Continuous subgroups of the fundamental groups of physics. II. The similitude group*

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All subalgebras of the similitude algebra (the algebra of the Poincaré group extended by dilatations) are classified into conjugacy classes under transformations of the similitude group. Use is made of the classification of all subalgebras of the Poincaré algebra, carried out in a previous article. The results are presented in tables listing representatives of each class and their basic properties.

1. INTRODUCTION

This article is the second in a series of papers devoted to a study of the subgroup structure of Lie groups of fundamental importance in physics. In the first article, [1] further to be referred to as I, we presented a general method for classifying Lie subalgebras of Lie algebras with nontrivial ideals. The method, making use of cohomology theory, was then applied to classify all continuous subgroups of the Poincaré group (inhomogeneous Lorentz group) and of the homogeneous similitude group, i. e., the Lorentz group extended by dilatations.

In this paper we make use of the previous results to provide a classification of all continuous subgroups of the similitude group, i. e., the Poincaré group extended by dilatations.

Let us mention in passing that related problems were treated in two other previous articles. In one of them[2] we found all maximal solvable subgroups of the pseudounitary groups $SU(p,q)$ and all continuous subgroups of $SU(2,1)$. In the other[3] we discussed all maximal solvable subgroups of the pseudoorthogonal groups $SO(p,q)$.

The similitude group SG, also called the Weyl group, [4] is of considerable interest in elementary particle physics, the general theory of relativity and other fields of physics. Its importance in high energy physics is largely related to the phenomenon of scaling in deep inelastic scattering and thus to short distance behavior in elementary particle theory. For information on various approaches to scale invariance we refer to recent reviews and some of the original articles (some of them also treat the more general conformal group of space–time[5–10]). The similitude group also underlies Weyl's unified field theory[4] and can figure as a gauge group for field theories involving gravitation. [11,12]

The similitude group is an 11-parameter Lie group containing the Poincaré group as an invariant subgroup. In itself it is the largest nontrivial continuous subgroup of the conformal group of space–time.

The motivation for our interest in subgroups of Lie groups was given, e. g., in our previous articles. [1–3] Let us just mention several points. In a situation where the similitude group is an invariance group of a physical system a classification of its subgroups provides a classification of possible symmetry breaking interactions (or boundary conditions). If we are interested in the representation theory of the group SG, then each chain of subgroups will provide us with a different basis for the representations (at least those subgroups the algebras of which have enveloping algebras with nontrivial centers). Thus, if we wish to use the representation theory of SG to provide expansions of physical quantities like scattering amplitudes, we will find that different chains of subgroups provide us with different expansions having different possible applications. (This problem for the Lorentz and Galilei groups is treated in detail in the review. [13]) Different subgroups of the similitude group may be of special relevance for the construction of elementary particle dynamics in certain frames of reference (see the discussion of the infinite momentum frame and its relation to an 8-parameter subgroup of the Poincaré group[14]).

In Sec. 2 of this article we review some known results on the similitude group in order to establish notation (which is consistent with that used in I) and then discuss the method used to obtain all classes of subalgebras of the similitude algebra S (up to conjugation under the similitude group itself). In Sec. 3 we obtain our main results, i. e., a list of representatives of each conjugacy class of subalgebras of S, summarized in Tables. Section 4 is devoted to the conclusions and future outlook.

2. METHOD FOR CLASSIFYING THE SUBALGEBRAS OF THE SIMILITUDE ALGEBRA

A. The similitude group and its algebra

The similitude group SG can be defined as the group of Lorentz transformations, translations and dilatations of Minkowski space, i. e., the transformations

$$x'_\mu = h\Lambda_{\mu\nu}x_\nu + a_\mu, \quad \mu,\nu = 0,1,2,3, \tag{1}$$

where h is a real positive number, $\Lambda_{\mu\nu}$ are matrix elements of an $O(3,1)$ matrix and a_μ are real numbers. The vectors $x \equiv \{x_0, x_1, x_2, x_3\}$ are real vectors in the four-dimensional Minkowski space with metric $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$.

We shall, however, make use of a different representation of SG, remembering that SG is a subgroup of the conformal group of space—time, i. e., the group of all transformations of x_μ, leaving the element ds^2 form-invariant: $ds^2 \to h\, ds'^2$. This group is isomorphic to $SU(2,2)$ (for reviews see, e.g., Refs. 5—7 and 10). We shall use a somewhat nonstandard realization of $SU(2,2)$, already introduced earlier,[1–3] namely the group of transformations G of a four-dimensional complex vector space satisfying

$$GJG^{+} = J, \tag{2}$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

(the cross on G implies Hermitian conjugation). Elements X of the Lie algebra of $SU(2,2)$ in this realization satisfy

$$X^{+}J + JX = 0 \tag{4}$$

and the general element of the algebra can be written as

$$X = \begin{pmatrix} \alpha & \beta & \epsilon & ia \\ \gamma & \delta & ib & -\epsilon^* \\ \xi & ic & -\delta^* & -\beta^* \\ id & -\xi^* & -\gamma^* & -\alpha^* \end{pmatrix}, \quad \alpha - \alpha^* + \delta - \delta^* = 0, \tag{5}$$

where Greek letters denote complex numbers, italic letters real ones, and the stars indicate complex conjugation. If we now consider the subalgebra of (5) leaving a two-dimensional vector space

$$z = \begin{pmatrix} \mu \\ \nu \\ 0 \\ 0 \end{pmatrix}$$

invariant, we obtain an 11-parameter subalgebra

$$S = \begin{pmatrix} d+c+iq & \alpha & \gamma & ia \\ \beta & d-c-iq & ib & -\gamma^* \\ 0 & 0 & -d+c-iq & -\alpha^* \\ 0 & 0 & -\beta^* & -d-c+iq \end{pmatrix}. \tag{6}$$

It is easy to verify that this algebra is isomorphic to that of the similitude algebra, i. e., its structure is

$$S = D \,\square\, (LSL(2,C) \,\square\, LT_4), \tag{7}$$

where $\square$ indicates a semidirect sum, D generates dilatations, LT_4 four-dimensional translations, and $LSL(2,C)$ is the algebra of the special linear group $SL(2,C)$.

For our purposes a convenient basis for the similitude algebra S is provided by the following matrices.

Dilatations:

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{8}$$

The homogeneous Lorentz transformations ($LSL(2,C)$):

$$B_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$
$$B_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{9}$$
$$B_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad B_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \end{pmatrix}.$$

Translations:

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$X_3 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

The dilatations satisfy the commutation relations

$$[D, B_i] = 0, \quad i = 1, \ldots, 6; \quad [D, X_a] = 2X_a, \quad a = 1, \ldots, 4, \tag{11}$$

and the translations commute:

$$[X_a, X_b] = 0, \quad a, b = 1, \ldots, 4. \tag{12}$$

All other commutation relations are given in Table I.

