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CONDITIONS FOR VOTING EQUILIBRIA IN CONTINUOUS VOTER DISTRIBUTIONS*

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Abstract. This paper extends Plott's necessary and sufficient conditions for the existence of majority rule equilibria to the case where there is a continuous distribution of voters. Plott's theorem extends in a natural way to this setting: It is shown that a point, x^*, is a majority rule equilibrium if and only if, for every measurable cone originating at the origin, the measure of the voters whose gradients (at x^*) lie in the cone is equal to the measure of the voters whose gradients lie in the negative cone.

1. Introduction.

Plott's (1967) well-known necessary and sufficient conditions for equilibrium in majority rule spatial voting games show that an equilibrium (or Condorcet point) exists only under the most restrictive circumstances. Briefly, assuming that (1) individual utility functions are differentiable, (2) the number of voters is finite, and (3) at most one voter's ideal point is located at the presumed equilibrium, Plott establishes that a "local" equilibrium exists if and only if all voters (excluding any at the equilibrium) can be paired in such a way that the contract curve of each pair passes through the equilibrium. Equivalently, voters can be paired such that the utility gradients of voters in each pair point in exactly opposite directions. Sloss (1973) extends this result to deal with nondifferentiable utility functions.

Here we focus on assumption (2) above and generalize Plott's analysis to continuous voter distributions. While empirical preference distributions are, of course, discrete, a substantial literature models them as continuous. For example, Downs (1957) and Tullock (1967) discuss, in very unformalized terms, the question of equilibria for continuous voter distributions. More recent literature formalizes some of these ideas, but assumes specific functional forms for individual utility, and generally gives only *sufficient* conditions for existence of equilibria (see Davis, Hinich and Ordeshook (1970), and Riker and Ordeshook (1973, Chaps. 11–12) for reviews of this literature). These models assume individual preferences are a function of distance (not necessarily Euclidean) from some ideal point, and the basic results are that a strong type of symmetry in the distribution of ideal points is sufficient for the existence of equilibria. Only when all preferences are a function of Euclidean distance have necessary and sufficient conditions been found for the infinite voter model. The condition is the existence of a total median in the distribution of ideal points (Davis, Degroot, and Hinich (1972)).

The main result of this paper shows that Plott's Theorem extends, in a natural way, to the continuous voter case. The above results then follow directly from this extension of Plott's Theorem. Further, this theorem gives both necessary and sufficient conditions for equilibria for more general types of utility functions than have heretofore been dealt with in the continuous voter case. Also, we prove somewhat weaker sufficient conditions than do either Plott or Sloss (although, since the equilibria considered here are global rather than local, we impose a stronger assumption on the utility functions—pseudoconcavity). Finally, we show, via several examples, that if assumption (3) is not satisfied, Plott's conditions, as well as our generalization of them, are no longer necessary for equilibria.

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2. Definitions and assumptions. Let Ω denote a set of *voters*, and X a *policy space*. Elements of Ω are *voters*, denoted α, β, etc., and elements of X are *alternatives*, denoted x, y, etc. Any subset of Ω is a *coalition*, designated A, B, C, etc. Now, for each $\alpha \in \Omega$, we assume a *utility function* $U_\alpha : X \to R$. This yields a binary relation $\geqq_\alpha \subseteq X \times X$, as follows: For any $x, y \in X$

$$x \geqq_\alpha y \quad \Leftrightarrow \quad U_\alpha(x) \geqq U_\alpha(y). \tag{2.1}$$

Note that $\geqq_\alpha$ is transitive and complete, for all $\alpha \in \Omega$. The derived relations $>_\alpha$ and $\sim_\alpha$ are defined in the usual manner. Further, for any coalition $B \subseteq \Omega$, we say B prefers x to y, written $x >_B y \Leftrightarrow x >_\alpha y$ for all $\alpha \in B$. Similarly, $x \geqq_B y \Leftrightarrow x \geqq_\alpha y$ for all $\alpha \in B$.

