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Immersing almost geodesic surfaces in a closed hyperbolic three manifold

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Abstract

Let $\mathbf{M}^3$ be a closed hyperbolic three manifold. We construct closed surfaces that map by immersions into $\mathbf{M}^3$ so that for each, one the corresponding mapping on the universal covering spaces is an embedding, or, in other words, the corresponding induced mapping on fundamental groups is an injection.

1. Introduction

The purpose of this paper is to prove the following theorem.

THEOREM 1.1. *Let* $\mathbf{M}^3 = \mathbb{H}^3/\mathcal{G}$ *denote a closed hyperbolic three manifold where* $\mathcal{G}$ *is a Kleinian group, and let* $\varepsilon > 0$. *Then there exists a Riemann surface* $S_\varepsilon = \mathbb{H}^2/F_\varepsilon$ *where* F_ε *is a Fuchsian group and a* $(1+\varepsilon)$*-quasiconformal map* $g : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$, *such that the quasifuchsian group* $g \circ F_\varepsilon \circ g^{-1}$ *is a subgroup of* $\mathcal{G}$. (*Here we identify the hyperbolic plane* $\mathbb{H}^2$ *with an oriented geodesic plane in* $\mathbb{H}^3$ *and the circle* $\partial\mathbb{H}^2$ *with the corresponding circle on the sphere* $\partial\mathbb{H}^3$.)

Remark. In the above theorem the Riemann surface S_ε has a pants decomposition where all the cuffs have a fixed large length and they are glued by twisting for $+1$.

One can extend the map g to an equivariant diffeomorphism of the hyperbolic space. This extension defines the map $f : S_\varepsilon \to \mathbf{M}^3$, and the surface $f(S_\varepsilon) \subset \mathbf{M}^3$ is an immersed $(1+\varepsilon)$-quasigeodesic surface. In particular, the surface $f(S_\varepsilon)$ is essential which means that the induced map $f_* : \pi_1(S_\varepsilon) \to \pi_1(\mathbf{M}^3)$ is an injection. We summarize this in the following theorem.

THEOREM 1.2. *Let* $\mathbf{M}^3$ *be a closed hyperbolic three manifold. Then we can find a closed hyperbolic surface* S *and a continuous map* $f : S \to \mathbf{M}^3$ *such that the induced map between fundamental groups is injective.*

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Let S be an oriented closed topological surface with a given pants decomposition $\mathcal{C}$, where $\mathcal{C}$ is a maximal collection of disjoint (unoriented) simple closed curves that cut S into the corresponding pairs of pants. Let $f : S \to \mathbf{M}^3$ be a continuous map and let $\rho_f : \pi_1(S) \to \pi_1(\mathbf{M}^3)$ be the induced map between the fundamental groups. Assume that ρ_f is injective on $\pi_1(\Pi)$ for every pair of pants Π from the pants decomposition of S. Then to each curve $C \in \mathcal{C}$ we can assign a complex half-length $\mathbf{hl}(C) \in (\mathbb{C}/2\pi i\mathbb{Z})$ and a complex twist-bend $s(C) \in \mathbb{C}/(\mathbf{hl}(C)\mathbb{Z} + 2\pi i\mathbb{Z})$. We prove the following in Section 2.

THEOREM 1.3. *There are universal constants $\widehat{\varepsilon}, K_0 > 0$ such that the following holds. Let ε be such that $\widehat{\varepsilon} > \varepsilon > 0$. Suppose $(S, \mathcal{C})$ and $f : S \to \mathbf{M}^3$ are as above, and for every $C \in \mathcal{C}$, we have*

$$|\mathbf{hl}(C) - \frac{R}{2}| < \varepsilon, \text{ and } |s(C) - 1| < \frac{\varepsilon}{R}$$

for some $R > R(\varepsilon) > 0$. Then ρ_f is injective and the map $\partial\widetilde{f} : \partial\widetilde{S} \to \partial\widetilde{\mathbf{M}^3}$ extends to a $(1 + K_0\varepsilon)$-quasiconformal map from $\partial\mathbb{H}^3$ to itself. (Here $\widetilde{S}$ and $\widetilde{\mathbf{M}^3}$ denote the corresponding universal covers.)

It then remains to construct such a pair $(f, (S, \mathcal{C}))$. If Π is a (flat) pair of pants, we say $f : \Pi \to \mathbf{M}^3$ is a skew pair of pants if ρ_f is injective, and $f(\partial\Pi)$ is the union of three closed geodesics. Suppose we are given a collection $\{f_\alpha : \Pi_\alpha \to \mathbf{M}^3\}_{\alpha \in A}$ of skew pants, and suppose for the sake of simplicity that no f_α maps two components of $\partial\Pi$ to the same geodesic.

For each closed geodesic γ in $\mathbf{M}^3$, we let $A_\gamma = \{\alpha \in A : \gamma \in f_\alpha(\partial\Pi_\alpha)\}$. Given permutations $\sigma_\gamma : A_\gamma \to A_\gamma$ for all such γ, we can build a closed surface in $\mathbf{M}^3$ as follows. For each (f_α, Π_α), we make two pairs of skew pants in $\mathbf{M}^3$, identical except for their orientations. For each γ, we connect via the permutation σ_γ the pants that induce one orientation on γ to the pants that induce the opposite orientation on γ. We show in Section 3 that if the pants are "evenly distributed' around each geodesic γ, then we can build a surface this way that satisfies the hypotheses of Theorem 1.3.

We can make this statement more precise as follows: for each $\gamma \in f_\alpha(\partial\Pi_\alpha)$, we define an unordered pair $\{n_1, n_2\} \in N^1(\gamma)$, the unit normal bundle to γ. The two vectors satisfy $2(n_1 - n_2) = 0$ in the torus $\mathbb{C}/(2\pi i\mathbb{Z} + l(\gamma)\mathbb{Z})$, where $l(\gamma)$ is the complex length of γ. So we write

$$\text{foot}_\gamma(\Pi_\alpha) \equiv \text{foot}_\gamma(f_\alpha, \Pi_\alpha) = \{n_1, n_2\} \in N^1(\sqrt{\gamma}) = \mathbb{C}/(2\pi i\mathbb{Z} + \mathbf{hl}(\gamma)\mathbb{Z}).$$

We let $\text{foot}(A) = \{\text{foot}_\gamma(\Pi_\alpha) : \alpha \in A, \gamma \in \partial\Pi_\alpha\}$. (Properly speaking, $\text{foot}(A)$ is a labeled set (or a multiset) rather than a set; see Section 3 for details.) We then define $\text{foot}_\gamma(A) = \text{foot}(A)|_{N^1(\sqrt{\gamma})}$. We let $\tau : N^1(\sqrt{\gamma}) \to N^1(\sqrt{\gamma})$ be defined by $\tau(n) = n + 1 + i\pi$. If for each γ we can define a permutation $\sigma_\gamma \colon A_\gamma \to A_\gamma$ such that

$$|\operatorname{foot}_\gamma(\Pi_{\sigma_\gamma(\alpha)}) - \tau(\operatorname{foot}_\gamma(\Pi_\alpha))| < \frac{\varepsilon}{R},$$

and $|\mathbf{hl}(\gamma) - \frac{R}{2}| < \varepsilon$ for all $\gamma \in \partial\Pi_\alpha$, then the resulting surface will satisfy the assumptions of Theorem 1.3. The details of the above discussion are carried out in Section 3.

In Section 4 we construct the measure on skew pants that after rationalisation will give us the collection Π_α we mentioned above. This is the heart of the paper. Showing that there exists a single skew pants that satisfies the first inequality in Theorem 1.3 is a nontrivial theorem, and the only known proofs use the ergodicity of either the horocyclic or the frame flow. This result was first formulated and proved by L. Bowen [1], where he used the horocyclic flow to construct such skew pants. Our construction is different. We use the frame flow to construct a measure on skew pants whose equidistribution properties follow from the exponential mixing of the frame flow. This exponential mixing is a result of Moore [10]; see also [11]. (It has been shown by Brin and Gromov [2] that for a much larger class of negatively curved manifolds the frame flow is strong mixing.) The detailed outline of this construction is given at the beginning of Section 4.

We point out that Cooper-Long-Reid [5] proved the existence of essential surfaces in cusped finite volume hyperbolic three manifolds. Lackenby [9] proved the existence of such surfaces in all closed hyperbolic three manifolds that are arithmetic.

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2. **Quasifuchsian representation of a surface group**

2.1. *The Complex Fenchel-Nielsen coordinates.* Below we define the Complex Fenchel-Nielsen coordinates. For a very detailed account, we refer to [12] and [8]. Originally the coordinates were defined in [13] and [8].

A word on notation. By $d(X, Y)$ we denote the hyperbolic distance between sets $X, Y \subset \mathbb{H}^3$. If $\gamma^* \subset \mathbb{H}^3$ is an oriented geodesic and $p, q \in \gamma^*$, then $\mathbf{d}_{\gamma^*}(p, q)$ denotes the signed real distance between p and q. Let α^*, β^* be two oriented geodesics in $\mathbb{H}^3$, and let γ^* be the geodesic that is orthogonal to both α^* and β^*, with an orientation. Let $p = \alpha^* \cap \gamma^*$ and $q = \beta^* \cap \gamma^*$.

Let u be the tangent vector to α^* at p, and let v be the tangent vector to β^* at q. We let u' be the parallel transport of u to q. By $\mathbf{d}_{\gamma^*}(\alpha^*, \beta^*)$ we denote the complex distance between α^* and β^* measured along γ^*. The real part is given by $\mathrm{Re}(\mathbf{d}_{\gamma^*}(\alpha^*, \beta^*)) = \mathbf{d}_{\gamma^*}(p, q)$. The imaginary part $\mathrm{Im}(\mathbf{d}_{\gamma^*}(\alpha^*, \beta^*))$ is the oriented angle from u' to v, where the angle is oriented by γ^* which is orthogonal to both u' and v. The complex distance is well defined (mod $2k\pi i$), $k \in \mathbb{Z}$. In fact, every identity we write in terms of complex distances is therefore assumed to be true (mod $2k\pi i$). We have the following identities: $\mathbf{d}_{\gamma^*}(\alpha^*, \beta^*) = -\mathbf{d}_{\gamma^*}(\beta^*, \alpha^*)$, $\mathbf{d}_{-\gamma^*}(\alpha^*, \beta^*) = -\mathbf{d}_{\gamma^*}(\alpha^*, \beta^*)$, and $\mathbf{d}_{\gamma^*}(-\alpha^*, \beta^*) = \mathbf{d}_{\gamma^*}(\alpha^*, \beta^*) + i\pi$.

We let $\mathbf{d}(\alpha^*, \beta^*)$ (without a subscript for $\mathbf{d}$) denote the unsigned complex distance equal to $\mathbf{d}_{\gamma^*}(\alpha^*, \beta^*)$ modulo $\langle z \to -z \rangle$. We will write $\mathbf{d}(\alpha^*, \beta^*) \in (\mathbb{C}/2\pi i\mathbb{Z})/\mathbb{Z}_2$, where $\mathbb{Z}_2$ of course stands for $\langle z \to -z \rangle$. We observe that $\mathbf{d}(\alpha^*, \beta^*) = \mathbf{d}(\beta^*, \alpha^*) = \mathbf{d}(-\alpha^*, -\beta^*) = \mathbf{d}(-\beta^*, -\alpha^*)$.

For a loxodromic element $A \in \mathbf{PSL}(2, \mathbb{C})$, by $\mathbf{l}(A)$ we denote its complex translation length. The number $\mathbf{l}(A)$ has a positive real part and it is defined (mod $2k\pi i$), $k \in \mathbb{Z}$. By γ^* we denote the oriented axis of A, where γ^* is oriented so that the attracting fixed point of A follows the repelling fixed point.

Let Π^0 be a topological pair of pants (a three holed sphere). We consider Π^0 as a manifold with boundary; that is, we assume that Π^0 contains its cuffs. We say that a pair of pants in a closed hyperbolic three manifold $\mathbf{M}^3$ is an injective homomorphism $\rho : \pi_1(\Pi^0) \to \pi_1(\mathbf{M}^3)$, up to conjugacy. This induces a representation

$$\rho : \pi_1(\Pi^0) \to \mathbf{PSL}(2, \mathbb{C}),$$

up to conjugacy, which in general we also call a free-floating pair of pants. A pair of pants in $\mathbf{M}^3$ is determined by (and determines) a continuous map $f : \Pi^0 \to \mathbf{M}^3$, up to homotopy, and free-floating pair of pants likewise determines a map

$$f : \Pi^0 \to \mathbb{H}^3/\rho(\pi_1(\Pi^0)) = M_\rho,$$

up to homotopy.

Suppose $\rho : \pi_1(\Pi^0) \to \mathbf{PSL}(2, \mathbb{C})$ is a free-floating pair of pants, and suppose $\rho = f_*$, where $f : \Pi^0 \to M_\rho$. We orient the components C_i of $\partial\Pi^0$ so that Π^0 is on the left of each C_i. For each i, there is a unique oriented closed geodesic γ_i in M_ρ freely homotopic to $f(C_0)$. Now let a_i be the simple nonseparating arc on Π^0 connecting C_{i-1} and C_{i+1}. (We take the subscript (mod 3).) We can homotop f so that f maps each C_i to γ_i and maps a_i to an arc η_i from γ_{i-1} to γ_{i+1} that is orthogonal at its endpoints to γ_{i-1} and γ_{i+1}.

While such an f is not unique, the 1-complex made of the γ_i and the η_i together divide $f(\Pi^0)$ into two singular regions whose boundaries are geodesic right-angled hexagons. Because the geometry of each of these two hexagons is determined by these unsigned complex distances $\mathbf{d}_{\eta_i}(\gamma_{i-1}, \gamma_{i+1})$, the two right-angled hexagons are isometric.

Let us fix for the moment $i \in \{0, 1, 2\}$. We then orient η_{i-1} and η_{i+1} to point away from γ_i (so the signed complex distance $\mathbf{d}_{\eta_{i\pm1}}(\gamma_i, \gamma_{i\mp1})$ has positive real part). Recall that $\mathbf{d}_{\gamma_i}(\eta_{i-1}, \eta_{i+1})$ denotes the signed complex distance from η_{i-1} to η_{i+1}, along γ_i. Because the two hexagons are isometric,

$$\mathbf{d}_{\gamma_i}(\eta_{i-1}, \eta_{i+1}) = \mathbf{d}_{\gamma_i}(\eta_{i+1}, \eta_{i-1}).$$

We let

$$\mathbf{hl}(\gamma_i) = \mathbf{d}_{\gamma_i}(\eta_{i-1}, \eta_{i+1}).$$

We can also think of this definition on the universal cover $\mathbb{H}^3$ as follows. We conjugate ρ so that there is a lift $\widetilde{\gamma}_i$ of γ_i to $\mathbb{H}^3 = \{(x, y, z) : z > 0\}$ that connects 0 and ∞. We let $A_{\gamma_i} \in \mathbf{PSL}(2, \mathbb{C})$ be such that $\gamma_i = \widetilde{\gamma}_i / \langle A_{\gamma_i} \rangle$. Then $A_{\gamma_i} : \mathbb{H}^3 \to \mathbb{H}^3$ extends to map $\widehat{\mathbb{C}} = \partial \mathbb{H}^3$ to itself by $z \mapsto e^{\mathbf{l}(\gamma_i)} \cdot z$.

Moreover, the lifts of η_{i-1} and η_{i+1} that intersect $\widetilde{\gamma}_i$ will alternate along $\widetilde{\gamma}_i$ (so we can define $\mathbf{d}_{\gamma_i}(\eta_{i-1}, \eta_{i+1})$ as $\mathbf{d}_{\widetilde{\gamma}_i}(\widetilde{\eta}_{i-1}, \widetilde{\eta}_{i+1})$, where $\widetilde{\eta}_{i-1}$ is a lift of η_{i-1} that intersects $\widetilde{\gamma}_i$ and $\widetilde{\eta}_{i+1}$ is the next lift of η_{i+1} along $\widetilde{\gamma}_i$). If we define $\sqrt{A_{\gamma_i}} \in \mathbf{PSL}(2, \mathbb{C})$ so that it maps $z \mapsto e^{\mathbf{hl}(\gamma_i)} \cdot z$, then it will map the lifts of η_{i-1} to the lifts of η_{i+1}, and vice verse.

Moreover, the unit normal bundle $N^1(\widetilde{\gamma}_i)$ is a torsor for $\mathbb{C}^* \equiv \mathbb{C}/2\pi i\mathbb{Z}$, and the unit normal bundle $N^1(\gamma_i)$ is a torsor for

$$\mathbb{C}^* / \langle A_{\gamma_i} \rangle = \mathbb{C}/2\pi i\mathbb{Z} + \mathbf{l}(\gamma_i) \cdot \mathbb{Z}.$$

Remark. Let G be a group, and let X be a space on which G acts. We say that X is a torsor for G (or that X is a G-torsor) if, for any two elements x_1 and x_2 of X, there exists a unique group element $g \in G$ with $g(x_1) = x_2$.

By a mild abuse of notation, we let

$$N^1(\sqrt{\gamma_i}) = N^1(\widetilde{\gamma}_i) / \left\langle \sqrt{A_{\gamma_i}} \right\rangle.$$

This is a torsor for

$$\mathbb{C}^* / \left\langle \sqrt{A_{\gamma_i}} \right\rangle = \mathbb{C}/2\pi i\mathbb{Z} + \mathbf{hl}(\gamma_i) \cdot \mathbb{Z}.$$

For $i \neq j$, $i, j = 0, 1, 2$, we let $n(i, j) \in N^1(\gamma_i)$ be the unit vector at $\gamma_i \cap \eta_j$ pointing along η_j. Then $\sqrt{A_{\gamma_i}}$ interchanges $n(i, i-1)$ and $n(i, i+1)$, so we can think of the unordered pair $\{n(i, i-1), n(i, i+1)\}$ as an element of $N^1(\sqrt{\gamma_i})$. We call this element $\mathrm{foot}_{\gamma_i}(\rho)$ or $\mathrm{foot}_{\gamma_i}(f)$, where $f : \Pi^0 \to M_\rho$ is a map whose homotopy class is determined by ρ.

If $\rho : \pi_1(\Pi^0) \to \mathbf{PSL}(2, \mathbb{C})$ is a representation for which $\mathbf{hl}(C) \in \mathbb{R}^+$ for each $C \in \partial \Pi^0$, then, after conjugation, $\rho(\pi_1(\Pi^0)) \in \mathbf{PSL}(2, \mathbb{R}) < \mathbf{PSL}(2, \mathbb{C})$, and $\mathbb{H}^2 / \rho(\pi_1(\Pi^0))$ is a topological pair of pants (homeomorphic to the interior of Π^0). Also the converse is true: if we are given $\rho : \pi_1(\Pi^0) \to \mathbf{PSL}(2, \mathbb{R})$ and

$\mathbb{H}^2/\rho(\pi_1(\Pi^0))$ is homeomorphic to the interior of Π^0, then $\mathbf{hl}(C) \in \mathbb{R}^+$ for each cuff $C \in \partial\Pi^0$.

Now suppose that S^0 is a closed surface (of genus at least 2) and $\mathcal{C}^0$ a maximal set of simple closed curves on S^0. (The curves in $\mathcal{C}^0$ are disjoint, nonisotopic and nontrivial.) By $\mathcal{C}^*$ we denote the set of oriented curves from $\mathcal{C}^0$. (Each curve is taken with both orientations.) A pair of pants Π for $(S^0, \mathcal{C}^0)$ is the closure of a component of $S^0 \setminus \bigcup \mathcal{C}^0$, and a marked pair of pants is a pair (Π, C), where $C \in \mathcal{C}^*$ is an oriented closed curve such that $C \in \partial\Pi$, and C lies to the left of Π. For any marked pair of pants (Π, C), there is a unique marked pair of pants (Π', C') such that $C' = -C$ (where $-C$ denotes the curve C but with the opposite orientation). We observe in passing that Π can be equal to Π'.

Now suppose that

$$\rho : \pi_1(S^0) \to \mathbf{PSL}(2, \mathbb{C})$$

is a representation that is discrete and faithful when restricted to $\pi_1(\Pi)$ for each pair of pants Π in $S^0 \setminus \bigcup \mathcal{C}^0$. By M_ρ we again denote the quotient $\mathbb{H}^3/\rho(\pi_1(S^0))$. Suppose that $\rho = f_*$ for some continuous map $f : S^0 \to M_\rho$. Then for each marked pair of pants (Π, C), we let γ be the oriented geodesic freely homotopic to $f(C)$. As before, we define $\mathbf{hl}_\Pi(\gamma)$ using $f|_\Pi$.

Let (Π', C') be the marked pair of pants such that $C' = -C$. Then $\mathbf{hl}_\Pi(C) = \mathbf{hl}_{\Pi'}(C)$ or $\mathbf{hl}_\Pi(C) = \mathbf{hl}_{\Pi'}(C) + i\pi$. In the former case, $\langle \sqrt{A_\gamma} \rangle = \langle \sqrt{A_{\gamma'}} \rangle$, so $N^1(\sqrt{\gamma}) = N^1(\sqrt{\gamma'})$ literally. In this case we write $\mathbf{hl}(C) = \mathbf{hl}_\Pi(C) = \mathbf{hl}_{\Pi'}(C)$.

Definition 2.1. Let S^0 and $\mathcal{C}^0$ be as above. We say that a representation

$$\rho : \pi_1(S^0) \to \mathbf{PSL}(2, \mathbb{C})$$

is viable if

- ρ is discrete and faithful when restricted to $\pi_1(\Pi)$ for each pair of pants Π in $S^0 \setminus \bigcup \mathcal{C}^0$;
- $\mathbf{hl}(C) = \mathbf{hl}_\Pi(C) = \mathbf{hl}_{\Pi'}(C)$ for each $C \in \mathcal{C}^0$, where Π and Π' are two pairs of pants that contain C.

Given a viable representation $\rho : \pi_1(S^0) \to \mathbf{PSL}(2, \mathbb{C})$, we let

$$s(C) = \mathrm{foot}_\gamma(\rho|_\Pi) - \mathrm{foot}_{\gamma'}(\rho|_{\Pi'}) - i\pi.$$

Then $s(C) \in \mathbb{C}/2\pi i\mathbb{Z} + \mathbf{hl}(C) \cdot \mathbb{Z}$. If we reverse the roles of (Π, C) and (Π', C'), we negate the difference of the two feet, but we also reverse the orientation of γ, so we get the same element $s(C) \in \mathbb{C}/2\pi i\mathbb{Z} + \mathbf{hl}(C) \cdot \mathbb{Z}$. The coordinates $(\mathbf{hl}(C), s(C))$ are called the reduced complex Fenchel-Nielsen coordinates for ρ.

The following is the main result of this section and it will be used later in the paper.

THEOREM 2.1. *Let $0 < \varepsilon < \widehat{\varepsilon}$, where $\widehat{\varepsilon} > 0$ is a universal constant. Then there exists $R_0 = R_0(\varepsilon) > 0$ such that the following holds. Let S^0 be a closed topological surface with a pants decomposition $\mathcal{C}^0$. Suppose that $\rho : \pi_1(S^0) \to$ $\mathbf{PSL}(2, \mathbb{C})$ is a viable representation such that*

$$|\mathbf{hl}(C) - \frac{R}{2}| < \varepsilon, \text{ and } |s(C) - 1| < \frac{\varepsilon}{R}$$

for some $R > R_0 > 0$. Then there exists a viable representation $\rho_0 : \pi_1(S^0) \to$ $\mathbf{PSL}(2, \mathbb{C})$ such that $\mathbf{hl}(C) = R$ and $s(C) = 1$ for all $C \in \mathcal{C}^0$ and a K-quasi-symmetric map $h : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$ so that $h^{-1}\rho_0(\pi_1(S^0))h = \rho(\pi_1(S^0))$, where $K = K(\varepsilon)$ and $K(\varepsilon) \to 1$ uniformly when $\varepsilon \to 0$. In particular, the representation ρ is injective and the group $\rho(\pi_1(S^0))$ is quasifuchsian.

2.2. *Holomorphic families of representations.* In this subsection we state Theorem 2.2 that will imply Theorem 2.1. The rest of Section 2 is devoted to proving Theorem 2.2.

Fix a closed surface S^0 with a pants decomposition $\mathcal{C}^0$. Fix a pair of pants Π from $S^0 \setminus \mathcal{C}^0$, and let $C_0, C_1, C_2 \in \mathcal{C}^0$ denote the cuffs of Π. The inclusion $\Pi \to S^0$ induces an embedding $\pi_1(\Pi) \to \pi_1(S^0)$. (Such embedding is well defined up to conjugation.) Let $c_0, c_1 \in \pi_1(\Pi) \subset \pi_1(S^0)$ be elements in the conjugacy classes corresponding to C_0 and C_1 respectively.

Let $\rho : \pi_1(S^0) \to \mathbf{PSL}(2, \mathbb{C})$ be a viable representation. After conjugating ρ by an element of $\mathbf{PSL}(2, \mathbb{C})$, we may assume that the axis of $\rho(c_0)$ is the geodesic in $\mathbb{H}^3$ that connects 0 and ∞ (such that 0 is the repelling point) and that the point $1 \in \partial\mathbb{H}^3$ is the repelling point of $\rho(c_1)$. (Such a conjugation exists since ρ is viable and the restriction of ρ to $\pi_1(\Pi)$ is injective.) Such ρ is said to be normalized. (The normalization depends on the choice of c_0 and c_1 but we suppress this.)

Let $R > 0$, and let Ω denote the set of all pairs (z_C, w_C), $C \in \mathcal{C}^0$, where for each C, we have

(1) $z_C \in \mathbb{C}/2\pi i\mathbb{Z}$ and $|z_C - \frac{R}{2}| < 1$,
(2) $w_C \in \mathbb{C}/2\pi i\mathbb{Z} + z_C \cdot \mathbb{Z}$ and $|s(C) - 1| < \frac{1}{R}$.

For simplicity we let $z = (z_C)_{C \in \mathcal{C}^0}$ and $w = (w_C)_{C \in \mathcal{C}^0}$. It follows from [8] and [12] that when R is large enough (say $R > 2$), for each $(z, w) \in \Omega$ there exists a normalized viable representation $\rho : \pi_1(S^0) \to \mathbf{PSL}(2, \mathbb{C})$ such that $\mathbf{hl}(C) = z_C$ and $s(C) = w_C$.

Remark. The representation ρ is not unique since $(\mathbf{hl}(C), s(C))$ are the reduced complex Fenchel-Nielsen coordinates and they determine the normalized representation only if we specify the marking of the cuffs. (That is, a normalized viable representation is uniquely determined by the choice of the (nonreduced) Fenchel-Nielsen coordinates.)

Suppose that we are given a normalized viable representation $\rho' : \pi_1(S^0) \to \mathbf{PSL}(2,\mathbb{C})$ such that $|\mathbf{hl}(C) - \frac{R}{2}| < 1$ and $|s(C) - 1| < \frac{1}{R}$, where $(\mathbf{hl}(C), s(C))$ are the reduced complex Fenchel-Nielsen coordinates for ρ'. Let $z'_C = \mathbf{hl}(C)$ and $w'_C = s(C)$. Then $(z', w') \in \Omega$. It then follows from [8] and [12] that for each $(z, w) \in \Omega$, there exists a unique normalized viable representation $\rho_{z,w} : \pi_1(S^0) \to \mathbf{PSL}(2,\mathbb{C})$ such that

- $z_C = \mathbf{hl}(C)$ and $w_C = s(C)$, where $(\mathbf{hl}(C), s(C))$ are the reduced complex Fenchel-Nielsen coordinates for $\rho_{z,w}$;
- the family of representations $\rho_{z,w}$ varies holomorphically in (z, w);
- $\rho' = \rho_{z',w'}$.

Definition 2.2. For $C \in \mathcal{C}^0$ let $\zeta_C, \eta_C \in \mathbb{D}$, where $\mathbb{D}$ denotes the unit disc in the complex plane. Let $\tau \in \mathbb{D}$ be a complex parameter. Fix $R > 1$ and let $\mathbf{hl}(C)(\tau) = \frac{1}{2}(R + \tau\zeta_C)$ and $s(C)(\tau) = 1 + \frac{\tau\eta_C}{R}$. By ρ_τ we denote the corresponding normalized viable representation with the reduced Fenchel-Nielsen coordinates $(\mathbf{hl}(C)(\tau), s(C)(\tau))$. Note that ρ_τ depends on ζ_C, η_C but we suppress this.

It follows that ρ_τ depends holomorphically on τ. The remainder of this section is devoted to proving the following theorem.

THEOREM 2.2. *There exist constants $\widehat{R}, \widehat{\varepsilon} > 0$, such that the following holds. Let S^0 be any closed topological surface with a pants decomposition $\mathcal{C}^0$, and fix $\zeta_C, \eta_C \in \mathbb{D}$ for $C \in \mathcal{C}^0$. Then for every $R \geq \widehat{R}$ and $|\tau| < \widehat{\varepsilon}$, the group $\rho_\tau(\pi_1(S^0))$ is quasifuchsian and the induced quasisymmetric map $f_\tau : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$ (that conjugates $\rho_0(\pi_1(S^0))$ to $\rho_\tau(\pi_1(S^0))$) is $K(\tau)$-quasisymmetric, where*

$$K(\tau) = \frac{\widehat{\varepsilon} + |\tau|}{\widehat{\varepsilon} - |\tau|}.$$

2.3. *Notation and the brief outline of the proof of Theorem* 2.2. The following notation remains valid through the section. Fix S^0, $\mathcal{C}^0$ and $\zeta_C, \eta_C \in \mathbb{D}$ as above. Denote by $\mathcal{C}_\tau(R)$ the collection of translation axes in $\mathbb{H}^3$ of all the elements $\rho_\tau(c)$, where $c \in \pi_1(S^0)$ is in the conjugacy class of some cuff $C \in \mathcal{C}^0$. Fix two such axes $C(\tau)$ and $\widehat{C}(\tau)$, and let $O(\tau)$ be their common orthogonal in $\mathbb{H}^3$. Since $C(\tau)$ and $\widehat{C}(\tau)$ vary holomorphically in τ, so does $O(\tau)$. (This means that the endpoints of $O(\tau)$ vary holomorphically on $\partial\mathbb{H}^3$.) Note that the endpoints of $O(\tau)$ might not belong to the limit set of the group $\rho_\tau(\pi_1(S^0))$.

