Dehn surgeries on knots in product manifolds

Yi NI
Department of Mathematics, Caltech, MC 253-37
1200 E California Blvd, Pasadena, CA 91125
Email: yni@caltech.edu

Abstract

We show that if a surgery on a knot in a product sutured manifold yields the same product sutured manifold, then this knot is a 0– or 1–crossing knot. The proof uses techniques from sutured manifold theory.

Dedicated to the memory of Professor Andrew Lange

1 Introduction

An interesting problem on Dehn surgery is: when does a surgery on a knot yield a manifold homeomorphic to the original ambient manifold? The most famous result in this direction is the Knot Complement Theorem proved by Gordon and Luecke [8]: when the ambient manifold is S^3, only the unknot admits surgeries which yield S^3.

In this paper, we are going to study this problem for knots in surfaces times an interval. Our main result is as follows.

Theorem 1.1. *Suppose F is a compact surface, $K \subset F \times I$ is a knot. Suppose α is a nontrivial slope on K, and $N(\alpha)$ is the manifold obtained from $F \times I$ via the α–surgery on K. If the pair $(N(\alpha), (\partial F) \times I)$ is homeomorphic to the pair $(F \times I, (\partial F) \times I)$, then one can isotope K such that its image on F under the natural projection*

$$p\colon\ F \times I \to F$$

has either no crossing or exactly one crossing.

The slope α can be determined as follows. Let λ_b be the "blackboard" frame of K associated with the previous projection. Namely, λ_b is the frame specified by the surface F. When the projection has no crossing, $\alpha = \frac{1}{n}$ for some integer n with respect to λ_b; when the minimal projection has exactly one crossing, $\alpha = \lambda_b$.

It is easy to see the surgeries in the statement of Theorem 1.1 do not change the homeomorphism type of the pair $(F \times I, (\partial F) \times I)$. In fact, when K is a 0–crossing knot, it is clear that the $\frac{1}{n}$–surgery preserves the homeomorphism

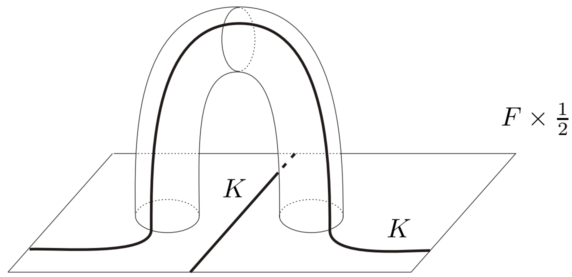

Figure 1: A local picture of the crossing

type of the pair. When K is a 1–crossing knot, we can add a one-handle to $F \times \frac{1}{2}$ near the crossing to get a Heegaard surface F' for $F \times I$. K can be embedded into F' as in Figure 1. F' splits $F \times I$ into two parts U_0, U_1, where U_0 is $F \times [0, \frac{1}{2}]$ with a one-handle added to $F \times \frac{1}{2}$, and U_1 is $F \times [\frac{1}{2}, 1]$ with a one-handle added to $-F \times \frac{1}{2}$. The embedding of K can be chosen such that K goes through each of the two one-handles exactly once. Now the blackboard frame λ_b is the frame specified by F', and the λ_b–surgery on F' cancels each one-handle with a two-handle. Hence the new pair is still homeomorphic to $(F \times I, (\partial F) \times I)$.

Definition 1.2. Notations are as in the previous theorem. Fix a product structure on $(\partial F) \times I$. Up to an isotopy relative to $(\partial F) \times I$, this product structure uniquely extends to a product structure $\mathcal{P}$ on $F \times I$ and a product structure $\mathcal{P}_\alpha$ on $N(\alpha)$. (This fact can be proved using Alexander's trick.) Identify F with $F \times 1$. Let $i, i_\alpha \colon F \times 0 \to F \times 1$ be the natural identity maps with respect to $\mathcal{P}$ and $\mathcal{P}_\alpha$, respectively. We call

$$\varphi_\alpha = i \circ i_\alpha^{-1} \colon F \to F$$

the map induced by the α–surgery. This map φ_α fixes ∂F pointwise, and is unique up to an isotopy relative to ∂F. Hence φ_α can be viewed as an element in the mapping class group $\mathcal{MCG}(F, \partial F)$.

The definition of the map φ_α is justified by the following lemma.