In general, the set Ω may be either finite or infinite, but we assume that there is a finite measure, μ, on a σ algebra of sets, $\underset{\sim}{B}$ in Ω, so that for any $B \in \underset{\sim}{B}$, $\mu(B)$ represents the size of the coalition B. Since μ is a finite measure, we set $\mu(\Omega) = N$. We also assume that for any $x, y \in X$, that $\{a \in \Omega \mid x \geqq_\alpha y\} \in \underset{\sim}{B}$. If Ω is finite, $\mu(B)$ is the counting measure, and corresponds to the number of voters in B. Using the shorthand $\mu[x >_\alpha y] = \mu\{\alpha \in \Omega \mid x >_\alpha y\}$, the *plurality preference relation* over X, $\geqq$, is then

$$x \geqq y \quad \Leftrightarrow \quad \mu[x >_\alpha y] \geqq \mu[y >_\alpha x].$$

This relation, in turn, defines a Condorcet point:

DEFINITION 2.1. $x \in X$ is a *Condorcet point* $\Leftrightarrow$ $x \geqq y$ for all $y \in X$.

Of course, a Condorcet point may not in general exist. An important definition in establishing conditions for existence is the following:

DEFINITION 2.2. For any $\alpha, \beta \in \Omega$, the *contract set* $C(\alpha, \beta)$, is the set $C(\alpha, \beta) = \{x \in X \mid \text{for all } y \in X,\ y >_\alpha x \Rightarrow x >_\beta y \text{ and } y >_\beta x \Rightarrow x >_\alpha y\}$.

In the usual terminology, $C(\alpha, \beta)$ corresponds to the set of points that are Pareto optimal for α and β. See e.g., McKelvey and Wendell (1976). We now state a sufficient condition for a point to be a Condorcet point.

THEOREM 1. *A sufficient condition for a point $x^* \in X$ to be a Condorcet point is that there exists some measure preserving transformation $T: \Omega \to \Omega$ such that*

$$x^* \in \bigcap_{\alpha \in \Omega} C(\alpha, T\alpha).$$

Proof. Let x^* and T satisfy the conditions of the theorem. We must show that, for all $x \in X$, $\mu[x^* >_\alpha x] \geqq \mu[x >_\alpha x^*]$. Let $B = \{\alpha \mid x >_\alpha x^*\}$ and let $A = T(B) = \{T\alpha \mid \alpha \in B\}$. Since $B \in \underset{\sim}{B}$ and T is measure preserving, $A \in \underset{\sim}{B}$ and $\mu(A) = \mu(B)$. Further since $x^* \in C(\alpha, T\alpha)$ for any $\alpha \in B$, it follows that $\beta \in A \Rightarrow x^* >_\beta x$. So $A \subseteq \{\alpha \in \Omega \mid x^* >_\alpha x\}$. Hence $\mu[x^* >_\alpha x] \geqq \mu[A] = \mu[B] = \mu[x >_\alpha x^*]$ and it follows that $x^* \geqq x$. So x^* is a Condorcet point. Q.E.D.

3. Necessary conditions. Note that the sufficiency condition requires no assumptions on the nature of individual utility functions nor even on the topological structure of X. To formulate necessary conditions, we consider only the case where $X \subseteq \mathcal{R}^n$, with X open and convex. Further, we concern ourselves only with the case when global characteristics of the space, such as contract sets and Condorcet points, can be described in terms of local properties of individual utility functions. Pseudoconcavity of individual utility functions is enough to guarantee this equivalence. We develop this in more detail.