Let $C_0(0), C_1(0), \ldots, C_{n+1}(0)$ be the ordered collection of geodesics from $\mathcal{C}_0(R)$ that $O(0)$ intersects (and in this order) and so that $C_0(0) = C(0)$ and $C_{n+1}(0) = \widehat{C}(0)$. The geodesic segment on $O(0)$ between $C_0(0)$ and $C_{n+1}(0)$ intersects $n \geq 0$ other geodesics from $\mathcal{C}_0(R)$. (Until the end of this section, n will have the same meaning.) We orient $O(0)$ so that it goes from $C_0(0)$ to

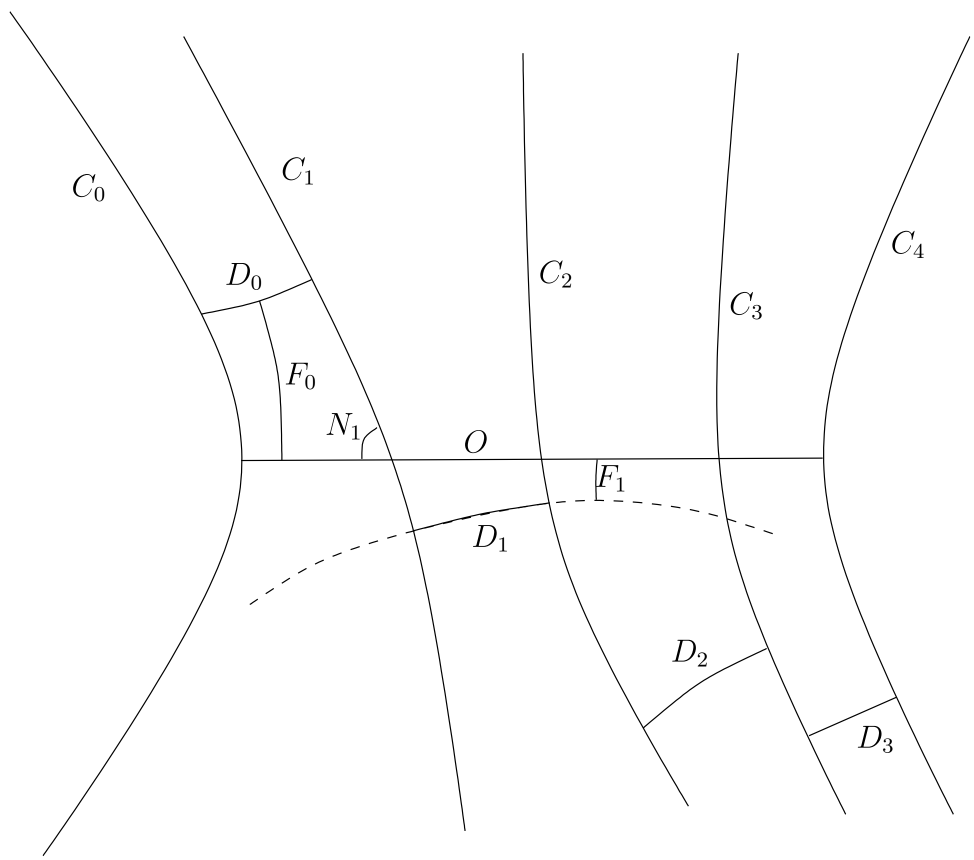

Figure 1. The geodesics O, C_i, N_i, F_i, and D_i

$C_{n+1}(0)$. We orient each $C_i(0)$ so that the angle from $O(0)$ to $C_i(0)$ is positive. (Recall that we fix in advance an orientation on the initial plane $\mathbb{H}^2 \subset \mathbb{H}^3$ so this angle is positive with respect to this orientation of the plane $\mathbb{H}^2$.) Then the oriented geodesics $C_i(\tau)$ vary holomorphically in τ.

Let $N_i(\tau)$ be the common orthogonal between $O(\tau)$ and $C_i(\tau)$ that is oriented so that the imaginary part of the complex distance $\mathbf{d}_{N_i(\tau)}(O(\tau), C_i(\tau))$ is positive. Let $D_i(\tau)$, $i = 0, \ldots, n$ be the common orthogonal between $C_i(\tau)$ and $C_{i+1}(\tau)$ that is oriented so that the angle from $D_i(0)$ to $C_i(0)$ is positive. Also, let $F_i(\tau)$ be the common orthogonal between $O(\tau)$ and $D_i(\tau)$ for $i = 0, \ldots, n$. We orient $F_i(\tau)$ so that the angle from $O(0)$ to $F_i(0)$ is positive. Observe that $F_0(\tau) = C_0(\tau)$ and $F_n(\tau) = C_{n+1}(\tau)$.

For simplicity, in the rest of this section we suppress the dependence on τ; that is, we write $C_i(\tau) = C_i$, $O(\tau) = O$ and so on. However, we still write $C_i(0)$, $O(0)$, to distinguish the case $\tau = 0$.

For Theorem 2.2, we need to estimate the quasisymmetric constant of the map f_τ, when τ belongs to some small, but definite neighbourhood of the origin in $\mathbb{D}$. In order to do that we want to estimate the derivative (with respect to τ) $\mathbf{d}'_O(C_0, C_{n+1})$ of the complex distance $\mathbf{d}_O(C_0, C_{n+1})$ between any two geodesics $C_0, C_{n+1} \in \mathcal{C}_\tau(R)$. We will compute an upper bound of $|\mathbf{d}'_O(C_0, C_{n+1})|$ in terms of $\mathbf{d}_O(C_0, C_{n+1})$. This will lead to an inductive type of argument that will finish the proof. We will offer more explanations as we go along.

2.4. *The Kerckhoff-Series-Wolpert type formula.* In [12], C. Series has derived the formula for the derivative of the complex translation length of a (not necessarily simple) closed curve on S^0 under the representation ρ_τ. Using the same method (word by word) one can obtain the appropriate formula for the derivative of the complex distance $\mathbf{d}_{O(\tau)}(C_0(\tau), C_{n+1}(\tau))$.

THEOREM 2.3. *Letting $'$ denote the derivative with respect to τ, we have*

$$\mathbf{d}'_O(C_0, C_{n+1}) = \sum_{i=0}^{n} \cosh(\mathbf{d}_{F_i}(O, D_i))\mathbf{d}'_{D_i}(C_i, C_{i+1}) + \sum_{i=1}^{n} \cosh(\mathbf{d}_{N_i}(O, C_i))\mathbf{d}'_{C_i}(D_{i-1}, D_i). \tag{1}$$

Proof. For each $i = 1, \dots, n$, consider the skew right-angled hexagon with sides $O, F_i, D_i, C_i, D_{i-1}, F_{i-1}$. Since each hexagon varies holomorphically in τ, we have the following derivative formula in each hexagon (this is the formula (7) in [12]):

$$\begin{aligned} \mathbf{d}'_O(F_{i-1}, F_i) = {} & \cosh(\mathbf{d}_{N_i}(O, C_i))\mathbf{d}'_{C_i}(D_{i-1}, D_i) \\ & + \cosh(\mathbf{d}_{F_{i-1}}(O, D_{i-1}))\mathbf{d}'_{D_{i-1}}(F_{i-1}, C_i) \\ & + \cosh(\mathbf{d}_{F_i}(O, D_i))\mathbf{d}'_{D_i}(C_i, F_i). \end{aligned} \tag{2}$$

The following relations (3), (4), and (5) are direct corollaries of the identities $F_0 = C_0$ and $F_n = C_{n+1}$. We have

$$\sum_{i=1}^{n} \mathbf{d}_O(F_{i-1}, F_i) = \mathbf{d}_O(C_0, C_{n+1}). \tag{3}$$

Also

$$\mathbf{d}'_{D_0}(F_0, C_1) = \mathbf{d}'_{D_0}(C_0, C_1) \tag{4}$$

and

$$\mathbf{d}'_{D_n}(C_n, F_n) = \mathbf{d}'_{D_n}(C_n, C_{n+1}). \tag{5}$$

Also for $1 = 1, \dots, n$, we observe the identity

$$\mathbf{d}_{D_i}(C_i, F_i) + \mathbf{d}_{D_i}(F_i, C_{i+1}) = \mathbf{d}_{D_i}(C_i, C_{i+1}).$$

Putting all this together and summing up formula (2), for $i = 1, \dots, n$ we obtain (1). $\square$

Let H be a consistently oriented skew right-angled hexagon with sides L_k, $k \in \mathbb{Z}$, and $L_k = L_{k+6}$. Set $\sigma(k) = \mathbf{d}_{L_k}(L_{k-1}, L_{k+1})$. Recall the cosine formula

$$\cosh(\sigma(k)) = \frac{\cosh(\sigma(k+3)) - \cosh(\sigma(k+1))\cosh(\sigma(k-1))}{\sinh(\sigma(k+1))\sinh(\sigma(k-1))}.$$

Assume that $\sigma(2j+1) = \frac{1}{2}(R + a_{2j+1}) + i\pi$, $j = 0, 1, 2$, and $a_{2j+1} \in \mathbb{D}$. A hexagon with this property is called a *thin hexagon*. From the cosine formula for a skew right-angled hexagon, we have (see also Lemma 5.1 in [1])

$$\sigma(2j) = 2e^{\frac{1}{4}[-R + a_{2j+3} - a_{2j+1} - a_{2j-1}]} + i\pi + O(e^{-\frac{3R}{4}}). \tag{6}$$

From the pentagon formula, the hyperbolic distance between opposite sides in the hexagon can be estimated as (see Lemma 5.4 in [1] and Lemma 2.1 in [12])

$$\frac{R}{4} - 10 < d(L_k, L_{k+3}) < \frac{R}{4} + 10 \tag{7}$$

for R large enough.

LEMMA 2.1. *Suppose that* $|\mathbf{d}_O(C_0, C_{n+1})| < \frac{R}{5}$. *Then for R large enough, the following estimate holds*:

$$|\mathbf{d}'_O(C_0, C_{n+1})| \le 20e^{-\frac{R}{4}} \sum_{i=0}^{n} e^{d(O,D_i)} + \frac{n}{R}\left(\max_{1 \le i \le n} e^{d(O,C_i)}\right).$$

Proof. Let γ be the geodesic segment on $O(0)$ that runs between $C_j(0)$ and $C_{j+1}(0)$. Then γ is a lift of a geodesic arc connecting two cuffs in the pair of pants whose all three cuffs have length R. Since the length of γ is at most $\frac{R}{5}$, we have from (7) that γ connects two different cuffs in this pair of pants and is freely homotopic to the shortest orthogonal arc between these two cuffs in this pair of pants. This implies that there exists $\widetilde{C} \in \mathcal{C}(R)$ such that the hexagon determined by C_j, C_{j+1} and $\widetilde{C}$ is a thin hexagon. Then D_i is a side of this hexagon since it is the common orthogonal for C_i and C_{i+1}. Taking into account that the orientation of D_i that comes from this hexagon is opposite to the one we defined above in terms of O and applying (6), we obtain

$$\mathbf{d}_{D_j}(C_j, C_{j+1}) = 2e^{\frac{1}{4}[-R + \widetilde{\zeta}\tau - \zeta_j\tau - \zeta_{j+1}\tau]} + O(e^{-\frac{3R}{4}}), \tag{8}$$

where $\zeta_j, \zeta_{j+1}, \widetilde{\zeta} \in \mathbb{D}$ are the complex numbers associated to the corresponding $C \in \mathcal{C}^0$ in Definition 2.2. Differentiating the cosine formula for the skew right-angled hexagon, we get

$$|\mathbf{d}'_{D_j}(C_j, C_{j+1})| < 20e^{-\frac{R}{4}}$$

for R large enough. (Here we use $|\zeta_C|, |\tau| < 1$.)

On the other hand, $\mathbf{d}_{C_j}(D_{j-1}, D_j) = 1 + \frac{\tau\eta_{j-1}}{R}$, where $\eta_{j-1} \in \mathbb{D}$ is the corresponding number. Differentiating this identity gives $|\mathbf{d}'_{C_j}(D_{j-1}, D_j)| \leq \frac{1}{R}$. (We use $|\eta_{j-1}| < 1$.) Combining these estimates with the equality of Theorem 2.3 proves the lemma. □

2.5. *Preliminary estimates.* The purpose of the next two subsections is to estimate the two terms on the right-hand side of the inequality of Lemma 2.1 in terms of the complex distance $\mathbf{d}_O(C_0, C_{n+1})$. We will show that

$$|\mathbf{d}'_O(C_0, C_{n+1})| \leq CF(\mathbf{d}_O(C_0, C_{n+1})),$$

where C is a constant and F is the function $F(x) = xe^x$. We will obtain this estimate under some natural assumptions (see Assumption 2.1 below).

Let α, β be two oriented geodesics in $\mathbb{H}^3$ such that $d(\alpha, \beta) > 0$, and let O be their common orthogonal (with either orientation). Let $q_0 = \beta \cap O$. Let $t \in \mathbb{R}$, and let $q : \mathbb{R} \to \beta$ be the parametrization by arc length such that $q(0) = q_0$. The following trigonometric formula follows directly from the cosh and sinh rules for right-angled triangles in the hyperbolic plane (the planar case of this formula was stated in Lemma 2.4.7 in [4]):

$$\sinh^2(d(q(t), \alpha)) = \sinh^2(d(\alpha, \beta))\cosh^2(t) + \sinh^2(t)\sin^2(\mathrm{Im}[\mathbf{d}_O(\alpha, \beta)]). \tag{9}$$

This yields the following inequality, which will suffice for us:

$$\sinh(d(\alpha, \beta))\cosh(t) \leq \sinh(d(q(t), \alpha)). \tag{10}$$

From this we derive

$$|t| \leq d(q(t), \alpha) - \log d(\alpha, \beta) \quad \text{for every } t \in \mathbb{R} \tag{11}$$

and

$$|t| \leq \log d(q(t), \alpha) + 1 - \log d(\alpha, \beta) \quad \text{when } d(q(t), \alpha) \leq 1. \tag{12}$$

Let $\gamma = \gamma(\tau)$, $\tau \in \mathbb{D}$, be an oriented geodesic in $\mathbb{H}^3$ that varies continuously in τ and such that $\gamma(0)$ belongs to the plane $\mathbb{H}^2 \subset \mathbb{H}^3$ that contains the lamination $\mathcal{C}_0(R)$. (The common orthogonal O from the previous subsection is an example of γ but there is no need to restrict ourselves to O in order to prove the estimates below.) Let $C_1(0), \dots, C_k(0)$ be an ordered subset of geodesics from $\mathcal{C}_0(R)$ that $\gamma(0)$ consecutively intersects. (This means that the segment of $\gamma(0)$ between $C_i(0)$ and $C_{i+1}(0)$ does not intersect any other geodesic from $\mathcal{C}_0(R)$.) Orient each C_i so that the angle from $\gamma(0)$ to $C_i(0)$ is positive. Let N_i be the common orthogonal between γ and C_i, and let $z_i = N_i \cap C_i$ and $z'_i = N_i \cap \gamma$ (see Figure 2). Let D_i, $i = 1, \dots, k$ be the common orthogonal between C_i and C_{i+1}, and let $w_i^- = D_i \cap C_i$ and $w_i^+ = D_i \cap C_{i+1}$. As long as the distance between z_i and z_{i+1} is at most $\frac{R}{5}$, then (as seen in the previous

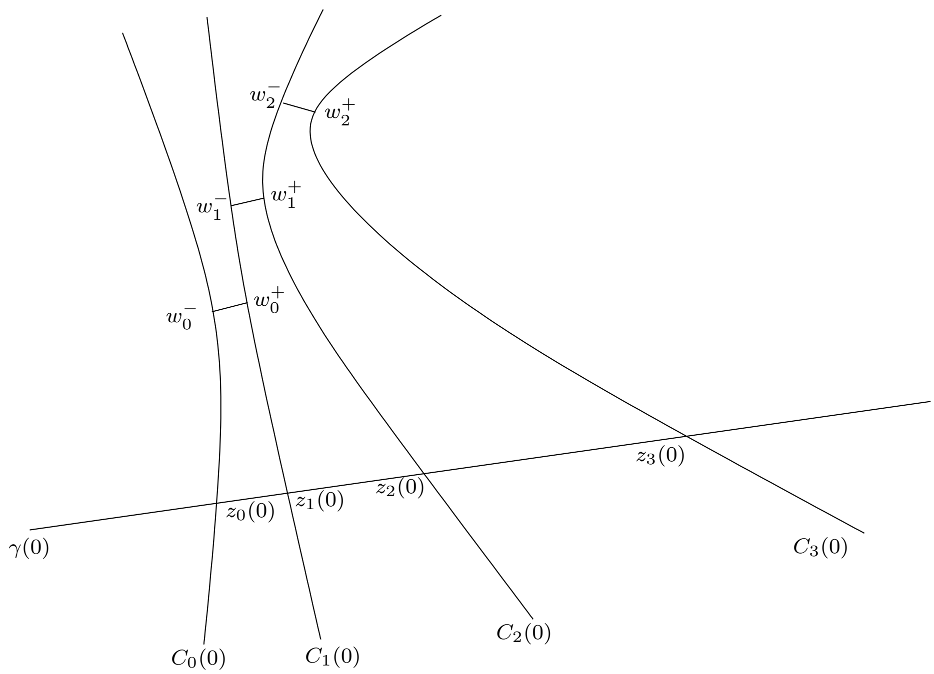

Figure 2. The z's and the w's

subsection), for R large enough, we have

$$\mathbf{d}_{D_i}(C_i, C_{i+1}) = (2 + o(1))e^{-\frac{R}{4}+\tau\mu} \leq e^{-\frac{R}{4}+2}, \tag{13}$$

where $\mu \in \mathbb{C}$ and $|\mu| \leq \frac{3}{4}$ (see (8)). Then it follows from the definition of $\mathcal{C}_\tau(R)$ that

$$\mathbf{d}_{C_i}(w_{i-1}^+, w_i^-) = 1 + \mathrm{Re}\Big[\frac{\tau\eta}{R} + j\frac{(R+\tau\zeta)}{2}\Big] \tag{14}$$

for some $j \in \mathbb{Z}$, where $\eta = \eta_C$ and $\zeta = \zeta_C$ are the complex numbers from the unit disc that correspond to the cuff in $C \in \mathcal{C}^0$ whose lift is $C_i(0)$. Here $\mathbf{d}_{C_i}(w_{i-1}^+, w_i^-)$ denotes the signed hyperbolic distance.

LEMMA 2.2. *Assume that* $d(z_i, z_{i+1}) < e^{-5}$ *for* $i = 1, \dots, k-1$. *Set* $a_i = \mathbf{d}_{C_i}(z_i, w_i^-)$. *Then for* R *large enough, the following inequalities hold*:

(1) $a_{i+1} - a_i < 1 + e^{-1}$, $i = 1, \dots, k-2$;
(2) $k < R$.

Proof. Since the distance between each pair z_i and z_{i+1} is at most e^{-5}, applying (12) and (13) to all pairs $\alpha = C_i$ and $\beta = C_{i+1}$ yields the inequality

$$d(z_i, w^+_{i-1}), d(z_i, w^-_i) \leq \frac{R}{4} - 2 \tag{15}$$

for each $i = 1, \dots, k-1$. By the triangle inequality, we have

$$\mathbf{d}_{C_i}(w^+_{i-1}, w^-_i) \leq \frac{R}{2} - 4. \tag{16}$$

On the other hand, from (14) we obtain

$$|j|\left(1 - \frac{|\tau\zeta|}{R}\right) \leq \frac{2}{R}\left(\mathbf{d}_{C_i}(w^+_{i-1}, w^-_i) + 1 + \frac{|\tau\eta|}{R}\right) \leq \frac{2}{R}\left(\frac{R}{2} - 4 + 2\right).$$

Since $|\tau|, |\zeta|, |\eta| < 1$ and from (16), we get

$$|j| \leq \frac{1 - \frac{4}{R}}{1 - \frac{1}{R}},$$

which shows that $j = 0$ in (14).

From (15) we have

$$|a_i| < \frac{R}{4}. \tag{17}$$

We write (using the triangle inequality)

$$a_{i+1} - a_i - 1 \leq d(w^-_i, w^+_i) + d(z_i, z_{i+1}) + |d(w^+_i, w^-_{i+1}) - 1|.$$

By (13) we have

$$d(w^-_i, w^+_i) \leq e^{-\frac{R}{4}+2}.$$

The assumption of the lemma is $d(z_i, z_{i+1}) \leq e^{-5}$. It follows from (14) (and the established fact that in this case $j = 0$) that

$$|d(w^+_i, w^-_{i+1}) - 1| \leq |\operatorname{Re}\left(\frac{\tau\eta}{R}\right)| \leq \frac{1}{R}.$$

Therefore

$$a_{i+1} - a_i - 1 < e^{-1},$$

which proves the first part of the lemma.

From (17) we have $-\frac{R}{4} < a_1$, which implies that $a_{k-1} > (k-1)(1-e^{-1}) - \frac{R}{4}$. Again from (17) we have $a_{k-1} < \frac{R}{4}$, which proves

$$k < \frac{R}{2(1-e^{-1})} + 1 < R. \qquad \square$$

The following lemma is a corollary of the previous one.

Lemma 2.3. *Let γ be a geodesic segment in $\mathbb{H}^2$ that is transverse to the lamination $\mathcal{C}_0(R)$. For R large enough, the number of geodesics from $\mathcal{C}_0(R)$ that γ intersects is at most $(2+R)e^5|\gamma|$.*

Proof. As above, denote by $C_i(0)$, $i = 1, \ldots, k$, the geodesics from $\mathcal{C}_0(R)$ that γ intersects. Using the above notation, let $j_1, \ldots, j_l \in \{0, .., k\}$, be such that $d(z_{j_i}(0), z_{j_i+1}(0)) > e^{-5}$. Then

$$l < \frac{|\gamma|}{e^{-5}} = e^5|\gamma|.$$

By definition, the open segment between $z_{j_i}(0)$ and $z_{j_i+1}(0)$ does not intersect any geodesics from $\mathcal{C}_0(R)$.

On the other hand, by the previous lemma the number of geodesics from $\mathcal{C}_0(R)$ that the subsegment of γ between $z_{j_i+1}(0)$ and $z_{j_{i+1}}(0)$ intersects is at most R (because the distance between any $z_i(0)$ and $z_{i+1}(0)$ in this range is at most e^{-5}). Since there are at most l such segments, we have that the total number of geodesics from $\mathcal{C}_0(R)$ that γ intersects is at most $2l + lR < (2 + R)e^5|\gamma|$. □

2.6. *Estimating the derivative* $|\mathbf{d}'_O(C_0, C_{n+1})|$. We now combine the notation of the previous two subsections (and set $\gamma = O$). In the following lemmas we prove estimates for the two terms on the right-hand side in the inequality of Lemma 2.1, which are independent of R.

We first estimate the second term in the inequality of Lemma 2.1.

Lemma 2.4. *We have*

$$\frac{n}{R}\left(\max_{1\le i\le n} e^{d(O,C_i)}\right) \le 1000 d(C_0(0), C_{n+1}(0))\left(\max_{1\le i\le n} e^{d(O,C_i)}\right),$$

where n is the number of geodesics that $O(0)$ intersects between $C_0(0)$ and $C_{n+1}(0)$.

Proof. From Lemma 2.3 we have

$$n \le (2 + R)e^5 d(C_0(0), C_{n+1}(0)) < 1000 R d(C_0(0), C_{n+1}(0)),$$

which proves the lemma. □

We now bound the first term in the inequality of Lemma 2.1 under the following assumption.

Assumption 2.1. *Assume that for some $\tau \in \mathbb{D}$, the following estimates hold for $i = 0, \ldots, n + 1$:*

$$d(z_i, z_i(0)),\ d(O, C_i) < \frac{1}{4}e^{-5}.$$

We have

Lemma 2.5. *Under Assumption* 2.1 *and for R large enough, we have*

$$20e^{-\frac{R}{4}} \sum_{i=0}^{n+1} e^{d(O,D_i)} \le 10^8 d(C_0(0), C_{n+1}(0)) e^{d(C_0(0), C_{n+1}(0))}.$$

Proof. Recall $z_i' = N_i \cap O$. (Note $z_0 = z_0'$ and $z_{n+1} = z_{n+1}'$ since O is the common orthogonal between C_0 and C_{n+1}.) Observe that

$$d(O, D_i) \le d(z_i', z_i) + d(z_i, w_i^-) = d(O, C_i) + |a_i| < 1 + |a_i|. \tag{18}$$

It follows from (11) that

$$|a_i| = d(z_i, w_i^-) \le d(z_i, C_{i+1}) - \log d(C_i, C_{i+1}).$$

We observe the estimate $d(z_i, C_{i+1}) \le d(z_i, z_{i+1})$. On the other hand, by (13) we have

$$\mathbf{d}_{D_i}(C_i, C_{i+1}) = (2 + o(1))e^{-\frac{R}{4}+\tau\mu},$$

so for R large enough (such that $|o(1)| < 1$) we find that (using the estimate $|\tau\mu| < 1$)

$$d(C_i, C_{i+1}) \ge e^{-\frac{R}{4}-1};$$

that is, $-\log d(C_i, C_{i+1}) \le \frac{R}{4} + 1$. It follows that

$$|a_i| \le d(z_i, z_{i+1}) + \frac{R}{4} + 1.$$

From

$$|d(z_i, z_{i+1}) - d(z_i(0), z_{i+1}(0))| \le d(z_i, z_i(0)) + d(z_{i+1}, z_{i+1}(0)) \le \frac{e^{-5}}{2} \tag{19}$$

and $d(z_i(0), z_{i+1}(0)) \le d(C_0(0), C_{n+1}(0))$, we obtain

$$|a_i| < \frac{R}{4} + d(C_0(0), C_{n+1}(0)) + 2. \tag{20}$$

Let $j_1, \dots, j_l \in \{1, .., n-1\}$, be such that $d(z_{j_i}, z_{j_i+1}) > e^{-5}$. (Note that $l = l(\tau)$ depends on τ.) Set $j_0 = 0$ and $j_{l+1} = n$. From (19), we have $d(z_{j_i}(0), z_{j_i+1}(0)) > \frac{e^{-5}}{2}$ for each $1 \le i \le l$. The intervals $(z_i(0), z_{i+1}(0))$ partition the arc between $z_0(0)$ and $z_{n+1}(0)$, so we get

$$l < \frac{d(C_0(0), C_{n+1}(0))}{\frac{e^{-5}}{2}} = 2e^5 d(C_0(0), C_{n+1}(0)). \tag{21}$$

Let $0 \le i \le l+1$. For $j_i + 1 \le t < j_{i+1}$, we have $d(z_t, z_{t+1}) \le e^{-5}$. It follows from Lemma 2.2 that

$$\frac{1}{2} < a_{t+1} - a_t.$$

We see that in this interval the sequence a_t is an increasing sequence. Combining this with (20) and (18), we obtain

$$\begin{aligned} \sum_{t=j_i+1}^{j_{i+1}} e^{d(O,D_t)} &\le 2e^{\frac{R}{4}+d(C_0(0),C_{n+1}(0))+3} \sum_{t=0}^{\infty} e^{-\frac{t}{2}} \\ &< 200e^{\frac{R}{4}+d(C_0(0),C_{n+1}(0))}. \end{aligned} \tag{22}$$

We have

$$\sum_{i=0}^{n+1} e^{d(O,D_i)} \leq (l+1) \max_{i=0,\dots,n+1} e^{d(O,D_i)} + \sum_{i=0}^{l+1} \sum_{t=j_i+1}^{j_{i+1}} e^{d(O,D_t)}.$$

By (18) and (20) we have

$$e^{d(O,D_i)} \leq e^{\frac{R}{4}+d(C_0(0),C_{n+1}(0))+2}.$$

Also, by (21) and (22) we have

$$\sum_{i=0}^{l+1} \sum_{t=j_i+1}^{j_{i+1}} e^{d(O,D_t)} \leq \left(2e^5 d(C_0(0),C_{n+1}(0))+1\right) \times 200 e^{\frac{R}{4}+d(C_0(0),C_{n+1}(0))}$$
$$< 10^6 d(C_0(0),C_{n+1}(0)) e^{\frac{R}{4}+d(C_0(0),C_{n+1}(0))}.$$

Combining all this gives

$$20e^{-\frac{R}{4}} \sum_{i=0}^{n+1} e^{d(O,D_i)} \leq 10^8 d(C_0(0),C_{n+1}(0)) e^{d(C_0(0),C_{n+1}(0))}.$$ □

The previous two lemmas together with Lemma 2.1 imply

LEMMA 2.6. *Under Assumption* 2.1 *and assuming that* $d(C_0,C_{n+1}) < \frac{R}{5}$, *for* R *large enough we have*

$$|\mathbf{d}'_O(C_0,C_{n+1})| < 10^9 d(C_0(0),C_{n+1}(0)) e^{d(C_0(0),C_{n+1}(0))}.$$

Proof. By Lemma 2.1 the estimate

$$|\mathbf{d}'_O(C_0,C_{n+1})| \leq 20e^{-\frac{R}{4}} \sum_{i=0}^{n} e^{d(O,D_i)} + \frac{n}{R}\left(\max_{1\leq i\leq n} e^{d(O,C_i)}\right)$$

holds for R large enough. (Recall that n is the number of geodesics that $O(0)$ intersects between $C_0(0)$ and $C_{n+1}(0)$.) By Lemma 2.4 we have

$$\frac{n}{R}\left(\max_{1\leq i\leq n} e^{d(O,C_i)}\right) \leq 1000 d(C_0(0),C_{n+1}(0)) \left(\max_{1\leq i\leq n} e^{d(O,C_i)}\right).$$

By Assumption 2.1 we have that

$$d(O,C_i) \leq \frac{1}{4}e^{-5}$$

for every $0 \leq i \leq n+1$, so we obtain

$$\frac{n}{R}\left(\max_{1\leq i\leq n} e^{d(O,C_i)}\right) \leq 3000 d(C_0(0),C_{n+1}(0)).$$

On the other hand, by Lemma 2.5 we have

$$20e^{-\frac{R}{4}} \sum_{i=0}^{n+1} e^{d(O,D_i)} \leq 10^8 d(C_0(0),C_{n+1}(0)) e^{d(C_0(0),C_{n+1}(0))}.$$

Putting the above estimates together proves the lemma. □

2.7. *The family of surfaces* $\mathbf{S}(R)$. We will consider geodesic laminations on a closed hyperbolic surface, and on its universal cover, the hyperbolic plane, which we will identify with the unit disk. By recording the endpoints of the leaves of a lamination of the unit disk, we can think of the lamination as a symmetric subset of $\partial\mathbb{D} \times \partial\mathbb{D}$, and by adding the diagonal, we obtain a closed subset of $\partial\mathbb{D} \times \partial\mathbb{D}$. The Hausdorff topology on such closed subsets will give us what we will call the Hausdorff topology on geodesic laminations of the unit disk.