The usual physical notation is different and less convenient for our purposes. Throughout the article we shall use the generators B_i and X_a. Their relation to the usual generators of rotations L_i, proper Lorentz transformations (boosts) K_i ($i = 1,2,3$) and translations P_μ ($\mu = 0,1,2,3$) is

$$B_1 = 2L_3, \quad B_2 = -2K_3, \quad B_3 = -L_2 - K_1,$$
$$B_4 = L_1 - K_2, \quad B_5 = L_2 - K_1, \quad B_6 = L_1 + K_2, \tag{13}$$

$$X_1 = \tfrac{1}{2}(P_0 - P_3), \quad X_2 = P_2, \quad X_3 = -P_1, \quad X_4 = \tfrac{1}{2}(P_0 + P_3). \tag{14}$$

The commutation relations for the usual physical generators are

$$[L_i, L_k] = \epsilon_{ikl}L_l, \quad [K_i, K_k] = -\epsilon_{ikl}L_l, \quad [L_i, K_k] = \epsilon_{ikl}K_l,$$
$$[L_i, P_0] = 0, \quad [L_i, P_k] = \epsilon_{ikl}P_l, \tag{15}$$
$$[K_i, P_0] = P_i, \quad [K_i, P_k] = \delta_{ik}P_0,$$
$$(i,k,l) = (1,2,3).$$

An element of the similitude group itself can in the considered realization be written as $G = \exp S$, where S is given by (6), i. e.,

$$G = \begin{pmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{pmatrix} \tag{16}$$

and condition (2) implies that the 2×2 matrices G_{ik}

TABLE I. Commutation relations for the Poincaré algebra.

	B_1	B_2	B_3	B_4	B_5	B_6	X_1	X_2	X_3	X_4
B_1	0	0	$2B_4$	$-2B_3$	$-2B_6$	$2B_4$	0	$2X_3$	$-2X_2$	0
B_2	0	0	$2B_3$	$2B_4$	$-2B_5$	$-2B_6$	$2X_1$	0	0	$-2X_4$
B_3	$-2B_4$	$-2B_3$	0	0	B_2	B_1	0	0	$2X_1$	X_3
B_4	$2B_3$	$-2B_4$	0	0	B_1	$-B_2$	0	$-2X_1$	0	$-X_2$
B_5	$2B_6$	$2B_5$	$-B_2$	$-B_1$	0	0	X_3	0	$2X_4$	0
B_6	$-2B_5$	$2B_6$	$-B_1$	B_2	0	0	X_2	$2X_4$	0	0

satisfy

$$G_{22}J_1G_{11}^{+} = J_1, \quad G_{12}J_1G_{11}^{+} + G_{11}J_1G_{12}^{+} = 0, \tag{17}$$

with

$$J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus an element of the similitude group can be written as

$$g = \begin{pmatrix} \alpha & \beta & \epsilon & \xi \\ \gamma & \delta & \mu & \nu \\ 0 & 0 & \alpha^*/\Delta & -\beta^*/\Delta \\ 0 & 0 & -\gamma^*/\Delta & \delta^*/\Delta \end{pmatrix}, \quad \Delta = \alpha\delta - \beta\gamma = \Delta^*, \tag{18}$$

with G_{12} satisfying

$$\begin{aligned} &\alpha^*\xi + \alpha\xi^* + \beta^*\epsilon + \beta\epsilon^* = 0, \\ &\gamma^*\xi + \delta^*\epsilon + \beta\mu^* + \alpha\nu^* = 0, \\ &\gamma^*\nu + \gamma\nu^* + \delta^*\mu + \delta\mu^* = 0. \end{aligned} \tag{19}$$

B. Classification of subalgebras of the similitude algebra

In paper I we have provided a list of representatives of all conjugacy classes of subalgebras of the Poincaré group Lie algebra. The results were summarized in three tables. The first of these (Table II of I) presents all subalgebras of the algebra of $SL(2,C)$ and hence all continuous subgroups of the homogeneous Lorentz group (these were known previously[15,16]). Table III of I presents all subalgebras of the Poincaré algebra P that split over their intersection with the translations LT_4 (i.e., the bases for these algebras can be written in a form containing elements of the type B_i and X_a only). Table IV of I lists representatives of all subalgebras of P that do not split over their intersections with LT_4 (i.e., their bases will always contain elements of the type $B_i + c_{ia}X_a$ where c_{ia} are real constants that are not all equal to zero and cannot be transformed into zero by an inner automorphism of the Poincaré group).

In this paper we take the results of I and build them up into a list of all subalgebras of the similitude algebra (up to conjugation under the similitude group). We use a related notation for the subalgebras of S, namely, $S_{j,k}$ where j runs from 1 to 15 and indicates the subalgebra F_j of $LSL(2,C)$ that has been extended to $S_{j,k}$ by translations and dilatations. The label k simply distinguishes different subalgebras obtained from the same F_j.

The procedure consists of several steps:

1. Find representatives of all conjugacy classes of subalgebras of the Poincaré algebra, that are not equivalent under the similitude group. To do this we must merely remember that the dilatation generator D commutes with all generators B_i of $SL(2,C)$, but multiplies translations by a constant [see formula (11)]. The transformation $\exp tD$ in the group SG will thus multiply all generators X_a by a constant, leaving B_i invariant. The matrix D itself is not of form (18), but we shall include it in our group of automorphisms, to simplify subalgebras of S. It corresponds to total inversion (parity times time-reversal) and is contained in the similitude group, but not in the component connected to identity. It follows that the extension of the Poincaré group by dilatations leads to the coalescence of certain nonsplitting subalgebras of the Poincaré algebra. Indeed, we have, e.g.,

$$D(B_1 - X_1)D^{-1} = B_1 + X_1, \tag{20}$$

$$e^{xD}(B_2 + x^2X_3)e^{-xD} = B_2 + X_3. \tag{21}$$

Thus the algebras $B_1 \pm X_1$, while inequivalent under the Poincaré group, are conjugated under the similitude group. Similarly, the continuous set of Poincaré subalgebras $B_2 + x^2X_3$, coalesces into one subalgebra $B_2 + X_3$.

In Table II of the following section we list conjugacy classes of subalgebras of P (and of S) that are inequivalent with respect to the similitude group.

2. Subalgebras of the similitude algebra containing D as a generator. Any subalgebra of this type has the form

$$D \dotplus P_{j,k}, \tag{22}$$

where $P_{j,k}$ is a subalgebra of the Poincaré algebra. It follows from the above discussion that we thus obtain subalgebras of S if and only if $P_{j,k}$ is a splitting subalgebra of P (Table III of I) and that each splitting subalgebra of P provides a different subalgebra (22) of S.

3. Subalgebras of S not contained in the Poincaré algebra and not containing any conjugate of D under SG, such that the intersection with the Poincaré algebra splits over the translations. Choosing one generator of such an algebra as

$$D + \Sigma a_\mu B_\mu + \Sigma x_a X_a, \quad (1 \le \mu \le 6, \ 1 \le a \le 4), \tag{23}$$

there has to be at least one a_μ or x_a nonzero even after SG-conjugation. The other generators $\{B_\mu, X_a\}$ form one of the splitting subalgebras $P_{j,k}$ of the Poincaré algebra listed in Table III of I. To find all these subalgebras of S we consider each splitting subalgebra $P_{j,k}$ of P, add to it a generator (23) with a_μ and x_a so chosen that we obtain an algebra. The element (23) is then simplified using the normalizer of $P_{j,k}$ in the Poincaré group and possibly further transformations involving D and normalizing (leaving invariant) the subalgebra $P_{j,k}$.