If U_α is differentiable, we let $\nabla U_\alpha(x)$ represent the gradient vector at the point $x \in X$. The function U_α is said to be *pseudoconcave* if it is differentiable and, for each $x^* \in X$,

$$\nabla U_\alpha(x^*)(x - x^*) \leqq 0 \quad \Rightarrow \quad U_\alpha(x) \leqq U_\alpha(x^*) \tag{3.1}$$

for all $x \in X$. See, e.g., Mangasarian (1969) for properties of pseudoconcave functions. In particular, any pseudoconcave function is also strictly quasiconcave (Mangasarian, (1969, p. 143)). That is, for any $x, y \in X$, and $0 < \lambda < 1$,

$$U_\alpha(x) < U_\alpha(y) \quad \Rightarrow \quad U_\alpha(x) < U_\alpha(\lambda x + (1-\lambda)y). \tag{3.2}$$

Further, if all U_α are pseudoconcave, then for any $x \in X$, $y \in \mathscr{R}^n$, it follows that sets of the form $\{\alpha | \nabla U_\alpha(x) \cdot y > 0\}$, $\{\alpha | \nabla U_\alpha(x) \cdot y \geqq 0\}$, $\{\alpha | \nabla U_\alpha(x) \cdot y = 0\}$, etc., are measurable. To see this, note, by pseudoconcavity,

$$\{\alpha | \nabla U_\alpha(x) \cdot y > 0\} = \bigcup_{i=i_0}^{\infty} \left\{\alpha \in \Omega \,\middle|\, \left(x + \frac{1}{i} y\right) >_\alpha x\right\}, \tag{3.3}$$

where i_0 is chosen so that $x + (1/i_0)\, y \in X$.

By assumption, the sets on the right-hand side are measurable, hence the set on the left-hand side is also measurable. Next, we can write $\{\alpha | \nabla U_\alpha(x) \cdot y \geqq 0\} = \Omega - \{\alpha | \nabla U_\alpha(x) \cdot (-y) > 0\}$ and $\{\alpha | \nabla U_\alpha(x) \cdot y = 0\} = \{\alpha | \nabla U_\alpha(x) \cdot y \geqq 0\} \cap \{\alpha | \nabla U_\alpha(x) \cdot (-y) \geqq 0\}$ to establish that sets of these forms are measurable.

Now if U_α and U_β are both pseudoconcave with unique maxima,

$$x \in C(\alpha, \beta) \quad \Leftrightarrow \quad k \nabla U_\alpha(x) = -l \nabla U_\beta(x) \tag{3.4}$$

for some $k \geqq 0$, $l \geqq 0$, not both zero (c.f., McKelvey and Wendell (1976)). So the contract set can be described in terms of local properties of the U_α. Similarly, a local characterization of Condorcet points can be given in terms of a "Plott equilibrium":

DEFINITION 3.1. If all U_α are differentiable, a point $x^* \in X$ is a *Plott equilibrium* if for all $y \in \mathscr{R}^n$, $\mu\{\alpha | y \cdot u_\alpha > 0\} \leqq N/2$ where $u_\alpha = \nabla U_\alpha(x^*)$. Equivalently, this can be written $\mu\{\alpha | y \cdot u_\alpha > 0\} \leqq \mu\{\alpha | y \cdot u_\alpha \leqq 0\}$ for all $y \in \mathscr{R}^n$.

The equivalence of a Plott equilibrium and a Condorcet point is not assured, however, for general measures on Ω (even if all U_α are pseudoconcave). The following example given by Sloss (1973), serves as a counterexample to this equivalence: Let $X = R^2$, $\Omega = \{1, 2, \cdots\}$, and $U_\alpha(x) = -\|x - x_\alpha\|^2$, where $x_\alpha = (0, 1/\alpha)$. Then, if the measure on Ω is the counting measure, but if $|\Omega|$ is infinite, $x^* = (0, 0)$ is a Condorcet point. At x^*, though, $\nabla U_\alpha(x^*) = (0, 2/\alpha)$, so all voters prefer to move in the direction $(0, 1)$. This counterexample clearly rests on the assumption that the measure on Ω is not finite. If the measure is finite, we can establish the equivalence of a Condorcet point and a Plott equilibrium. Thus, we have

LEMMA 1. *If U_α is pseudoconcave for all $\alpha \in \Omega$, and if μ is a finite measure on Ω, then x^* is a Condorcet point $\Leftrightarrow$ x^* is a Plott equilibrium.*