Definition 2.3. Let $R > 1$, and let $P(R)$ be the pair of pants whose all three cuffs have the length R. We define the surface $\mathbf{S}(R)$ to be the genus two surface that is obtained by gluing two copies of $P(R)$ alongside the cuffs with the twist parameter equal to $+1$. (These are the Fenchel-Nielsen coordinates for $\mathbf{S}(R)$.) The surface $\mathbf{S}(R)$ can also be obtained by first doubling $P(R)$ and then applying the right earthquake of length 1, where the lamination that supports the earthquake is the union of the three cuffs of $P(R)$.

By $\mathrm{Orb}(R)$ we denote the quotient orbifold of the surface $\mathbf{S}(R)$ (the quotient of $\mathbf{S}(R)$ by the group of automorphisms of $\mathbf{S}(R)$). Observe that the Riemann surface $\mathbb{H}^2/\rho_0(\pi_1(S^0))$ is a regular finite degree cover of the orbifold $\mathrm{Orb}(R)$. In particular, there exists a Fuchsian group $G(R)$ such that $\mathrm{Orb}(R) = \mathbb{H}^2/G(R)$ and that $\rho_0(\pi_1(S^0)) < G(R)$ is a finite index subgroup. It is important to point out that the lamination $\mathcal{C}_0(R)$ is invariant under the group $G(R)$. In fact, one can define the group $G(R)$ as the group of all elements of $\mathbf{PSL}(2,\mathbb{R})$ that leave invariant the lamination $\mathcal{C}_0(R) \subset \mathbb{H}^2$. Observe that the group $G(R)$ acts transitively on the geodesics from $\mathcal{C}_0(R)$; that is, the $G(R)$-orbit of a geodesic from $\mathcal{C}_0(R)$ is equal to $\mathcal{C}_0(R)$.

Although the marked family of surfaces $S(R)$ (marked by its Fenchel-Nielsen coordinates defined above) tends to ∞ in the Teichmüller space of genus two surfaces, the unmarked family $S(R)$ stays in some compact set in the moduli space of genus two surfaces. We prove this fact below.

LEMMA 2.7. *For R large enough, the length of the shortest closed geodesic on the surface* $\mathbf{S}(R)$ *is at least* e^{-5}.

Proof. Suppose that the length of the shortest closed geodesic on $\mathbf{S}(R)$ is less than e^{-5}, and let γ be a lift of this geodesic to $\mathbb{H}^2$. (This geodesic is transverse to the lamination $\mathcal{C}_0(R)$ because otherwise $\gamma \in \mathcal{C}_0(R)$, which implies that the length of the shortest closed geodesic on $\mathbf{S}(R)$ is equal to R.) Then by Lemma 2.2 every subsegment of γ can intersect at most R geodesics from $\mathcal{C}_0(R)$, which means that γ intersects at most R geodesics from $\mathcal{C}_0(R)$. This is impossible since γ is a lift of a closed geodesic that is transverse to $\mathcal{C}_0(R)$

so it has to intersect infinitely many geodesics from $\mathcal{C}_0(R)$. This proves the lemma. □

The conclusion is that the family of (unmarked) Riemann surfaces $\mathbf{S}(R)$ stays in some compact set in the moduli space of genus two surfaces. One can describe the accumulation set of the family $\mathbf{S}(R)$ in the moduli space as follows. Let P be a pair of pants that is decomposed into two ideal triangles so that all three shears between these two ideal triangles are equal to 1. Then all three cuffs have the length equal to 2. Let $\mathbf{S}_t$, $t \in [0,1]$ be the genus two Riemann surface that is obtained by gluing one copy of P onto another copy of P (along the three cuffs) and twisting by $+2t$ along each cuff. The "circle" of surfaces $\mathbf{S}_t$ is the accumulation set of $\mathbf{S}(R)$ when $R \to \infty$. Note that the edges of the ideal triangles that appear in the pants P are the limits of the (R long) cuffs from the pairs of pants $P(R)$.

Then we have the induced circle of orbifolds Orb_t. Let G_t be a circle of Fuchsian groups such that $\mathrm{Orb}_t = \mathbb{H}^2/G_t$. By $\mathcal{C}_{0,t}$ we denote the lamination in $\mathbb{H}^2$ that is the lift of the corresponding ideal triangulation on $\mathbf{S}_t$. Then up to a conjugation by elements of $\mathbf{PSL}(2,\mathbb{R})$, the circle of groups G_t is the accumulation set of the groups $G(R)$ when $R \to \infty$, and the circle of laminations $\mathcal{C}_{0,t}$ is the accumulation set of the laminations $\mathcal{C}_0(R)$. We observe that the group G_t acts transitively on $\mathcal{C}_{0,t}$.

2.8. *Quasisymmetric maps and hyperbolic geometry.* In this subsection we state and prove a few preparatory statements about quasisymmetric maps and the complex distances between geodesics in $\mathbb{H}^3$, culminating in Theorem 2.5.

Definition 2.4. We say that a geodesic lamination λ on $\mathbb{H}^2$ is nonelementary if neither of the following holds:

(1) There exists $z \in \partial\mathbb{H}^2$ that is an endpoint of every leaf of λ.
(2) There exists a geodesic $O \subset \mathbb{H}^2$ that is orthogonal to every leaf of λ.

Of course, λ has at least three elements if λ is nonelementary. Moreover, if λ is nonelementary, then there is a sublamination $\lambda' \subset \lambda$ such that λ' contains exactly three geodesics and such that λ' is nonelementary.

Let λ be a geodesic lamination, all of whose leaves have disjoint closures. By $\partial\lambda$ we denote the union of the endpoints of leaves from λ. We let $\iota_\lambda : \partial\lambda \to \partial\lambda$ be the involution such that ι_λ exchanges the two points of $\partial\alpha$ for every leaf $\alpha \in \lambda$.

We say that a quasisymmetric map $g : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$ is K-quasisymmetric if for every 4 points on $\partial\mathbb{H}^2$ with cross ratio equal to -1, the cross ratio of the image four points is within $\log K$ hyperbolic distance of -1 for the hyperbolic metric on $\mathbb{C} \setminus \{0,1,\infty\}$. (Observe that a map is K-quasisymmetric if and only if it has a K'-quasiconformal extension to $\partial\mathbb{H}^3$ for some $K' > 1$.)

If α and β are oriented geodesics in $\mathbb{H}^3$ by $\mathbf{d}(\alpha, \beta)$, we denote their unsigned complex distance.

LEMMA 2.8. *Suppose that λ is nonelementary and $f : \partial\lambda \to \partial\mathbb{H}^3$ is such that*

$$\mathbf{d}(f(\alpha), f(\beta)) = \mathbf{d}(\alpha, \beta)$$

for all $\alpha, \beta \in \lambda$. Then there is a unique Möbius transformation T such that either

(1) *$T = f$ on $\partial\lambda$, or*
(2) *$T = f \circ \iota_\lambda$ on $\partial\lambda$.*

The second case can only occur when all the leaves of λ have disjoint closures. We will prove two special cases of Lemma 2.8 before we prove the lemma.

If the endpoints of α are x and y and α is oriented from x to y, then we write $\partial\alpha = (x, y)$. The following lemma is elementary.

LEMMA 2.9. *For every $d \in \mathbb{C}/2\pi i\mathbb{Z}/\mathbb{Z}_2$, with $d \neq 0$, there exists a unique $s \in \mathbb{C}/2\pi i\mathbb{Z}$ such that for two oriented geodesics α and β we have $\mathbf{d}(\alpha, \beta) = d$ if and only if $\partial\beta = (x, y)$ and $y = T_{s,\alpha}(x)$, where $T_{s,\alpha}$ is the translation by s along α.*

PROPOSITION 2.1. *Suppose that $\alpha_0, \alpha_1, \alpha_2$ are oriented geodesics in $\mathbb{H}^3$ for which $\mathbf{d}(\alpha_i, \alpha_j) \neq 0$ for $i \neq j$, and suppose that $\alpha_0, \alpha_1, \alpha_2$ do not have a common orthogonal. Suppose $\alpha'_0, \alpha'_1, \alpha'_2$ are such that $\mathbf{d}(\alpha_i, \alpha_j) = \mathbf{d}(\alpha'_i, \alpha'_j)$. Then we can find a unique $T \in \mathbf{PSL}(2, \mathbb{C})$ that satisfies one of the two conditions*

(1) *$T(\alpha_i) = \alpha'_i$, $i = 0, 1, 2$;*
(2) *$T(\alpha_i) = -\alpha'_i$, $i = 0, 1, 2$, where $-\alpha'_i$ is α'_i with the orientation reversed.*

Proof. Given α_i and α'_i satisfying the hypotheses of the proposition, we can assume that $\alpha_i = \alpha'_i$ for $i = 0, 1$. Let $d_i = \mathbf{d}(\alpha_i, \alpha_2)$, and let $T_i = T_{d_i,\alpha_i}$ as in Lemma 2.9. Then by Lemma 2.9, for any β for which $\mathbf{d}(\alpha_i, \beta) = d_i$, we have $T_i(x) = y$ where $\partial\beta = (x, y)$. Thus $(T_1^{-1} \circ T_0)(x) = x$. Since $T_1 \neq T_0$ (because $\alpha_0 \neq \alpha_1$), we see that the equation $\mathbf{d}(\alpha_i, \beta) = d_i$ (in β) has at most as many solutions as the equation $(T_1^{-1} \circ T_0)(x) = x$, $x \in \partial\mathbb{H}^2$. Therefore $\mathbf{d}(\alpha_i, \beta) = d_i$ has at most two solutions, and it has at most one solution if $T_1^{-1} \circ T_0$ has a unique fixed point on $\partial\mathbb{H}^2$.

On the other hand, we let Q be the Möbius transformation such that $Q(\alpha_i) = -\alpha_i$ for $i = 0, 1$. (Such Q exists since $\mathbf{d}(\alpha_i, \alpha_j) \neq 0$ for $i \neq j$.) Let $\widehat{\alpha}_2 = -Q(\alpha_2)$. Then $\mathbf{d}(\alpha_i, \widehat{\alpha}_2) = \mathbf{d}(\alpha_i, \alpha_2)$ for $i = 0, 1$. Therefore $\widehat{\alpha}_2 \neq \alpha_2$ since α_0, α_1 and α_2 do not have a common orthogonal. We conclude that $\alpha'_2 = \alpha_2$ or $\alpha'_2 = \widehat{\alpha}_2$. □

Proposition 2.2. *Suppose that distinct geodesics α_0 and α_1 in $\mathbb{H}^2$ have a common endpoint $x \in \partial\mathbb{H}^2$, and let β be another geodesic in $\mathbb{H}^2$ such that x is not an endpoint of β. Set $E = \partial\alpha_0 \cup \partial\alpha_1 \cup \partial\beta$. Let $f : E \to \partial\mathbb{H}^3$ be such that $\mathbf{d}(f(\alpha_i), f(\beta)) = \mathbf{d}(\alpha_i, \beta)$, $i = 0, 1$. Then there exists a unique Möbius transformation T such that $f = T$ on E.*

Proof. We can assume that the restriction of f to $\partial\alpha_0 \cup \partial\alpha_1$ is the identity. If $\partial\beta \subset \partial\alpha_0 \cup \partial\alpha_1$, then $|E| = 3$ and we are finished. If $\partial\beta \cap \partial\alpha_0 = \{y\}$ for some y, then we can write $\partial\beta = (y, z)$ (or (z, y)), and then $\partial f(\beta) = (y, z')$ (or respectively (z', y)). But then $z = z'$ because $\mathbf{d}(f(\alpha_1), f(\beta)) = \mathbf{d}(\alpha_1, \beta)$ (here we use Lemma 2.9); likewise if $\partial\beta \cap \partial\alpha_1 \neq \emptyset$.

If $\partial\beta \cap (\partial\alpha_0 \cup \partial\alpha_1) = \emptyset$, then by Lemma 2.9, $\partial\beta = (y, z)$ and $\partial f(\beta) = (y', z')$ and $z = T_0(y) = T_1(y)$, $z' = T_0(y') = T_1(y')$, where T_i translates along α_i, and then $T_0^{-1} \circ T_1$ has x as one of its fixed points, so the other must be y, so $y' = y$, so $f(\beta) = \beta$. $\square$

Now we are ready to prove Lemma 2.8.

Proof. First suppose that λ has two distinct leaves α, β with a common endpoint x. Then there is a unique $T \in \mathbf{PSL}(2, \mathbb{C})$ for which $T = f$ on $\partial\alpha \cup \partial\beta$. By Proposition 2.2 we have $T(\gamma) = f(\gamma)$, whenever $\gamma \in \lambda$ and x does not belong to $\partial\gamma$. Because λ is nonelementary, we can find at least one such γ.

Now suppose $\delta \in \lambda$ and $x \in \partial\delta$. We want to show $T(\delta) = f(\delta)$. We can find $T' \in \mathbf{PSL}(2, \mathbb{C})$ such that $T' = f$ on $\partial\alpha \cup \partial\delta$. By Proposition 2.2, $T'(\gamma) = f(\gamma)$, so T and T' agree on $\partial\alpha \cup \partial\delta$, so $T = T'$, so $f(\delta) = T(\delta)$, and we are done.

Now suppose that any two distinct leaves of λ have disjoint closures. Then we can find three leaves α_i, $i = 0, 1, 2$, with no common orthogonal (because λ is nonelementary). By Proposition 2.1 we can find a unique $T \in \mathbf{PSL}(2, \mathbb{C})$ such that $T = f$ on $E = \bigcup_{i=0}^{2} \partial\alpha_i$, or $T = f \circ \iota_\lambda$ on E. In the latter case we can replace f with $f \circ \iota_\lambda$. In either case we can assume that T is not the identity.

Now given any $\beta \in \lambda$, we want to show that $f(\beta) = \beta$. For $i = 1, 2$, let Q_i be the 180 degree rotation around O_i, the common orthogonal to α_0 and α_i. If $f(\beta) \neq \beta$, then $f(\beta) = -Q_i(\beta)$ for $i = 1, 2$, and so $Q_0^{-1} \circ Q_1$ fixes the endpoints of β. But $Q_0^{-1} \circ Q_1$ fixes the endpoints of α_0, and $\beta \neq \alpha_0$, so this is impossible. So $f(\beta) = \beta$ for every β, and we are finished. $\square$

We observe that Lemma 2.8 holds even if we do not require the lamination to be closed.

Definition 2.5. Let λ be a geodesic lamination on $\mathbb{H}^2$. An effective radius for λ is a number $M > 0$ such that every open hyperbolic disc of radius M in $\mathbb{H}^2$ intersects λ in a (not necessarily closed) nonelementary sublamination.

We observe that the condition that the intersection of λ and the open disc centred at z of radius M is nonelementary is open in both z and λ. The following proposition follows easily from this observation.

PROPOSITION 2.3. *Let Λ be a family of geodesic laminations on $\mathbb{H}^2$ such that*

(1) *if $\lambda \in \Lambda$ and $g \in \mathbf{PSL}(2,\mathbb{R})$, then $g(\lambda) \in \Lambda$;*
(2) *Λ is closed (and hence compact) in the Hausdorff topology on the space of geodesic laminations modulo $\mathbf{PSL}(2,\mathbb{R})$;*
(3) *if $\lambda \in \Lambda$, then λ is nonelementary.*

Then we can find $M > 0$ such that M is an effective radius for every $\lambda \in \Lambda$.

We call such a family a closed invariant family of nonelementary laminations. For any $R_1 > 0$, we let $\Lambda(R_1)$ be the closure of $\bigcup_{R \geq R_1} \mathcal{C}_0(R)$ under properties 1 and 2 in Proposition 2.3. We observe that taking the Hausdorff closure just adds the translates of all the $\mathcal{C}_{0,t}$ under $\mathbf{PSL}(2,\mathbb{R})$, where $\mathcal{C}_{0,t}$ was defined in the previous subsection. Hence $\Lambda(R_1)$ is a closed invariant family of nonelementary laminations.

We say that a lamination λ is *unflippable* if it has two distinct leaves with a common endpoint or if the involution ι_λ is not continuous. The latter occurs if and only if there is a point of $\partial\lambda$ that is the limit of a sequence leaves of λ whose diameter go to zero (or λ has two distinct leaves with a common endpoint). This will always occur when λ is invariant by a nonelementary Fuchsian group G and λ has a recurrent (or closed) leaf in $\mathbb{H}^2/G$. In particular, a nonempty lamination λ that is invariant under a cocompact group is unflippable (and nonelementary). We conclude that all of the laminations in $\Lambda(R_1)$ are unflippable.

We can now prove that a quasisymmetric map that locally preserves complex distances on an unflippable lamination is Möbius.

PROPOSITION 2.4. *Suppose that λ is an unflippable nonelementary lamination. Suppose that M is an effective radius for λ and $f : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$ is a continuous embedding such that $\mathbf{d}(f(\alpha), f(\beta)) = \mathbf{d}(\alpha, \beta)$, for all $\alpha, \beta \in \lambda$, such that $d(\alpha, \beta) \leq 3M$. Then f is the restriction of a Möbius transformation.*

Proof. For $z \in \mathbb{H}^2$, let D_z be the open disc of radius M centred at z, and let λ_z be the leaves of λ that meet D_z. Because M is an effective radius, λ_z is nonelementary. Therefore there is a unique $T_z \in \mathbf{PSL}(2,\mathbb{C})$ such that either $T_z = f$ on $\partial\lambda_z$ or $T_z = f \circ \iota_\lambda$ on $\partial\lambda_z$. Now if $d(z, z') \leq M$, then $\mathbf{d}(f(\alpha), f(\beta)) = \mathbf{d}(\alpha, \beta)$ for all $\alpha, \beta \in \lambda_z \cup \lambda_{z'}$, and $\lambda_z \cup \lambda_{z'}$ is nonelementary, so $T_z = T_{z'}$. We conclude that there is one $T \in \mathbf{PSL}(2,\mathbb{C})$ such that $T = f$ or

$T = f \circ \iota_\lambda$ on all of $\partial\lambda$. But in the latter case, ι_λ would be continuous, which is impossible since λ is unflippable. □

We now characterize the sequences of K-quasiconformal maps whose dilatations do not go to 1.

LEMMA 2.10. *Let $K_1 > K > 1$. Suppose that for $m \in \mathbb{N}$, $f_m : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$ is K_1-quasisymmetric but not K-quasisymmetric. Then, after passing to a subsequence if necessary, we have that there exist $h_m, q_m \in \mathbf{PSL}(2,\mathbb{C})$ such that $q_m \circ f_m \circ h_m \to f_\infty : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$ is a K_1-quasisymmetric map and f_∞ is not a restriction of a Möbius transformation on $\partial\mathbb{H}^2$.*

Proof. Fix $a, b, c, d \in \partial\mathbb{H}^2$ such that the cross ratio of these four points is equal to 1. Since f_m is not K-quasisymmetric, there exist points $a_m, b_m, c_m, d_m \in \partial\mathbb{H}^2$ whose cross ratio is equal to one and such that the cross ratio of the points $f_m(a_m), f_m(b_m), f_m(c_m), f_m(d_m) \in \partial\mathbb{H}^3$ stays outside some closed disc U centred at the point $1 \in \mathbb{C}$ for every m. We let h_m be the Möbius transformation that maps a, b, c, d to a_m, b_m, c_m, d_m. We then choose $q_m \in \mathbf{PSL}(2,\mathbb{C})$ such that $q_m \circ f_m \circ h_m$ fixes the points a, b, c. Then for each m, the map $q_m \circ f_m \circ h_m$ is K_1-quasisymmetric and it fixes the points a, b, c.

The standard normal family argument states that given $L > 1$, a sequence of L-quasisymmetric maps that all fix the same three distinct points, converges uniformly to a L-quasisymmetric map (after passing onto a subsequence if necessary). Therefore, we have $q_m \circ f_m \circ h_m \to f_\infty$. Moreover the cross ratio of the points $f_\infty(a), f_\infty(b), f_\infty(c), f_\infty(d)$ lies outside the disc U, and so we conclude that f_∞ is not a Möbius transformation on $\partial\mathbb{H}^2$. □

We can now conclude the constant of quasisymmetry for f is close to 1 when f changes the complex distance of neighbouring geodesics a sufficiently small amount.

THEOREM 2.4. *Let Λ be a closed invariant family of unflippable nonelementary laminations, and let $K_1 \geq K > 1$. Then there exist $\delta = \delta(K_1, K, \Lambda) > 0$ and $T = T(\Lambda)$ such that the following holds. If $\lambda \in \Lambda$ and $f : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$ is a K_1-quasisymmetric map, and*

$$|\mathbf{d}(f(\alpha), f(\beta)) - \mathbf{d}(\alpha, \beta)| \leq \delta$$

for all $\alpha, \beta \in \lambda$ such that $d(\alpha, \beta) \leq T$, then f is K-quasisymmetric.

Proof. By Proposition 2.3, we can find $M = M(\Lambda) > 0$ such that M is an effective radius for every $\lambda \in \Lambda$. We let $T = 3M$. Suppose that there is no good δ. Then we can find $\lambda_m \in \Lambda$, f_m (for $m \in \mathbb{N}$) such that

$$|\mathbf{d}(f(\alpha), f(\beta)) - \mathbf{d}(\alpha, \beta)| \to 0, m \to \infty,$$

uniformly for all $\alpha, \beta \in \lambda_m$ for which $d(\alpha, \beta) \leq T$, but for which f is not K-quasisymmetric. Passing to a subsequence and applying Lemma 2.10, we obtain $\lambda_m \to \lambda_\infty \in \Lambda$ and $f_m \to f_\infty : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$ such that f_∞ is a K_1-quasisymmetric map that is not a Möbius transformation on $\partial\mathbb{H}^2$. Moreover $\mathbf{d}(f_\infty(\alpha), f_\infty(\beta)) = \mathbf{d}(\alpha, \beta)$ for all $\alpha, \beta \in \lambda_\infty$ with $d(\alpha, \beta) \leq T = 3M$. Then by Proposition 2.4, f_∞ is a Möbius transformation, a contradiction. □

We can now derive a corollary, which is our object for this section.

THEOREM 2.5. *Let $K_1 \geq K > 1$, and let $R_1 = 10$. There exists $\delta_1 = \delta_1(K, K_1) > 0$ and a universal constant T_1 such that the following holds. Suppose that $R \geq R_1$, $f : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$ is a K_1-quasisymmetric map, and*

$$|\mathbf{d}(f(\alpha), f(\beta)) - \mathbf{d}(\alpha, \beta)| \leq \delta_1$$

for all $\alpha, \beta \in \mathcal{C}_0(R)$ such that $d(\alpha, \beta) \leq T_1$. Then f is K-quasisymmetric.

This follows immediately from Theorem 2.4, because $\Lambda(R_1)$ is a closed invariant family of unflippable noninvariant laminations. Observe that $T_1 = 3M_1$ is a universal constant, where M_1 is the effective radius of every lamination in $\Lambda(R_1)$.

2.9. *Proof of Theorem* 2.2. In this section we will verify that Assumption 2.1 holds when the quasisymmetry constant for f_τ is close to 1. This will permit us, thanks to Lemma 2.6, to verify the hypotheses of Theorem 2.5 and thereby improve the quasisymmetry constant for f_τ. We thus obtain an inductive argument for Theorem 2.2.

This lemma is an abstraction of its corollary, Corollary 2.1, where A, B, C will be $C_0(0), C_i(0), C_{n+1}(0)$.

LEMMA 2.11. *For all $\delta_2, T_1 > 0$ we can find $K > 1$ such that if*

(1) *A,B,C are oriented geodesics in $\mathbb{H}^2$, $d(A, C) > 0$, and B separates A and C;*
(2) *$d(A, C) \leq T_1$;*
(3) *O is the common orthogonal for A and C;*
(4) *$x = A \cap O$, $y = B \cap O$;*
(5) *$f : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$ is K-quasisymmetric;*
(6) *$\partial A' = f(\partial A)$, $\partial B' = f(\partial B)$, and $\partial C' = f(\partial C)$ (taking into account the order of the endpoints);*
(7) *O' is the common orthogonal to A' and C', and $x' = A' \cap O'$;*
(8) *N is the common orthogonal to O' and B', and $y' = N \cap O'$;*

then $d(O', B') \leq \delta_2$ and $|\mathbf{d}_{O'}(x', y') - \mathbf{d}(x, y)| \leq \delta_2$.

Proof. First suppose that $d(A, C)$ is small, say $d(A, C) \leq T_2$ for some $T_2 > 0$, and f is 2-quasisymmetric. Then by applying a Möbius transformation to the range and domain of f, we can assume that $\partial A = \partial A' = \{0, \infty\}$ and $1 \in \partial O$, $1 \in \partial O'$ (and hence $\partial O = \partial O' = \{-1, 1\}$). Note that while $f(0) = 0$ and $f(\infty) = \infty$, $f(1)$ is not necessarily equal to 1. It follows that $\partial C = \{c, \frac{1}{c}\}$, for c real and small (we can assume $c > 0$), and $\partial C' = \{c', \frac{1}{c'}\}$, where c' is small and $c' = f(c)$, $\frac{1}{c'} = f(\frac{1}{c})$.

We let $\partial B = \{b_0, b_1\}$, where $b_0, \frac{1}{b_1} \in (0, c)$. Then $|f(b_0)| < 10|c'|$ and $|f(b_1)| > \frac{1}{10}|\frac{1}{c'}|$ because f is 2-quasisymmetric and f fixes $0, \infty$. Therefore, by choosing T_2 to be small enough we can arrange that $d(O', B')$, $d(x, y)$ and $d(x', y')$ are as small as we want, so we conclude that for every $\delta_2 > 0$, there exists $T_2 > 0$ such that if $d(A, C) \leq T_2$ and f is 2-quasisymmetric, then

$$d(O', B'), |d(x, y) - d(x', y')| < \delta_2.$$

So we need only show that for every δ_2 and T_1, there exists $K > 1$ such that if $d(A, C) \in [T_2, T_1]$, where $T_2 = T_2(\delta_2)$, and all other hypotheses hold, then

$$d(O', B'), |d(x, y) - d(x', y')| < \delta_2. \tag{23}$$

Suppose that this statement is false. Then we can find a sequence of A_n, B_n, C_n, and f_n for which f_n is K_n-quasisymmetric, $K_n \to 1$, but for which (23) does not hold. Then normalizing and passing to a subsequence we obtain A, B, C in the limit, and $f_n \to \mathrm{id}$. So $A'_n \to A' = A$, $B'_n \to B' = B$, and $C'_n \to C' = C$. Moreover, because the common orthogonal to two geodesics varies continuously when the complex distance is nonzero, $O_n \to O$ and $O'_n \to O'$, so $d(O', B'_n) \to 0$ and $\mathbf{d}(B'_n, O) \neq 0$. Also $N'_n \to N$, $(x_n, y_n, x'_n, y'_n) \to (x, y, x', y')$, and $x' = x$, $y' = y$, so

$$|d(x'_n, y'_n) - d(x_n, y_n)| \to 0.$$

We conclude that (23) holds for large enough n, a contradiction. □

Assume that for some $\tau \in \mathbb{D}$, the representation $\rho_\tau : \pi_1(S^0) \to \mathbf{PSL}(2, \mathbb{C})$ is quasifuchsian, and let $f_\tau : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$ be the normalised equivariant quasisymmetric map (that conjugates $\rho_0(\pi_1(S^0))$ to $\rho_\tau(\pi_1(S^0))$).