Lemma 1.3. *Let $Y(\alpha)$ be the manifold obtained from $F \times S^1$ by α–surgery on K. Then $Y(\alpha)$ can be obtained from $F \times I$ by identifying $(x, 0)$ with $(\varphi_\alpha(x), 1)$ for any $x \in G$.*

Proof. The manifold $F \times S^1$ is obtained from $F \times I$ by identifying y with $i(y)$ for each $y \in F \times 0$. Let $y = (x, 0)$ with respect to the product structure $\mathcal{P}_\alpha$ on $N(\alpha)$, then $i_\alpha(y) = (x, 1)$ with respect to $\mathcal{P}_\alpha$. We then have

$$i(y) = \varphi_\alpha(x, 1) = (\varphi_\alpha(x), 1),$$

since we identify F with $F \times 1$ in the above definition. Hence $(x, 0)$ is identified with $(\varphi_\alpha(x), 1)$ in $Y(\alpha)$ for each $x \in F$. □

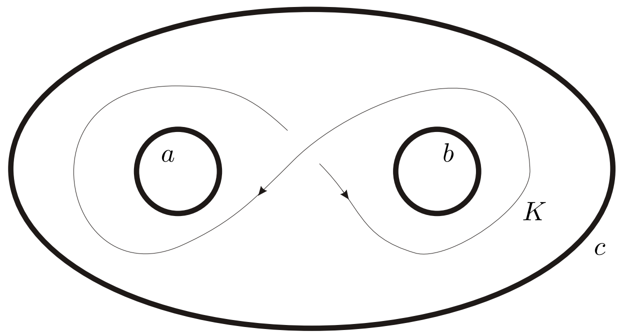

Figure 2: A 1–crossing knot

Proposition 1.4. *Notations are as in Theorem 1.1. When the projection of* K *has no crossing and* $\alpha = \frac{1}{n}$,

$$\varphi_\alpha = \tau^n,$$

where τ *is the righthand Dehn twist along* $K \subset F$. *When the minimal projection of* K *has exactly one crossing, let* a, b, c *be the simple closed curves obtained by resolving the crossing in two different ways as in Figure 2 and let* τ_a, τ_b, τ_c *be the righthand Dehn twists along* a, b, c. *Then*

$$\varphi_\alpha = \tau_a^2 \tau_b^2 \tau_c^{-1}$$

when the crossing is positive, and

$$\varphi_\alpha = \tau_a^{-2} \tau_b^{-2} \tau_c$$

when the crossing is negative.

This paper can be compared with Ni [9]. In fact, Theorem 1.4 in [9] can be restated in a form similar to Theorem 1.1.

Theorem 1.5. *Suppose* F *is a compact surface,* $K \subset F \times I$ *is a knot and* α *is a slope on* K. *Let* $N(\alpha)$ *be the manifold obtained by the* α*–surgery on* K. *If* $F \times \{0\}$ *is not Thurston norm minimizing in* $H_2(N(\alpha), (\partial F) \times I)$, *then there is an ambient isotopy of* $F \times I$ *which takes* K *to a curve in* $F \times \{\frac{1}{2}\}$. *Moreover,* α *is the frame on* K *specified by* $F \times \{\frac{1}{2}\}$.

The proof of Theorem 1.5 uses Gabai's sutured manifold theory [2, 3, 4] and an argument due to Ghiggini [7]. Using a different method, Scharlemann and Thompson [12] get the same conclusion of Theorem 1.5 under the assumption that $F \times \{0\}$ is compressible in $N(\alpha)$.

This paper is organized as follows. In Section 2, we give some preliminaries on sutured manifold theory and foliations, as well as a characterization of one-crossing knot projections. In Section 3, we study some warm-up cases. In Section 4, we use the argument in the proof of Theorem 1.5 to reduce our problem to the case where F is a pair of pants. In Section 5, we study this case by analyzing the map induced by surgery and using a variant of the argument in Ni [9].

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2 Preliminaries

In this section, we are going to review the sutured manifold theory introduced by Gabai in [2]. We also state a uniqueness result for the Euler classes of taut foliations of fibred manifolds. In addition, we define "double primitive" knots in $F \times I$ and show that they are exactly the knots with a projection consisting of only one crossing.

2.1 Sutured manifold decompositions

Definition 2.1. A *sutured manifold* (M, γ) is a compact oriented 3–manifold M together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. The core of each component of $A(\gamma)$ is a *suture*, and the set of sutures is denoted by $s(\gamma)$.

Every component of $R(\gamma) = \partial M - \mathrm{int}(\gamma)$ is oriented. Define $R_+(\gamma)$ (or $R_-(\gamma)$) to be the union of those components of $R(\gamma)$ whose normal vectors point out of (or into) M. The orientations on $R(\gamma)$ must be coherent with respect to $s(\gamma)$, hence every component of $A(\gamma)$ lies between a component of $R_+(\gamma)$ and a component of $R_-(\gamma)$.