Proof. Assume x^* is a Condorcet point, but that it is not a Plott equilibrium. Then for some $y \in \mathscr{R}^n$, $\mu\{\alpha | y \cdot u_\alpha > 0\} > N/2$. (Here $u_\alpha = \nabla U_\alpha(x^*)$, as before.) Let $C = \{\alpha | y \cdot u_\alpha > 0\}$. Pick i_0 so that $x^* + (1/i_0)\, y \in X$. For $i \geqq i_0$, let

$$A_i = \left\{\alpha \,\middle|\, U_\alpha\left(x^* + \frac{1}{i} y\right) > U_\alpha(x^*)\right\}. \tag{3.5}$$

By assumption these sets are in the σ-algebra B. By differentiability of the U_α, it follows for every $\alpha \in C$ there exists an integer $i \geqq i_0$ such that $\alpha \in A_i$. Further, by pseudoconcavity and (3.2), the A_i form an increasing nested sequence. So there exists a J such that for $i > J$, $\mu(A_i) > \frac{1}{2}N$. Hence for these values of i, $x^* + (1/i)y > x^*$, contrary to the assumption that x^* is a Condorcet point.

Sloss (1973, Thrm. 4.4, p. 66) proves the converse, so the two equilibrium notions are equivalent in the case of finite measure spaces and pseudoconcave utility functions. Q.E.D.

Before we prove the main theorem of this paper, we require an additional lemma that is central to its proof. First, given a vector $y \in \mathcal{R}^n$, we define:

$$H_y = \{x \in \mathcal{R}^n | y' \cdot x = 0\},$$
$$(3.6) \qquad H_y^+ = \{x \in \mathcal{R}^n | y' \cdot x > 0\},$$
$$H_y^- = \{x \in \mathcal{R}^n | y' \cdot x < 0\}.$$

Further,

$$(3.7) \qquad \underline{H} = \{\underline{H}_y | y \in \mathcal{R}^n\}, \quad \underline{H}^+ = \{\underline{H}_y^+ | y \in \mathcal{R}^n\},$$

and we let $\underline{\underline{B}}$ be the σ-algebra generated by $\underline{H}^+$. So sets in $\underline{\underline{B}}$ are cones with vertex at the origin. Then we have

LEMMA 2. *Let λ be a finite measure on $\underline{\underline{B}}$; then if $\lambda(H_y^+) = \lambda(H_y^-)$ for all $y \in \mathcal{R}^n$, $\lambda(A) = \lambda(-A)$ for all $A \in \underline{\underline{B}}$.*[1]

Proof. This is proven by Overdijk, Simons and Steutel (1977).

Now, given $x^* \in X$, the measure μ generates a natural measure μ^*, on $\underline{\underline{B}}$ as follows. For any $y \in \mathcal{R}^n$, set

$$(3.8) \qquad \mu^*(H_y^+) = \mu\{\alpha | \nabla U_\alpha(x^*) \cdot y > 0\} = \mu\{\alpha | \nabla U_\alpha(x^*) \in H_y^+\}.$$

It follows, for any $A \in \underline{\underline{B}}$, that

$$(3.9) \qquad \mu^*(A) = \mu\{\alpha | \nabla U_\alpha(x^*) \in A\}.$$

We now have the following theorem giving necessary and sufficient conditions for a point to be a Condorcet point.

THEOREM 2. *Assume all U_α are pseudoconcave, μ is a finite measure on Ω, and that $\mu\{\alpha | \nabla U_\alpha(x^*) = 0\} = 0$. Then a necessary and sufficient condition for x^* to be a Condorcet point is that*

$$\mu^*(A) = \mu^*(-A)$$

for all $A \in \underline{\underline{B}}$.