Here we show that Assumption 2.1 holds if f_τ is sufficiently close to being conformal.

Corollary 2.1. *Given T_1 we can find $K_1 > 1$ such that if f_τ is K_1-quasisymmetric, then the following holds. Let $C_0(0), C_{n+1}(0)$ be geodesics in $\mathcal{C}_0(R)$ such that $d(C_0(0), C_{n+1}(0)) \leq T_1$, and let $C_i(0) \in \mathcal{C}_0(R)$, $i = 1, \dots, n$, denote the intermediate geodesics. Also, $O(0), O(\tau)$, $z_i(0), z_i$ and $C_i(\tau)$ are defined as usual. Then*

$$|d(z_i, z_{i+1}) - d(z_i(0), z_{i+1}(0))| < \frac{e^{-5}}{4}$$

and $d(O, C_i) \leq \frac{e^{-5}}{4}$.

Proof. We apply the previous lemma with $\delta_2 = \frac{e^{-5}}{16}$. Then $d(O, C_i) < \frac{e^{-5}}{16}$. Furthermore,

$$|d(z'_0, z'_i) - d(z_0(0), z_i(0))| < \frac{e^{-5}}{16}$$

and

$$|d(z'_0, z'_{i+1}) - d(z_0(0), z_{i+1}(0))| < \frac{e^{-5}}{16},$$

so

$$|d(z'_i, z'_{i+1}) - d(z_i(0), z_{i+1}(0))| < \frac{e^{-5}}{8}.$$

Moreover

$$d(z_i, z_{i+1}) \leq d(z'_i, z'_{i+1}) + d(O, C_i) + d(O, C_{i+1}),$$

and therefore

$$|d(z_i, z_{i+1}) - d(z_i(0), z_{i+1}(0))| < \frac{e^{-5}}{4}. \qquad \square$$

We are now ready to complete the proof of Theorem 2.2. Let $R > R_1 = 10$. Since the space of quasifuchsian representations of the group $\pi_1(S^0)$ is open (in the space of all representations), there exists $0 < \varepsilon_1 < 1$ so that the disc $\mathbb{D}(0, \varepsilon_1)$ (of radius ε_1 and centred at 0) is the maximal disc such that f_τ is K_1-quasisymmetric on all of $\mathbb{D}(0, \varepsilon_1)$, where K_1 is the constant from Corollary 2.1. We can choose such ε_1 to be positive because the map f_0 is 1-quasisymmetric and given any $K > 1$ we can find an open neighbourhood of 0 in the τ plane such that in that neighbourhood we have that every f_τ is K-quasisymmetric.

By that corollary, Assumption 2.1 holds for f_τ for all $\tau \in \mathbb{D}(0, \varepsilon_1)$. Let $C_0(0), C_{n+1}(0) \in \mathcal{C}_0(R)$ be such that $d(C_0(0), C_0(n+1)) \leq T_1$, where T_1 is the constant from Theorem 2.5. From Lemma 2.6, for R large enough and for every $\tau \in \mathbb{D}(0, \varepsilon_1)$, we have

$$|\mathbf{d}'_O(C_0, C_{n+1})| \leq 10^9 T_1 e^{T_1}.$$

This yields

$$|\mathbf{d}_O(C_0, C_{n+1}) - \mathbf{d}_{O(0)}(C_0(0), C_{n+1}(0))| \leq 10^9 \varepsilon_1 T_1 e^{T_1} \tag{24}$$

for every $\tau \in \mathbb{D}(0, \varepsilon_1)$.

Let $0 < \delta_1 = \delta_1(\sqrt{K_1}, K_1)$ be the corresponding constant from Theorem 2.5. We show

$$\varepsilon_1 \geq \frac{\delta_1}{10^9 T_1 e^{T_1}}.$$

Assume that this is not the case. Then from (24) we have that for every $\tau \in \mathbb{D}(0, \varepsilon_1)$, the map f_τ is $\sqrt{K_1}$-quasisymmetric (and hence for $\tau \in \overline{\mathbb{D}(0, \varepsilon_1)}$). This implies that f_τ is K_1-quasisymmetric for every $\tau \in \mathbb{D}(0, \varepsilon)$ for some $\varepsilon > \varepsilon_1$. But this contradicts the assumption that $\mathbb{D}(0, \varepsilon_1)$ is the maximal disc so that every f_τ is K_1-quasisymmetric.

Set

$$\widehat{\varepsilon} = \frac{\delta_1}{10^9 T_1 e^{T_1}}.$$

Then for every $\tau \in \mathbb{D}(0, \widehat{\varepsilon})$ and for R large enough, the map f_τ is K_1-quasisymmetric.

We prove the other estimate in Theorem 2.2 as follows. First of all, by the Slodkowski extension theorem (for the statement and proof of this theorem see [6]), we can extend the maps f_τ to quasiconformal maps of the sphere $\partial\mathbb{H}^3$ such that the Beltrami dilatation

$$\mu_\tau(z) = \frac{\overline{\partial} f_\tau}{\partial f_\tau}(z)$$

varies holomorphically in τ for every fixed $z \in \partial\mathbb{H}^3$. Observe that $\mu_0(z) = 0$, and $|\mu_\tau(z)| < 1$ for every τ and z. (Recall that the absolute value of the Beltrami dilatation of any quasiconformal map is less than 1.) For a fixed z, we then apply the Schwartz lemma to the function $\mu_\tau(z)$, and this yields the desired estimate from Theorem 2.2.

3. Surface group representations in $\pi_1(\mathbf{M}^3)$

3.1. *Labelled collection of oriented skew pants.* From now on $\mathbf{M}^3 = \mathbb{H}^3/\mathcal{G}$ is a fixed closed hyperbolic three manifold and $\mathcal{G}$ a suitable Kleinian group. By Γ^* and Γ we denote respectively the collection of oriented and unoriented closed geodesics in $\mathbf{M}^3$. By $-\gamma^*$ we denote the opposite orientation of an oriented geodesic $\gamma^* \in \Gamma^*$.

Let Π^0 be a topological pair of pants. Recall (from the beginning of Section 2) that a pair of pants in a closed hyperbolic three manifold $\mathbf{M}^3$ is an injective homomorphism $\rho : \pi_1(\Pi^0) \to \pi_1(\mathbf{M}^3)$, up to conjugacy. A pair of pants in $\mathbf{M}^3$ is determined by (and determines) a continuous map $f : \Pi^0 \to \mathbf{M}^3$, up to homotopy. Moreover, the representation ρ induces a representation

$$\rho : \pi_1(\Pi^0) \to \mathbf{PSL}(2, \mathbb{C}),$$

up to conjugacy.

Fix an orientation and a base point on Π^0. We equip Π^0 with an orientation preserving homeomorphism $\omega : \Pi^0 \to \Pi^0$ of order three that permutes the cuffs and let $\omega^i(C)$, $i = 0, 1, 2$, denote the oriented cuffs of Π^0. We may assume that the base point of Π^0 is fixed under ω. By $\omega : \pi_1(\Pi^0) \to \pi_1(\Pi^0)$ we also denote the induced isomorphism of the fundamental group. (Observe that the homeomorphism $\omega : \Pi^0 \to \Pi^0$ has a fixed point that is the base point for Π^0 so the isomorphism of the fundamental group is well defined.) Choose $c \in \pi_1(\Pi^0)$ to be an element in the conjugacy class that corresponds to the cuff C such that $\omega^{-1}(c) c \omega(c) = \text{id}$.

Definition 3.1. Let $\rho : \pi_1(\Pi^0) \to \mathbf{PSL}(2,\mathbb{C})$ be a faithful representation. We say that ρ is an admissible representation if $\rho(\omega^i(c))$ is a loxodromic Möbius transformation, and

$$\mathbf{hl}(\omega^i(C)) = \frac{\mathbf{l}(\omega^i(C))}{2},$$

where $\mathbf{l}(\omega^i(C))$ is chosen so that $-\pi < \operatorname{Im}(\mathbf{l}(\omega^i(C))) \leq \pi$.

Definition 3.2. Let $\rho : \pi_1(\Pi^0) \to \mathcal{G}$ be an admissible representation. A skew pants Π is the conjugacy class $\Pi = [\rho]$. The set of all skew pants is denoted by $\mathbf{\Pi}$.

For $\Pi \in \mathbf{\Pi}$, we define $\mathcal{R}(\Pi) \in \mathbf{\Pi}$ as follows. Let $\rho : \pi_1(\Pi^0) \to \mathcal{G}$ be a representation such that $[\rho] = \Pi$, and set $\rho(\omega^i(c)) = A_i \in \mathcal{G}$. Define the representation $\rho_1 : \pi_1(\Pi^0) \to \mathcal{G}$ by $\rho_1(\omega^{-i}(c)) = A_i^{-1}$. We verify that ρ_1 is well defined, and we let $\mathcal{R}(\Pi) = [\rho_1]$. The mapping $\mathcal{R} : \mathbf{\Pi} \to \mathbf{\Pi}$ is a fixed point free involution.

For $\Pi \in \mathbf{\Pi}$ such that $\Pi = [\rho]$, we let $\gamma^*(\Pi, \omega^i(c)) \in \Gamma^*$ denote the oriented geodesic that represents the conjugacy class of $\rho(\omega^i(c))$. Observe the identity $\gamma^*(\mathcal{R}(\Pi), \omega^i(c)) = -\gamma^*(\Pi, \omega^{-i}(c))$. The set of pairs (Π, γ^*), where $\gamma^* = \gamma^*(\Pi, \omega^i(c))$, for some $i = 0, 1, 2$, is called the set of marked skew pants and denoted by $\mathbf{\Pi}^*$.

There is the induced (fixed point free) involution $\mathcal{R} : \mathbf{\Pi}^* \to \mathbf{\Pi}^*$, given by $\mathcal{R}(\Pi, \gamma^*(\Pi, \omega^i(c))) = (\mathcal{R}(\Pi), \gamma^*(\mathcal{R}(\Pi), \omega^{-i}(c)))$. Another obvious mapping $\operatorname{rot} : \mathbf{\Pi}^* \to \mathbf{\Pi}^*$ is given by $\operatorname{rot}(\Pi, \gamma^*(\Pi, \omega^i(c))) = (\Pi, \gamma^*(\Pi, \omega^{i+1}(c)))$.

Definition 3.3. Let $\mathcal{L}$ be a finite set of labels. We say that a map $\operatorname{lab} : \mathcal{L} \to \mathbf{\Pi}^*$ is a legal labeling map if the following holds:

(1) there exists an involution $\mathcal{R}_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$ such that $\mathcal{R}(\operatorname{lab}(a)) = \operatorname{lab}(\mathcal{R}_{\mathcal{L}}(a))$,
(2) there is a bijection $\operatorname{rot}_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$ such that $\operatorname{rot}(\operatorname{lab}(a)) = \operatorname{lab}(\operatorname{rot}_{\mathcal{L}}(a))$.

Example. Let $\mathbb{N}\mathbf{\Pi}$ denote the collection of all formal sums of oriented skew pants from $\mathbf{\Pi}$ over nonnegative integers. We say that $W \in \mathbb{N}\mathbf{\Pi}$ is symmetric if $W = n_1(\Pi_1 + \mathcal{R}(\Pi_1)) + n_2(\Pi_2 + \mathcal{R}(\Pi_2)) + \cdots + n_m(\Pi_m + \mathcal{R}(\Pi_m))$, where n_i are positive integers and $\Pi_i \in \mathbf{\Pi}$. Every symmetric W induces a canonical legal labeling defined as follows. The corresponding set of labels is $\mathcal{L} = \{(j,k) : j = 1, 2, \ldots, 2(n_1 + n_2 + \cdots + n_m); k = 0, 1, 2\}$. (Observe that the set $\mathcal{L}$ has $6(n_1 + \cdots + n_m)$ elements.) Set $\operatorname{lab}(j,k) = (\Pi_s, \gamma^*(\Pi_s, \omega^k(c)))$ if j is odd and if $2(n_1 + \cdots + n_{s-1}) < j \leq 2(n_1 + \cdots + n_s)$. Set $\operatorname{lab}(j,k) = (\mathcal{R}(\Pi_s), \gamma^*(\mathcal{R}(\Pi_s), \omega^{-k}(c)))$ if j is even, and $2(n_1 + \cdots + n_{s-1}) < j \leq 2(n_1 + \cdots + n_s)$. The bijection $\mathcal{R}_{\mathcal{L}}$ is given by $\mathcal{R}_{\mathcal{L}}(j,k) = (j + \delta(j), k)$, where $\delta(j) = +1$ if j is even and $\delta(j) = -1$ if j is odd. The bijection $\operatorname{rot}_{\mathcal{L}}$ is defined accordingly.

Definition 3.4. Let $\sigma : \mathcal{L} \to \mathcal{L}$ be an involution. We say that σ is admissible with respect to a legal labeling lab if the following holds. Let $a \in \mathcal{L}$, and let $\text{lab}(a) = (\Pi_1, \gamma^*)$ for some $\Pi_1 \in \mathbf{\Pi}$, where $\gamma^* = \gamma(\Pi_1, \omega^i(c))$ for some $i \in \{0, 1, 2\}$. Then $\text{lab}(\sigma(a)) = (\Pi_2, -\gamma^*)$, where $\Pi_2 \in \mathbf{\Pi}$ is some other skew pants.

Observe that every legal labeling has an admissible involution $\sigma : \mathcal{L} \to \mathcal{L}$, given by $\sigma(a) = \mathcal{R}_{\mathcal{L}}(a)$.

Suppose that we are given a legal labeling $\text{lab} : \mathcal{L} \to \mathbf{\Pi}^*$ and an admissible involution $\sigma : \mathcal{L} \to \mathcal{L}$. We construct a closed topological surface S^0 (not necessarily connected) with a pants decomposition $\mathcal{C}^0$, and a representation $\rho_{\text{lab},\sigma} : \pi(S^0) \to \mathcal{G}$ as follows. Each element of $\mathcal{L}$ determines an oriented cuff in $\mathcal{C}^0$. Each element in the orbit space $\mathcal{L}/\operatorname{rot}_{\mathcal{L}}$ gives a copy of the oriented topological pair of pants Π^0. The pairs of pants are glued according to the instructions given by σ, and this defines the representation $\rho_{\text{lab},\sigma}$. One can check that after we glue the corresponding pairs of pants we construct a closed surface S^0. Moreover, S^0 is connected if and only if the action of the group of bijections $\langle \mathcal{R}_{\mathcal{L}}, \operatorname{rot}_{\mathcal{L}}, \sigma \rangle$ is minimal on $\mathcal{L}$ (that is $\mathcal{L}$ is the smallest invariant set under the action of this group).

Let $a \in \mathcal{L}$. Then $(\Pi, \gamma^*) = \text{lab}(a)$ and $(\Pi_1, -\gamma^*) = \text{lab}(\sigma(a))$ for some skew pants $\Pi, \Pi_1 \in \mathbf{\Pi}$. Also $\gamma^* = \gamma^*(\Pi, \omega^i(c))$ and $-\gamma^* = \gamma^*(\Pi_1, \omega^j(c))$. Set

$$\mathbf{hl}(a) = \mathbf{hl}(\omega^i(C)),$$

where the half-length $\mathbf{hl}(\omega^i(C))$ is computed for the representation that corresponds to the skew pants Π.

It follows from our definition of admissible representations that $\mathbf{hl}(a) = \mathbf{hl}(\sigma(a))$. Set $\mathbf{l}(a) = \mathbf{l}(\omega^i(C))$. Then $\mathbf{l}(a) = \mathbf{l}(\sigma(a))$ and

$$\mathbf{hl}(a) = \frac{\mathbf{l}(a)}{2}.$$

3.2. *The unit normal bundle of a closed geodesic.* Next, we discuss in more details the structure of the unit normal bundle $N^1(\gamma)$ of a closed geodesic $\gamma \subset \mathbf{M}^3$. (For the readers convenience we will repeat several definitions given at the beginning of Section 2.) The bundle $N^1(\gamma)$ has an induced differentiable structure and it is diffeomorphic to a torus. Elements of $N^1(\gamma)$ are pairs (p, v), where $p \in \gamma$ and v is a unit vector at p that is orthogonal to γ. The disjoint union of all the bundles is denoted by $N^1(\Gamma)$.

Fix an orientation γ^* on γ. Consider $\mathbb{C}$ as an additive group and for $\zeta \in \mathbb{C}$, let $\mathcal{A}_\zeta : N^1(\gamma) \to N^1(\gamma)$ be the mapping given by $\mathcal{A}_\zeta(p, v) = (p_1, v_1)$ where p_1 and v_1 are defined as follows. Let $\widetilde{\gamma^*}$ be a lift of γ^* to $\mathbb{H}^3$ and let $(\widetilde{p}, \widetilde{v}) \in N^1(\widetilde{\gamma})$ be a lift of (p, v). We may assume that $\widetilde{\gamma^*}$ is the geodesic between $0, \infty \in \partial\mathbb{H}^3$. Let $A_\zeta \in \mathbf{PSL}(2,\mathbb{C})$ be given by $A_\zeta(w) = e^\zeta w$ for $w \in \partial\mathbb{H}^3$. Set $(\widetilde{p}_1, \widetilde{v}_1) = A_\zeta(\widetilde{p}, \widetilde{v})$. Then $(\widetilde{p}_1, \widetilde{v}_1)$ is a lift of (p_1, v_1).

If $\mathcal{A}^1_\zeta : N^1(\gamma) \to N^1(\gamma)$ and $\mathcal{A}^2_\zeta : N^1(\gamma) \to N^1(\gamma)$ are the actions that correspond to different orientations on γ, then on $N^1(\gamma)$ we have $\mathcal{A}^1_\zeta = \mathcal{A}^2_{-\zeta} = (\mathcal{A}^2_\zeta)^{-1}$, $\zeta \in \mathbb{C}$. Unless we specify otherwise, by $\mathcal{A}_\zeta$ we denote either of the two actions.

The group $\mathbb{C}$ acts transitively on $N^1(\gamma)$. Let $\mathbf{l}(\gamma)$ be the complex translation length of γ such that $-\pi < \operatorname{Im}(\mathbf{l}(\gamma)) \le \pi$. (By definition $\operatorname{Re}(\mathbf{l}(\gamma)) > 0$.) Then $A_{\mathbf{l}(\gamma)} = \operatorname{id}$ on $N^1(\gamma)$. This implies that the map $A_{\frac{\mathbf{l}(\gamma)}{2}}$ is an involution which enables us to define the bundle $N^1(\sqrt{\gamma}) = N^1(\gamma)/A_{\frac{\mathbf{l}(\gamma)}{2}}$. The disjoint union of all the bundles is denoted by $N^1(\sqrt{\Gamma})$.

The additive group $\mathbb{C}$ acts on $N^1(\sqrt{\gamma})$ as well. There is a unique complex structure on $N^1(\sqrt{\gamma})$ so that the action $\mathcal{A}_\zeta$ is by biholomorphic maps. With this complex structure, we have

$$N^1(\sqrt{\gamma}) \equiv \mathbb{C}/\left(\frac{\mathbf{l}(\gamma)}{2}\mathbb{Z} + 2\pi i\mathbb{Z}\right).$$

The corresponding Euclidean distance on $N^1(\sqrt{\gamma})$ is denoted by dis. Then for $|\zeta|$ small, we have $\operatorname{dis}((p,v),(\mathcal{A}_\zeta(p,v))) = |\zeta|$. There is also the induced map $\mathcal{A}_\zeta : N^1(\sqrt{\Gamma}) \to N^1(\sqrt{\Gamma})$, $\zeta \in \mathbb{C}$, where the restriction of $\mathcal{A}_\zeta$ on each torus $N^1(\sqrt{\gamma})$ is defined above.

Let $(\Pi, \gamma^*) \in \mathbf{\Pi}^*$, and let γ^*_k be such that $(\Pi, \gamma^*_k) = \operatorname{rot}^k(\Pi, \gamma^*)$, $k = 1, 2$. Let δ^*_k be an oriented geodesic (not necessarily closed) in $\mathbf{M}^3$ such that δ^*_k is the common orthogonal of γ^* and γ^*_k and so that a lift of δ^*_k is a side in the corresponding skew right-angled hexagon that determines Π (see Section 2). The orientation on δ^*_k is determined so that the point $\delta^*_k \cap \gamma^*_k$ comes after the point $\delta^*_k \cap \gamma^*$. Let $p_k = \delta^*_k \cap \gamma^*$, and let v_k be the unit vector v_k at p_k that has the same direction as δ^*_k. Since the pants Π is the conjugacy class of an admissible representation in sense of Definition 3.1, we observe that $\mathcal{A}_{\frac{\mathbf{l}(\gamma)}{2}}$ exchanges (p_1, v_1) and (p_2, v_2), and so the class $[(p_k, v_k)] \in N^1(\sqrt{\gamma})$ does not depend on $k \in \{1, 2\}$. Define the map

$$\operatorname{foot} : \mathbf{\Pi}^* \to N^1(\sqrt{\Gamma})$$

by

$$\operatorname{foot}_\gamma(\Pi) = \operatorname{foot}(\Pi, \gamma^*) = [(p_k, v_k)] \in N^1(\gamma).$$

Observe that $\operatorname{foot}(\Pi, \gamma^*) = \operatorname{foot}(\mathcal{R}(\Pi, \gamma^*))$.

Let S^0 be a topological surface with a pants decomposition $\mathcal{C}^0$, and let $\rho : \pi_1(S^0) \to \mathcal{G}$ be a representation such that the restriction of ρ on the fundamental group of each pair of pants satisfies the assumptions of Definition 3.1. (Recall that $\mathcal{G}$ is the Kleinian group such that $\mathbf{M}^3 \equiv \mathbb{H}^3/\mathcal{G}$.) Let Π^0_i, $i = 1, 2$

be two pairs of pants from the pants decomposition of S^0 that both have a given cuff $C \in \mathcal{C}^0$ in its boundary. By (Π_1, γ^*) and $(\Pi_2, -\gamma^*)$ we denote the corresponding marked pants in $\mathbf{\Pi}^*$. Let $s(C)$ denote the corresponding reduced complex Fenchel-Nielsen coordinate for ρ. Let $\mathcal{A}^1_\zeta$ be the action on $N^1(\sqrt{\gamma})$ that corresponds to the orientation γ^*. Fix $\zeta_0 \in \mathbb{C}$ to be such that

$$\mathcal{A}^1_{\zeta_0}(\text{foot}(\Pi_1, \gamma^*)) = \text{foot}(\Pi_2, -\gamma^*).$$

Such ζ_0 is uniquely determined up to a translation from the lattice $\frac{\mathbf{l}(\gamma)}{2}\mathbb{Z} + 2i\pi\mathbb{Z}$. If $\mathcal{A}^2_\zeta$ is the other action, then we have

$$\mathcal{A}^2_{\zeta_0}(\text{foot}(\Pi_2, -\gamma^*)) = (\Pi_1, \gamma^*),$$

since $\mathcal{A}^1_\zeta \circ \mathcal{A}^2_\zeta = \text{id}$. That is, the choice of ζ_0 does not depend on the choice of the action $\mathcal{A}_\zeta$. Then $s(C) \in \mathbb{C}/(\frac{\mathbf{l}(\gamma)}{2}\mathbb{Z} + 2\pi i\mathbb{Z})$ and

$$s(C) = (\zeta_0 - i\pi), \ \left(\text{mod } \frac{\mathbf{l}(\gamma)}{2}\mathbb{Z} + 2\pi i\mathbb{Z}\right). \tag{25}$$

The rest of the paper is devoted to proving the following theorem.

Theorem 3.1. *There exist constants* $\mathbf{q} > 0$ *and* $K > 0$ *such that for every* $\varepsilon > 0$ *and for every* $R > 0$ *large enough, the following holds. There exist a finite set of labels* $\mathcal{L}$, *a legal labeling* lab : $\mathcal{L} \to \mathbf{\Pi}$, *and an admissible involution* $\sigma : \mathcal{L} \to \mathcal{L}$ *such that for every* $a \in \mathcal{L}$, *we have*

$$|\mathbf{hl}(a) - \frac{R}{2}| < \varepsilon,$$

and

$$\text{dis}(\mathcal{A}_{1+i\pi}(\text{foot}(\text{lab}(a))), \text{foot}(\text{lab}(\sigma(a)))) \le KRe^{-\mathbf{q}R},$$

where dis *is the Euclidean distance on* $N^1(\sqrt{\gamma})$.

Remark. The constant $\mathbf{q}$ depends on the manifold $\mathbf{M}^3$. In fact, it only depends on the first eigenvalue for the Laplacian on $\mathbf{M}^3$.

Given this theorem we can prove Theorem 1.1 as follows. We saw that every legal labeling together with an admissible involution yields a representation $\rho(\text{lab}, \sigma) : \pi_1(S^0) \to \mathcal{G}$, where $\mathcal{G}$ is the corresponding Kleinian group and S^0 is a closed topological surface. (If S^0 is not connected, we pass onto a connected component.) By the above discussion the reduced complex Fenchel-Nielsen coordinates $(\mathbf{hl}(C), s(C))$ satisfy the assumptions of Theorem 2.1. (Observe that $KRe^{-\mathbf{q}R} = o(\frac{1}{R})$, when $R \to \infty$.) Then Theorem 1.1 follows from Theorem 2.1.

3.3. *Transport of measure*. Let (X, d) be a metric space. By $\mathcal{M}(X)$ we denote the space of positive, finite Borel measures on X with compact support. For $A \subset X$ and $\delta > 0$, let

$$\mathcal{N}_\delta(A) = \{x \in X : \text{ there exists } a \in A \text{ such that } d(x, a) \le \delta\}$$

be the δ-neighbourhood of A.

Definition 3.5. Let $\mu, \nu \in \mathcal{M}(X)$ be two measures such that $\mu(X) = \nu(X)$, and let $\delta > 0$. Suppose that for every Borel set $A \subset X$, we have $\mu(A) \le \nu(\mathcal{N}_\delta(A))$. Then we say that μ and ν are δ-equivalent measures.

It appears that the definition is asymmetric in μ and ν. But this is not the case. For any Borel set $A \subset X$, the above definition yields $\nu(A) \le \nu(X) - \nu\big(\mathcal{N}_\delta(X \setminus \mathcal{N}_\delta(A))\big) \le \mu(X) - \mu(X \setminus \mathcal{N}_\delta(A)) = \mu(\mathcal{N}_\delta(A)$. This shows that the definition is in fact symmetric in μ and ν.

The following propositions follow from the definition of equivalent measures.

Proposition 3.1. *Suppose that μ and ν are δ-equivalent. Then for any $K > 0$, the measures $K\mu$ and $K\nu$ are δ-equivalent. If, in addition, we assume that measures ν and η are δ_1-equivalent, then μ and η are $(\delta + \delta_1)$-equivalent.*

Proposition 3.2. *Let (T, Λ) be a measure space, and let $f_i : T \to X$, $i = 1, 2$, be two maps such that $d(f_1(t), f_2(t)) \le \delta$ for almost every $t \in T$. Then the measures $(f_1)_*\Lambda$ and $(f_2)_*\Lambda$ are δ-equivalent.*

In the remainder of this subsection we prove two theorems, each representing a converse of the previous proposition in a special case. The following theorem is a converse of Proposition 3.2 in the special case of discrete measures.

Theorem 3.2. *Suppose that A and B are finite sets with the same number of elements and equipped with the standard counting measures Λ_A and Λ_B respectively. Suppose that there are maps $f : A \to X$ and $g : B \to X$ such that the measures $f_*\Lambda_A$ and $g_*\Lambda_B$ are δ-equivalent for some $\delta > 0$. Then one can find a bijection $h : A \to B$ such that $d(g(h(a)), f(a)) \le \delta$ for every $a \in A$.*

Proof. We use the Hall's marriage theorem, which states the following. Suppose that $\mathrm{Rel} \subset A \times B$ is a relation. For every $Q \subset A$ we let

$$\mathrm{Rel}(Q) = \{b \in B : \text{ there exists } a \in Q \text{ such that } (a, b) \in \mathrm{Rel}\}.$$

If $|\mathrm{Rel}(Q)| \ge |Q|$ for every $Q \subset A$, then there exists an injection $h : A \to B$ such that $(a, h(a)) \in \mathrm{Rel}$ for every $a \in A$. This is Hall's marriage theorem. In the general case of this theorem the sets A and B need not have the same number of elements. However, in our case they do, so the map h is a bijection.

Define $\mathrm{Rel} \subset A \times B$ by saying that $(a, b) \in \mathrm{Rel}$ if $d(f(a), g(b)) \le \delta$. Then

$$\mathrm{Rel}(Q) = \{b \in B : \text{ there exists } a \in Q \text{ such that } d(f(a), g(b)) \le \delta\}$$

for every $Q \subset A$. Therefore $|\mathrm{Rel}(Q)| = (g_*\Lambda_B)(\mathcal{N}_\delta(f(Q))) \ge (f_*\Lambda_A)(f(Q)) = |Q|$, since $f_*\Lambda_A$ and $g_*\Lambda_B$ are δ-equivalent. This means that the hypothesis of the Hall's marriage theorem is satisfied, and one can find the bijection $h : A \to B$ such that $d(g(h(a)), f(a)) \le \delta$. □

Let $a, b \in \mathbb{C}$ be two complex numbers such that $T(a,b) = \mathbb{C}/(a\mathbb{Z} + ib\mathbb{Z})$ is a torus. We let $z = x + iy$ denote a point in $\mathbb{C}$. (Sometimes we use (x, y) to denote a point in $\mathbb{R}^2 \equiv \mathbb{C}$.)