Definition 2.2. Let S be a compact oriented surface with connected components $S_1, \dots, S_n$. We define

$$x(S) = \sum_i \max\{0, -\chi(S_i)\}.$$

Let M be a compact oriented 3–manifold, A be a compact codimension–0 submanifold of ∂M. Let $h \in H_2(M, A)$. The *Thurston norm* $x(h)$ of h is defined to be the minimal value of $x(S)$, where S runs over all the properly embedded surfaces in M with $\partial S \subset A$ and $[S] = h$.

Definition 2.3. Let (M, γ) be a sutured manifold, and S a properly embedded surface in M, such that no component of ∂S bounds a disk in $R(\gamma)$ and no component of S is a disk with boundary in $R(\gamma)$. Suppose that for every component λ of $S \cap \gamma$, one of 1)–3) holds:

1) λ is a properly embedded non-separating arc in γ.

2) λ is a simple closed curve in an annular component A of γ in the same homology class as $A \cap s(\gamma)$.

3) λ is a homotopically nontrivial curve in a toral component T of γ, and if δ is another component of $T \cap S$, then λ and δ represent the same homology class in $H_1(T)$.

Then S is called a *decomposing surface*, and S defines a *sutured manifold decomposition*

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma'),$$

where $M' = M - \mathrm{int}(\mathrm{Nd}(S))$ and

$$\begin{aligned} \gamma' &= (\gamma \cap M') \cup \mathrm{Nd}(S'_+ \cap R_-(\gamma)) \cup \mathrm{Nd}(S'_- \cap R_+(\gamma)), \\ R_+(\gamma') &= ((R_+(\gamma) \cap M') \cup S'_+) - \mathrm{int}(\gamma'), \\ R_-(\gamma') &= ((R_-(\gamma) \cap M') \cup S'_-) - \mathrm{int}(\gamma'), \end{aligned}$$

where S'_+ (S'_-) is that component of $\partial \mathrm{Nd}(S) \cap M'$ whose normal vector points out of (into) M'.

Definition 2.4. A sutured manifold (M, γ) is *taut*, if M is irreducible and $R(\gamma)$ is Thurston norm minimizing in $H_2(M, \gamma)$.

Suppose S is a decomposing surface in (M, γ), S decomposes (M, γ) to (M', γ'). S is *taut* if (M', γ') is taut.

Definition 2.5. Suppose

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$$

is a taut decomposition, by [2] we can extend this decomposition to a sutured manifold hierarchy of (M, γ), from which we can construct a taut foliation $\mathscr{F}$ of M, such that $R(\gamma)$ consists of compact leaves of $\mathscr{F}$. We then call $\mathscr{F}$ a *foliation induced by* S. Moreover, when $R_+(\gamma)$ is homeomorphic to $R_-(\gamma)$, from M we can obtain a manifold Y with boundary consisting of tori by gluing $R_+(\gamma)$ to $R_-(\gamma)$ via a homeomorphism. $\mathscr{F}$ then becomes a taut foliation $\mathscr{F}_1$ of Y. We also say that $\mathscr{F}_1$ is a *foliation induced by* S.

Definition 2.6. A decomposing surface is called a *product disk*, if it is a disk which intersects $s(\gamma)$ in exactly two points. A decomposing surface is called a *product annulus*, if it is an annulus with one boundary component in $R_+(\gamma)$, and the other boundary component in $R_-(\gamma)$.

We recall the main result in Gabai [3], which has been intensively used in Ni [9]. Note that the result is not stated in its original form, but it is contained in the argument in [3]. See also [9, Theorem 2.8] for a sketch of the proof.

Definition 2.7. An *I-cobordism* between closed connected surfaces T_0 and T_1 is a compact 3–manifold V such that $\partial V = T_0 \cup T_1$ and for $i = 0, 1$ the induced maps $j_i\colon H_1(T_i) \to H_1(V)$ are injective.

Definition 2.8. Suppose M is a 3-manifold, T is a toral component of ∂M. If all tori in M which are I-cobordant to T in M must be parallel to T, then we say M is *T–atoroidal*.

Theorem 2.9 (Gabai)**.** *Let (M, γ) be a taut sutured 3–manifold. T is a toral component of γ, S is a decomposing surface such that $S \cap T = \emptyset$, and the decomposition*

$$(M, \gamma) \overset{S}{\rightsquigarrow} (M_1, \gamma_1)$$

is taut. Suppose M is T–atoroidal, then for any slope α on T except at most one slope, the decomposition after Dehn filling

$$(M(\alpha), \gamma \backslash T) \overset{S}{\rightsquigarrow} (M_1(\alpha), \gamma_1 \backslash T)$$

is taut. □

A special case of the above theorem is the case $\gamma = \partial M$, which is the original form in [3].