4. Subalgebras of S not contained in the Poincaré algebra and not containing any conjugate of D under SG, such that the intersection with the Poincaré algebra does not split over the translations. We choose one generator of each of these subalgebras in the form (23), the others $\{B_a + \Sigma x_{ak} X_k, X_i\}$ form one of the nonsplitting subalgebras $\tilde{P}_{j,k}$ of P listed in Table IV of I. To find all such subalgebras of S we consider each nonsplitting subalgebra $\tilde{P}_{j,k}$ of P separately and choose a_μ and x_a in (23) in the most general manner that forms an algebra with $\tilde{P}_{j,k}$. The element (23) is then simplified, using the normalizer of $\tilde{P}_{j,k}$ in the Poincaré group, supplemented by elements involving D and also normalizing $\tilde{P}_{j,k}$ (i.e., we use the normalizer of $\tilde{P}_{j,k}$ in the similitude group). This method provides a list of all subalgebras $S_{j,k}$ of S.

Several comments are in order.

1. The subalgebras of S obtained by applying the above steps 3 and 4 correspond to a generalization of the "Goursat twist"[17–19] method for obtaining subgroups of a group that is in itself the direct product of two subgroups [e.g., $O(4)$ as $O(3)\times O(3)$].

2. We could have applied directly the general method developed in I for classifying subalgebras of a given algebra. The Poincaré algebra would then have served as a nonabelian invariant subalgebra whose subalgebras are known. In this particular case we found the method described above to be more convenient.

3. COMPLETE LIST OF CONJUGACY CLASSES OF SUBALGEBRAS OF THE SIMILITUDE ALGEBRA

A. Subalgebras of the Poincaré algebra P as subalgebras of the similitude algebra

All subalgebras $P_{j,k}$ listed in Table III of I split over their intersection with the translations. These subalgebras are not affected by dilations. Hence Table II of I also provides a list of representatives of conjugacy classes of subalgebras of the similitude algebra S and no two entries are conjugate to each other under the similitude group. We shall not reproduce the table here but only refer to I. For the purposes of this article all subalgebras $P_{j,k}$ of Table III of I will be denoted $S_{j,k}$ (same value of j and k).

Table IV of I, listing all nonsplitting subalgebras $\tilde{P}_{j,k}$ of P is modified when conjugacy is considered under the similitude group. In view of formulas of the type (20) and (21) many classes coalesce. Thus Table IV of part I is replaced by the following Table II.

The first column in Table II introduces a notation for the subalgebra, the second tells us from which subalgebra of $LSL(2,C)$ it was obtained, the third lists the subalgebras $P_{j,k}$ of P that coalesce to form the same subalgebra of S up to SG conjugacy. The fourth column gives the generators of $S_{j,k}$ and the last one its dimension (over the real numbers).

B. Subalgebras of the similitude algebra containing D as a generator

A complete list of such algebras is obtained by taking each splitting subalgebra of the Poincaré group and adding D to the basis. Thus, we take all algebras listed in Table III of I and add D to them. No other subalgebras of S, containing D as a basis element exist. Again, we shall not reproduce this table and refer the reader to I. We thus obtain subalgebras which we denote

$$S_{1,3}, S_{1,4};\quad S_{2,5}, -, S_{2,8};\quad S_{3,5}, -, S_{3,8};\quad S_{4,5}, -, S_{4,8};$$
$$S_{5,5} - S_{5,8};\quad S_{6,8} - S_{6,11};\quad S_{7,8} - S_{7,12};\quad S_{8,18} - S_{8,27};$$

TABLE II. List of subalgebras $S_{j,k}$ that are nonsplitting subalgebras of the Poincaré algebra and that are nonconjugate under the similitude group.

Notation	F_j	$\tilde{P}_{j,k}$	Generators of $S_{j,k}$	$\dim_R S_{j,k}$
$S_{6,5}$	F_6	$\tilde{P}_{6,5}, \tilde{P}_{6,6}$	$B_1+X_4, B_3, B_4, X_1, X_2, X_3$	6
$S_{6,6}$		$\tilde{P}_{6,7}, \tilde{P}_{6,8}$	$B_1, B_3+X_2, B_4+X_3, X_1$	4
$S_{6,7}$		$\tilde{P}_{6,9}, \tilde{P}_{6,10}$	B_1+X_1, B_3, B_4	3
$S_{7,6}$	F_7	$\tilde{P}_{7,6}$	$B_2+X_3, B_3, B_4, X_1, X_2$	5
$S_{7,7}$		$\tilde{P}_{7,7}$	B_2+X_3, B_3, B_4, X_1	4
$S_{8,10}$	F_8	$\tilde{P}_{8,10}$	$B_2+X_2, B_3, X_1, X_3, X_4$	5
$S_{8,11}$		$\tilde{P}_{8,11}$	B_2+X_3, B_3, X_1, X_2	4
$S_{8,12}$		$\tilde{P}_{8,12}$	B_2+X_2, B_3, X_1, X_3	4
$S_{8,13}$		$\tilde{P}_{8,13}$	$B_2+X_2, B_3, X_1, X_2+bX_3, b\neq 0$	4
$S_{8,14}$		$\tilde{P}_{8,14}$	B_2+X_3, B_3, X_1	3
$S_{8,15}$		$\tilde{P}_{8,15}$	B_2+X_2, B_3, X_1	3
$S_{8,16}$		$\tilde{P}_{8,16}$	$B_2+X_2+bX_3, B_3, X_1, b\neq 0$	3
$S_{8,17}$		$\tilde{P}_{8,17}$	B_2+X_2, B_3	2
$S_{10,6}$	F_{10}	$\tilde{P}_{10,6}$	$B_3+X_4, B_4, X_1, X_2, X_3$	5
$S_{10,7}$		$\tilde{P}_{10,7}$	$B_3+X_4, B_4+X_3, X_1, X_2$	4
$S_{10,8}$		$\tilde{P}_{10,8}$	B_3+X_4, B_4, X_1, X_2	4
$S_{10,9}$		$\tilde{P}_{10,9}, \tilde{P}_{10,10}$	B_3, B_4+X_3, X_1, X_2	4
$S_{10,10}$		$\tilde{P}_{10,11}$	B_3, B_4+X_2, X_1	3
$S_{10,11}$		$\tilde{P}_{10,12}, \tilde{P}_{10,13}$	$B_3+X_2, B_4+bX_2+X_3, X_1, b\neq 0$	3
$S_{10,12}$		$\tilde{P}_{10,14}, \tilde{P}_{10,15}$	B_3+X_2, B_4+X_3, X_1	3
$S_{10,13}$		$\tilde{P}_{10,16}$	B_3, B_4+X_2	2
$S_{12,11}$	F_{12}	$\tilde{P}_{12,11}, \tilde{P}_{12,12}$	B_1+X_4, X_1, X_2, X_3	4
$S_{12,12}$		$\tilde{P}_{12,13}$	$B_1+X_1+X_4, X_1-X_4, X_2, X_3$	4
$S_{12,13}$		$\tilde{P}_{12,14}$	$B_1+X_1-X_4, X_1+X_4, X_2, X_3$	4
$S_{12,14}$		$\tilde{P}_{12,15}, \tilde{P}_{12,16}$	B_1+X_4, X_2, X_3	3
$S_{12,15}$		$\tilde{P}_{12,17}$	$B_1+X_1+X_4, X_2, X_3$	3
$S_{12,16}$		$\tilde{P}_{12,18}$	$B_1+X_1-X_4, X_2, X_3$	3
$S_{12,17}$		$\tilde{P}_{12,19}, \tilde{P}_{12,20}$	B_1+X_4, X_1	2
$S_{12,18}$		$\tilde{P}_{12,21}$	$B_1+X_1+X_4, X_1-X_4$	2
$S_{12,19}$		$\tilde{P}_{12,22}$	$B_1+X_1-X_4, X_1+X_4$	2
$S_{12,20}$		$\tilde{P}_{12,23}, \tilde{P}_{12,24}$	B_1+X_4	1
$S_{12,21}$		$\tilde{P}_{12,25}$	$B_1+X_1+X_4$	1
$S_{12,22}$		$\tilde{P}_{12,26}$	$B_1+X_1-X_4$	1
$S_{13,10}$	F_{13}	$\tilde{P}_{13,10}$	B_2+X_2, X_1, X_3, X_4	4
$S_{13,11}$		$\tilde{P}_{13,11}$	B_2+X_2, X_1, X_3	3
$S_{13,12}$		$\tilde{P}_{13,12}$	B_2+X_2, X_1, X_4	3
$S_{13,13}$		$\tilde{P}_{13,13}$	B_2+X_2, X_1	2
$S_{13,14}$		$\tilde{P}_{13,14}$	B_2+X_2, X_3	2
$S_{13,15}$		$\tilde{P}_{13,15}$	B_2+X_2	1
$S_{14,10}$	F_{14}	$\tilde{P}_{14,10}$	B_3+X_4, X_1, X_2, X_3	4
$S_{14,11}$		$\tilde{P}_{14,11}, \tilde{P}_{14,12}$	B_3+X_2, X_1, X_3, X_4	4
$S_{14,12}$		$\tilde{P}_{14,13}$	B_3+X_4, X_1, X_2	3
$S_{14,13}$		$\tilde{P}_{14,14}$	B_3+X_4, X_1, X_3	3
$S_{14,14}$		$\tilde{P}_{14,15}, \tilde{P}_{14,16}$	B_3+X_2, X_1, X_3	3
$S_{14,15}$		$\tilde{P}_{14,17}$	$B_3+X_4, X_1, X_2+bX_3, b\neq 0$	3
$S_{14,16}$		$\tilde{P}_{14,18}, \tilde{P}_{14,19}$	$B_3+X_2, X_1, X_2+bX_3, b\neq 0$	3
$S_{14,17}$		$\tilde{P}_{14,20}$	B_3+X_4, X_1	2
$S_{14,18}$		$\tilde{P}_{14,21}, \tilde{P}_{14,22}$	B_3+X_2, X_1	2
$S_{14,19}$		$\tilde{P}_{14,23}$	B_3+X_4, X_2	2
$S_{14,20}$		$\tilde{P}_{14,24}$	B_3+X_4	1
$S_{14,21}$		$\tilde{P}_{14,25}, \tilde{P}_{14,26}$	B_3+X_2	1