Let ϕ be a positive C^0 function on $T(a,b)$. As usual, by $\phi(x,y)\,dx\,dy$ we denote the corresponding two form on the torus $T(a,b)$. By λ_ϕ we denote the measure on $T(a,b)$ given by

$$\lambda_\phi(A) = \int_A \phi(x,y)\,dx\,dy$$

for any measurable set A. We abbreviate this equation $d\lambda_\phi = \phi\,dx\,dy$. By λ we denote the standard Lebesgue measure on $T(a,b)$; that is, $\lambda = \lambda_\phi$ for $\phi \equiv 1$.

In the following lemma we show that any C^0 measure that is close to the Lebesgue measure is obtained by transporting the Lebesgue measure by a diffeomorphism that is C^0 close to the identity.

LEMMA 3.1. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a C^0 function on $\mathbb{C}$ that is well defined on the quotient $T(1,1) = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, and such that*

(1) *for some $0 < \delta < \frac{1}{3}$, we have*

$$1 - \delta \le g(x,y) \le 1 + \delta$$

for all $(x,y) \in \mathbb{R}^2$;

(2) *the following equality holds:*

$$\int_0^1 \int_0^1 g(x,y)\,dx\,dy = 1.$$

Then we can find a C^1 diffeomorphism $h : T(1,1) \to T(1,1)$ such that

(1) *$g(x,y) = \mathrm{Jac}(h)(x,y)$, that is $g(x,y)\,dx\,dy = h^*(dx\,dy)$, where $h^*(dx\,dy)$ is the pull-back of the two form $dx\,dy$ by the diffeomorphism h and $\mathrm{Jac}(h)$ is the Jacobian of h.*

(2) *The inequality*

$$|h(z) - z| \le 4\delta$$

holds for every $z = x + iy \in \mathbb{C}$.

Proof. We define the map $h : \mathbb{R}^2 \to \mathbb{R}^2$ by $h(x,y) = (h_1(x,y), h_2(x,y))$, where

$$h_1(x,y) = \int_0^x \left(\int_0^1 g(s,t)\,dt \right) ds$$

and

$$h_2(x,y) = \frac{\int_0^y g(x,t)\,dt}{\int_0^1 g(x,t)\,dt}.$$

Since $g(x+1,y) = g(x,y+1) = g(x,y)$, we find that $h(x+1,y) - h(x,y) = (1,0)$ and $h(x,y+1) - h(x,y) = (0,1)$, so h descends to a map from $T(1,1)$ to itself.

Furthermore, we find that

$$\frac{\partial h_1}{\partial x} = \int_0^1 g(x,t)\,dt; \quad \frac{\partial h_1}{\partial y} = 0$$

and

$$\frac{\partial h_2}{\partial y} = \frac{g(x,y)}{\int_0^1 g(x,t)\,dt},$$

which is sufficient to conclude that

$$\mathrm{Jac}(h)(x,y) = g(x,y).$$

Therefore, the map $h : T(1,1) \to T(1,1)$ is a local diffeomorphism and thus a covering map of degree n, where

$$n = \int_{T(1,1)} \mathrm{Jac}(x,y)\,dx\,dy.$$

Since $\mathrm{Jac}(h)(x,y) = g(x,y)$ and

$$\int_{T(1,1)} g(x,y)\,dx\,dy = 1,$$

it follows that $n = 1$; that is, h is a diffeomorphism.

On the other hand, for $x, y \in [0,1]$,

$$|h_1(x,y) - x| \le \delta x \le \delta$$

and

$$h_2(x,y) - y \le \frac{y(1+\delta)}{1-\delta} - y \le 3\delta y \le 3\delta,$$

since $\delta < \frac{1}{3}$, and

$$y - h_2(x,y) \le y - \frac{y(1-\delta)}{1+\delta} \le 2\delta y \le 2\delta.$$

Therefore, $|h_2(x,y) - y| \le 3\delta$. Combining the estimates for $|h_1(x,y) - x|$ and $|h_2(x,y) - y|$, we find that

$$|h(z) - z| \le |h_1(x,y) - x| + |h_2(x,y) - y| \le 4\delta.$$

This completes the proof. □

The following theorem is a corollary of the of the previous lemma.

Theorem 3.3. *Let $\mu \in \mathcal{M}(T(a,b))$ be a measure whose Radon-Nikodym derivative $\frac{d\mu}{d\lambda}(z)$ is a C^0 function on the torus $T(a,b)$ such that for some $K > 0$ and $\frac{1}{3} > \delta > 0$, we have $\mu(T(a,b)) = K\lambda(T(a,b))$ and*

$$K \le \left|\frac{d\mu}{d\lambda}\right| \le K(1+\delta), \quad \textit{everywhere on } T(a,b).$$

Then μ is $4\delta(|a| + |b|)$-equivalent to the measure $K\lambda$.

Proof. By μ we also denote the lift of the corresponding measure to the universal cover $\mathbb{C}$. Then $d\mu = g_1(x,y)dx\,dy$, where $g_1(x,y) = \frac{d\mu}{d\lambda}(x,y)$ is the Radon-Nikodym derivative. The function g_1 is C^0 on $\mathbb{C}$, and g_1 is well defined on the quotient $\mathbb{C}/(a\mathbb{Z}+bi\mathbb{Z}) = T(a,b)$.

Let $L : T(1,1) \to T(a,b)$ be the standard affine map. Let

$$g(x,y) = \frac{1}{K}(g_1 \circ L)(x,y).$$

Then $g(x,y)$ satisfies the assumptions of the previous lemma. Let h be the corresponding diffeomorphism from Lemma 3.1, and let $h_1 = L \circ h \circ L^{-1}$. Then $(h_1)^*\mu = K\lambda$ on $T(a,b)$. Since the affine map L is $(|a|+|b|)$ bi-Lipschitz we conclude that

$$|h_1(z) - z| \leq 4\delta(|a|+|b|)$$

for every $z \in \mathbb{C}$, so μ is $4\delta(|a|+|b|)$-equivalent to $K\lambda$. □

3.4. *Measures on skew pants and the $\widehat{\partial}$ operator.*

Definition 3.6. By $\mathcal{M}_0^{\mathcal{R}}(\mathbf{\Pi})$ we denote the space of positive Borel measures with finite support on the set of oriented skew pants $\mathbf{\Pi}$ such that the involution $\mathcal{R} : \mathbf{\Pi} \to \mathbf{\Pi}$ preserves each measure in $\mathcal{M}_0^{\mathcal{R}}(\mathbf{\Pi})$. By $\mathcal{M}_0(N^1(\sqrt{\Gamma}))$ we denote the space of positive Borel measures with compact support on the manifold $N^1(\sqrt{\Gamma})$. (A measure from $\mathcal{M}_0(N^1(\sqrt{\Gamma}))$ has a compact support if and only if its support is contained in at most finitely many tori $N^1(\sqrt{\gamma}) \subset N^1(\sqrt{\Gamma})$.)

We define the operator

$$\widehat{\partial} : \mathcal{M}_0^{\mathcal{R}}(\mathbf{\Pi}) \to \mathcal{M}_0(N^1(\sqrt{\Gamma}))$$

as follows. The set $\mathbf{\Pi}$ is a countable set, so every measure from $\mu \in \mathcal{M}_0^{\mathcal{R}}(\mathbf{\Pi})$ is determined by its value $\mu(\Pi)$ on every $\Pi \in \mathbf{\Pi}$. Let $\Pi \in \mathbf{\Pi}$, and let $\gamma_i^* \in \Gamma^*$, $i = 0,1,2$, denote the corresponding oriented geodesics so that $(\Pi, \gamma_i^*) \in \mathbf{\Pi}^*$. Let $\alpha_i^{\Pi} \in \mathcal{M}_0(N^1(\sqrt{\Gamma}))$ be the atomic measure supported at the point $\operatorname{foot}(\Pi, \gamma_i^*) \in N^1(\sqrt{\gamma_i})$, where the mass of the atom is equal to 1. Let

$$\alpha^{\Pi} = \sum_{i=0}^{2} \alpha_i^{\Pi},$$

and define

$$\widehat{\partial}\mu = \sum_{\Pi \in \mathbf{\Pi}} \mu(\Pi)\alpha^{\Pi}.$$

We call this the $\widehat{\partial}$ operator on measures. The total measure of $\widehat{\partial}\mu$ is three times the total measure of μ.

Let $\alpha \in \mathcal{M}_0(N^1(\sqrt{\Gamma}))$. Choose $\gamma^* \in \Gamma^*$, and recall the action $\mathcal{A}_\zeta : N^1(\sqrt{\Gamma}) \to N^1(\sqrt{\Gamma})$, $\zeta \in \mathbb{C}$. Let $(\mathcal{A}_\zeta)_*\alpha$ denote the push-forward of the measure α. We say that α is δ-symmetric if the measures α and $(\mathcal{A}_\zeta)_*\alpha$ are δ-equivalent for every $\zeta \in \mathbb{C}$.

THEOREM 3.4. *There exists* $\mathbf{q} > 0$ *and* $D_1, D_2 > 0$, *so that for every* $1 \geq \varepsilon > 0$ *and every* $R > 0$ *large enough, there exists a measure* $\mu \in \mathcal{M}_0^{\mathcal{R}}(\mathbf{\Pi})$ *with the following properties. If* $\mu(\Pi) > 0$ *for some* $\Pi \in \mathbf{\Pi}$, *then the half-lengths* $\mathbf{hl}(\omega^i(C))$ *that correspond to the skew pants* Π *satisfy the inequality*

$$\left|\mathbf{hl}(\omega^i(C)) - \frac{R}{2}\right| \leq \varepsilon.$$

There exists a measure $\beta \in \mathcal{M}_0(N^1(\sqrt{\Gamma}))$ *such that the measure* $\widehat{\partial}\mu$ *and* β *are* $D_1 e^{-\frac{R}{8}}$*-equivalent and such that for each torus* $N^1(\sqrt{\gamma})$, *there exists a constant* $K_\gamma \geq 0$ *so that*

$$K_\gamma \leq \left|\frac{d\beta}{d\lambda}\right| \leq K_\gamma(1 + D_2 e^{-\mathbf{q}R}), \textit{almost everywhere on}\, N^1(\sqrt{\gamma}),$$

where λ *is the standard Lebesgue measure on the torus* $N^1(\sqrt{\gamma}) = \mathbb{C}/\big(\frac{\mathbf{l}(\gamma)}{2}\mathbb{Z} + 2i\pi\mathbb{Z}\big)$.

Remark. This theorem holds in two dimensions as well. That is, in the statement of the above theorem we can replace a closed hyperbolic three manifold $\mathbf{M}^3$ with any hyperbolic closed surface.

We prove this theorem in the next section. But first we prove Theorem 3.1 assuming Theorem 3.4.

PROPOSITION 3.3. *There exist* $\mathbf{q} > 0$, $D > 0$, *so that for every* $1 \geq \varepsilon > 0$ *and every* $R > 0$ *large enough, there exists a measure* $\mu \in \mathcal{M}_0^{\mathcal{R}}(\mathbf{\Pi})$ *with the following properties*:

(1) $\mu(\Pi)$ *is a rational number for every* $\Pi \in \mathbf{\Pi}$.
(2) *If* $\mu(\Pi) > 0$ *for some* $\Pi \in \mathbf{\Pi}$, *then the half-lengths* $\mathbf{hl}(\omega^i(C))$ *that correspond to the skew pants* Π *satisfy the inequality* $|\mathbf{hl}(\omega^i(C)) - \frac{R}{2}| \leq \varepsilon$.
(3) *The measures* $\widehat{\partial}\mu$ *and* $(\mathcal{A}_{1+i\pi})_*\widehat{\partial}\mu$ *are* $DRe^{-\mathbf{q}R}$*-equivalent.*

Proof. Assume the notation and the conclusions of Theorem 3.4. First we show that the measures $\widehat{\partial}\mu$ and $(\mathcal{A}_{1+i\pi})_*\widehat{\partial}\mu$ are $DRe^{-\mathbf{q}R}$-equivalent. Let $\gamma \in \Gamma$ be a closed geodesic such that $\beta(N^1(\sqrt{\gamma})) > 0$; that is, the support of β has a nonempty intersection with the torus

$$N^1(\sqrt{\gamma}) \equiv \mathbb{C}/\left(\frac{\mathbf{l}(\gamma)}{2}\mathbb{Z} + 2\pi i\mathbb{Z}\right).$$

The Lebesgue measure λ on $N^1(\sqrt{\gamma})$ is invariant under the action $\mathcal{A}_\zeta$. This, together with Theorem 3.3, implies that for any $\zeta \in \mathbb{C}$, the measure $(\mathcal{A}_\zeta)_*\beta$ is $\big(2\pi + \big|\frac{\mathbf{l}(\gamma^*)}{2}\big|\big)D_2 e^{-\mathbf{q}R}$-equivalent with the measure $K'\lambda$ for some $K' > 0$, where D_2 is from the previous theorem. Since $\left|\frac{\mathbf{l}(\gamma^*)}{2}\right| \leq \frac{R}{2}+1$, we have that the measures $(\mathcal{A}_\zeta)_*\beta$ and $K'\lambda$ are $C_1 R e^{-\mathbf{q}R}$-equivalent for some $C_1 > 0$.

On the other hand, the measures $(\mathcal{A}_\zeta)_*\beta$ and $(\mathcal{A}_\zeta)_*\widehat{\partial}\mu$ are $D_1 e^{-\frac{R}{8}}$-equivalent. From Proposition 3.1 we conclude that the measures $(\mathcal{A}_\zeta)_*\widehat{\partial}\mu$ and $K'\lambda$ are $D_2(Re^{-\mathbf{q}R}+e^{-\frac{R}{8}})$-equivalent, for every $\zeta \in \mathbb{C}$, and for some constant $D_2 > 0$. Again, since λ is invariant under $\mathcal{A}_\zeta$ and from Proposition 3.1, we conclude that $\widehat{\partial}\mu$ is $DRe^{-\mathbf{q}R}$-symmetric for some constant $D > 0$. (This assumption can be made without loss of generality.) In particular, we have that the measures $\widehat{\partial}\mu$ and $(\mathcal{A}_{1+i\pi})_*\widehat{\partial}\mu$ are $DRe^{-\mathbf{q}R}$-equivalent.

Both measures $\widehat{\partial}\mu$ and $(\mathcal{A}_{1+i\pi})_*\widehat{\partial}\mu$ are atomic (with finitely many atoms), so it follows from the definition that the measures $\widehat{\partial}\mu$ and $(\mathcal{A}_{1+i\pi})_*\widehat{\partial}\mu$ are $DRe^{-\mathbf{q}R}$-equivalent if and only if a finite system of linear inequalities with integer coefficients has a real valued solution. Then the standard rationalization procedure (see [7, Prop. 2.4] and [3]) implies that this system of equations has a rational solution, so we may assume that that the measure μ from Theorem 3.4 has rational weights. This proves the proposition. □

3.5. *Proof of Theorem* 3.1. First we make several observations about an arbitrary measure $\nu \in \mathcal{M}_0^{\mathcal{R}}(\mathbf{\Pi})$. The measure ν is supported on finitely many skew pants $\Pi \in \mathbf{\Pi}$. Moreover, $\nu(\Pi) = \nu(\mathcal{R}(\Pi))$ for every $\Pi \in \mathbf{\Pi}$. Let $\mathbf{\Pi}^+$ and $\mathbf{\Pi}^-$ be disjoint subsets of $\mathbf{\Pi}$ such that $\mathbf{\Pi}^+ \cup \mathbf{\Pi}^- = \mathbf{\Pi}$, and $\mathcal{R}(\mathbf{\Pi}^+) = \mathbf{\Pi}^-$. (There are many such decompositions of $\mathbf{\Pi}$.) Let ν^+ and ν^- denote the restrictions of ν on the sets $\mathbf{\Pi}^+$ and $\mathbf{\Pi}^-$ respectively. Then $\widehat{\partial}\nu^+ = \widehat{\partial}\nu^-$ and $\widehat{\partial}\nu = 2\widehat{\partial}\nu^-$. (This follows from the fact that $\operatorname{foot}(\Pi,\gamma^*) = \operatorname{foot}(\mathcal{R}(\Pi),-\gamma^*)$.) Therefore, if the measure $\widehat{\partial}\nu$ is δ-symmetric then so are the measures $\widehat{\partial}\nu^+$ and $\widehat{\partial}\nu^-$.

Let μ be the measure from Proposition 3.3. Then μ has rational weights. We multiply μ by a large enough integer and obtain the measure μ' such that the weights $\mu'(\Pi)$ are even numbers, $\Pi \in \mathbf{\Pi}$. Then $\widehat{\partial}\mu'$ and $(\mathcal{A}_{1+i\pi})_*\widehat{\partial}\mu'$ are $DRe^{-\mathbf{q}R}$-equivalent. For simplicity, we set $\mu = \mu'$.

Since μ is invariant under reflection and the weights are even integers, we see that $\mu \in \mathbb{N}\mathbf{\Pi}$ is a $\mathcal{R}$-symmetric formal sum. Let $\operatorname{lab} : \mathcal{L} \to \mathbf{\Pi}^*$ denote the corresponding legal labeling (see the example at the beginning of this section). It remains to define an admissible involution $\sigma : \mathcal{L} \to \mathcal{L}$.

Fix $\gamma^* \in \Gamma^*$. Let $X^+ \subset \mathcal{L}$ such that $a \in X^+$ if $\operatorname{lab}(a) = (\Pi, \gamma^*)$, where $\Pi \in \mathbf{\Pi}^+$. Define X^- similarly, and let $f^{+/-} : X^{+/-} \to \mathbf{\Pi}^*$ denote the corresponding restriction of the labeling map lab on the set $X^{+/-}$. (Observe that $f^+ = \mathcal{R} \circ f^- \circ \mathcal{R}_{\mathcal{L}}$.)

Denote by α^+ the restriction of $\widehat{\partial}\mu^+$ on $N^1(\sqrt{\gamma})$. (Define α^- similarly.) Observe that $\alpha^+ = \alpha^-$. Then by the definition of $\mathcal{L}$, the measure $\alpha^{+/-}$ is the $\widehat{\partial}$ of the push-forward of the counting measure on $X^{+/-}$ by the map $f^{+/-}$.

Define $g : X^- \to N^1(\sqrt{\gamma})$ by $g = \mathcal{A}_{1+i\pi} \circ f^-$. Then the measure $(\mathcal{A}_{1+i\pi})_*\alpha^-$ is the push-forward of the counting measure on X^- by the map g. Since α^+ and $(\mathcal{A}_{1+i\pi})_*\alpha^-$ are $2DRe^{-\mathbf{q}R}$-equivalent, by Theorem 3.2 there is a bijection $h : X^+ \to X^-$ such that $\operatorname{dis}(g(h(b)), f^+(b)) \leq 2DRe^{-\mathbf{q}R}$ for any $b \in X^+$. (Recall that dis denotes the Euclidean distance on $N^1(\sqrt{\gamma^*})$.)

We define $\sigma : X^+ \cup X^- \to X^+ \cup X^-$ by $\sigma(x) = h(x)$ for $x \in X^+$ and $\sigma(x) = h^{-1}(x)$ for $x \in X^-$. The map $\sigma : (X^+ \cup X^-) \to (X^+ \cup X^-)$ is an involution. By varying γ^* we construct the involution $\sigma : \mathcal{L} \to \mathcal{L}$. It follows from the definitions that σ is admissible and that the pair (lab, σ) satisfies the assumptions of Theorem 3.1.

4. Measures on skew pants and the frame flow

We start by outlining the construction of the measures from Theorem 3.4. Fix a sufficiently small number $\varepsilon > 0$, and let $r \gg 0$ denote any large enough real number. Set $R = 2(r - \log \frac{4}{3})$. We let $\mathbf{\Pi}_{\varepsilon,R}$ be the set of skew pants Π in $\mathbf{M}^3$ for which $|\mathbf{hl}(\delta) - \frac{R}{2}| < \varepsilon$ for all $\delta \in \partial\Pi$. In this section we will construct a measure μ on $\mathbf{\Pi}_{D\varepsilon,R}$ (for some universal constant $D > 0$) and a measure β_δ on each $N^1(\sqrt{\delta})$ such that for r large enough, we have

$$\left| K(\delta) \frac{d\beta_\delta}{d\mathrm{Eucl}_\delta} - 1 \right| \leq e^{-\mathbf{q}r},$$

and the measures $\widehat{\partial}\mu|_{N^1(\sqrt{\delta})}$ and β_δ are $Ce^{-\frac{r}{4}}$ equivalent, where Eucl_δ is the Euclidean measure on $N^1(\sqrt{\delta})$, the unique probability measure invariant under $\mathbb{C}/(2\pi i\mathbb{Z} + l(\delta)\mathbb{Z})$ action.

Let $\mathcal{F}(\mathbb{H}^3)$ denote the set of (unit) 2-frames $F_p = (p, u, n)$, where $p \in \mathbb{H}^3$ and the unit tangent vectors u and n are orthogonal at p. By $\mathbf{g}_t$, $t \in \mathbb{R}$, we denote the frame flow that acts on $\mathcal{F}(\mathbb{H}^3)$ and by Λ the invariant Liouville measure on $\mathcal{F}(\mathbb{H}^3)$. We then define a bounded nonnegative affinity function $\mathbf{a} = \mathbf{a}_{\varepsilon,r} : \mathcal{F}(\mathbb{H}^3) \times \mathcal{F}(\mathbb{H}^3) \to \mathbb{R}$ with the following properties (for r large enough):

(1) $\mathbf{a}(F_p, F_q) = \mathbf{a}(F_q, F_p)$ for every $F_p, F_q, \in \mathcal{F}(\mathbb{H}^3)$.
(2) $\mathbf{a}(A(F_p), A(F_q)) = \mathbf{a}(F_p, F_q)$ for every $A \in \mathbf{PSL}(2, \mathbb{C})$.
(3) If $\mathbf{a}(F_p, F_q) > 0$, and $F_p = (p, u, n)$ and $F_q = (q, v, m)$, then

$$|d(p, q) - r| < \varepsilon,$$

$$\Theta(n \,@\, q, m) < \varepsilon,$$

$$\Theta(u, v(p, q)) < Ce^{-\frac{r}{4}},\ \Theta(v, v(q, p)) < Ce^{-\frac{r}{4}},$$

where $\Theta(x, y)$ denotes the unoriented angle between vectors x and y, and $v(p, q)$ denotes the unit vector at p that is tangent to the geodesic segment from p and q. Here $n@q$ denotes the parallel transport of n along the geodesic segment from p (where n is based) to q.

(4) For every co-compact group $G < \mathbf{PSL}(2,\mathbb{C})$, we have

$$\left| \sum_{A \in G} \mathbf{a}(F_p, A(F_q)) - \frac{1}{\Lambda(\mathcal{F}(\mathbb{H}^3)/G)} \right| < e^{-\mathbf{q}_G r}.$$

The last property will follow from the exponential mixing of the frame flow on $\mathcal{F}(\mathbb{H}^3)/G$.

Now let $F_p = (p,u,n)$ and $F_q = (q,v,m)$ be two 2-frames in $\mathcal{F}(\mathbf{M}^3) = \mathcal{F}(\mathbb{H}^3)/\mathcal{G}$, where $\mathbf{M}^3$ is a closed hyperbolic three manifold and $\mathcal{G}$ is the corresponding Kleinian group. Let γ be a geodesic segment in $\mathbf{M}^3$ between p and q. We let $\widetilde{F_p}$ be an arbitrary lift of F_p to $\mathcal{F}(\mathbb{H}^3)$, and let $\widetilde{F_q}$ be the lift of F_q along γ. We let $\mathbf{a}_\gamma(F_p,F_q) = \mathbf{a}(\widetilde{F_p},\widetilde{F_q})$. By properties (1) and (2) this is well defined. Moreover, for any $F_p, F_q \in \mathcal{F}(\mathbf{M}^3)$,

$$\left| \sum_{\gamma} \mathbf{a}_\gamma(F_p, F_q) - \frac{1}{\Lambda(\mathcal{F}(\mathbf{M}^3))} \right| < e^{-\mathbf{q} r} \tag{26}$$

by property (4).

We define $\omega : \mathcal{F}(\mathbb{H}^3) \to \mathcal{F}(\mathbb{H}^3)$ by $\omega(p,u,n) = (p,\omega(u),n)$, where $\omega(u)$ is equal to u rotated around n for $\frac{2\pi}{3}$, using the right-hand rule. Observe that ω^3 is the identity and we let $\omega^{-1} = \overline{\omega}$. To any frame F we associate the tripod $T = (F,\omega(F),\omega^2(F))$, and likewise to any frame F we associate the "anti-tripod" $\overline{T} = (F,\overline{\omega}(F),\overline{\omega}^2(F))$. We have the similar definitions for frames in $\mathcal{F}(\mathbf{M}^3)$.

Let θ-graph be the 1-complex comprising three 1-cells (called h_0,h_1,h_2) each connecting two 0-cells (called $\underline{p}$ and $\underline{q}$). A connected pair of tripods is a pair of frames $F_p = (p,u,n)$, $F_q = (q,v,m)$ from $\mathcal{F}(\mathbf{M}^3)$, and three geodesic segments γ_i, $i = 0,1,2$, that connect p and q in $\mathbf{M}^3$. We abbreviate $\gamma = (\gamma_0,\gamma_1,\gamma_2)$, and we let

$$\mathbf{b}_\gamma(T_p,T_q) = \prod_{i=0}^{2} \mathbf{a}_{\gamma_i}(\omega^i(F_p), \overline{\omega}^i(F_q)).$$

We say (T_p,T_q,γ) is a well-connected pair of tripods along the triple of segments γ if $\mathbf{b}_\gamma(T_p,T_q) > 0$.

For any connected pair of tripods (T_p,T_q,γ), there is a continuous map from the θ-graph to $\mathbf{M}^3$ that is obvious up to homotopy (map $\underline{p}$ to p and $\underline{q}$ to q, and h_i to γ_i). If (T_p,T_q,γ) is a well-connected pair of tripods, then this map will be injective on the fundamental group $\pi_1(\theta - \text{graph})$. Moreover, the resulting pair of skew pants Π has the half-lengths $D\varepsilon$ close to $\frac{R}{2}$, where $R = 2(r - \log\frac{4}{3})$ (then the cuff lengths of the skew pants Π are close to R) and D is a universal constant. Recall that the collection of skew pants whose half-lengths are $D\varepsilon$ close to $\frac{R}{2}$ (for some large R and fixed ε) is called $\mathbf{\Pi}_{D\varepsilon,R}$.

We write $\Pi = \pi(T_p, T_q, \gamma)$, so π maps well-connected pairs of tripods to pairs of skew pants in $\mathbf{\Pi}_{D\varepsilon,R}$. We define the measure $\widetilde{\mu}$ on well-connected tripods by

$$d\widetilde{\mu}(T_p, T_q, \gamma) = \mathbf{b}_\gamma(T_p, T_q)\, d\lambda_T(T_p, T_q, \gamma),$$

where $\lambda_T(T_p, T_q, \gamma)$ is the product of the Liouville measure Λ (for $\mathcal{F}(\mathbf{M}^3)$) on the first two terms and the counting measure on the third term. The measure λ_T is infinite but $\mathbf{a}_\gamma(T_p, T_q)$ has compact support, so $\widetilde{\mu}$ is finite. We define the measure μ on $\mathbf{\Pi}_{D\varepsilon,R}$ by $\mu = \pi_* \widetilde{\mu}$. This is the measure from Theorem 3.4.

It remains to construct the measure β_δ and show the $Ce^{-\frac{r}{4}}$-equivalence of β_δ and $\widehat{\partial}\mu|_{N^1(\sqrt{\delta})}$. To any frame F we associate the bipod $B = (F, \omega(F))$, and likewise to any frame F we associate the "anti-bipod" $\overline{B} = (F, \overline{\omega}(F))$. We have the similar definitions for frames in $\mathcal{F}(\mathbf{M}^3)$.

We say that $(B_p, B_q, \gamma_0, \gamma_1)$ is a well-connected pair of bipods along the pair of segments γ_0 and γ_1 if

$$\mathbf{a}_{\gamma_0}(F_p, F_q)\mathbf{a}_{\gamma_1}(\omega(F_p), \overline{\omega}(F_q)) > 0.$$

Then the closed curve $\gamma_0 \cup \gamma_1$ is homotopic to a closed geodesic in $\mathbf{M}^3$. Given a closed geodesic $\delta \in \Gamma$ we let S_δ be the set of well-connected bipods $(B_p, B_q, \gamma_0, \gamma_1)$ such that $\gamma_0 \cup \gamma_1$ is homotopic to δ. (Note that S_δ is an open subset of the space of connected bipods which is the space of quadruples $(B_p, B_q, \gamma_0, \gamma_1)$, where B_p and B_q are tripods and γ_0 and γ_1 are geodesic segments in $\mathbf{M}^3$ connecting the points p and q.) The set S_δ of connected bipods carries the natural measure λ_B which is the product of the Liouville measures on the first two terms and the counting measure on the third and fourth.

Remark. One can show that if ε is small in terms of the injectivity radius of $\mathbf{M}^3$, then for two bipods B_p and B_q in $\mathcal{F}(\mathbf{M}^3)$ there exists at most one pair of segments (γ_0, γ_1) such that $(B_p, B_q, \gamma_0, \gamma_1)$ is a well-connected pair of bipods and that $\gamma_0 \cup \gamma_1$ is homotopic to δ. However, we do not use this.

Next, we define the action of the torus $\mathbb{C}/(2\pi i\mathbb{Z} + l(\delta)\mathbb{Z})$ on S_δ that leaves the measure λ_B invariant.