2.2 Euler classes of foliations

We will need the Euler classes of foliations.

Definition 2.10. Suppose Y is a compact 3–manifold with ∂Y consisting of tori. $\mathscr{P}$ is an oriented plane field transverse to ∂Y. Let $T(\partial Y)$ be the tangent plane field of ∂Y. The line field $\mathscr{P} \cap T(\partial Y)$ has a natural orientation induced by the orientations of $\mathscr{P}$ and $T(\partial Y)$, thus it has a nowhere vanishing section $v \subset \mathscr{P}|_{\partial Y}$. Then one can define the *relative Euler class*

$$e(\mathscr{P}) \in H^2(Y, \partial Y)$$

of $\mathscr{P}$ to be the obstruction to extending v to a nowhere vanishing section of $\mathscr{P}$. When $\mathscr{F}$ is a foliation of Y that is transverse to ∂Y, let $T\mathscr{F}$ be the tangent plane field of $\mathscr{F}$ and let $e(\mathscr{F}) = e(T\mathscr{F})$.

Definition 2.11. Suppose C is a properly embedded curve in a compact surface F. We say C is *efficient* in F if

$$|C \cap \delta| = |[C] \cdot [\delta]|, \quad \text{for each boundary component } \delta \text{ of } F.$$

Suppose S is a properly embedding surface in compact 3–manifold Y with boundary consisting of tori. We say S is *efficient* in Y if $S \cap T$ consists of coherently oriented parallel essential curves for each boundary component T of Y.

Proposition 2.12. *Suppose Y is a compact 3–manifold that fibres over S^1. Let G be a fibre of the fibration $\mathscr{E}$. Suppose $\mathscr{F}$ is a taut foliation of Y which is transverse to ∂Y such that G is a leaf of $\mathscr{F}$. Then*

$$e(\mathscr{F}) = e(\mathscr{E}) \in H^2(Y, \partial Y)/\text{Tors}.$$

Proof. This result follows easily from the fact that the Floer homology of a fibred manifold is "monic". Using this approach, one can even prove that the two Euler classes are equal in $H^2(Y, \partial Y)$. Here we will present a more geometric proof.

In order to prove the desired result, we only need to show that

$$\langle e(\mathscr{F}), h\rangle = \langle e(\mathscr{E}), h\rangle \tag{1}$$

for any $h \in H_2(Y, \partial Y)$. When $h = [G]$, we have

$$\langle e(\mathscr{F}), [G]\rangle = \langle e(\mathscr{E}), [G]\rangle = \chi(G). \tag{2}$$

In general, suppose $\overline{U} \subset Y$ is a proper surface representing h such that $\overline{U} \pitchfork G$. We can choose the representative $\overline{U}$ such that $\overline{U}$ is efficient in Y. Then $\overline{U} \cap G$ can also be made efficient in G. Cutting Y open along G, we get $G \times I$. Let $U \subset G \times I$ be the proper surface obtained by cutting $\overline{U}$ open along $C = \overline{U} \cap G$. Let $C_0, C_1 \subset G$ be proper oriented curves such that

$$-C_0 \times 0 = (\partial U) \cap (G \times 0), \quad C_1 \times 1 = (\partial U) \cap (G \times 1).$$

Since C_0 and C_1 are homologous efficient curves in G relative to ∂G, as in the proof of [3, Lemma 0.6], we can find compact subsurfaces $V_1, V_2, \dots, V_n$ and efficient curves

$$C_0 = \gamma_0, \gamma_1, \dots, \gamma_{n-1}, \gamma_n = C_1$$

in G, such that

$$\overline{\partial V_i \backslash (\partial G)} = \gamma_i \cup (-\gamma_{i-1}).$$

Let $W_i = \overline{G \backslash V_i}$. Perturbing the surface

$$\bigcup_{i=1}^{n} \Big((-\gamma_{i-1} \times [\frac{i-1}{n}, \frac{i}{n}]) \cup (V_i \times \frac{i}{n}) \Big) \tag{3}$$

slightly, we get a proper surface $V \subset G \times I$, such that

$$(\partial V) \cap (G \times 0) = -C_0 \times 0, \quad (\partial V) \cap (G \times 1) = C_1 \times 1.$$

Similarly, perturbing the surface

$$\bigcup_{i=1}^{n} \Big((\gamma_{i-1} \times [\frac{i-1}{n}, \frac{i}{n}]) \cup (W_i \times \frac{i}{n}) \Big) \tag{4}$$

slightly, we have a proper surface $W \subset G \times I$, such that

$$(\partial W) \cap (G \times 0) = C_0 \times 0, \quad (\partial W) \cap (G \times 1) = -C_1 \times 1.$$