$$S_{9,7}-S_{9,12};\quad S_{10,14}-S_{10,18};\quad S_{11,7}-S_{11,12};$$

$$S_{12,23}-S_{12,32};\quad S_{13,16}-S_{13,24};\quad S_{14,22}-S_{14,30};\quad S_{15,12}-S_{15,22}. \tag{24}$$

Note that algebra $S_{15,22}$ is generated by D alone.

C. Subalgebras of S that are not contained in the Poincaré algebra do not contain any SG-conjugate of D and are such that the intersection with the Poincaré algebra splits over the translations

We consider each subalgebra $P_{j,k}$ of Table III of I, add the generator $\tilde{D}=D+a_\mu B_\mu+x_a X_a$ to it and find a_μ and x_a in such a manner as to obtain an algebra. We put $a_\mu=x_a=0$ for those generators B_μ and X_a, that are contained in $P_{j,k}$. This algebra must then be simplified using transformations contained in $\mathrm{Nor}_{SG}P_{j,k}$ (normalizer of $P_{j,k}$ in the similitude group).

In view of the fact that transformations of D by translations produce all expressions

$$\tilde{D}=D+\Sigma_a x_a X_a \tag{25}$$

it follows that for no SG-conjugate of $\tilde{D}\neq D$ can we have $a_\mu=0$, $\mu=1,\ldots,6$.

We consider several examples to illustrate our method and then list all subalgebras of this type in Table III.

The algebras $P_{1,k}$ and $P_{2,k}$ (derived from F_1 and F_2) of Table III in I cannot be extended in this way (i. e., $a_\mu=x_a=0$). Consider those derived from F_3. The generators of the homogeneous part of $P_{3,j}$ are B_1, B_3-B_5, B_4+B_6. Hence we could have

$$\tilde{D}=D+aB_2+b(B_3+B_5)+c(B_4-B_6)+x_aX_a.$$

Commuting $\tilde{D}$ with B_1, we obtain $b=c=0$, commuting with B_3-B_5, we find $a=0$. Now consider, e. g., $P_{3,4}$, not containing any translations. Commuting $\tilde{D}$ with B_1, B_3-B_5 and B_4+B_6, we find $x_2=x_3=0$, $x_1=x_4$, i. e.,

$$D+x(X_1+X_4),\quad B_1,\quad B_3-B_5,\quad B_4+B_6 \tag{26}$$

form an algebra. However, the transformation $\exp[-\frac{1}{2}x(X_1+X_4)]$ is in the normalizer of $P_{3,4}$ and we have

$$\exp[-\tfrac{1}{2}x(X_1+X_4)][D+x(X_1+X_4)]\exp[\tfrac{1}{2}x(X_1+X_4)]=D,$$

so that the algebra (26) is conjugate to one of the splitting subalgebras of (24) (and will hence not figure in Table III).

As a further example, consider the algebras $P_{10,k}$ of Table III of paper I, derived from F_{10}. The generators of the homogeneous part of $P_{10,k}$ (i. e., of F_{10}) are B_3 and B_4. Putting $\tilde{D}=D+a_1B_1+a_2B_2+a_5B_5+a_6B_6+x_aX_a$ and commuting with B_3 and B_4, we find $a_5=a_6=0$. Algebra $P_{10,1}$ is thus extended to

$$\tilde{D}=D+aB_1+bB_2, B_3, B_4, X_1, X_2, X_3, X_4,$$
$$-\infty<a<\infty,\quad -\infty<b<\infty,\quad a^2+b^2\neq 0.$$

The algebra $P_{10,2}$, on the other hand, leads to

$$\tilde{D}=D+aB_1+bB_2+xX_4, B_3, B_4, X_1, X_2, X_3.$$

The transformation $\exp yX_4$ leaves $P_{10,2}$ invariant but takes x into zero in $\tilde{D}$, if we put $y=x/2(1-b)$ for $b\neq 1$. For $b=1$, on the other hand, the transformation $\exp yD$ with $e^y=x^{-1/2}$ for $x>0$ or $D\exp yD$ with $e^y=(-x)^{-1/2}$ for $x<0$ will take x into 1. We thus obtain from $P_{10,2}$ two types of subalgebras of S:

$$D+aB_1+bB_2, B_3, B_4, X_1, X_2, X_3, \tag{27}$$
$$-\infty<a<\infty,\quad -\infty<b<\infty,\quad a^2+b^2\neq 0,$$

and

$$D+aB_1+B_2+X_4, B_3, B_4, X_1, X_2, X_3,\quad -\infty<a<\infty. \tag{28}$$

We proceed quite analogously with all subalgebras of Table III of I. The results are summarized in Table III.

TABLE III. Subalgebras of S that are not contained in P, do not contain any SG-conjugate of D and are such that their intersection with P splits over the translations.