Let $\mathbb{T}_\delta$ be the open solid torus cover associated to δ (so δ lifts to a closed geodesic $\widetilde{\delta}$ in $\mathbb{T}_\delta$). Given a pair of well-connected bipods in S_δ, each bipod lifts in a unique way to a bipod in $\mathcal{F}(\mathbb{T}_\delta)$ such that the pair of the lifted bipods is well connected in $\mathbb{T}_\delta$. We denote by $\widetilde{S}_\delta$ the set of such lifts, so $\widetilde{S}_\delta$ is in one-to-one correspondence with S_δ. There is a natural action of the torus $\mathbb{C}/(2\pi i\mathbb{Z} + l(\delta)\mathbb{Z})$ on both $N^1(\delta)$ $(= N^1(\widetilde{\delta}))$ and on $\mathcal{F}(\mathbb{T}_\delta)$, and hence on $\widetilde{S}_\delta$ as well. Since $\widetilde{S}_\delta$ and S_δ are in one-to-one correspondence, we have the induced action of $\mathbb{C}/(2\pi i\mathbb{Z} + l(\delta)\mathbb{Z})$ on S_δ. This action leaves invariant the measure λ_B on S_δ.

For either choice of $\mathbf{hl}(\delta)$ there is a natural action of $\mathbb{C}/(2\pi i\mathbb{Z}+l(\delta)\mathbb{Z})$ on $N^1(\sqrt{\delta})$ via $\mathbb{C}/(2\pi i\mathbb{Z}+\mathbf{hl}(\delta)\mathbb{Z})$. We define in Section 4.7 a map $\mathbf{f}_\delta : S_\delta \to N^1(\sqrt{\delta})$ with two important properties. The first one is that $\mathbf{f}_\delta$ is equivariant with respect to the action of $\mathbb{C}/(2\pi i\mathbb{Z}+l(\delta)\mathbb{Z})$. The second property is as follows.

Let C_δ be the set of well-connected tripods (T_p, T_q, γ) for which $\gamma_0 \cup \gamma_1$ is homotopic to δ, and let $\chi : C_\delta \to S_\delta$ be the forgetting map, so $\chi(T_p, T_q, \gamma_0, \gamma_1, \gamma_2) = (B_p, B_q, \gamma_0, \gamma_1)$. Then for any pair of well-connected tripods $T = (T_p, T_q, \gamma) \in C_\delta$,

$$|\mathbf{f}_\delta(\chi(T)) - \mathrm{foot}_\delta(\pi(T))| < Ce^{-\frac{r}{4}}, \tag{27}$$

where $\pi(T)$ is the skew pants defined above. (Recall that the map $\mathrm{foot}_\delta(\Pi)$ that associates the foot to a pair of marked skew pants (Π, δ), $\delta \in \partial\Pi$, was defined in Section 3.) In other words, the map $\mathbf{f}_\delta$ predicts feet of the skew pants $\pi(T)$ (just by knowing the pair of well-connected bipods $\chi(T)$) up to an error of $Ce^{-\frac{r}{4}}$. This $Ce^{-\frac{r}{4}}$ comes from property (3) of the affinity function $\mathbf{a}$ defined above.

There are two more natural measures on S_δ. The first is $\chi_*(\widetilde{\mu}|_{C_\delta})$. The second is ν_δ, defined on S_δ by

$$d\nu_\delta(B_p, B_q, \gamma_0, \gamma_1) = \mathbf{a}_{\gamma_0}(F_p, F_q)\mathbf{a}_{\gamma_1}(\omega(F_p), \overline{\omega}(F_q))\, d\lambda_B(F_p, F_q, \gamma_0, \gamma_1),$$

where we recall that $\lambda_B(F_p, F_q, \gamma_0, \gamma_1)$ is the product of the Liouville measure on the first two terms and the counting measure on the last two. The two measures satisfy the fundamental inequality

$$\left| \frac{d\chi_*(\widetilde{\mu}|_{C_\delta})}{d\nu_\delta(B_p, B_q, \gamma_0, \gamma_1)} - \frac{1}{\Lambda(\mathcal{F}(\mathbf{M}^3))} \right| < Ce^{-\mathbf{q}r} \tag{28}$$

because the total affinity between $\omega^2(F_p)$ and $\overline{\omega}^2(F_q)$ (summing over all positive connections γ_2) is exponentially close to $\frac{1}{\Lambda(\mathcal{F}(\mathbf{M}^3))}$ by the inequality (26) above.

Moreover, since λ_B and the product $\mathbf{a}_{\gamma_0}(F_p, F_q)\mathbf{a}_{\gamma_1}(\omega(F_p), \overline{\omega}(F_q))$ are both invariant under the action of $\mathbb{C}/(2\pi i\mathbb{Z}+l(\delta)\mathbb{Z})$, we see that ν_δ is also invariant under the action of $\mathbb{C}/(2\pi i\mathbb{Z}+l(\delta)\mathbb{Z})$. Therefore $(\mathbf{f}_\delta)_*\nu_\delta$ is as well because $\mathbf{f}_\delta$ is $\mathbb{C}/(2\pi i\mathbb{Z}+l(\delta)\mathbb{Z})$ equivariant. It follows that $(\mathbf{f}_\delta)_*(\nu_\delta) = K_\delta \mathrm{Eucl}_\delta$ for some constant K_δ. Therefore, by (47),

$$\left| \frac{d(\mathbf{f}_\delta)_*(\widetilde{\mu}|_{C_\delta})}{dK'_\delta \mathrm{Eucl}_\delta} - 1 \right| < Ce^{-\mathbf{q}r}, \tag{29}$$

where $K'_\delta = K_\delta/\Lambda(\mathcal{F}(\mathbf{M}^3))$.

This measure $(\mathbf{f}_\delta)_*(\widetilde{\mu}|_{C_\delta})$ is our desired measure β_δ; it is $Ce^{-\frac{r}{4}}$-equivalent to the measure $\widehat{\partial}\mu|_{N^1(\sqrt{\delta})}$ because the later measure is just $(\mathrm{foot}_\delta)_*\pi_*(\widetilde{\mu}|_{C_\delta})$, and as we already said

$$|\mathbf{f}_\delta(\chi(T)) - \mathrm{foot}_\delta(\pi(T))| < Ce^{-\frac{r}{4}}$$

for every tripod T in C_δ.

We define $\mathbf{a}(F_p, F_q)$ in Section 4.4, and we prove that the skew pants $\pi(T_p, T_q, \gamma) \in \mathbf{\Pi}_{D\varepsilon,R}$ (for some universal constant $D > 0$) when $\mathbf{b}_\gamma(T_p, T_q) > 0$ in Sections 4.5 and 4.6, using preliminaries developed in Sections 4.1 and 4.2. We define $\mathbf{f}_\delta$ and prove (27) in Section 4.7, using preliminaries developed in Section 4.3. Finally we prove equation (29) in Section 4.8.

4.1. *The Chain Lemma.* Let $T^1(\mathbb{H}^3)$ denote the unit tangent bundle. Elements of $T^1(\mathbb{H}^3)$ are pairs (p, u), where $p \in \mathbb{H}^3$ and $u \in T^1_p(\mathbb{H}^3)$. For $u, v \in T^1_p(\mathbb{H}^3)$, we let $\mathbf{\Theta}(u, v)$ denote the unoriented angle between u and v. The function $\mathbf{\Theta}$ takes values in the interval $[0, \pi]$. For $a, b \in \mathbb{H}^3$, we let $v(a, b) \in T^1_a(\mathbb{H}^3)$ denote the unit vector at a that points toward b. If $v \in T^1_a(\mathbb{H}^3)$, then $v @ b \in T^1_b(\mathbb{H}^3)$ denotes the vector parallel transported to b along the geodesic segment connecting a and b. By (a, b, c) we denote the hyperbolic triangle with vertices $a, b, c \in \mathbb{H}^3$. For two points $a, b \in \mathbb{H}^3$, we let $|ab| = d(a, b)$.

Proposition 4.1. *Let $a, b, c \in \mathbb{H}^3$ and $v \in T^1_a(\mathbb{H}^3)$. Then the inequalities*

$$\mathbf{\Theta}(v @ b @ c @ a, v) \leq \operatorname{Area}(abc) \leq |bc|$$

hold, where $\operatorname{Area}(abc)$ *denotes the hyperbolic area of the triangle* (a, b, c).

Proof. It follows from the Gauss-Bonnet theorem that the inequality $\mathbf{\Theta}(v@ b @ c @ a, v) \leq \operatorname{Area}(abc)$ holds for every $v \in T^1_a(\mathbb{H}^3)$. Moreover, if v is in the plane of the triangle (a, b, c), then the equality $\mathbf{\Theta}(v @ b @ c @ a, v) = \operatorname{Area}(abc)$ holds.

We now prove that in every hyperbolic triangle the length of a side is greater than the area of the triangle; that is, we prove $|bc| \geq \operatorname{Area}(abc)$. Consider the geodesic ray that starts at b and that contains a, and let $a' \in \partial\mathbb{H}^2$ be the point where this ray hits the ideal boundary. Then the triangle (a, b, c) is contained in the triangle (a', b, c), so it suffices to show that $\operatorname{Area}(a', b, c) \leq |bc|$. Thus we may assume that the vertex a is a point on $\partial\mathbb{H}^2$.

Considering the standard model of the upper half-plane $\mathbb{H}^2 = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$, we can assume that $a = \infty$ and that the geodesic segment (bc) lies on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. By the first part of the proposition we know that $\operatorname{Area}(abc)$ is equal to α, where α is the unoriented angle between the Euclidean lines l_b and l_c, where l_b contains 0 and b and l_c contains 0 and c ($0 \in \mathbb{C}$ denotes the origin). Since b and c lie on the unit circle, we have that α is also equal to the Euclidean length of the arc of the unit circle between b and c. On the other hand, the hyperbolic length of this arc (which is the geodesic segment (bc) between b and c in the hyperbolic metric) is strictly larger than α because the density of the hyperbolic metric is $y^{-1}|dz|$, which is greater than 1 on the unit circle. We have

$$\operatorname{Area}(abc) \leq \alpha \leq |bc|,$$

which proves the proposition. $\square$

The following claim will be used in the proof of Theorem 4.1 below.

CLAIM 4.1. *Let $a, b, c \in \mathbb{H}^3$. Then the inequality*

$$\mathbf{\Theta}(v(c,a), v(b,a)@c) \le \mathbf{\Theta}(v(a,b), v(a,c)) + \text{Area}(abc) \le \mathbf{\Theta}(v(a,b), v(a,c)) + |bc|$$

holds.

Proof. We have

$$\begin{aligned}\mathbf{\Theta}(v(c,a), v(b,a) @ c) &= \mathbf{\Theta}(v(c,a) @ a, v(b,a) @ c @ a) \\ &= \mathbf{\Theta}(-v(a,c), v(b,a) @ c @ a) \\ &\le \mathbf{\Theta}(-v(a,c), -v(a,b)) + \mathbf{\Theta}(-v(a,b), v(b,a) @ c @ a) \\ &= \mathbf{\Theta}(v(a,c), v(a,b)) + \mathbf{\Theta}(-v(a,b) @ b, v(b,a) @ c @ a @ b) \\ &= \mathbf{\Theta}(v(a,c), v(a,b)) + \mathbf{\Theta}(v(b,a), v(b,a) @ c @ a @ b).\end{aligned}$$

By the previous proposition we have $\mathbf{\Theta}(v(b,a), v(b,a)@c@a@b) \le \text{Area}(abc) \le |bc|$, and we are finished. □

The following two propositions are elementary and follow from the cosh rule for hyperbolic triangles.

PROPOSITION 4.2. *Let (a,b,c) be a hyperbolic triangle such that $|ab| = l_1$ and $|bc| = \eta$. Then for l_1 large and η small enough, the inequality*

$$\mathbf{\Theta}(v(a,b), v(a,c)) \le D\eta e^{-l_1}$$

holds for some constant $D > 0$.

PROPOSITION 4.3. *Let (a,b,c) be a hyperbolic triangle, and set $|ab| = l_1$, $|cb| = l_2$ and $|ac| = l$. Let $\eta = \pi - \mathbf{\Theta}(v(b,a), v(b,c))$. Then for l_1 and l_2 large, we have*

(1) $|(l-(l_1+l_2)) + \log 2 - \log(1+\cos\eta)| \le De^{-2\min\{l_1,l_2\}}$ *for any* $0 \le \eta \le \frac{\pi}{2}$,
(2) $|l - (l_1 + l_2)| \le D\eta$ *for η small,*
(3) $\mathbf{\Theta}(v(a,c), v(a,b)) \le D\eta e^{-l_1}$ *for η small,*
(4) $\mathbf{\Theta}(v(c,a), v(c,b)) \le D\eta e^{-l_2}$ *for η small*

for some constant $D > 0$.

The following theorem (the "Chain Lemma") allows us to estimate the geometry of a segment that is formed from a chain of long segments that nearly meet at their endpoints. It will be used in Section 4.2 to estimate the complex length of a closed geodesic formed from a closed chain of such segments.

THEOREM 4.1. *Suppose that $a_i, b_i \in \mathbb{H}^3$, $i = 1, \dots, k$, and*

(1) $|a_i b_i| \ge Q$;
(2) $|b_i a_{i+1}| \le \varepsilon$;
(3) $\mathbf{\Theta}(v(b_i, a_i) @ a_{i+1}, -v(a_{i+1}, b_{i+1})) \le \varepsilon$.

Suppose also that $n_i \in T^1_{a_i}(\mathbb{H}^3)$ *is a vector at* a_i *normal to* $v(a_i, b_i)$ *and*

$$\mathbf{\Theta}(n_i \,@\, b_i, n_{i+1} \,@\, b_i) \leq \varepsilon.$$

Then for ε *small and* Q *large and some constant* $D > 0$,

$$\left| |a_1 b_k| - \sum_{i=1}^{k} |a_i b_i| \right| \leq kD\varepsilon, \tag{30}$$

$$\mathbf{\Theta}(v(a_1, b_k), v(a_1, b_1)) < kD\varepsilon e^{-Q} \text{ and } \mathbf{\Theta}(v(b_k, a_1), v(b_k, a_k)) < kD\varepsilon e^{-Q}, \tag{31}$$

$$\mathbf{\Theta}(v(a_1, a_k), v(a_1, b_1)) < 2kD\varepsilon e^{-Q} \text{ if } k > 1, \tag{32}$$

$$\mathbf{\Theta}(n_k, n_1 \,@\, a_k) \leq 5k\varepsilon. \tag{33}$$

We can think of the sequence of geodesic segments from a_i to b_i as forming an "ε-chain," and we can think of the broken segment connecting $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ as the concatenation of the ε-chain, and the geodesic segment from a_1 to b_k (or a_k) as the geodesic representative of the concatenation. Then the Chain Lemma is describing the relationship between the concatenation of an ε-chain and its geodesic representative, and also estimating the discrepancy between parallel transport along the concatenation and transport along its geodesic representative.

Proof. By induction. Suppose that the statement is true for some $k \geq 1$. We need to prove the above inequalities for $k + 1$.

We first prove inequalities (31) and (32). By the triangle inequality we have

$$\mathbf{\Theta}(v(a_1,b_k), v(a_1,b_{k+1})) \leq \mathbf{\Theta}(v(a_1, b_k), v(a_1,a_{k+1})) + \mathbf{\Theta}(v(a_1,a_{k+1}), v(a_1,b_{k+1})).$$

By Proposition 4.2 we have $\mathbf{\Theta}(v(a_1, b_k), v(a_1, a_{k+1})) \leq D_1 \varepsilon e^{-Q}$, where D_1 is the constant from Proposition 4.2. By (34) and Proposition 4.3 we have $\mathbf{\Theta}(v(a_1, a_{k+1}), v(a_1, b_{k+1})) \leq 2D_2 \varepsilon e^{-Q}$, where D_2 is from Proposition 4.3. Together this shows

$$\mathbf{\Theta}(v(a_1, b_k), v(a_1, b_{k+1})) \leq D\varepsilon e^{-Q}.$$

Then by the triangle inequality and the induction hypothesis, we have

$$\begin{aligned}\mathbf{\Theta}(v(a_1, b_{k+1}), v(a_1, b_1)) &\leq \mathbf{\Theta}(v(a_1,b_k), v(a_1,b_1)) + \mathbf{\Theta}(v(a_1, b_k), v(a_1, b_{k+1})) \\ &\leq kD\varepsilon e^{-Q} + D\varepsilon e^{-Q} = (k+1)D\varepsilon e^{-Q},\end{aligned}$$

which proves the first inequality in (31). The second one follows by symmetry. Inequality (32) follows from (31) and Proposition 4.2.

Next, we prove inequality (30). By the triangle inequality, we have

$$\mathbf{\Theta}(v(a_1, a_{k+1}), v(a_1, b_k)) \leq \mathbf{\Theta}(v(a_1, a_{k+1}), v(a_1, b_1)) + \mathbf{\Theta}(v(a_1, b_1), v(a_1, b_k)),$$

and then applying (31) and (32), we get

$$\mathbf{\Theta}(v(a_1, a_{k+1}), v(a_1, b_k)) \leq 2(k+1)D\varepsilon e^{-Q} + kD\varepsilon e^{-Q} < \varepsilon$$

for Q large enough. Then by Claim 4.1, we have

$$\mathbf{\Theta}(v(a_{k+1}, a_1), v(b_k, a_1) @ a_{k+1}) \leq \mathbf{\Theta}(v(a_1, a_{k+1}), v(a_1, b_k)) + |b_k a_{k+1}| \leq 2\varepsilon.$$

Combining this inequality with assumption (3) of the theorem, and by the triangle inequality we obtain $\mathbf{\Theta}(v(a_{k+1}, a_1), -v(a_{k+1}, b_{k+1})) \leq 3\varepsilon$. Therefore the inequality

$$\pi - \mathbf{\Theta}(v(a_{k+1}, a_1), v(a_{k+1}, b_{k+1})) \leq 3\varepsilon \tag{34}$$

holds. (Observe that the same inequality holds for all $1 \leq i \leq k$.)

It follows from Proposition 4.3 and (34) that $\Big||a_1 a_{k+1}| + |a_{k+1} b_{k+1}| - |a_1 b_{k+1}|\Big| \leq 3D_1\varepsilon$, where D_1 is the constant from Proposition 4.3. Since by the triangle inequality

$$\Big||a_1 b_k| - |a_1 a_{k+1}|\Big| \leq \varepsilon,$$

we obtain

$$\Big||a_1 b_k| + |a_{k+1} b_{k+1}| - |a_1 b_{k+1}|\Big| \leq D\varepsilon.$$

This proves the induction step for the inequality (30).

It remains to prove (33). Using the induction hypothesis and the assumptions in the statement of this theorem, we obtain the following string of inequalities:

$$\begin{aligned}
\mathbf{\Theta}(n_{k+1}, n_1 @ a_{k+1}) &= \mathbf{\Theta}(n_{k+1} @ b_k, n_1 @ a_{k+1} @ b_k) \\
&\leq \mathbf{\Theta}(n_{k+1} @ b_k, n_k @ b_k) + \mathbf{\Theta}(n_k @ b_k, n_1 @ a_{k+1} @ b_k) \\
&\leq \varepsilon + \mathbf{\Theta}(n_k @ b_k, n_1 @ a_{k+1} @ b_k) \\
&\leq \varepsilon + \mathbf{\Theta}(n_k @ b_k, n_1 @ a_k @ b_k) \\
&\quad + \mathbf{\Theta}(n_1 @ a_k @ b_k, n_1 @ a_{k+1} @ b_k) \\
&\leq (5k+1)\varepsilon + \mathbf{\Theta}(n_1 @ a_k @ b_k, n_1 @ a_{k+1} @ b_k) \\
&\leq (5k+1)\varepsilon + \mathbf{\Theta}(n_1 @ a_k @ b_k, n_1 @ b_k) \\
&\quad + \mathbf{\Theta}(n_1 @ b_k, n_1 @ a_{k+1} @ b_k).
\end{aligned}$$

By (34) we have

$$\mathbf{\Theta}(n_1 @ a_k @ b_k, n_1 @ b_k) \leq 3\varepsilon,$$

and by Claim 4.1 we have

$$\mathbf{\Theta}(n_1 @ b_k, n_1 @ a_{k+1} @ b_k) \leq \varepsilon.$$

Combining these estimates gives

$$\mathbf{\Theta}(n_{k+1}, n_1 @ a_{k+1}) \leq (5k+5)\varepsilon,$$

which proves the induction step for (33). ∎

4.2. *Corollaries of the Chain Lemma.* For $X \in \mathbf{PSL}(2,\mathbb{C})$, we write $X(z) = \frac{az+b}{cz+d}$, where $ad-bc=1$. The following proposition will provide the bridge between the Chain Lemma and Lemma 4.1.

PROPOSITION 4.4. *Let $p, q \in \mathbb{H}^3$ and $A \in \mathbf{PSL}(2,\mathbb{C})$ be such that $A(p)=q$. Suppose that for every $u \in T^1_p(\mathbb{H}^3)$ we have $\mathbf{\Theta}(A(u), u \,@\, q) \le \varepsilon$. Then for ε small enough and $d(p,q)$ large enough, and for some constant $D>0$, we have*

1. *the transformation A is loxodromic*;
2. $|\mathbf{l}(A) - d(p,q)| \le D\varepsilon$;
3. *if* $\operatorname{axis}(A)$ *denotes the axis of A, then $d(p, \operatorname{axis}(A)), d(q, \operatorname{axis}(A)) \le D\varepsilon$.*

Proof. We may assume that the points p and q lie on the geodesic that connects 0 and ∞, such that q is the point with coordinates $(0,0,1)$ in $\mathbb{H}^3$, and p is $(0,0,x)$ for some $0<x<1$. Let $B \in \mathbf{PSL}(2,\mathbb{C})$ be given by $B(z)=Kz$, where $\log K = d(p,q)$. Since K is a positive number, it follows that for every $u \in T^1_p(\mathbb{H}^3)$, the identity $B(u) = u \,@\, q$.

Let $A = C \circ B$, where $C \in \mathbf{PSL}(2,\mathbb{C})$ fixes the point $(0,0,1) \in \mathbb{H}^3$. It follows that for every $u \in T^1_{(0,0,1)}(\mathbb{H}^3)$, we have $\mathbf{\Theta}(u, C(u)) \le \varepsilon$. This implies that for some $a,b,c,d \in \mathbb{C}$, $ad-bc=1$, we have

$$C(z) = \frac{az+b}{cz+d},$$

and $|a-1|, |b|, |c|, |d-1| \le D_1\varepsilon$ for some constant $D_1>0$. Then

$$A(z) = \frac{a\sqrt{K}z + \frac{b}{\sqrt{K}}}{\sqrt{K}cz + \frac{d}{\sqrt{K}}},$$

and we find

$$\operatorname{tr}(A) = a\sqrt{K} + \frac{d}{\sqrt{K}},$$

where $\operatorname{tr}(A)$ denotes the trace of A. Since $|a-1|, |d-1| \le D_1\varepsilon$, we see that for K large enough, the real part of the trace $\operatorname{tr}(A)$ is a positive number >2, which shows that A is loxodromic. On the other hand, $\operatorname{tr}(A) = 2\cosh(\frac{\mathbf{l}(A)}{2})$. This shows that $|\mathbf{l}(A) - \log K| \le D_2\varepsilon$ for some constant $D_2>0$.

Let $z_1, z_2 \in \overline{\mathbb{C}}$ denote the fixed points of A. We find

$$z_{1,2} = \frac{(a-\frac{d}{K}) \pm \sqrt{(a-\frac{d}{K})^2 + \frac{4bc}{K}}}{2c}.$$

Then for K large enough, we have

$$|z_1| \le \varepsilon, \ \ |z_2| \ge \frac{3}{\varepsilon}.$$

This shows that $d(q, \operatorname{axis}(A)) = d((0,0,1), \operatorname{axis}(A)) \le D_3\varepsilon$, for some constant $D_3>0$. The inequality $d(p, \operatorname{axis}(A)) \le D_3\varepsilon$ follows by symmetry. □

The following lemma is a corollary of Theorem 4.1 and the previous proposition. It provides an estimate for the complex length of the closed geodesic that is freely homotopic to the concatenation of a closed chain of geodesic segments.

LEMMA 4.1. *Let* $a_i, b_i \in \mathbb{H}^3$, $i \in \mathbb{Z}$ *such that*

(1) $|a_i b_i| \geq Q$;
(2) $|b_i a_{i+1}| \leq \varepsilon$;
(3) $\mathbf{\Theta}(v(b_i, a_i) \,@\, a_{i+1}, -v(a_{i+1}, b_{i+1})) \leq \varepsilon$.

Suppose also that $n_i \in T^1_{a_i}(\mathbb{H}^3)$ *is a vector at* a_i *normal to* $v(a_i, b_i)$ *and*

$$\mathbf{\Theta}(n_i \,@\, b_i, n_{i+1} \,@\, b_i) \leq \varepsilon.$$

Suppose there exists $A \in \mathbf{PSL}(2,\mathbb{C})$ *and* $k > 0$ *be such that* $A(a_i) = a_{i+k}$, $A(b_i) = b_{i+k}$, *and* $A(n_i) = n_{i+k}$, $i \in \mathbb{Z}$. *Then for* ε *small and* Q *large*, A *is a loxodromic transformation and*

$$\left| \mathbf{l}(A) - \sum_{i=0}^{k} |a_i b_i| \right| \leq k D \varepsilon \tag{35}$$

for some constant $D>0$. *Moreover*, $a_i, b_i \in \mathcal{N}_{Dk\varepsilon}(\text{axis}(A))$, *where* $\mathcal{N}_{Dk\varepsilon}(\text{axis}(A)) \subset \mathbb{H}^3$ *is the* $Dk\varepsilon$ *neighbourhood of* $\text{axis}(A)$.

We can think of taking $a_i, b_i \in \mathbb{H}^3/A$ (or even in some hyperbolic 3-manifold N) and $i \in \mathbb{Z}/k\mathbb{Z}$. We must then describe the geodesic segments from a_i to b_i, which we will use to determine $v(b_i, a_i)$ and $n_i \,@\, b_i$, and so forth. (As long as the injectivity radius of N is greater than ε, there are unique choices of geodesic segments from b_i to a_{i+1} with length less than ε.) We then think of this sequence of segments as a "closed ε-chain" and $\text{axis}(A)/A$ as its geodesic representative.

Proof. Let $v_0 = v(a_0, b_0)$. Observe that $A(v_0) = v(a_k, b_k)$. First we show that the inequality $\mathbf{\Theta}(A(v_0), v_0 \,@\, a_k) \leq 4\varepsilon$ holds for Q large enough.

Recall that for Q large enough, the inequality (34) holds (see the proof of Theorem 4.1); that is, we have

$$\pi - \mathbf{\Theta}(v(a_k, a_0), v(a_k, b_k)) \leq 3\varepsilon.$$

Since $\mathbf{\Theta}(v(a_k, a_0), -v(a_k, b_k)) = \mathbf{\Theta}(v(a_0, a_k), v(a_k, b_k) \,@\, a_0)$, we have

$$\mathbf{\Theta}(v(a_0, a_k), v(a_k, b_k) \,@\, a_0) \leq 3\varepsilon.$$

On the other hand, it follows from (32) that for Q large enough, we have

$$\mathbf{\Theta}(v(a_0, a_k), v(a_0, b_0)) \leq \varepsilon,$$

so by the triangle inequality we obtain

$$\begin{aligned}\boldsymbol{\Theta}(v(a_0,b_0),v(a_k,b_k)\,@\,a_0) &\le \boldsymbol{\Theta}(v(a_0,a_k),v(a_0,b_0)) \\ &\quad+ \boldsymbol{\Theta}(v(a_0,a_k),v(a_k,b_k)\,@\,a_0) \\ &\le 4\varepsilon.\end{aligned}$$

Since $v(a_k,b_k)\,@\,a_0\,@\,a_k = v(a_k,b_k)$, we find $\boldsymbol{\Theta}(v(a_0,b_0),v(a_k,b_k)\,@\,a_0) = \boldsymbol{\Theta}(v_0\,@\,a_k,A(v_0))$. Thus we have proved the inequality $\boldsymbol{\Theta}(v_0\,@\,a_k,A(v_0)) \le 4\varepsilon$.

Next, from (33) we find $\boldsymbol{\Theta}(n_k,n_0\,@\,a_k) \le 4k\varepsilon$. Since v_0 is normal to n_0, and the parallel transport preserves angles, it follows that

$$\boldsymbol{\Theta}(u\,@\,a_k,A(u)) \le 4kD\varepsilon, \text{ for every vector } u \in T^1_{a_0}(\mathbb{H}^3). \tag{36}$$

On the other hand, the inequality

$$\left| d(a_0,A(a_0)) - \sum_{i=0}^{k} |a_ib_i| \right| \le kD\varepsilon \tag{37}$$

follows from (30). The lemma now follows from Proposition 4.4. ☐

4.3. *Preliminary propositions.* In this subsection we will prove two results in hyperbolic geometry (Lemmas 4.2 and 4.3) that we will use in Section 4.7. The following proposition is elementary.