Let $\overline{V} \subset Y$ be the proper surface obtained from V by identifying $C_0 \times 0$ and $C_1 \times 1$ with $C \subset G \subset Y$. Similarly, define $\overline{W} \subset Y$. Note that

$$[\overline{V}] - [\overline{U}] = [V \cup (-U)] \in H_2(Y, \partial Y).$$

Perturbing $V \cup (-U)$ slightly, we get a properly immersed surface in Y which is disjoint from the fibre G. So $[V \cup (-U)] = m[G]$ for some integer m. Using (2), in order to check (1) for $h = [\overline{U}]$, we only need to check it for $h = [\overline{V}]$.

Since $\mathscr{F}$ is taut, by Thurston [13, Corollary 1] we have

$$\begin{aligned} \chi(\overline{V}) &\leq \langle e(\mathscr{F}), [\overline{V}] \rangle, \\ \chi(\overline{W}) &\leq \langle e(\mathscr{F}), [\overline{W}] \rangle. \end{aligned}$$

Adding the two inequalities together, we get

$$\chi(\overline{V}) + \chi(\overline{W}) \leq \langle e(\mathscr{F}), [\overline{V}] + [\overline{W}] \rangle. \tag{5}$$

By the constructions (3), (4), the result of doing oriented cut-and-pastes to $\overline{V}$ and $\overline{W}$ is n copies of G. So the left hand side of (5) is $n\chi(G)$, while the right hand side is $\langle e(\mathscr{F}), n[G] \rangle = n\chi(G)$. So the equality holds. In particular, we should have

$$\chi(\overline{V}) = \langle e(\mathscr{F}), [\overline{V}] \rangle.$$

The same argument shows that

$$\chi(\overline{V}) = \langle e(\mathscr{E}), [\overline{V}] \rangle,$$

so (1) holds for $h = [\overline{V}]$. Since we have checked (1) for all elements $h \in H_2(Y, \partial Y)$, $e(\mathscr{F})$ is equal to $e(\mathscr{E})$ up to a torsion element in $H^2(Y, \partial Y)$. □

2.3 A characterization of one-crossing knot projections

In this subsection, we will give a characterization of one-crossing knot projections in terms of double primitive knots. This fact is not used in the current paper, but it is useful to bare it in mind.

Definition 2.13. Let $F' \subset F \times I$ be a connected surface of genus $g(F)+1$, and $\partial F' = (\partial F) \times \frac{1}{2}$. Suppose F' is a Heegaard surface. Namely, F' splits $F \times I$ into two parts U_0 and U_1, such that U_0 is homeomorphic to $(F \times [0, \frac{1}{2}]) \cup H_1$, and U_2 is homeomorphic to $(F \times [\frac{1}{2}, 1]) \cup H_2$, where H_1 is a one-handle with feet on $F \times \frac{1}{2}$ and H_2 is a one-handle with feet on $-F \times \frac{1}{2}$. A knot $K \subset F \times I$ is a *double primitive* knot if it is isotopic to a curve on F' which goes through each of H_1, H_2 exactly once.

Lemma 2.14. *A knot $K \subset F \times I$ is double primitive if and only if it has a projection which has only one crossing.*

Proof. If a knot has a one-crossing projection, then it is double primitive as shown in the introduction. Now assume K is double primitive, then K is embedded into a Heegaard surface F' as in the above definition.

We claim that F' is stabilized. Namely, there is a compressing disk $D_0 \subset U_0$ and a compressing disk $D_1 \subset U_1$ such that $|(\partial D_0) \cap (\partial D_1)| = 1$. When F is closed, this follows from the theorem of Scharlemann and Thompson [11] that the Heegaard splittings of $F \times I$ are standard. When F is not closed, let R be the torus with one hole, we can glue a copy of R to each component of ∂F, then F becomes a closed surface G and F' becomes a Heegaard surface G' in $G \times I$. Using Scharlemann and Thompson's theorem, G' is stabilized, hence there are compressing disks D_0 and D_1 in the two compression bodies separated by G', such that $|(\partial D_0) \cap (\partial D_1)| = 1$. Using standard arguments we can isotope D_0 and D_1 to be disjoint from the copies of $R \times I$, so $D_0 \subset U_0$, $D_1 \subset U_1$, thus our conclusion follows.