Notations	F_j	$P_{j,k}$	$\tilde{D}$	Generators of $P_{j,k}$	$\dim_R S_{j,k}$
$S_{5,9}$	F_5	$P_{5,1}$	$D+aB_2,\ a\neq 0$	$\cos cB_1+\sin cB_2, B_3, B_4, X_1, X_2, X_3, X_4,\ 0<c<\pi,\ c\neq\pi/2$	8
$S_{5,10}$		$P_{5,2}$	$D+aB_2,\ a\neq 0$	$\cos cB_1+\sin cB_2, B_3, B_4, X_1, X_2, X_3,\ 0<c<\pi,\ c\neq\pi/2$	7
$S_{5,11}$		$P_{5,3}$	$D+aB_2,\ a\neq 0$	$\cos cB_1+\sin cB_2, B_3, B_4, X_1,\ 0<c<\pi,\ c\neq\pi/2$	5
$S_{5,12}$		$P_{5,4}$	$D+aB_2,\ a\neq 0$	$\cos cB_1+\sin cB_2, B_3, B_4,\ 0<c<\pi,\ c\neq\pi/2$	4
$S_{6,12}$	F_6	$P_{6,1}$	$D+aB_2,\ a\neq 0$	$B_1, B_3, B_4, X_1, X_2, X_3,$	8
$S_{6,13}$		$P_{6,2}$	$D+aB_2,\ a\neq 0$	$B_1, B_3, B_4, X_1, X_2, X_3,$	7
$S_{6,14}$		$P_{6,2}$	$D+B_2+X_4$	$B_1, B_3, B_4, X_1, X_2, X_3$	7
$S_{6,15}$		$P_{6,3}$	$D+aB_2,\ a\neq 0$	B_1, B_3, B_4, X_4	5
$S_{6,16}$		$P_{6,4}$	$D+aB_2,\ a\neq 0$	B_1, B_3, B_4	4
$S_{6,17}$		$P_{6,4}$	$D-B_2+X_1$	B_1, B_3, B_4	4
$S_{7,13}$	F_7	$P_{7,1}$	$D+aB_1,\ a\neq 0$	$B_2, B_3, B_4, X_1, X_2, X_3, X_4$	8
$S_{7,14}$		$P_{7,2}$	$D+aB_1,\ a\neq 0$	$B_2, B_3, B_4, X_1, X_2, X_3$	7
$S_{7,15}$		$P_{7,4}$	$D+aB_1,\ a\neq 0$	B_2, B_3, B_4, X_1	5
$S_{7,16}$		$P_{7,5}$	$D+aB_1,\ a\neq 0$	B_2, B_3, B_4	4
$S_{10,19}$	F_{10}	$P_{10,1}$	$D+aB_1+bB_2,\ a^2+b^2\neq 0$	$B_3, B_4, X_1, X_2, X_3, X_4$	7
$S_{10,20}$		$P_{10,2}$	$D+aB_1+bB_2,\ a^2+b^2\neq 0$	$B_3, B_4, X_1, X_2, X_3,$	6
$S_{10,21}$		$P_{10,2}$	$D+aB_1+B_2+X_4, -\infty<a<\infty$	B_3, B_4, X_1, X_2, X_3	6
$S_{10,22}$		$P_{10,3}$	$D+aB_2,\ a\neq 0$	B_3, B_4, X_1, X_2	5
$S_{10,23}$		$P_{10,4}$	$D+aB_1+bB_2,\ a^2+b^2\neq 0$	B_3, B_4, X_1	4
$S_{10,24}$		$P_{10,5}$	$D+aB_1+bB_2,\ a^2+b^2\neq 0$	B_3, B_4	3
$S_{10,25}$		$P_{10,5}$	$D+aB_1-B_2+X_1, -\infty<a<\infty$	B_3, B_4	3

TABLE III. (*Continued*)

