find an expression for ε_s^*,

$$\varepsilon_s^*(\zeta, a^*, q^*) = \frac{1}{32\zeta^3}\Bigg\{2\zeta\Big[8\zeta^2 - 32a^*\zeta^{7/2} + (a^{*2} - q^*)(45 - 135\zeta + 60\zeta^2 + 30\zeta^3) + 24a^{*2}\zeta^4 + 8a^{*2}\zeta^5 - 48a^{*2}\zeta^6\Big] + 45(a^{*2} - q^*)(1-2\zeta)^2 \ln(1-2\zeta)\Bigg\}. \tag{22}$$

5. APPLICATIONS

In order to reduce the number of parameters necessary when fitting a neutron star observation, the following procedure can be used. For a given value of M, $R_{\rm eq}$, and f, one can calculate the parameter ϵ_0 via Equations (6) and (7) as

$$\epsilon_0 = f\left(\frac{GM}{R_{\rm eq}^3}\right)^{-1/2}. \tag{23}$$

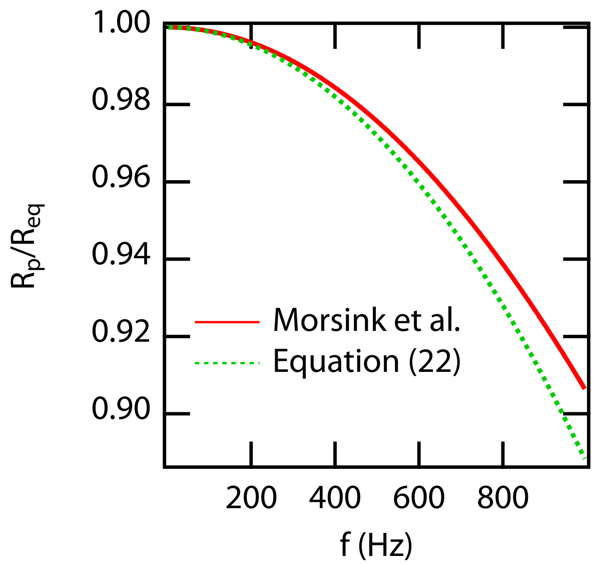

FIG. 4.— Comparison of the ratio of polar radius to equatorial radius found in this paper and the ratio found by Morsink et al. (2007)

The spin parameter a can then be found from Equation (15),

$$a = \epsilon_0\left[1.1035 - 2.146\zeta + 4.5756\zeta^2\right]. \tag{24}$$

The fit between moment of inertia and the quadrupole moment can then be used to write

$$q = a^2 \exp\Bigg[-2.014 + 0.601\ln\left(\frac{a}{\epsilon_0}\zeta^{-3/2}\right) + 1.10\ln\left(\frac{a}{\epsilon_0}\zeta^{-3/2}\right)^2 - 0.412\ln\left(\frac{a}{\epsilon_0}\zeta^{-3/2}\right)^3 + 0.0459\ln\left(\frac{a}{\epsilon_0}\zeta^{-3/2}\right)^4\Bigg]. \tag{25}$$

The parameters $a^* \equiv a/\epsilon_0$ and $q^* \equiv q/\epsilon_0^2$ can then be used in Equations (20) or (22) to find the eccentricity or ellipticity parameter of the neutron star surface in the appropriate spacetime. As defined in Equation (22), the ellipticity of the neutron star is given in Hartle-Thorne coordinates. In order to convert to the commonly used Boyer-Lindquist coordinate system, the following transformation can be applied:

$$R_{\rm BL}(R_{\rm HT}, \theta) = R_{\rm HT} - \frac{a^2\left(\frac{GM}{c^2}\right)^2}{2R_{\rm HT}^3}\Bigg[\left(R_{\rm HT} + 2\frac{GM}{c^2}\right)\left(R_{\rm HT} - \frac{GM}{c^2}\right) - \cos^2(\theta)\left(R_{\rm HT} - 2\frac{GM}{c^2}\right)\left(R_{\rm HT} + 3\frac{GM}{c^2}\right)\Bigg] \tag{26}$$

where $R_{\rm HT}$ is the radial coordinate in the Hartle-Thorne coordinate system and $R_{\rm BL}$ is the radial coordinate in the Boyer-Lindquist coordinate system (Hartle & Thorne 1968).