Since $g(F') = g(F) + 1$, after compressing F' along D_0 we get a surface homeomorphic to F (and hence parallel to $F \times 0$ in $F \times I$). So F' is obtained from $F \times \frac{1}{2}$ by adding a one-handle, and D_1 is a disk whose boundary goes through the one-handle exactly once. Now the local picture of F' looks exactly like in Figure 1. The knot K goes through the one-handle once and intersects ∂D_1 once, so there is a crossing near D_1 and no crossing elsewhere. $\square$

3 Warm-up cases

In this section, we are going to prove some easy cases of our main theorem. When F is a disk or sphere, our result follows from Gordon and Luecke's Knot Complement Theorem [8]. When F is an annulus, we have the following lemma.

Lemma 3.1. *Theorem 1.1 is true when F is an annulus.*

Proof. Let $\mathcal{M}$ be the meridian of the solid torus $V = F \times I$, and let $\mathcal{L}$ be the frame of V specified by ∂F. By Gabai [5], if K is nontrivial, then K is a 0– or 1–bridge braid in $F \times I$.

Capping off one boundary component of F with a disk, we get a disk D. Let λ be the Seifert frame of K in $D \times I$ and let μ be the meridian of K.

If K is the core of V, then the surgery preserves the homeomorphism type of $(F \times I, (\partial F) \times I)$ if and only if the slope is $\mu + n\lambda$ for some integer n.

From now on we assume the braid index of K is greater than 1.

If K is a 0–bridge braid, then K is isotopic to $p\mathcal{L} + q\mathcal{M}$ on ∂V for some $p, q \in \mathbb{Z}$. Let Λ be the frame on K specified by ∂V, then $\Lambda = pq\mu + \lambda$. A surgery on K yields a solid torus if and only if the slope α of the surgery satisfies that $\Delta(\alpha, \Lambda) = 1$, namely, when the slope α is $\mu + n\Lambda$ for some integer n. Now $p\alpha = p\mu + pn\Lambda$ is homologous to $\mathcal{M} + pn(p\mathcal{L} + q\mathcal{M})$ in $V \backslash K$, so the meridian of the new ambient solid torus after surgery is $(1 + pqn)\mathcal{M} + p^2 n\mathcal{L}$. Since the surgery preserves the homeomorphism type of the pair $(F \times I, (\partial F) \times I)$, we must have $\Delta((1+pqn)\mathcal{M}+p^2n\mathcal{L}, \mathcal{L}) = 1$, thus $1+pqn = \pm 1$. Since $p > 1, n \neq 0$, we have $(p, q, n) = (2, 1, -1)$ or $(2, -1, 1)$. When $(p, q) = (2, 1)$, the slope α on K is

$$\mu + n(pq\mu + \lambda) = (1 + pqn)\mu + n\lambda,$$

which is 1 with respect to the frame λ, and the meridian of the new ambient solid torus is $\mathcal{M} + 4\mathcal{L}$; when $(p, q) = (2, -1)$, the slope α on K is -1, and the meridian of the new ambient solid torus is $\mathcal{M} - 4\mathcal{L}$. We can check α is the blackboard frame.

If K is a 1–bridge braid, then K is determined by 3 parameters ω, b, t by Gabai [6]. Here $\omega > 0$ is the braid index, $1 \leq b \leq \omega - 2$, $t \equiv r \pmod{\omega}$ for some integer r with $1 \leq r \leq \omega - 2$. Since the α–surgery yields a solid torus, by [6, Lemma 3.2] the slope of the surgery is $\lambda - (t\omega + d)\mu$, where $d \in \{b, b+1\}$. So $t\omega + d = \pm 1$, which is impossible for any ω, b, t satisfying the previous restrictions. □

Lemma 3.2. *In the above lemma, let $\varphi_\alpha \in \mathcal{MCG}(F, \partial F)$ be the map induced by the α–surgery. If K is the core of $F \times I$ and $\alpha = \frac{1}{n}$, then $\varphi_\alpha = \tau^n$, where τ is the right hand Dehn twist in F; if K is the $(2, \pm 1)$–cable in $F \times I$, then $\varphi_\alpha = \tau^{\pm 4}$.*

Proof. When K is the core of $F \times I$, the conclusion is well-known. When K is the $(2, \pm 1)$–cable, then from the proof of the previous lemma we know that the meridian of the new ambient solid torus is $\mathcal{M} \pm 4\mathcal{L}$, hence the conclusion follows from the first case. □

The following lemma is obvious.