Proof. For any $\alpha \in \Omega$ write $u_\alpha = \nabla U_\alpha(x^*)$. Then x^* is a Condorcet point if and only if for any x, $\mu\{\alpha | x >_\alpha x^*\} \leqq \mu\{\alpha | x^* >_\alpha x\}$. But, by Lemma 1, this is true if and only if for all x, $\mu\{\alpha | (x - x^*)' \cdot u_\alpha > 0\} \leqq \mu\{\alpha | (x - x^*)' \cdot u_\alpha \leqq 0\}$. Equivalently, setting $y = x - x^*$, we have for all y, $\mu\{\alpha | y' \cdot u_\alpha > 0\} \leqq \mu\{\alpha | y' \cdot u_\alpha \leqq 0\}$. Using the notation defined above, it follows that x^* is a Condorcet point if and only if

$$(3.10) \qquad \mu^*(H_y^+) \leqq \mu^*(H_y \cup H_y^-) \quad \text{for all } y.$$

We wish to show now that (3.10) is equivalent to

$$(3.11) \qquad \mu^*(H_y^+) = \mu^*(H_y^-) \quad \text{for all } y.$$

Obviously, (3.11) implies (3.10), so we must only show that (3.10) implies (3.11).

If $\{z_i\}_{i=1}^\infty$ is a sequence of vectors in $\mathcal{R}^n$ such that any n vectors in the sequence are linearly independent then for any $\varepsilon > 0$ the inequality $\mu^*(H_{z_i}) > \varepsilon$ holds for at most

[1] This lemma was proved first by Peter Ungar, but we cite here Overdijk et al.'s proof since it is considerably shorter than the original.

finitely many i. To prove this, we let $\mathcal{N}$ be the natural numbers, and for any finite subset $S \subseteq \mathcal{N}$, let $|S|$ be the number of elements in S. For all $i, j \in \mathcal{N}$, define $H_i = H_{z_i}$, and

$$H_{ij} = \{x \in H_i \,|\, |\{k \in \mathcal{N} | k \leqq 1 \text{ and } x \in H_k\}| = j\}. \tag{3.12}$$

Then, by construction, for all $i \in \mathcal{N}$, H_i is partitioned by $\{H_{ij}\}_{j=1}^{\infty}$. But by assumption, for any nonzero vector x, at most $n-1$ vectors z_k satisfy $x \cdot z_k = 0$. That is, x is in at most $n-1$ different H_k. So $H_{ij} = \phi$ for $j \geqq n$, and H_i is actually partitioned by $\{H_{ij}\}_{j=1}^{n-1}$. In particular, this implies, for all $i \in \mathcal{N}$,

$$\mu^*(H_i) = \sum_{j=1}^{n-1} \mu^*(H_{ij}). \tag{3.13}$$

Further, for any $j \in \mathcal{N}$, H_{ij} and $H_{i'j}$ are disjoint for any $i \neq i'$. Hence, for any $K \in \mathcal{N}$, $\bigcup_{i=1}^{K} H_{ij} \subseteq \mathcal{R}^n$ is a disjoint union. So

$$\sum_{i=1}^{K} \mu^*(H_{ij}) \leqq \mu^*(\mathcal{R}^n) = N. \tag{3.14}$$

Now, using (3.13) and (3.14), for any $K \in \mathcal{N}$ we have

$$\sum_{i=1}^{K} \mu^*(H_i) = \sum_{i=1}^{K} \sum_{j=1}^{n-1} \mu^*(H_{ij}) = \sum_{j=1}^{n-1} \sum_{i=1}^{K} \mu^*(H_{ij})$$

$$\leqq \sum_{j=1}^{n-1} N = (n-1)N. \tag{3.15}$$

That is

$$\sum_{i=1}^{\infty} \mu^*(H_i) \leqq (n-1)N. \tag{3.16}$$

From this, it follows that $\mu^*(H_{z_i}) > \varepsilon$ can hold for at most finitely many i, as we wished to show. Thus, if $v_1, v_2, \cdots$ is a sequence of vectors such that any n are linearly independent then $\mu^*(H_{v_i}) \to 0$ and consequently, from (3.10)

$$\mu^*(H_{v_i}^+) \to \tfrac{1}{2}N. \tag{3.17}$$

Now, introduce coordinates so that $y = (1, 0, \cdots, 0)$. Let $v = (1, v_{12}, \cdots, v_{1n})$, $v_2 = (1, v_{22}, v_{23}, \cdots, v_{2n}), \cdots$ be a sequence of vectors such that

(3.18)
(i) $\lim_{i \to \infty} \frac{v_{i,j+1}}{v_{i,j}} = 0$ for $j = 1, 2, \cdots, n-1$;
(ii) $v_{ij} > 0$ for all i, j;
(iii) any n of the vectors v_i are linearly independent.