In the context of modeling pulse profiles, Morsink et al. (2007) similarly reduce the parameter space by finding

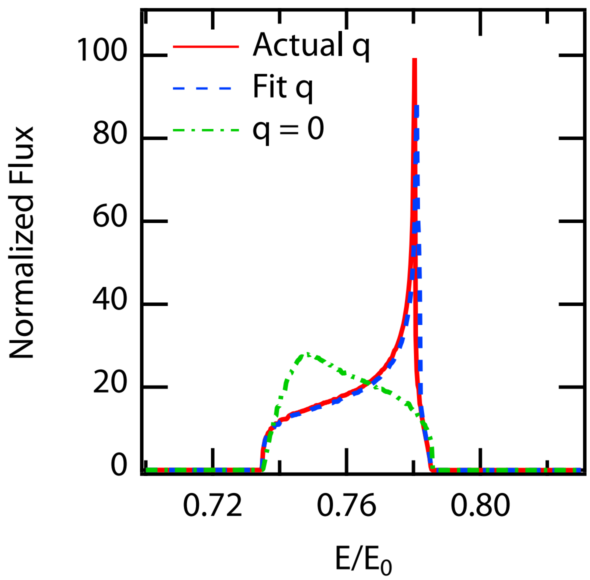

FIG. 5.— Simulated line profiles of an emission line from a neutron-star surface. The solid line shows a profile calculated using parameters from a numerical simulation of an AP4 star with a mass of 1.40 $M_\odot$ and a spin frequency of 700 Hz. The dashed line shows a star with the same mass, spin frequency, and radius (10.18 km), but with the quadrupole moment, spin parameter, and eccentricity determined by our fits. For reference, the dash-dotted line shows a profile with identical parameters but with the quadrupole moment set to zero. The photon energy at infinity and in the local Lorentz frame are denoted by E and E_0, respectively.

an empirical description of the oblate shape of spinning neutron stars that is accurate for multiple equations of state. They find that compact objects can be divided into two broad classes with different oblateness at high spin frequencies. Normal neutron stars and hybrid quark stars follow one relation, while color-flavor–locked stars exhibit a different behavior. In both cases, Morsink et al. (2007) find that the deviation of the stellar surface from the spherical shape is proportional to the square of the spin frequency, with some additional correction at fourth order in the spin.

The empirical model of Morsink et al. (2007) for calculating the shape of normal neutron stars should agree with the analytic formula we find above when compared in the same coordinate system. Morsink et al. (2007) define the shape of the stellar surface in the Schwarzschild coordinate system. Since the Boyer-Lindquist coordinate system reduces to the Schwarzschild coordinate system in the limit of zero spin, we use Equation (22) to calculate the ellipticity in Hartle-Thorne coordinates and apply the change of coordinates described in Equation (26). Figure 4 shows the predicted ratio of the polar to the equatorial radius in the model of Morsink et al. (2007) as well as the analytic relation described above for a range of spin frequencies. In both models a neutron star with a mass of 1.4 $M_\odot$ and a radius of 10 km was used. The deviation derived here of the empirical model of Morsink et al. (2007) and the analytic formula is of order 1% in the range of observed spins.

The neutron-star shape and quadrupole moment play an important role in the profiles of lines that originate on neutron-star surfaces. Bauböck et al. (2013) showed that, at low inclinations, the quadrupole moment can cause anomalously narrow features to appear even for neutron stars spinning at moderate rates. In order to test whether the fits proposed in this work are precise enough to accurately model line profiles, we compared the profile calculated with the parameters predicted by a numerical simulation to one using the parameters from our fits. We show the result in Figure 5. For this example, we chose a model where the fits have large residuals, especially for the quadrupole moment, which provides the dominant contribution to the profile shape (Bauböck et al 2013). Even in this case, the narrow profile is recovered, and the difference in the resulting profiles is negligible.

6. CONCLUSIONS

We have demonstrated that several macroscopic parameters of spinning neutron stars can be approximated with high accuracy using relations that depend only on their masses, radii, and spin frequencies, but that are practically independent of the equation of state. These fits enable measurements of neutron-star masses and radii using X-ray spectroscopy, timing observations of pulse profiles, and gravitational-wave observations of neutron stars spinning at moderate frequencies.

Future detectors such as NICER, LOFT, and Advanced LIGO will soon allow for more precise measurements of neutron-star parameters than have been possible to date. Using these observations to constrain the equation of state of the dense matter found in neutron star cores requires that the parameter space be reduced in order to determine the mass and radius with the highest precision. The relations demonstrated above allow this reduction of the parameter space independent of the equation of state, making possible more precise measurement of the equation of state of neutron-star cores.

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