$S_{11,13}$	F_{11}	$P_{11,1}$	$D+aB_1,\ a>0$	$\cos cB_1+\sin cB_2, X_1, X_2, X_3, X_4,\quad 0<c<\pi,\ c\neq\pi/2$	6
$S_{11,14}$		$P_{11,2}$	$D+aB_1,\ a\neq 0$	$\cos cB_1+\sin cB_2, X_1, X_2, X_3,\quad 0<c<\pi,\ c\neq\pi/2$	5
$S_{11,15}$		$P_{11,3}$	$D+aB_1,\ a>0$	$\cos cB_1+\sin cB_2, X_2, X_3,\quad 0<c<\pi,\ c\neq\pi/2$	4
$S_{11,16}$		$P_{11,4}$	$D+aB_1,\ a>0$	$\cos cB_1+\sin cB_2, X_1, X_4,\quad 0<c<\pi,\ c\neq\pi/2$	4
$S_{11,17}$		$P_{11,5}$	$D+aB_1,\ a\neq 0$	$\cos cB_1+\sin cB_2, X_1,\quad 0<c<\pi,\ c\neq\pi/2$	3
$S_{11,18}$		$P_{11,6}$	$D+aB_1,\ a>0$	$\cos cB_1+\sin cB_2,\quad 0<c<\pi,\ c\neq\pi/2$	2
$S_{12,33}$	F_{12}	$P_{12,1}$	$D+aB_2,\ a>0$	B_1, X_1, X_2, X_3, X_4	6
$S_{12,34}$		$P_{12,2}$	$D+aB_2,\ a\neq 0$	B_1, X_1, X_2, X_3	5
$S_{12,35}$		$P_{12,2}$	$D+B_2+X_4$	B_1, X_1, X_2, X_3	5
$S_{12,36}$		$P_{12,5}$	$D+aB_2, a>0$	B_1, X_1, X_4	4
$S_{12,37}$		$P_{12,6}$	$D+aB_2,\ a>0$	B_1, X_2, X_3	4
$S_{12,38}$		$P_{12,6}$	$D+B_2+X_4$	B_1, X_2, X_3	4
$S_{12,39}$		$P_{12,7}$	$D+aB_2,\ a\neq 0$	B_1, X_1	3
$S_{12,40}$		$P_{12,7}$	$D+B_2+X_4$	B_1, X_1	3
$S_{12,41}$		$P_{12,10}$	$D+aB_2,\ a>0$	B_1	2
$S_{12,42}$		$P_{12,10}$	$D+B_2+X_4$	B_1	2
$S_{13,25}$	F_{13}	$P_{13,1}$	$D+aB_1,\ a>0$	B_2, X_1, X_2, X_3, X_4	6
$S_{13,26}$		$P_{13,2}$	$D+aB_1,\ a\neq 0$	B_2, X_1, X_2, X_3	5
$S_{13,27}$		$P_{13,4}$	$D+aB_1,\ a>0$	B_2, X_1, X_4	4
$S_{13,28}$		$P_{13,6}$	$D+aB_1,\ a>0$	B_2, X_2, X_3	4
$S_{13,29}$		$P_{13,7}$	$D+aB_1,\ a\neq 0$	B_2, X_1	3
$S_{13,30}$		$P_{13,9}$	$D+aB_1,\ a>0$	B_2	2
$S_{14,31}$	F_{14}	$P_{14,1}$	$D+aB_2,\ a\neq 0$	B_3, X_1, X_2, X_3, X_4	6
$S_{14,32}$		$P_{14,1}$	$D+B_4$	B_3, X_1, X_2, X_3, X_4	6
$S_{14,33}$		$P_{14,2}$	$D+aB_2,\ a\neq 0$	B_3, X_1, X_2, X_3	5
$S_{14,34}$		$P_{14,2}$	$D+B_2+X_4$	B_3, X_1, X_2, X_3	5
$S_{14,35}$		$P_{14,2}$	$D+B_4$	B_3, X_1, X_2, X_3	5
$S_{14,36}$		$P_{14,3}$	$D+aB_2, a\neq 0$	B_3, X_1, X_3, X_4	5
$S_{14,37}$		$P_{14,4}$	$D+aB_2',\ a\neq 0$	B_3, X_1, X_2	4
$S_{14,38}$		$P_{14,4}$	$D+B_4$	B_3, X_1, X_2	4
$S_{14,39}$		$P_{14,5}$	$D+aB_2, a\neq 0$	B_3, X_1, X_3	4
$S_{14,40}$		$P_{14,5}$	$D+B_2+X_4$	B_3, X_1, X_3	4
$S_{14,41}$		$P_{14,5}$	$D+B_4$	B_3, X_1, X_3	4
$S_{14,42}$		$P_{14,6}$	$D+aB_2,\ a\neq 0$	$B_3, X_1, X_2+cX_3,\ c\neq 0$	4
$S_{14,43}$		$P_{14,6}$	$D+B_4$	$B_3, X_1, X_2+cX_3,\ c\neq 0$	4
$S_{14,44}$		$P_{14,7}$	$D+aB_2,\ a\neq 0$	B_3, X_1	3
$S_{14,45}$		$P_{14,7}$	$D+B_4$	B_3, X_1	3
$S_{14,46}$		$P_{14,8}$	$D+aB_2,\ a\neq 0$	B_3, X_2	3
$S_{14,47}$		$P_{14,8}$	$D-B_2+X_1$	B_3, X_2	3
$S_{14,48}$		$P_{14,9}$	$D+aB_2,\ a\neq 0$	B_3	2
$S_{14,49}$		$P_{14,9}$	$D-B_2+X_1$	B_3	2
$S_{14,50}$		$P_{14,9}$	$D+B_4$	B_3	2
$S_{15,23}$	F_{15}	$P_{15,1}$	$D+a(\cos cB_1+\sin cB_2),\ a>0,\ 0\leq c<\pi$	X_1, X_2, X_3, X_4	5
$S_{15,24}$		$P_{15,1}$	$D+B_3$	X_1, X_2, X_3, X_4	5
$S_{15,25}$		$P_{15,2}$	$D+a(\cos cB_1+\sin cB_2),\ a>0,\ 0\leq c<2\pi$	X_1, X_2, X_3	4
$S_{15,26}$		$P_{15,2}$	$D+B_3$	X_1, X_2, X_3	4
$S_{15,27}$		$P_{15,3}$	$D+aB_1,\ a>0$	X_1-X_4, X_2, X_3	4
$S_{15,28}$		$P_{15,4}$	$D+aB_1,\ a>0$	X_1+X_4, X_2, X_3	4
$S_{15,29}$		$P_{15,4}$	$D+a(B_4-B_6),\ a>0$	X_1+X_4, X_2, X_3	4
$S_{15,30}$		$P_{15,4}$	$D+B_1+B_3+B_5$	X_1+X_4, X_2, X_3	4
$S_{15,31}$		$P_{15,5}$	$D+aB_2,\ a\neq 0$	X_1, X_2	3
$S_{15,32}$		$P_{15,5}$	$D+(\cos cB_3+\sin cB_4),\quad 0\leq c<\pi$	X_1, X_2	3
$S_{15,33}$		$P_{15,6}$	$D+a(\cos cB_1+\sin cB_2),\quad 0\leq c<\pi$	X_1, X_4	3
$S_{15,34}$		$P_{15,7}$	$D+a(\cos cB_1+\sin cB_2),\quad 0\leq c<\pi$	X_2, X_3	3
$S_{15,35}$		$P_{15,8}$	$D+a(\cos cB_1+\sin cB_2)\quad 0\leq c<2\pi$	X_1	2
$S_{15,36}$		$P_{15,8}$	$D+B_3$	X_1	2
$S_{15,37}$		$P_{15,9}$	$D+aB_1,\ a>0$	X_1+X_4	2
$S_{15,38}$		$P_{15,10}$	$D+aB_1,\ a>0$	X_1-X_4	2
$S_{15,39}$		$P_{15,10}$	$D+a(B_3+B_5),\ a>0$	X_1-X_4	2
$S_{15,40}$		$P_{15,10}$	$D+B_1-B_3-B_5$	X_1-X_4	2
$S_{15,41}$		$P_{15,11}$	$D+a(\cos cB_1+\sin cB_2), a>0,\ 0\leq c<\pi$	0	1
$S_{15,42}$		$P_{15,11}$	$D+B_3$	0	1

In the first column the symbol $S_{j,k}$ indicates that this is the kth algebra obtained as an extension of F_j by dilations and translations. The second column lists the subalgebras F_j and the third column gives $P_{j,k'}$, i.e., the subalgebra of the Poincaré algebra that we are adding the generator $\tilde{D}$ to. All generators of $S_{j,k}$ are in columns 4 and 5. The dimension $\dim_R S_{j,k}$ of $S_{j,k}$ over the field of real numbers is given in column 6.

TABLE IV. Subalgebras of S that are not contained in P, do not contain an SG-Conjugate of D and are such that their intersection with P does not split over the translations.

Notation	F_j	$\tilde{P}_{j,k}$	$\tilde{D}$	Generators of $\tilde{P}_{j,k}$	$\dim_R S_{j,k}$
$S_{6,18}$	F_6	$S_{6,5}$	$D+B_2+xX_4, -\infty<x<\infty$	$B_1+X_4, B_3, B_4, X_1, X_2, X_3$	7
$S_{6,19}$		$S_{6,6}$	$D+B_2$	$B_1, B_3+X_2, B_4+X_3, X_1$	5
$S_{6,20}$		$S_{6,7}$	$D-B_2+xX_1, -\infty<x<\infty$	B_1+X_1, B_3, B_4	4
$S_{10,26}$	F_{10}	$S_{10,6}$	$D+\frac{1}{2}B_2$	$B_3+X_4, B_4, X_1, X_2, X_3$	6
$S_{10,27}$		$S_{10,8}$	$D+\frac{1}{2}B_2$	B_3+X_4, B_4, X_1, X_2	5
$S_{10,28}$		$S_{10,9}$	$D+B_2$	B_3, B_4+X_3, X_1, X_2	5
$S_{10,29}$		$S_{10,10}$	$D+B_2$	B_3, B_4+X_2, X_1	4
$S_{10,30}$		$S_{10,11}$	$D+B_2$	$B_3+X_2, B_4+bX_2+X_3, X_1,\ b\neq 0$	4
$S_{10,31}$		$S_{10,12}$	$D+aB_1+B_2, -\infty<a<\infty$	B_3+X_2, B_4+X_3, X_1	4
$S_{10,32}$		$S_{10,13}$	$D+B_2$	$B_3, B_4+X_2,$	3
$S_{12,43}$	F_{12}	$S_{12,11}$	$D+B_2+xX_4, -\infty<x<\infty$	B_1+X_4, X_1, X_2, X_3	5
$S_{12,44}$		$S_{12,14}$	$D+B_2+xX_4, -\infty<x<\infty$	B_1+X_4, X_2, X_3	4
$S_{12,45}$		$S_{12,17}$	$D+B_2+xX_4, -\infty<x<\infty$	B_1+X_4, X_1	3
$S_{12,46}$		$S_{12,20}$	$D+B_2+xX_4, -\infty<x<\infty$	B_1+X_4	2
$S_{14,51}$	F_{14}	$S_{14,10}$	$D+\frac{1}{2}B_2$	B_3+X_4, X_1, X_2, X_3	5
$S_{14,52}$		$S_{14,11}$	$D+B_2$	B_3+X_2, X_1, X_3, X_4	5
$S_{14,53}$		$S_{14,12}$	$D+\frac{1}{2}B_2$	B_3+X_4, X_1, X_2	4
$S_{14,54}$		$S_{14,13}$	$D+\frac{1}{2}B_2$	B_3+X_4, X_1, X_3	4
$S_{14,55}$		$S_{14,14}$	$D+B_2+xX_4, -\infty<x<\infty$	B_3+X_2, X_1, X_3	4
$S_{14,56}$		$S_{14,15}$	$D+\frac{1}{2}B_2+b(B_4+aX_4), -\infty<b<\infty$	$B_3+X_4, X_1, X_2+aX_3, a\neq 0$	4
$S_{14,57}$		$S_{14,16}$	$D+bB_2+2a(b-1)X_4, -\infty<b<\infty$	$B_3+X_2, X_1, X_2+aX_3, a\neq 0$	4
$S_{14,58}$		$S_{14,17}$	$D+\frac{1}{2}B_2$	B_3+X_4, X_1	3
$S_{14,59}$		$S_{14,18}$	$D+B_2$	B_3+X_2, X_1	3
$S_{14,60}$		$S_{14,19}$	$D+\frac{1}{2}B_2$	B_3+X_4, X_2	3
$S_{14,61}$		$S_{14,20}$	$D+\frac{1}{2}B_2$	B_3+X_4	2
$S_{14,62}$		$S_{14,21}$	$D+B_2+b(B_4-X_3),\ b\geq 0$	B_3+X_2	2