PROPOSITION 4.5. *Let α be a geodesic in $\mathbb{H}^3$, and let $p_1,p_2 \in \mathbb{H}^3$ be two points such that $d(p_1,p_2) \le C$ and $d(p_i,\alpha) \ge s$, $i=1,2$, for some constants $C,s>0$. Let η_i be the oriented geodesic that contains p_i and is normal to α and that is oriented from α to p_i. Then there exists a constant $D>0$, that depends only on C, such that $|\mathbf{d}_\alpha(\eta_1,\eta_2)| \le De^{-s}$.*

Let α and β be two oriented geodesics in $\mathbb{H}^3$ such that $d(\alpha,\beta)>0$, and let γ be their common orthogonal that is oriented from α to β. We observe that both α and β are mapped to $-\alpha$ and $-\beta$ respectively, by a 180 degree rotation around γ. Let $t \in \mathbb{R}$ and let $q:\mathbb{R}\to\beta$ be parametrization by arc length such that $q_0(0)=\beta\cap\gamma$. Let $\delta(t)$ be the geodesic that contains $q_0(t)$ and is orthogonal to α, and is oriented from α to $q_0(t)$. The following proposition follows from the symmetry of α and β around γ. Recall that the complex distance is well defined (mod $2\pi i$), so we can always choose a complex distance such that its imaginary part is in the interval $(-\pi,\pi]$.

PROPOSITION 4.6. *Assume that $\alpha \ne \beta$. Then $\mathbf{d}(q_0(t_1),\alpha) = \mathbf{d}(q_0(t_2),\alpha)$ if and only if $|t_2|=|t_1|$. Moreover, if for some $t\in\mathbb{R}$, we can choose the complex distance $\mathbf{d}_\alpha(\delta(-t),\delta(t))$ such that*

$$-\pi < \operatorname{Im}\left(\mathbf{d}_\alpha(\delta(-t),\delta(t))\right) < \pi,$$

then

$$\mathbf{d}_\alpha(\delta(-t),\gamma) = \frac{1}{2}\mathbf{d}_\alpha(\delta(-t),\delta(t)). \tag{38}$$

Remark. Observe that if α and β do not intersect we can always choose the complex distance $\mathbf{d}_\alpha(\delta(-t),\delta(t))$ such that

$$-\pi < \operatorname{Im}\Big(\mathbf{d}_\alpha(\delta(-t),\delta(t))\Big) < \pi.$$

Assuming the above notation we have the following.

PROPOSITION 4.7. *Let $s(t) = d(q_0(t),\alpha)$. Suppose that $d(\alpha,\beta) \leq 1$. Then for $s(t)$ large enough, we have*

$$s(t+h) = s(t) + h + o(1) \text{ as } t \to \infty,$$
$$s(t+h) = s(t) - h + o(1) \text{ as } t \to \infty$$

for any $|h| \leq \frac{s(t)}{2}$.

Proof. By the triangle inequality, we have

$$\begin{aligned} s(t) &= d(q_0(t),\alpha) \\ &\leq d(q_0(t),q_0(0)) + d(q_0(0),\alpha) \\ &\leq |t| + 1 \end{aligned}$$

since $d(q_0(0),\alpha) = d(\alpha,\beta) \leq 1$. That is, we have

$$s(t) \leq |t| + 1. \tag{39}$$

It follows from (39) that $s(t)$ large implies that $|t|$ is large.

Recall the following formula (9) from Section 1:

$$\sinh^2(d(q_0(t),\alpha)) = \sinh^2(d(\alpha,\beta))\cosh^2(t) + \sin^2(\operatorname{Im}[\mathbf{d}_\gamma(\alpha,\beta)])\sinh^2(t).$$

Combining this with (39), we get

$$e^{2s(t)} = e^{2t}\Big(\sinh^2(d(\alpha,\beta)) + \sin^2(\operatorname{Im}[\mathbf{d}_\gamma(\alpha,\beta)])\Big) + O(1),$$

which proves the proposition. □

We can define the foot of the geodesic β on α as the normal to α pointing along γ. The lemma below estimates how the foot of β on α moves when β is moved (and β is very close to α).

Let $\varepsilon \in \mathbb{D}$ be a complex number, and let $r > 0$. Assume that

$$\mathbf{d}_\gamma(\alpha,\beta) = e^{-\frac{r}{2}+\varepsilon}. \tag{40}$$

Then there exists $\varepsilon_0 > 0$ such that for every for $|\varepsilon| < \varepsilon_0$, for every $r > 1$, and for every $t \in \mathbb{R}$, we can choose the complex distance $\mathbf{d}_\alpha(\delta(-t),\delta(t))$ such that

$$-\frac{\pi}{4} < \operatorname{Im}\mathbf{d}_\alpha(\delta(-t),\delta(t)) < \frac{\pi}{4}. \tag{41}$$

Let β_1 be another geodesic with a parametrization by arc length $q_1 : \mathbb{R} \to \beta_1$. We let γ_1 denote the common orthogonal between α and β_1 that is oriented from α to β_1. We have

LEMMA 4.2. *Assume that α and β satisfy* (40). *Let $C > 0$, and suppose that for some $t_1, t'_1, t_2, t'_2 \in \mathbb{R}$, where $t_1 < 0 < t_2$, we have*

(1) $d(q_1(t'_1), q_0(t_1)), d(q_1(t'_2), q_0(t_2)) \leq C$;
(2) $|d(q_0(t_1), \alpha) - d(q_0(t_2), \alpha)| \leq C$;
(3) $d(q_0(t_1), \alpha) > \frac{r}{4} - C$.

Then for $|\varepsilon| < \varepsilon_0$ and for r large, we have

$$\mathbf{d}_\alpha(\gamma, \gamma_1) \leq De^{-\frac{r}{4}}$$

for some constant $D > 0$, where D only depends on C.

Proof. The constants D_i defined below all depend only on C. From (40) we have $d(\alpha, \beta) < e^{-\frac{r}{2}+1}$. Since

$$d(q_1(t'_1), q_0(t_1)), d(q_1(t'_2), q_0(t_2)) \leq C,$$

it follows that for r large we have $d(\alpha, \beta_1) = o(1)$ and, in particular, we have $d(\alpha, \beta_1) < 1$. By the triangle inequality we obtain $|d(q_1(t'_1), \alpha) - d(q_1(t'_2), \alpha)| \leq D_1$. Then it follows from the previous proposition that the inequalities $|t_2 + t_1|, |t'_2 + t'_1| \leq D_2$ hold. This implies that $d(q_0(-t_1), q_1(-t'_1)) \leq D_3$.

Let $\delta_1(t)$ be the geodesic that contains $q_1(t)$ and is orthogonal to α, and is oriented from α to $q_1(t)$. Now we apply Proposition 4.5 and find that

$$|\mathbf{d}_\alpha(\delta(-t_1), \delta_1(-t'_1))| \leq D_4 e^{-\frac{r}{4}}.$$

Similarly

$$|\mathbf{d}_\alpha(\delta(t_1), \delta_1(t'_1))| \leq D_4 e^{-\frac{r}{4}}.$$

It follows from (41) and the above two inequalities that for r large, we can choose the complex distance $\mathbf{d}_\alpha(\delta_1(-t'_1), \delta_1(t'_1))$ such that

$$-\frac{\pi}{3} < \operatorname{Im} \mathbf{d}_\alpha(\delta_1(-t'_1), \delta_1(t'_1)) < \frac{\pi}{3}.$$

In particular, we can choose the complex distances

$$\mathbf{d}_\alpha(\delta(-t_1), \delta(t_1)) \text{ and } \mathbf{d}_\alpha(\delta_1(-t'_1), \delta_1(t'_1))$$

such that the corresponding imaginary parts belong to the interval $(-\pi, \pi)$, and such that

$$\Big|\mathbf{d}_\alpha(\delta(-t_1), \delta(t_1)) - \mathbf{d}_\alpha(\delta_1(-t'_1), \delta_1(t'_1))\Big| \leq 2D_4 e^{-\frac{r}{4}}.$$

The proof now follows from Proposition 4.6 and the triangle inequality. □

Lemma 4.3. *Let $A \in \mathbf{PSL}(2,\mathbb{C})$ be a loxodromic transformation with the axis γ. Let $p, q \in (\partial\mathbb{H}^3 \setminus endpoints(A))$, and denote by α_1 the oriented geodesic from p to q, and by α_2 the oriented geodesic from q to $A(p)$. We let δ_j be the common orthogonal between γ and α_j, oriented from γ to α_j. Then*

$$\mathbf{d}_\gamma(\delta_1,\delta_2) = (-1)^j \frac{\mathbf{l}(A)}{2} + k\pi i$$

for some $k \in \{0,1\}$ and some $j \in \{1,2\}$.

Alternatively, we can think of p and q as points on the ideal boundary of $\mathbb{H}^3/A$, and α_1 and α_2 as two geodesics from p to q, such that $\alpha_1 \cdot (\alpha_2)^{-1}$ is freely homotopic to the core curve of the solid torus $\mathbb{H}^3/A$.

Proof. Let α_3 be the oriented geodesic from $A(p)$ to $A(q)$, and let δ_3 be the common orthogonal between γ and α_3 (oriented from γ to α_3). Consider the right-angled hexagon H_1 with the sides $L_0 = \gamma$, $L_1 = \delta_1$, $L_2 = \alpha_1$, $L_3 = q$, $L_4 = \alpha_2$, and $L_5 = \delta_2$. Let H_2 be the right-angled hexagon with the sides $L'_0 = \gamma$, $L'_1 = \delta_3$, $L'_2 = \alpha_3$, $L'_3 = A(p)$, $L'_4 = \alpha_2$, and $L'_5 = \delta_2$. Note that H_1 is a degenerate hexagon since the common orthogonal between α_1 and α_2 has shrunk to a point on $\partial\mathbb{H}^3$. The same holds for H_2. We note that the cosh formula is valid in degenerate right-angled hexagons and every such hexagon is uniquely determined by the complex lengths of its three alternating sides.

Denote by σ_k and σ'_k the complex lengths of the sides L_k and L'_k respectively. By changing the orientations of the sides L_k and L'_k if necessary, we can arrange that $\sigma_1 = \sigma'_1$, $\sigma_5 = \sigma'_5$ and $\sigma_3 = \sigma'_3 = 0$ (see Section 2.2 in [12]). This shows that the hexagons H_1 and H_2 are isometric modulo the orientations of the sides, and this implies the equality $\sigma_0 = \sigma'_0$. On the other hand, changing orientations of the sides can change the complex length of a side by changing its sign and/or adding πi. This proves the lemma. □

4.4. *The two-frame bundle and the well-connected frames.* Let $\mathcal{F}(\mathbb{H}^3)$ denote the two frame bundle over $\mathbb{H}^3$. Elements of $\mathcal{F}(\mathbb{H}^3)$ are frames $F = (p,u,n)$, where $p \in \mathbb{H}^3$ and $u, n \in T^1_p(\mathbb{H}^3)$ are two orthogonal vectors at p (here $T^1(\mathbb{H}^3)$ denotes the unit tangent bundle). The group $\mathbf{PSL}(2,\mathbb{C})$ acts naturally on $\mathcal{F}(\mathbb{H}^3)$. For (p_i,u_i,n_i), $i = 1,2$, we define the distance function $\mathcal{D}$ on $\mathcal{F}(\mathbb{H}^3)$ by

$$\mathcal{D}((p_1,u_1,n_1),(p_2,u_2,n_2)) = d(p_1,p_2) + \mathbf{\Theta}(u'_1,u_2) + \mathbf{\Theta}(n'_1,n_2),$$

where $u'_1, n'_1 \in T^1_{p_2}(\mathbb{H}^3)$, are the parallel transports of u_1 and v_1 along the geodesic that connects p_1 and p_2. One can check that $\mathcal{D}$ is invariant under the action of $\mathbf{PSL}(2,\mathbb{C})$. (We do not claim that $\mathcal{D}$ is a metric on $\mathcal{F}(\mathbb{H}^3)$.) By $\mathcal{N}_\varepsilon(F) \subset \mathcal{F}(\mathbb{H}^3)$ we denote the ε ball around a frame $F \in \mathcal{F}(\mathbb{H}^3)$.

Recall the standard geodesic flow $\mathbf{g}_r : T^1(\mathbb{H}^3) \to T^1(\mathbb{H}^3)$, $r \in \mathbb{R}$. The flow action extends naturally on $\mathcal{F}(\mathbb{H}^3)$; that is, the map $\mathbf{g}_r : \mathcal{F}(\mathbb{H}^3) \to \mathcal{F}(\mathbb{H}^3)$

is given by $\mathbf{g}_r(p,u,n) = (p_1,u_1,n_1)$, where $(p_1,u_1) = \mathbf{g}_r(p,u)$ and n_1 is the parallel transport of the vector n along the geodesic that connects p and p_1. The flow $\mathbf{g}_r$ on $\mathcal{F}(\mathbb{H}^3)$ is called the frame flow. The space $\mathcal{F}(\mathbb{H}^3)$ is equipped with the Liouville measure Λ, which is invariant under the frame flow and under the $\mathbf{PSL}(2,\mathbb{C})$ action. Locally on $\mathcal{F}(\mathbb{H}^3)$, the measure Λ is the product of the standard Liouville measure for the geodesic flow and the Lebesgue measure on the unit circle.

Recall that $\mathbf{M}^3 = \mathbb{H}^3/\mathcal{G}$ denotes a closed hyperbolic three manifold, and $\mathcal{G}$ from now on denotes an appropriate Kleinian group. We identify the frame bundle $\mathcal{F}(\mathbf{M}^3)$ with the quotient $\mathcal{F}(\mathbb{H}^3)/\mathcal{G}$. The frame flow acts on $\mathcal{F}(\mathbf{M}^3)$ by the projection.

It is well known [2] that the frame flow is mixing on closed three manifolds of variable negative curvature. In the case of constant negative curvature the frame flow is known to be exponentially mixing. This was proved by Moore in [10] using representation theory (see also [11]). The proof of the following theorem follows from the spectral gap theorem for the Laplacian on closed hyperbolic manifold $\mathbf{M}^3$ and Proposition 3.6 in [10]. (We thank Livio Flaminio and Mark Pollicott for explaining this to us.)

THEOREM 4.2. *There exists a* $\mathbf{q} > 0$ *that depends only on* $\mathbf{M}^3$ *such that the following holds. Let* $\psi, \phi : \mathcal{F}(\mathbf{M}^3) \to \mathbb{R}$ *be two* C^1 *functions. Then for every* $r \in \mathbb{R}$, *the inequality*

$$\left| \Lambda(\mathcal{F}(\mathbf{M}^3)) \int\limits_{\mathcal{F}(\mathbf{M}^3)} (\mathbf{g}_r^*\psi)(x)\phi(x)\, d\Lambda(x) - \int\limits_{\mathcal{F}(\mathbf{M}^3)} \psi(x)\, d\Lambda(x) \int\limits_{\mathcal{F}(\mathbf{M}^3)} \phi(x)\, d\Lambda(x) \right| \leq Ce^{-\mathbf{q}|r|}$$

holds, where $C > 0$ *only depends on the* C^1 *norm of* ψ *and* ϕ.

Remark. In fact, one can replace the C^1 norm in the above theorem by the (weaker) Hölder norm (see [10]).

For two functions $\psi, \phi : \mathcal{F}(\mathbf{M}^3) \to \mathbb{R}$ we set

$$(\psi, \phi) = \int\limits_{\mathcal{F}(\mathbf{M}^3)} \psi(x)\phi(x)\, d\Lambda(x).$$

From now on, $r \gg 0$ denotes a large positive number that stands for the flow time of the frame flow. Also let $\varepsilon > 0$ denote a positive number that is smaller than the injectivity radius of $\mathbf{M}^3$. Then the projection map $\mathcal{F}(\mathbb{H}^3) \to \mathcal{F}(\mathbf{M}^3)$ is injective on every ε ball $\mathcal{N}_\varepsilon(F) \subset \mathcal{F}(\mathbb{H}^3)$.

Fix $F_0 \in \mathcal{F}(\mathbb{H}^3)$, and let $\mathcal{N}_\varepsilon(F_0) \subset \mathcal{F}(\mathbb{H}^3)$ denote the ε ball around the frame F_0. Choose a C^1 function $f_\varepsilon(F_0) : \mathcal{F}(\mathbb{H}^3) \to \mathbb{R}$ that is positive on

$\mathcal{N}_\varepsilon(F_0)$, supported on $\mathcal{N}_\varepsilon(F_0)$, and such that

$$\int_{\mathcal{F}(\mathbb{H}^3)} f_\varepsilon(F_0)(X)\, d\Lambda(X) = 1. \tag{42}$$

For every $F \in \mathcal{F}(\mathbb{H}^3)$, we define $f_\varepsilon(F)$ by pulling back $f_\varepsilon(F_0)$ by the corresponding element of $\mathbf{PSL}(2,\mathbb{C})$. For $F \in \mathcal{F}(\mathbf{M}^3)$, the function $f_\varepsilon(F) : \mathcal{F}(\mathbf{M}^3) \to \mathbb{R}$ is defined accordingly. (It is well defined since every ball $\mathcal{N}_\varepsilon(F) \subset \mathcal{F}(\mathbb{H}^3)$ embeds in $\mathcal{F}(\mathbf{M}^3)$.) Moreover, the equality (42) holds for every $f_\varepsilon(F)$.

The following definition tells us when two frames in $\mathcal{F}(\mathbb{H}^3)$ are well connected.

Definition 4.1. Let $F_j = (p_j, u_j, n_j) \in \mathcal{F}(\mathbb{H}^3)$, $j = 1, 2$, be two frames, and set $\mathbf{g}_{\frac{r}{4}}(p_j, u_j, n_j) = (\widehat{p}_j, \widehat{u}_j, \widehat{n}_j)$. Define

$$\mathbf{a}_{\mathbb{H}^3}(F_1, F_2) = \Big(\mathbf{g}^*_{\frac{r}{2}} f_\varepsilon(\widehat{p}_1, \widehat{u}_1, \widehat{n}_1), f_\varepsilon(\widehat{p}_2, -\widehat{u}_2, \widehat{n}_2)\Big).$$

We say that the frames F_1 and F_2 are (ε, r)-well connected (or just well connected if ε and r are understood) if $\mathbf{a}_{\mathbb{H}^3}(F_1, F_2) > 0$.

The preliminary flow by time $\frac{r}{4}$ to get $(\widehat{p}_j, \widehat{u}_j, \widehat{n}_j)$ is used to get the estimates needed for Propositions 4.8 and 4.9.

Definition 4.2. Let $F_j = (p_j, u_j, n_j) \in \mathcal{F}(\mathbf{M}^3)$, $j = 1, 2$, be two frames, and let γ be a geodesic segment in $\mathbf{M}^3$ that connects p_1 and p_2. Let $\widetilde{p}_1 \in \mathbb{H}^3$ be a lift of p_1, and let $\widetilde{p}_2$ denotes the lift of p_2 along γ. By $\widetilde{F_j} = (\widetilde{p}_j, \widetilde{u}_j, \widetilde{n}_j) \in \mathcal{F}(\mathbb{H}^3)$ we denote the corresponding lifts. Set $\mathbf{a}_\gamma(F_1, F_2) = \mathbf{a}_{\mathbb{H}^3}(\widetilde{F_1}, \widetilde{F_2})$. We say that the frames F_1 and F_2 are (ε, r)-well connected (or just well connected if ε and r are understood) along the segment γ if $\mathbf{a}_\gamma(F_1, F_2) > 0$.

The function $\mathbf{a}_\gamma(F_1, F_2)$ is the affinity function from the outline above.

Let $F_j = (p_j, u_j, n_j) \in \mathcal{F}(\mathbf{M}^3)$, $j = 1, 2$, and let $\mathbf{g}_{\frac{r}{4}}(p_j, u_j, n_j) = (p'_j, u'_j, n'_j)$. Define

$$\mathbf{a}(F_1, F_2) = \Big(\mathbf{g}^*_{-\frac{r}{2}} f_\varepsilon(p'_1, u'_1, n'_1), f_\varepsilon(p'_2, -u'_2, n'_2)\Big).$$

Then

$$\mathbf{a}(F_1, F_2) = \sum_\gamma \mathbf{a}_\gamma(F_1, F_2),$$

where γ varies over all geodesic segments in $\mathbf{M}^3$ that connect p_1 and p_2. (Only finitely many numbers $\mathbf{a}_\gamma(F_1, F_2)$ are nonzero.) One can think of $\mathbf{a}(F_1, F_2)$ as the total probability that the frames F_1 and F_2 are well connected, and $\mathbf{a}_\gamma(F_1, F_2)$ represents the probability that they are well connected along the segment γ. The following lemma follows from Theorem 4.2.

LEMMA 4.4. *Fix $\varepsilon > 0$. Then for r large and any $F_1, F_2 \in \mathcal{F}(\mathbf{M}^3)$, we have*

$$\mathbf{a}(F_1,F_2) = \frac{1}{\Lambda(\mathcal{F}(\mathbf{M}^3))}(1 + O(e^{-\mathbf{q}\frac{r}{2}})),$$

where $\mathbf{q} > 0$ is a constant that depends only on the manifold $\mathbf{M}^3$.

4.5. *The geometry of well-connected bipods.* There is natural order three homeomorphism $\omega : \mathcal{F}(\mathbb{H}^3) \to \mathcal{F}(\mathbb{H}^3)$ given by $\omega(p,u,n) = (p, \omega(u), n)$, where $\omega(u)$ is the vector in $T^1_p(\mathbb{H}^3)$; that is, orthogonal to n and such that the oriented angle (measured anticlockwise) between u and $\omega(u)$ is $\frac{2\pi}{3}$. (The plane containing the vectors u and $\omega(u)$ is oriented by the normal vector n.) An equivalent way of defining ω is by the right-hand rule. The homeomorphism ω commutes with the $\mathbf{PSL}(2,\mathbb{C})$ action and it is well defined on $\mathcal{F}(\mathbf{M}^3)$ by the projection. The distance function $\mathcal{D}$ on $\mathcal{F}(\mathbf{M}^3)$ is invariant under ω.

To every $F_p = (p,u,n) \in \mathcal{F}(\mathbf{M}^3)$ we associate the bipod $B_p = (F_p, \omega(F_p)$ and the anti-bipod $\overline{B}_p = (F_p, \overline{\omega}(F_p)$. (We recall that $\overline{\omega} = \omega^{-1}$.) We have the following definition.

Definition 4.3. Given two frames $F_p = (p,u,n) \in \mathcal{F}(\mathbf{M}^3)$ and $F_q = (q,v,m) \in \mathcal{F}(\mathbf{M}^3)$, let B_p and B_q denote the corresponding bipods. Let $\gamma = (\gamma_0, \gamma_1)$, be a pair of geodesic segments in $\mathbf{M}^3$, each connecting the points p and q. We say that the bipods B_p and B_q are (ε, r)-well connected along the pair of segments γ if the pairs of frames F_p and F_q, and $\omega(F_p)$ and $\overline{\omega}(F_q)$, are (ε, r)-well connected along the segments γ_0 and γ_1 respectively.

LEMMA 4.5. *Let $F_p = (p,u,n)$ and $F_q = (q,v,m)$ be two frames in $\mathbf{M}^3$. Suppose that the corresponding bipods B_p and B_q are (ε, r)-well connected along a pair of geodesic segments γ_0 and γ_1 that connect p and q in $\mathbf{M}^3$; that is, we assume $\mathbf{a}_{\gamma_0}(F_p,F_q) > 0$ and likewise $\mathbf{a}_{\gamma_1}(\omega(F_p), \overline{\omega}(F_q)) > 0$. Then for r large, the closed curve $\gamma_0 \cup \gamma_1$ is homotopic to a closed geodesic $\delta \in \Gamma$, and the following inequality holds*:

$$\left| \mathbf{l}(\delta) - 2r + 2\log\frac{4}{3} \right| \leq D\varepsilon$$

for some constant $D > 0$. Moreover,

$$d(p,\delta),\, d(q,\delta) \leq \log\sqrt{3} + D\varepsilon.$$

Proof. We define, for $i = 0,1$, $F_{\widehat{p_i}} = (\widehat{p}_i, \widehat{u}_i, \widehat{n}_i)$ by $\mathbf{g}_{\frac{r}{4}}(\omega^i(F_p))$. Likewise, we let $F_{\widehat{q_i}} = (\widehat{q}_i, \widehat{v}_i, \widehat{m}_i)$ by $\mathbf{g}_{\frac{r}{4}}(\overline{\omega}^i(F_q))$. Because $\omega^i(F_p)$ and $\overline{\omega}^i(F_q)$ are well connected, we can find $F_{p'_i} \in \mathcal{N}_\varepsilon(F_{\widehat{p_i}})$ and $F_{q'_i} \in \mathcal{N}_\varepsilon(F_{\widehat{q_i}})$ such that $\mathbf{g}_{\frac{r}{2}}(p'_i, u'_i, n'_i) = (q'_i, -v'_i, m'_i)$. Moreover, there is a homotopy condition that is satisfied, namely that the concatenation of the ε-chain

$$\left(\mathbf{g}_{[0,\frac{r}{4}]}(p_i,u_i), \mathbf{g}_{[0,\frac{r}{2}]}(p'_i,u'_i), \mathbf{g}_{[0,\frac{r}{4}]}(\widehat{q}_i, -\widehat{v}_i\right),$$

is homotopic rel endpoints to γ_i.

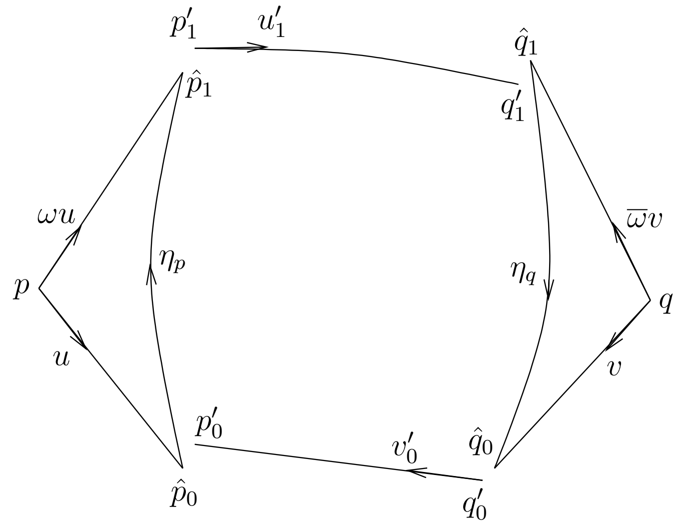

Figure 3. The closed ε-chain for two well-connected bipods

We let η_p be the geodesic segment from $\widehat{p}_0$ to $\widehat{p}_1$ that is homotopic rel endpoints to

$$\mathbf{g}_{[0,\frac{r}{4}]}(\widehat{p}_0, -\widehat{u}_0) \cdot \mathbf{g}_{[0,\frac{r}{4}]}(p, u).$$

Then $\widehat{n}_0$ and $\widehat{n}_1$ are parallel along η_p (because they are orthogonal to the plane of the immersed triangle we have formed), and the angle between η_p and $-\widehat{u}_i$ (at $\widehat{p}_i$) is less than $De^{-\frac{r}{4}}$. Moreover,

$$\left|\mathbf{l}(\eta_p) - \frac{r}{2} + \log\frac{4}{3}\right| \leq De^{-\frac{r}{4}}.$$

We likewise define η_q and make the same observation.

We refer the reader to Figure 3 for an illustration of our construction.

The segments η_p, $\mathbf{g}_{[0,\frac{r}{2}]}(p'_1, u'_1)$, η_q^{-1}, $\mathbf{g}_{[0,\frac{r}{2}]}(q'_0, v'_0)$, form a closed ε-chain, and we are therefore in a position to apply Lemma 4.1. We take

$$(a_0, b_0, a_1, b_1, a_2, b_2, a_3, b_3) = (\widehat{p}_0, \widehat{p}_1, p'_1, q'_1, \widehat{q}_1, \widehat{q}_0, q'_0, p'_0)$$

(and connect a_i to b_i by the aforementioned segments), and we let $(n_0, n_1, n_2, n_3) = (\widehat{n}_0, n'_1, m'_1, m'_0)$. We can easily verify that the hypotheses of Lemma 4.1 are satisfied, and we conclude that $\gamma_0 \cup \gamma_1$ is freely homotopic to a closed geodesic δ, and the following inequalities

$$\left|\mathbf{l}(\delta) - 2r + 2\log\frac{4}{3}\right| \leq D\varepsilon$$

and

$$d(\widehat{p}_i, \delta),\ d(\widehat{q}_i, \delta) \leq D\varepsilon$$

hold. It follows that the projection of p onto η_p is exponentially close to δ, and therefore

$$d(p,\delta),\ d(q,\delta) \leq \log\sqrt{3} + D\varepsilon. \qquad \square$$

4.6. *The geometry of well-connected tripods.* Let $P, P_1, P_2 \in \mathcal{F}(\mathbb{H}^3)$. We call P the reference frame and P_1, P_2 the moving frames. Let $F_1 \in \mathcal{F}(\mathbb{H}^3)$, and r large. Then the frame $F_2 = L(F_1, P_1, P_2, r)$ is defined as follows.

Let $\widetilde{F_1} = \mathbf{g}_{\frac{r}{4}}(F_1)$. Let $\widehat{F_1} \in \mathcal{N}_\varepsilon(\widetilde{F_1})$ denote the frame such that for some $M_1 \in \mathbf{PSL}(2,\mathbb{C})$, we have $M_1(P) = \widetilde{F_1}$ and $M_1(P_1) = \widehat{F_1}$. Set $\mathbf{g}_{\frac{r}{2}}(\widehat{F_1}) = (\widehat{q}, -\widehat{v}, \widehat{m})$ and $\widehat{F_2} = (\widehat{q}, \widehat{v}, \widehat{m})$. Let $\widetilde{F_2}$ denote the frame such that for some $M_2 \in \mathbf{PSL}(2,\mathbb{C})$, we have $M_2(P) = \widetilde{F_2}$ and $M_2(P_2) = \widehat{F_2}$. Set $\mathbf{g}_{-\frac{r}{4}}(\widetilde{F_2}) = F_2 = (q, v, m)$. Observe that the frame F_2 only depends on F_1, P_1, P_2, and r.