To satisfy condition (iii) v_i must not be in any one of the $i-1^{C}n-1$ planes spanned by $n-1$-tuples of previously selected vectors. Clearly there is no difficulty in constructing a sequence which satisfies (i), (ii) and this additional restriction.

The sets $H_{v_i}^+$ converge pointwise to $H_y^+ \cup P$, where P is the set of vectors which belong to H_y (i.e., have 0 first component) and whose first non-zero component is positive. It follows by Egoroff's Theorem (see Kingman and Taylor (1966, Thrm. 7.1, also Thrm. 7.2)) that the $H_{v_i}^+$ converge in measure to $H_y^+ \cup P$. So, using (3.17)

$$\mu^*(H_y^+) + \mu^*(P) = \lim \mu^*(H_{v_i}^+) = \tfrac{1}{2}N. \tag{3.19}$$

Applying the same argument with the sequence $w_i = (-1, v_{i2}, \cdots, v_{in})$ we get

$$\mu^*(H_y^-) + \mu^*(P) = \tfrac{1}{2}N. \tag{3.20}$$

The relation (3.11) is a consequence of the last two equations.

We can now apply Lemma 2 to conclude that (3.10) holds if and only if

$$\mu^*(A) = \mu^*(-A) \tag{3.21}$$

for all $A \in \underline{\underline{B}}$. Hence x^* is a Condorcet point if and only if (3.21) holds, as claimed in the theorem. Q.E.D.

4. Implications. Plott's theorem shows that in the finite voter case, if there is no more than one voter with ideal point at the point x^*, necessary and sufficient conditions for x^* to be a (local) Condorcet point are that it is possible to pair all voters such that if voters α and β are paired, then

$$\nabla U_\alpha(x^*) = -k \nabla U_\beta(x^*)$$

for some $k > 0$. Thus there must be perfect symmetry around the origin of the vectors $\nabla U_\alpha(x^*)$: For every gradient vector pointing in one direction there is another pointing in exactly the opposite direction.

Theorem 2 of the previous section is a natural extension of Plott's Theorem to the continuous voter case. The theorem says that x^* is a Condorcet point if and only if for every cone, $A \in \underline{\underline{B}}$, the measure (or proportion) of the voters with gradients in A must be exactly the same as the measure of the voters with gradients in $-A$. Note that if the set, Ω, of voters is finite, and μ is the counting measure, then Theorem 2 reduces to Plott's Theorem if we only consider sets, A, which are rays from the origin. The severity of the above conditions indicates that, as in the finite voter models, existence of equilibria in the continuous voter model would be a rare event.

Theorem 2 can be illustrated by a particular, but widely studied class of models—namely, the several extensions of the so called "Downsian" model of two party competition. In their simplest form, these models assume that for all voters, U_α is a function of Euclidean distance. Thus, for all $\alpha \in \Omega$, there is a $x_\alpha \in \mathscr{R}^n$, such that

$$U_\alpha(x) = -\|x - x_\alpha\|^2. \tag{4.1}$$

The vector x_α is called voter α's ideal point. In this case, it follows that, for $x^* \in \mathscr{R}^n$,

$$\nabla U_\alpha(x^*) = -2(x^* - x_\alpha). \tag{4.2}$$

So, if $A \in \underline{\underline{B}}$, then

$$\nabla U_\alpha(x^*) \in A \quad \Leftrightarrow \quad x_\alpha - x^* \in A \quad \Leftrightarrow \quad x_\alpha \in x^* + A. \tag{4.3}$$