D. Subalgebras of S that are not contained in the Poincaré algebra, do not contain any SG-conjugate of D and are such that the intersection with the Poincaré algebra does not split over the translations

We consider individually each subalgebra $S_{j,k}\equiv \tilde{P}_{j,k}$ of Table II of the present article, i.e., the algebras obtained from Table IV of I by using dilatations to make certain classes of subalgebras of P coalesce. To the generators of $S_{j,k}$ we again add a further operator $\tilde{D} = D + a_\mu B_\mu + x_a X_a$, putting the coefficient a_μ and x_a equal to zero if the corresponding B_μ or X_a figures in $\tilde{P}_{j,k}$ (we can set $a_\mu = 0$ if $B_\mu \in \tilde{P}_{j,k}$ or $B_\mu + y_{\mu k}X_k \in \tilde{P}_{j,k}$ where $y_{\mu k}$ are real constants). Restrictions on the possible values of a_μ and x_a are obtained by requiring that $\tilde{D} \dotplus \tilde{P}_{j,k}$ forms a Lie algebra. The element $\tilde{D}$ of the algebra is then simplified using transformations belonging to the normalizer of $\tilde{P}_{j,k}$ in the similitude group, i.e., the normalizer of $\tilde{P}_{j,k}$ in the Poincaré group, listed in Table IV of I, supplemented by the discrete element D in the similitude group and transformations of the type $\exp\{D + b_\mu B_\mu + y_a X_a\}$ with b_μ and y_a so chosen as to leave $\tilde{P}_{j,k}$ invariant.

We shall consider some examples and then list all subalgebras of S obtained in this manner in Table IV above.

Consider the algebras $S_{6,k}$ of Table II. The element $\tilde{D}$ can be of the form $D + aB_2 + bB_5 + cB_6 + x_\mu X_\mu$. Commuting with $B_1 + X_4$, B_1 or $B_1 + X_1$, as the case may be, we find $b = c = 0$. Consider first case $S_{6,5}$, i.e.,

$$\tilde{D} = D + aB_2 + xX_4,\ \ B_1 + X_4, B_3, B_4, X_1, X_2, X_3. \tag{29}$$

We have

$$[\tilde{D}, B_1 + X_4] = 2(1-a)X_4$$

and hence $a = 1$. Algebra (29) with $a = 1$, x arbitrary real should be further simplified, i.e., we must attempt to restrict further possible values of x. The normalizer of $S_{6,5}$ contains transformations generated by $D + B_2$, B_1 and X_4 (in addition to the inner automorphisms $\exp S_{6,5}$). Using the commutation relations of Table I, it is easy to see that none of these change the value of x and hence (29) cannot be further simplified. Similar results are obtained for $S_{6,6}$ and $S_{6,7}$ (see Table IV).

It can be verified directly that none of the algebras $S_{7,k}$ or $S_{8,k}$ of Table II can be extended by dilatations. Now consider algebras $S_{10,6}$ and $S_{10,8}$ involving $B_3 + X_4$, B_4, X_1, X_2 and in the case of $S_{10,6}$, also X_3. In the case $S_{10,6}$ we find that the most general operator $\tilde{D}$ forming a Lie algebra with $S_{10,6}$ is

$$\tilde{D} = D + \tfrac{1}{2}B_2 + xX_4.$$

The normalizer of $S_{10,6}$ is generated by B_3, B_4, X_1, X_2, X_3, X_4 and $D + \frac{1}{2}B_2$. We have

$$\exp(yX_4)\tilde{D}\exp(-yX_4) = D + \tfrac{1}{2}B_2$$

if we put $y = x$. We thus obtain a single algebra generated by

$$D + \tfrac{1}{2}B_2, B_3 + X_4, B_4, X_1, X_2, X_3.$$

Similarly, for $S_{10,8}$ we find that

$$\tilde{D} = D + \tfrac{1}{2}B_2 + xX_3$$

provides an extension for all x. However, $\exp yX_3$ belongs to the normalizer of $S_{10,8}$ and

$$\exp(yX_3)\tilde{D}\exp(-yX_3) = D + \tfrac{1}{2}B_2$$

if we put $y = x/2$. We again obtain a single algebra

$$D + \tfrac{1}{2}B_2, B_3 + X_4, B_4, X_1, X_2.$$

Continuing along the same lines we obtain the results presented in Table IV. The first column simply enumerates the subalgebras of this type, the second tells us which subalgebra of $LSL(2, C)$ they were derived from, the third lists their intersections with the Poincaré algebra using the notations of Table II, the fourth and fifth column give all the generators and the last column gives the dimensions of the subalgebras.

This completes the list of all conjugacy classes of subalgebras of the similitude algebra.

Since the subalgebras of the homogeneous similitude algebra (the algebra of the homogeneous Lorentz group extended by dilatations) represent separate interest we provide a separate table of these (Table V). We suggest the name "scaling group" for this group. In Table V we use somewhat different conventions than in the rest of this article, in order to be able to show the mutual inclusions of the subalgebras. In this table $B_x = \cos x B_1 + \sin x B_2$ with $0 \leq x < \pi$, i. e., we *include* the points $x = 0$ and $x = \pi/2$. Subgroups of $D \otimes SL(2, C)$ that are contained in $SL(2, C)$ are separated out graphically. The lines connect each subalgebra (or continuous subgroup) with its maximal subalgebras. A full line indicates that the inclusion holds always, a dotted line indicates inclusion for specified values of the parameters only. Note that the way of writing the subalgebras in Table V corresponds more directly to Sec. 4 of article I than to the conventions of the rest of the present article.