Recall from Section 3 that Π^0 denotes an oriented topological pair of pants equipped with a homeomorphism $\omega_0 : \Pi^0 \to \Pi^0$, of order three that permutes the cuffs. By $\omega_0^i(C)$, $i = 0,1,2$, we denote the oriented cuffs of Π^0. For each $i = 0,1,2$, we choose $\omega_0^i(c) \in \pi_1(\Pi^0)$ to be an element in the conjugacy class that corresponds to the cuff $\omega_0^i(C)$ such that $\omega_0^0(c)\omega_0^1(c)\omega_0^2(c) = \mathrm{id}$.

Fix a frame $P \in \mathcal{F}(\mathbb{H}^3)$, and fix six frames $P_i^j \in \mathcal{N}_\varepsilon(P)$, $i = 0,1,2$, $j = 1,2$, where $\mathcal{N}_\varepsilon(P)$ is the ε neighbourhood of P. Denote by (P_i^j) the corresponding six-tuple of frames. We define the representation

$$\rho(P_i^j) : \pi_1(\Pi^0) \to \mathbf{PSL}(2,\mathbb{C})$$

as follows.

Choose a frame $F_1^0 = (p, u, n) \in \mathbb{H}^3$, and let $F_2 = L(F_1^0, P_0^1, P_0^2, r)$. Denote by F_1^j, $j = 1,2$, given by $\omega(F_1^1) = L(\omega^{-1}(F_2), P_1^2, P_1^1, r)$ and $\omega^2(F_1^2) = L(\omega^{-2}(F_2), P_2^2, P_2^1, r)$. Let $A_i \in \mathbf{PSL}(2,\mathbb{C})$ given by $A_0(F_1^0) = F_1^1$, $A_1(F_1^1) = F_1^2$ and $A_2(F_1^2) = F_1^0$. Observe $A_2A_1A_0 = \mathrm{id}$. We define $\rho(P_i^j) = \rho$ by $\rho(\omega^i(c)) = A_i$. Up to conjugation in $\mathbf{PSL}(2,\mathbb{C})$, the representation $\rho(P_i^j)$ depends only on the six-tuple (P_i^j) and r. Observe that if $P_i^j = P$, for all i, j, then $\mathbb{H}^3/\rho(P_i^j)$ is a planar pair of pants whose all three cuffs have equal length, and the half-lengths of the cuffs that correspond to this representation are positive real numbers.

We will use the following lemma to show that the skew pants that corresponds to a pair of well-connected tripods (see the definition below) is indeed in $\mathbf{\Pi}_{D\varepsilon,R}$ for some universal constant $D > 0$.

Lemma 4.6. *Fix a frame $P \in \mathcal{F}(\mathbb{H}^3)$, and fix $P_i^j \in \mathcal{N}_\varepsilon(P)$, $i = 0,1,2$, $j = 1,2$. Set $\rho(P_i^j) = \rho$. Then*

$$\left|\mathbf{hl}(\omega_0^i(C)) - r + \log\frac{4}{3}\right| \leq D\varepsilon$$

for some constant $D > 0$, where $\mathbf{hl}(\omega_0^i(C))$ denotes the half-lengths that correspond to the representation ρ. In particular, the transformation $\rho(\omega_0^i(c))$ is loxodromic.

Proof. It follows from Lemma 4.5 that

$$\left|\mathbf{l}(\omega_0^i(C)) - 2r + 2\log\frac{4}{3}\right| \leq D\varepsilon,$$

where $\mathbf{l}(\omega_0^i(C))$ denotes the cuff length of $\rho(\omega_0^i(c)) = A_i$.

We have $\mathbf{hl}(\omega^i(C)) = \frac{\mathbf{l}(\omega^i(C))}{2} + k\pi i$ for some $k \in \{0,1\}$. It remains to show that $k = 0$.

Let $t \in [0,1]$, and let $P_i^j(t)$ be a continuous path in $\mathcal{N}_\varepsilon(P)$ such that $P_i^j(1) = P_i^j$, and $P_i^j(0) = P$. Set $\rho_t = \rho(P_i^j(t))$. Then for each t, we obtain the corresponding number $k(t) \in \{0,1\}$. Since $k(0) = 0$ and since $k(t)$ is continuous, we have $k(1) = k = 0$. □

To every frame $F \in \mathcal{F}(\mathbf{M}^3)$ we associate the tripod $T = \omega^i(F)$, $i = 0,1,2$, and the anti-tripod $\overline{T} = \overline{\omega}^i(F)$, $i = 0,1,2$, where $\overline{\omega} = \omega^{-1}$.

Definition 4.4. Given two frames $F_p = (p,u,n)$ and $F_q = (q,v,m)$ in $\mathcal{F}(\mathbf{M}^3)$, let $T_p = \omega^i(F_p)$ and $T_q = \omega^i(F_q)$, $i = 0,1,2$, be the corresponding tripods. Let $\gamma = (\gamma_0,\gamma_1,\gamma_2)$ be a triple of geodesic segments in $\mathbf{M}^3$, each connecting the points p and q. We say that the pair of tripods T_p and T_q is well connected along γ if each pair of frames $\omega^i(F_p)$ and $\omega^{-i}(F_q)$ is well connected along the segment γ_i.

Next we show that to every pair of well-connected tripods we can naturally associate a skew pants in the sense of Definition 3.2. Recall from Section 3 that Π^0 denotes an oriented topological pair of pants equipped with a homeomorphism $\omega_0 : \Pi^0 \to \Pi^0$ of order three that permutes the cuffs. By $\omega_0^i(C)$, $i = 0,1,2$, we denote the oriented cuffs of Π^0. For each $i = 0,1,2$, we choose $\omega_0^i(c) \in \pi_1(\Pi^0)$ to be an element in the conjugacy class that corresponds to the cuff $\omega_0^i(C)$ such that $\omega_0^0(c)\omega_0^1(c)\omega_0^2(c) = \mathrm{id}$.

Let $a, b \in \Pi^0$ be the fixed points of the homeomorphism ω_0. Let $\alpha_0 \subset \Pi^0$ be a simple arc that connects a and b, and set $\omega_0^i(\alpha_0) = \alpha_i$. The union of two different arcs α_i and α_j is a closed curve in Π^0 homotopic to a cuff. One can think of the union of these three segments as the spine of Π^0. Moreover, there is an obvious projection from Π^0 to the spine $\alpha_0 \cup \alpha_1 \cup \alpha_2$, and this projection is a homotopy equivalence.

Let $T_p = (p, \omega^i(u), n)$ and $T_q = (q, \omega^i(v), m)$, $i = 0,1,2$, be two tripods in $\mathcal{F}(\mathbf{M}^3)$, and let $\gamma = (\gamma_0,\gamma_1,\gamma_2)$ be a triple of geodesic segments in $\mathbf{M}^3$ each connecting the points p and q. One constructs a map ϕ from the spine of Π^0 to $\mathbf{M}^3$ by letting $\phi(a) = p$, $\phi(b) = q$ and by letting $\phi : \alpha_i \to \gamma_i$ be any homeomorphism. By precomposing this map with the projection from Π^0 to its spine we get a well-defined map $\phi : \Pi^0 \to \mathbf{M}^3$. By $\rho(T_p, T_q, \gamma) : \pi_1(\Pi^0) \to \mathcal{G}$ we denote the induced representation of the fundamental group of Π^0.

In principle, the representation $\rho(T_p, T_q, \gamma)$ can be trivial. However if the the tripods T_p and T_q are well connected along γ, we prove below that the representation $\rho(T_p, T_q, \gamma)$ is admissible (in sense of Definition 3.1) and that the conjugacy class $[\rho(T_p, T_q, \gamma)]$ is a skew pants in terms of Definition 3.2.

LEMMA 4.7. *Let T_p and T_q be two tripods that are well connected along a triple of segments γ, and set $\rho = \rho(T_p, T_q, \gamma)$. Then*

$$\left|\mathbf{hl}(\omega_0^i(C)) - r + \log\frac{4}{3}\right| \leq D\varepsilon$$

for some constant $D > 0$. In particular, the conjugacy class of transformations $\rho(\omega_0^i(C))$ is loxodromic.

Proof. Observe that there exist $P_i^j \in \mathcal{N}_\varepsilon(P)$ such that $\rho(P_i^j) = \rho(T_p, T_q, \gamma)$. The lemma follows from Lemma 4.6. □

Recall that $\mathbf{\Pi}_{\varepsilon,R}$ is the set of skew pants whose half-lengths are ε close to $\frac{R}{2}$ and that $R = 2(r - \log\frac{4}{3})$. If we write $\pi(T_p, T_q, \gamma) = [\rho(T_p, T_q, \gamma)]$, then by Lemma 4.7, π maps well-connected pairs of tripods to pairs of skew pants in $\mathbf{\Pi}_{D\varepsilon,R}$.

Definition 4.5. Let T_p and T_q be two tripods that are well connected along a triple of segments $\gamma = (\gamma_0, \gamma_1, \gamma_2)$. Set

$$\mathbf{b}_\gamma(T_p, T_q) = \prod_{i=0}^{i=2} \mathbf{a}_{\gamma_i}(\omega^i(F_p)), (\omega^{-i}(F_q)).$$

Observe that two tripods T_p and T_q are (ε, r)-well connected along a triple of geodesic segments γ if and only if $\mathbf{b}_\gamma(T_p, T_q) > 0$.

We define the space of well-connected tripods as the space of all triples (T_p, T_q, γ) such that the tripods T_p and T_q are well connected along γ. It follows from the exponential mixing statement that given any two tripods T_p and T_q, and for r large enough, there will exist at least one triple of segments γ so that T_p and T_q are well connected along γ. (In fact, it can be shown that there will be many such segments.)

We define the measure $\widetilde{\mu}$ on the set of well-connected tripods by

$$d\widetilde{\mu}(T_p, T_q, \gamma) = \mathbf{b}_\gamma(T_p, T_q)\, d\lambda_T(T_p, T_q, \gamma), \tag{43}$$

where $\lambda_T(T_p, T_q, \gamma)$ is the product of the Liouville measure Λ (for $\mathcal{F}(\mathbf{M}^3)$) on the first two terms and the counting measure on the third term. The measure λ_T is infinite (since there are infinitely many geodesic segments between any two points $p, q \in \mathbf{M}^3$), but $\mathbf{b}_\gamma(T_p, T_q)$ has compact support (that is, only finitely many such triples of connections γ are "good"), so $\widetilde{\mu}$ is finite.

Recall that $R = 2(r - \log \frac{4}{3})$ (see the discussion after Lemma 4.7 above). We define the measure μ on $\mathbf{\Pi}_{D\varepsilon,R}$ by $\mu = \pi_* \widetilde{\mu}$. This is the measure from Theorem 3.4. It follows from the construction that this measure is invariant under the involution $\mathcal{R} : \mathbf{\Pi} \to \mathbf{\Pi}$ (see Section 3 for the definition); that is, $\mu \in \mathcal{M}_0^{\mathcal{R}}(\mathbf{\Pi})$.

In order to prove Theorem 3.4 it remains to construct the corresponding measure $\beta \in \mathcal{M}_0(N^1(\sqrt{\Gamma}))$ and prove the stated properties.

4.7. *The "predicted foot" map* $\mathbf{f}_\delta$. By $F_p = (p, u, n)$ and $F_q = (q, v, m)$ we continue to denote two frames in $\mathcal{F}(\mathbf{M}^3)$. Suppose the frames $\omega^i(F_p)$ and $\overline{\omega}^i(F_q)$ are well connected along the geodesic segments γ_i, $i = 0, 1$. In our terminology this means that the bipods B_p and B_q are well connected along the segments γ_0 and γ_1. Let $\delta_2 \in \Gamma$ denote the closed geodesic in $\mathbf{M}^3$ freely homotopic to $\gamma_0 \cup \gamma_1$. We now associate the "geometric feet" to $(B_p, B_q, \gamma_0, \gamma_1)$.

We first define the geodesic ray $\alpha_p : [0, \infty) \to \mathbf{M}^3$ by $\alpha_p(0) = p$, $\alpha_p'(0) = \overline{\omega}(u)$, and we likewise define the geodesic ray $\alpha_q : [0, \infty) \to \mathbf{M}^3$ by $\alpha_q(0) = q$, $\alpha_q'(0) = \omega(v)$. Then for $t \in [0, \infty)$ and $i = 0, 1$, we let β_i^t be the geodesic segment homotopic relative endpoints to the piecewise geodesic arc $(\alpha_p[0, t])^{-1} \cdot \gamma_i \cdot \alpha_q[0, t])$. (The endpoints of both segments β_1^t and β_2^t are $\alpha_p(t)$ and $\alpha_q(t)$, and $\beta_i^0 = \gamma_i$.) We let β_i^∞ be the limiting geodesic of β_i^t, when $t \to \infty$. For each $t > 0$ and $i = 0, 1$, there is an obvious choice of common orthogonal from δ_2 to β_i^t, which varies continuously with $t \in [0, \infty]$. We let $f_i^t \in N^1(\delta_2)$ be the foot of this common orthogonal at δ_2, and we let $f_i = f_i^\infty$.

For a closed geodesic $\delta \in \Gamma$, let $\mathbb{T}_\delta$ denote the solid torus whose core curve is δ. As an alternative point of view, we can lift $\gamma_0 \cup \gamma_1$ to a closed curve in the solid torus $\mathbb{T}_{\delta_2}$. (There is a unique such lift to a closed curve in $\mathbb{T}_{\delta_2}$.) We can then lift F_p and F_q, and also $\alpha_p[0, \infty]$ and $\alpha_q[0, \infty]$, where $\alpha_p(\infty), \alpha_q(\infty) \in \partial \mathbb{T}_{\delta_2}$. Then we define β_i^t (and β_i^∞) as before, and there will be unique common orthogonals from (the lift of) δ_2 to β_i^t, $t \in [0, \infty]$.

By Lemma 4.3 we see that $\mathbf{d}_{\delta_2}(f_0, f_1) = \mathbf{hl}(\delta_2)$, so f_0 and f_1 represent the same point in $N^1(\sqrt{\delta_2})$. Therefore, we have defined the mapping

$$(B_p, B_q, \gamma_0, \gamma_1) \mapsto \mathbf{f}_{\delta_2}(B_p, B_q, \gamma_0, \gamma_1) \in N^1(\sqrt{\delta_2})$$

on the set of all well-connected bipods such that the $\gamma_0 \cup \gamma_1$ is homotopic to δ_2. We think of the vector $\mathbf{f}_{\delta_2}(B_p, B_q, \gamma_0, \gamma_1) \in N^1(\sqrt{\delta_2})$ as the geometric foot of $(B_p, B_q, \gamma_0, \gamma_1)$.

Assume now that we are given a third geodesic segment γ_2 between p and q (also known as the third connection) such that (T_p, T_q, γ) is a pair of well-connected tripods along the triple of segments $\gamma = (\gamma_0, \gamma_1, \gamma_2)$. Above, we have defined the skew pants $\Pi = \pi(T_p, T_q, \gamma)$ such that $\partial \Pi = \delta_0 + \delta_1 + \delta_2$, where δ_i is homotopic to $\gamma_{i-1} \cup \gamma_{i+1}$ (using the convention $\gamma_i = \gamma_{i+3}$).

Let $h_i \in N^1(\delta_2)$, $i = 0, 1$, denote the foot of the common orthogonal from δ_2 to δ_i. Recall that since $\mathbf{d}_{\delta_2}(h_0, h_1) = \mathbf{hl}(\delta_2)$, the projections of h_0 and h_1 to $N^1(\sqrt{\delta_2})$ agree and, as before, we let $\text{foot}_{\delta_2}(\Pi) \in N^1(\sqrt{\delta_2})$ denote this projection. We say that $\text{foot}_{\delta_2}(\Pi)$ is the foot of the skew pants Π on the cuff δ_2.

We will now verify that on $N^1(\sqrt{\delta_2})$ we have $\mathbf{d}_{\delta_2}(f_0, h_1) = \mathbf{d}_{\delta_2}(f_1, h_0) = O(e^{-\frac{r}{4}})$. This will imply that the pairs $\{h_0, h_1\}$ and $\{f_0, f_1\}$ project to vectors in $N^1(\sqrt{\delta_2})$ that are $e^{-\frac{r}{4}}$ close.

Proposition 4.8. *With the above notation we have that for r large and ε small, the inequalities*

$$\text{dis}(f_0, h_1),\ \text{dis}(f_1, h_0) \le De^{-\frac{r}{4}}$$

hold for some universal constant $D > 0$.

Proof. Assume that we are given a skew pants $\Pi = \pi(T_p, T_q, \gamma)$, where $\gamma = (\gamma_0, \gamma_1, \gamma_2)$ is a triple of good connections. Recall that δ_i is a cuff of Π that is homotopic to $\gamma_{i-1} \cup \gamma_{i+1}$. Then for $i = 0, 1$, the geodesics δ_2 and δ_i (or more precisely the appropriate lifts of δ_2 and δ_i to the solid torus cover corresponding to δ_2) satisfy (40).

On the other hand, since γ_2 is a good connection, and from the definition of a good connection between two frames, it follows that for some universal constant $E > 0$, the segment $\beta_0^{\frac{r}{4}}$ (considered in the solid torus cover $\mathbb{T}_{\delta_2}$) has the endpoints E close to δ_1. Similarly, the segment $\beta_1^{\frac{r}{4}}$ has the endpoints E close to δ_0. The inequality $\text{dis}(f_0, h_1), \text{dis}(f_1, h_0) \le De^{-\frac{r}{4}}$ now follows from Lemma 4.2. □

For each skew pants $\Pi = \pi(T_p, T_q, \gamma)$, we let

$$\mathbf{f}_{\delta_2}(\Pi) = \mathbf{f}_{\delta_2}(T_p, T_q, \gamma) = \mathbf{f}_{\delta_2}(B_p, B_q, \gamma_0, \gamma_1).$$

That is, we have defined the map $(\Pi, \delta^*) \mapsto \mathbf{f}_\delta(\Pi, \delta) \in N^1(\sqrt{\delta})$ on the set of all marked skew pants $\mathbf{\Pi}^*_{D\varepsilon,R}$ that contain the geodesic δ in its boundary. Recall that we have already defined the mapping $(\Pi, \delta^*) \mapsto \text{foot}_\delta(\Pi, \delta) \in N^1(\sqrt{\delta})$.

Proposition 4.9. *Let $(\pi(T_p, T_q, \gamma), \delta^*) \in \mathbf{\Pi}^*$. Then for r large and ε small, we have*

$$\mathbf{d}(\text{foot}_\delta(\pi(T_p, T_q, \gamma), \mathbf{f}_\delta(T_p, T_q, \gamma)) \le De^{-\frac{r}{4}}$$

for some constant $D > 0$.

Proof. It follows from Proposition 4.8. □

Given skew pants $\Pi = \pi(T_p, T_q, \gamma)$, the new foot $\mathbf{f}_{\delta_2}(T_p, T_q, \gamma)$ "predicts" the location of the old foot $\text{foot}_{\delta_2}(T_p, T_q, \gamma)$ (up to an exponentially small error in r) without knowing the third connection γ_2.

4.8. *The proof of Theorem* 3.4. Fix $\delta \in \Gamma$. For a given measure α on $N^1(\sqrt{\Gamma})$ we let α_δ denote the restriction of α on $N^1(\sqrt{\delta})$. It remains to construct the measure β on $N^1(\sqrt{\Gamma})$ from Theorem 3.4 and estimate the Radon-Nikodym derivative of β_δ with respect to the Euclidean measure on $N^1(\sqrt{\delta})$.

Recall that $(B_p, B_q, \gamma_0, \gamma_1)$ is a well-connected pair of bipods along the pair of segments γ_0 and γ_1 if

$$\mathbf{a}_{\gamma_0}(F_1, F_2)\mathbf{a}_{\gamma_1}(\omega(F_1), \overline{\omega}(F_2)) > 0.$$

We define the set S_δ by saying that $(F_p, F_q, \gamma_0, \gamma_1) \in S_\delta$ if $(B_p, B_q, \gamma_0, \gamma_1)$ is a well-connected pair of bipods along a pair of segments γ_0 and γ_1 such that $\gamma_0 \cup \gamma_1$ is homotopic to δ. In the previous subsection we have defined the map

$$\mathbf{f}_\delta : S_\delta \to N^1(\sqrt{\delta}).$$

Recall that that the bundle $N^1(\sqrt{\delta})$ has the natural $\mathbb{C}/(2\pi i\mathbb{Z} + l(\delta)\mathbb{Z})$ action by isometries. Now, we define the action of the torus $\mathbb{C}/(2\pi i\mathbb{Z} + l(\delta)\mathbb{Z})$ on S_δ so that the map $\mathbf{f}_\delta$ becomes equivariant with respect to the torus actions on S_δ and $N^1(\sqrt{\delta})$; that is, for each $\tau \in \mathbb{C}/(2\pi i\mathbb{Z} + l(\delta)\mathbb{Z})$, we have

$$\mathbf{f}_\delta(\tau + (B_p, B_q, \gamma_0, \gamma_1)) = \tau + \mathbf{f}_\delta(B_p, B_q, \gamma_0, \gamma_1), \tag{44}$$

where $\tau + (B_p, B_q, \gamma_0, \gamma_1)$ denotes the new element of S_δ (obtained after applying the action by τ to $(B_p, B_q, \gamma_0, \gamma_1)$).

Let $\mathbb{T}_\delta$ be the open solid torus cover associated to δ (so δ has a unique lift to a closed geodesic in $\mathbb{T}_\delta$ which we denote by $\widehat{\delta}(\delta)$). Given a pair of well-connected bipods in S_δ, each bipod lifts in a unique way to a bipod in $\mathcal{F}(\mathbb{T}_\delta)$ such that the pair of the lifted bipods is well connected in $\mathbb{T}_\delta$. We denote by $\widetilde{S}_\delta$ the set of such lifts, so $\widetilde{S}_\delta$ is in one-to-one correspondence with S_δ.

We observe that the group of automorphisms of the solid torus $\mathbb{T}_\delta$ is isomorphic to the group of isomorphisms of the unit normal bundle $N^1(\delta)$; that is, in turn isomorphic to $\mathbb{C}/(2\pi i\mathbb{Z} + l(\delta)\mathbb{Z})$ which acts on both $N^1(\delta)$ and on $\mathcal{F}^2(\mathbb{T}_\delta)$ so as to map $\widetilde{S}_\delta$ to itself. Since $\widetilde{S}_\delta$ and S_δ are in one-to-one correspondence, we have the induced action of $\mathbb{C}/(2\pi i\mathbb{Z} + l(\delta)\mathbb{Z})$ on S_δ. The equivariance (44) follows from the construction.

Let C_δ be the space of well-connected tripods (T_p, T_q, γ), where $\gamma = (\gamma_0, \gamma_1, \gamma_2)$, such that $\gamma_0 \cup \gamma_1$ is homotopic to δ. Let $\chi : C_\delta \to S_\delta$ be the forgetting map (the term forgetting map refers to forgetting the third connection γ_2), so $\chi(T_p, T_q, \gamma_0, \gamma_1, \gamma_2) = (B_p, B_q, \gamma_0, \gamma_1)$.

It follows from Proposition 4.9 that for any pair of well-connected tripods $T = (T_p, T_q, \gamma) \in C_\delta$, we have

$$|\mathbf{f}_\delta(\chi(T)) - \text{foot}_\delta(\pi(T, \gamma))| < Ce^{-\frac{r}{4}}, \tag{45}$$

where $\pi(T, \gamma)$ is the corresponding skew pants.

Next, we define the measure ν_δ on S_δ by

$$d\nu_\delta(B_p, B_q, \gamma_0, \gamma_1) = \mathbf{a}_{\gamma_0}(F_p, F_q)\mathbf{a}_{\gamma_1}(\omega(F_p), \overline{\omega}(F_q))\, d\lambda_B(B_p, B_q, \gamma_0, \gamma_1),$$

where λ_B is the measure on S_δ defined as the product of the Liouville measures on the first two terms and the counting measure on the other two terms.

We make two observations. The first one is that λ_B is invariant under the $\mathbb{C}/(2\pi i\mathbb{Z} + l(\delta)\mathbb{Z})$ action on S_δ. The second one is as follows. Let $\tau \in \mathbb{C}/(2\pi i\mathbb{Z} + l(\delta)\mathbb{Z})$. For $(B_p, B_q, \gamma_0, \gamma_1) \in S_\delta$, we let

$$(B_{p(\tau)}, B_{q(\tau)}, \gamma_0(\tau), \gamma_1(\tau)) = \tau + (B_p, B_q, \gamma_0, \gamma_1)$$

denote the corresponding element of S_δ. It follows from the definition of the affinity functions that

$$\mathbf{a}_{\gamma_0}(F_p, F_q)\mathbf{a}_{\gamma_1}(\omega(F_p), \overline{\omega}(F_q)) = \mathbf{a}_{\gamma_0(\tau)}(F_{p(\tau)}, F_{q(\tau)})\mathbf{a}_{\gamma_1(\tau)}(\omega(F_{p(\tau)}), \overline{\omega}(F_{q(\tau)}))$$

for any τ. These two observations show that the measure ν_δ is invariant under the $\mathbb{C}/(2\pi i\mathbb{Z} + l(\delta)\mathbb{Z})$ action on S_δ.

Since the map $\mathbf{f}_\delta$ is invariant under the $\mathbb{C}/(2\pi i\mathbb{Z}+l(\delta)\mathbb{Z})$ actions (see (44)), it follows from the above two observations that the measure $(\mathbf{f}_\delta)_*\nu_\delta$ is invariant under the $\mathbb{C}/(2\pi i\mathbb{Z}+l(\delta)\mathbb{Z})$ action on $N^1(\sqrt{\delta})$. Therefore, the measure $(\mathbf{f}_\delta)_*\nu_\delta$ is equal to a multiple of the Euclidean measure Eucl_δ on $N^1(\sqrt{\delta})$. We write

$$(\mathbf{f}_\delta)_*\nu_\delta = E_\delta \mathrm{Eucl}_\delta \tag{46}$$

for some constant $E_\delta \geq 0$.

The other natural measure on S_δ is defined as follows. Let $\chi : C_\delta \to S_\delta$ be the forgetting map (defined above). Recall that $\widetilde{\mu}$ is the measure (defined by (43) above) on the space of well-connected tripods given by

$$d\widetilde{\mu}(T_p, T_q, \gamma) = \mathbf{b}_\gamma(T_p, T_q)\, d\lambda_T(T_p, T_q, \gamma),$$

where $\lambda_T(T_p, T_q, \gamma)$ is the product of the Liouville measure Λ (for $\mathcal{F}(\mathbf{M}^3)$) on the first two terms and the counting measure on the third term. Then we get a new measure on S_δ by $\chi_*(\widetilde{\mu}|_{C_\delta})$, where $\widetilde{\mu}|_{C_\delta}$ is the restriction of $\widetilde{\mu}$ to the set C_δ.

The two measures satisfy

$$\left|\frac{d\chi_*(\widetilde{\mu}|_{C_\delta})}{d\nu_\delta}\right| = \sum_{\gamma_2} \mathbf{a}_{\gamma_2}(\omega^2(F_p), \overline{\omega}^2(F_q)) = \mathbf{a}(\omega^2(F_p), \overline{\omega}^2(F_q)).$$

But by the mixing we have

$$\mathbf{a}(\omega^2(F_p), \overline{\omega}^2(F_q)) = \frac{1}{\Lambda(\mathcal{F}(\mathbf{M}^3))}(1 + O(e^{-\mathbf{q}r})),$$

so we find that for some constant $C = C(\varepsilon, \mathbf{M}^3) > 0$, we have

$$\left| \frac{d\chi_*(\widetilde{\mu}|_{C_\delta})}{d\nu_\delta} - \frac{1}{\Lambda(\mathcal{F}(\mathbf{M}^3))} \right| < Ce^{-\mathbf{q}r},$$

which implies

$$\frac{1}{\Lambda(\mathcal{F}(\mathbf{M}^3))}(1 - Ce^{-\mathbf{q}r})\nu_\delta \leq \chi_*(\widetilde{\mu}|_{C_\delta}) \leq \frac{1}{\Lambda(\mathcal{F}(\mathbf{M}^3))}(1 + Ce^{-\mathbf{q}r})\nu_\delta. \tag{47}$$

Applying the mapping $(\mathbf{f}_\delta)_*$, and from (46), we obtain

$$\frac{E_\delta}{\Lambda(\mathcal{F}(\mathbf{M}^3))}(1 - Ce^{-\mathbf{q}r})\mathrm{Eucl}_\delta \leq \mathbf{f}_*(\chi_*(\widetilde{\mu}|_{C_\delta})) \leq \frac{E_\delta}{\Lambda(\mathcal{F}(\mathbf{M}^3))}(1 + Ce^{-\mathbf{q}r})\mathrm{Eucl}_\delta.$$

We let

$$\beta_\delta = \mathbf{f}_*(\chi_*(\widetilde{\mu}|_{C_\delta})).$$

It follows that the Radon-Nikodym derivative of β_δ satisfies desired inequality from Theorem 3.4. On the other hand, it follows from (45) that β_δ and $\widehat{\partial}\mu|_{N^1(\sqrt{\delta})}$ are $O(e^{-\frac{r}{4}})$ equivalent. This completes the proof.

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