Now Davis, Degroot and Hinich [1972] show that a necessary and sufficient condition for x^* to be a Condorcet point is that x^* be a total median of the distribution of the vectors x_α. Note that this follows from Theorem 2 and equation (4.3) by considering sets, A, which are half spaces of the form H_y^+. Davis, Degroot and Hinich then proceed to give conditions on the distribution of the x_α which are sufficient for x^* to be a total median. These conditions are that the distribution of x_α is symmetric about x^* in the following sense: For every Borel set A,

$$\mu\{\alpha \mid x_\alpha \in x^* + A\} = \mu\{\alpha \mid x_\alpha \in x^* - A\}. \tag{4.4}$$

From Theorem 2, of course, we see that this condition is stronger than need be. The condition is sufficient, but not necessary for x^* to be a Condorcet point. Necessary *and* sufficient conditions require only that (4.4) holds for $A \in \underline{\underline{B}}$. It follows, that for this simple model, equilibrium exists if and only if the distribution of ideal points is "weakly symmetric" about some point—the presumed equilibrium. Here, weak symmetry means that for any cone, say A, the proportion of voters with ideal points in the cone x^*+A equals the proportion with ideal points in the cone x^*-A.

It is important to emphasize, nevertheless, the importance of Plott's third precondition and the corresponding condition of our theorem that the set of voters with ideal points at x^* is of measure zero. It is straightforward to show, that if these conditions do not hold in their respective finite and continuous population cases, neither our theorem nor Plott's analysis establishes necessity. Suppose that U_α is given by simple Euclidean distances. In Fig. 1, with two voter's ideal points at x^*, x^* is a Condorcet point. Nevertheless, weak symmetry of the ideal points does not hold. Similarly, in Fig. 2, we suppose that voter ideal points are uniformly distributed over a

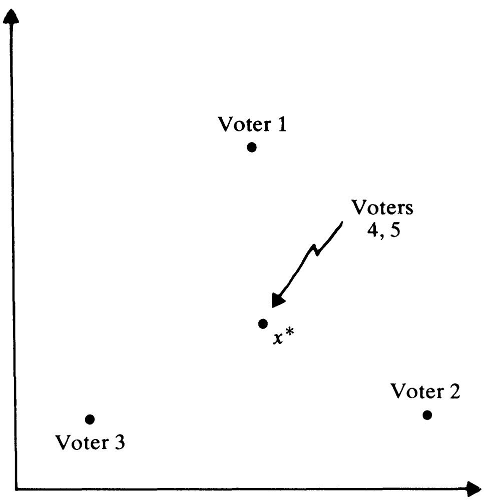

FIG. 1

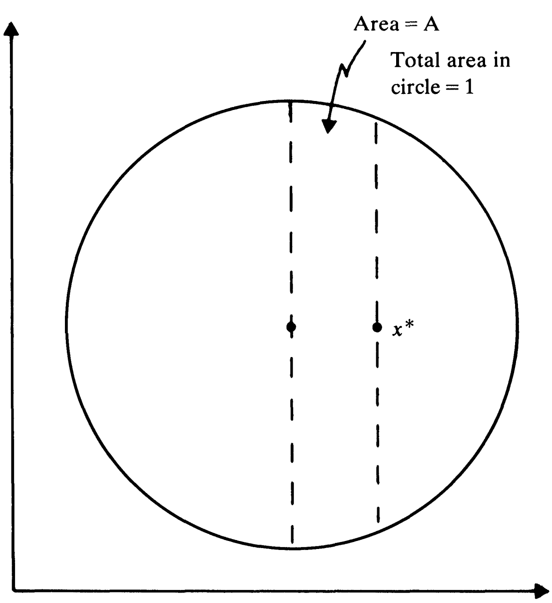

FIG. 2

circle of unit area with measure 1. The point x^* is a Condorcet point if we "pile up" additional voters with measure $2A$ at x^*. Again, then, an equilibrium exists even though weak symmetry does not hold.

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