4. CONCLUSIONS

The result of this paper is the complete classification of all subalgebras of the Lie algebra S of the similitude group SG. These subalgebras are of several types:

1. Subalgebras of S that are also subalgebras of the Poincaré algebra P and are splitting extensions of subalgebras of $LSL(2, C)$ by translations. Conjugacy classes of such algebras under the similitude group coincide with conjugacy classes under the Poincaré group. Representatives of all such algebras are listed in Table III of I and are not reproduced here. Their labels $S_{j,k}$ are obtained by setting j and k equal to the values they take in Table III of I.

2. Subalgebras of S that are also subalgebras of P and are nonsplitting extensions of $LSL(2, C)$ by translations. Many independent conjugacy classes under the Poincaré group coalesce under the similitude group. Representatives of all conjugacy classes of such algebras (under the similitude group) are given in Table II of this paper.

3. Subalgebras of S that contain D (the dilatation) as a generator. Representatives of all such algebras are obtained by taking Table III of I and adding the element D itself to the generators. We do not reproduce these subalgebras here; they are however assigned labels $S_{j,k}$ [see (24)].

4. Subalgebras of S such that (i) they contain an element $\tilde{D} = D + \Sigma a_\mu B_\mu + \Sigma x_a X_a$, but no SG-conjugate of D, (ii) their intersection with P splits over the translations. Representatives of all such algebras are listed in Table III above.

5. Subalgebras of S satisfying condition 4(i) above, but such that their intersection with P does not split over the translations. Representatives of all such algebras are listed in Table IV above.

The notations of this article are not entirely self-evident. It is, however, quite trivial to return to the usual physical notations. For the generators, indeed, the connection is given in formulas (13) and (14). Note that in our tables we have sometimes let the continuous parameters range through closed regions, e. g., $0 \leq c < 2\pi$ in $S_{15,25}$, sometimes through open ones like $0 < c < \pi/2$, $\pi/2 < c < \pi$ in $S_{5,9}$. In the last case the end points are separated out and listed separately. We could clearly have bunched more algebras together under one heading, but we did not find this appropriate, since the algebras, corresponding to the limiting values of the parameters often have quite specific properties.

The homogeneous similitude group $\exp D \otimes SL(2, C)$ is of separate interest and has already been treated in I. Indeed, in Table V of I we gave a complete list of subalgebras of $D \oplus LSL(2, C)$, obtained by using a version of the "Goursat twist method,"[17–19] also presented in I. The results of Table V of I are actually contained in the tables of this paper in a somewhat different, but equivalent form. Table V of the present paper is new and shows the mutual inclusions of various conjugacy classes of subgroups of the homogeneous similitude group.

Let us just mention some related work on the classification of continuous subgroups of real Lie groups. All one-dimensional subgroups of $U(p, q)$ and $SU(p, q)$ groups are known.[20] A classification of the real semisimple subgroups of real semisimple groups was performed.[21] Subgroups of the Poincaré group were also considered by other authors[22] and some work has been done on certain subgroups of the conformal group, Galilei group and others.[23]

In the following papers of this series we plan to provide similar lists of subalgebras and continuous subgroups for further groups of interest (de Sitter, conformal and others). We shall also return to the subgroups of the Poincaré and similitude groups and discuss some of their properties (mutual inclusions, isomorphisms, existence of Casimir operators, etc.).

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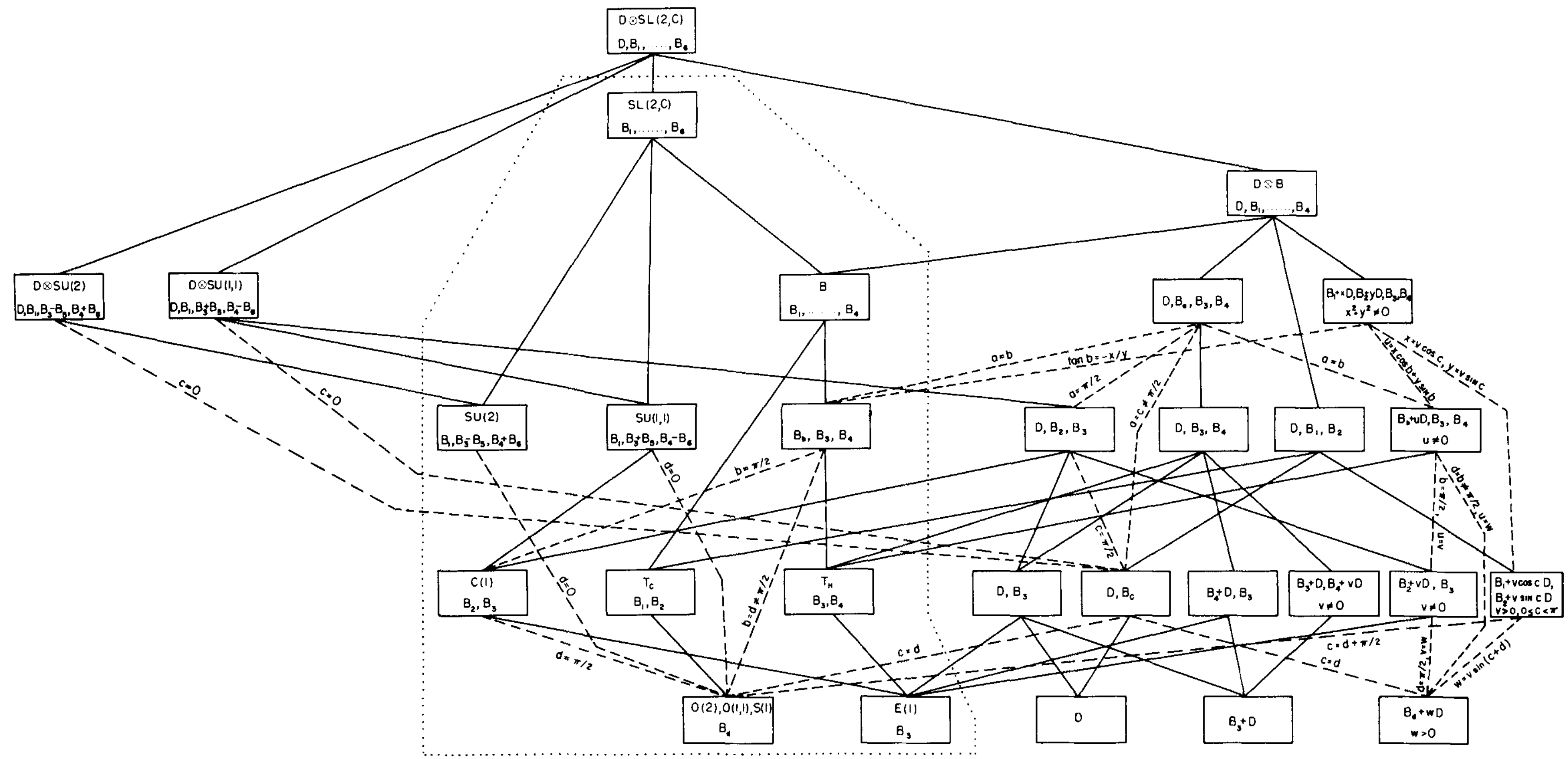
TABLE V. Continuous subgroups of the scaling group (homogeneous Lorentz group extended by dilatations).



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