Lemma 3.3. *Suppose $(C \times I) \subset (F \times I)$ is a product disk or product annulus, $(C \times I) \cap K = \emptyset$. Let F_1 be the surface obtained from F by cutting F open along C, let N_1 be the manifold obtained from $N = (F \times I) \backslash \mathrm{int}(\mathrm{Nd}(K))$ by cutting N open along $C \times I$. Then the pair $(N(\alpha), (\partial F) \times I)$ is homeomorphic to $(F \times I, (\partial F) \times I)$ if and only if the pair $(N_1(\alpha), (\partial F_1) \times I)$ is homeomorphic to $(F_1 \times I, (\partial F_1) \times I)$.* □

Lemma 3.4. *Theorem 1.1 is true when F is a torus.*

Proof. Let $C \subset F$ be a simple closed curve such that K is homologous to a multiple of C. Consider the homology class $[C \times I] \in H_2(F \times I, \partial(F \times I))$, then $[C \times I] \cdot [K] = 0$. It follows that $[C \times I]$ is also a homology class in $H_2((F \times I) \backslash K, \partial(F \times I))$.

Let $(S, \partial S) \subset ((F \times I) \backslash K, \partial(F \times I))$ be a taut surface representing $[C \times I]$. By Theorem 2.9, S remains taut in at least one of the original $F \times I$ and $N(\alpha) \cong F \times I$. Hence S must be a product annulus. Cutting $F \times I$ open along S, K becomes a knot in (annulus $\times I$). Now we can apply Lemma 3.1 and Lemma 3.3 to get our conclusion. □

Lemma 3.5. *If the conclusion of Theorem 1.1 holds for all knots whose exterior are $\partial(\mathrm{Nd}(K))$–atoroidal, then the conclusion holds for all knots in $F \times I$.*

Proof. By the assumption, we only need to consider the case where there is a torus in $N = F \times I \backslash \mathrm{int}(\mathrm{Nd}(K))$ which is I-cobordant but not parallel to $\partial \mathrm{Nd}(K)$. Let R be an “innermost” such torus.

By [9, Lemma 3.1], R bounds a solid torus U in $F \times I$, such that $K \subset U$. Since R is innermost in N, if a torus in $(F \times I) \backslash \mathrm{int}(U)$ is I-cobordant to $\partial U = R$, then this torus is parallel to R. Let V be the manifold obtained from U by α–surgery on K.

By Gabai [5], one of the following cases holds.

1) $V = D^2 \times S^1$. In this case K is a 0–bridge or 1–bridge braid in U, and the core K' of the surgery is also a 0–bridge or 1–bridge braid in V. Moreover, K and K' have the same braid index ω.
2) $V = (D^2 \times S^1)\#W$, where W is a closed 3–manifold and $1 < |H_1(W)| < \infty$.
3) V is irreducible and ∂V is incompressible.

Since $V \subset N(\alpha) \cong F \times I$, Cases 2) and 3) can not happen, so the only possible case is 1). Thus the core of U is a knot such that a surgery on the knot yields the pair $(N(\alpha), (\partial F) \times I)$ which is homeomorphic to $(F \times I, (\partial F) \times I)$. Moreover, $N \backslash \mathrm{int}(U)$ is ∂U–atoroidal. By our assumption, the core of U is a 0–crossing or 1–crossing knot in $F \times I$.

If the core of U is isotopic to $\eta \times \{\frac{1}{2}\}$ for some simple closed curve $\eta \subset F$, let $G \subset F$ be a tubular neighborhood of η, then K lies in $G \times I$ after an isotopy. Let $M = (G \times I)\backslash \mathrm{int}(\mathrm{Nd}(K))$. By Lemma 3.3, $(M(\alpha), (\partial G) \times I)$ is homeomorphic to $(G \times I, (\partial G) \times I)$. Applying Lemma 3.1, we find that K is the $(2, \pm 1)$–cable of the core of $G \times I$, and the slope α is the blackboard frame λ_b.

If the core of U is a 1–crossing knot, then the blackboard frame λ'_b on ∂U is the meridian of V, so λ'_b cobounds a punctured disk with ω oriented copies of α in $U \backslash \mathrm{int}(\mathrm{Nd}(K))$. Moreover, the meridian μ' on ∂U cobounds a punctured disk with ω oriented copies of μ in $U \backslash \mathrm{int}(\mathrm{Nd}(K))$. Since $[\lambda'_b] \cdot [\mu'] = 1$, considering the intersection of the two punctured disks we conclude that $\omega = 1$. Hence ∂U is parallel to $\partial \mathrm{Nd}(K)$, a contradiction. □

In light of the above lemma, from now on we assume the exterior of the knot K is $\partial(\mathrm{Nd}(K))$–atoroidal.

4 Comparing Euler classes of foliations

Let E be a maximal (up to isotopy) compact essential subsurface of F, such that K can be isotoped in $F \times I$ to be disjoint from $E \times I$. Let $G = \overline{F \backslash E}$.

The goal of this section is to prove the following proposition.

Proposition 4.1. *The subsurface G is either an annulus or a pair of pants.*

Let $T = \partial(\mathrm{Nd}(K))$, $\gamma = ((\partial G) \times I) \cup T$. Let

$$N = (F \times I)\backslash \mathrm{int}(\mathrm{Nd}(K)), M = (G \times I)\backslash \mathrm{int}(\mathrm{Nd}(K)).$$

Then the sutured manifold (M, γ) contains no product disks or product annuli. For a proper surface $S \subset M$, let $\partial_i(S) = S \cap (G \times i)$, $i = 0, 1$.

Let $X = (G \times S^1)\backslash \mathrm{int}(\mathrm{Nd}(K))$ be the manifold obtained from M by gluing $G \times 1$ to $G \times 0$ via the identity map of G. Suppose ξ is a slope on K. Let $N(\xi), M(\xi), X(\xi)$ be the manifolds obtained from N, M, X by ξ–filling on T, respectively. Let $K(\xi) \subset M(\xi)$ be the core of the new solid torus.

By Lemma 3.3, $X(\xi)$ is a surface bundle over S^1 with fibre G when $\xi = \infty$ or α. We then let $\mathscr{E}(\xi)$ be the fibration of $X(\xi)$.

Lemma 4.2. *$K \subset F \times I$ is as in Theorem 1.1. N is T–atoroidal. Suppose $S \subset M$ is a taut surface such that $S \cap T = \emptyset$ and there exists a curve $C \subset F$ with $\partial_0 S = -C \times 0, \partial_1 S = C \times 1$. Let $\overline{S} \subset X$ be the surface obtained from S by gluing $\partial_0 S$ to $\partial_1 S$ via the identity map. Let $\mathscr{F}$ be a taut foliation of X induced by S. Then*

$$\langle e(\mathscr{F}), [\overline{S}]\rangle = \langle e(\mathscr{E}(\xi)), [\overline{S}]\rangle = \chi(\overline{S})$$

for some $\xi \in \{\infty, \alpha\}$.

Proof. By Theorem 2.9, S remains taut in $M(\xi)$ for some $\xi \in \{\infty, \alpha\}$. Let $\mathscr{F}'$ be a taut foliation of $X(\xi)$ induced by S. By Proposition 2.12,

$$e(\mathscr{F}') = e(\mathscr{E}(\xi)) \in H^2(X(\xi), \partial X(\xi); \mathbb{Q}).$$

Since both $\mathscr{F}$ and $\mathscr{F}'$ are induced by S, we have

$$\begin{aligned} \chi(\overline{S}) &= \langle e(\mathscr{F}), [\overline{S}]\rangle \\ &= \langle e(\mathscr{F}'), [\overline{S}]\rangle \\ &= \langle e(\mathscr{E}(\xi)), [\overline{S}]\rangle. \end{aligned}$$

□

Proposition 4.3. *$K \subset F \times I$ is as in Theorem 1.1. N is T–atoroidal. The inclusion $K \subset G \times I$ induces a map*

$$i_*\colon\ H_1(K; \mathbb{Q}) \to H_1(G; \mathbb{Q}).$$

Let

$$\mathcal{V} = \{v \in H_1(G, \partial G; \mathbb{Q}) |\ v \cdot i_*[K] = 0\}.$$

Then the dimension of $\mathcal{V}$ is at most 1.

Let

$$\rho_\xi\colon\ H^2(X, \partial X; \mathbb{Q}) \to H^2(X(\xi), \partial X(\xi); \mathbb{Q})$$

be the map induced by the map of pairs

$$(X(\xi), \partial X(\xi)) \to (X(\xi), (\partial X(\xi)) \cup K(\xi)).$$

Lemma 4.4. *Notations are as in Proposition 4.3. If the dimension of $\mathcal{V}$ is greater than 1, then there exists a properly embedded surface $H \subset X$ such that*

1) *H is not a multiple of $[G]$,*
2) *$H \cap T = \emptyset$,*
3) *for any two elements $\varepsilon_\infty \in \rho_\infty^{-1}(e(\mathscr{E}(\infty))), \varepsilon_\alpha \in \rho_\alpha^{-1}(e(\mathscr{E}(\alpha)))$, we have*

$$\langle \varepsilon_\infty, [H]\rangle = \langle \varepsilon_\alpha, [H]\rangle.$$

Proof. There is a natural injective map

$$\sigma\colon\ H_1(G,\partial G)\to H_2(G\times S^1,\partial G\times S^1)$$

defined via multiplying with the S^1 factor. Moreover, all elements in $\sigma(\mathcal{V})$ are represented by surfaces which are disjoint from K, hence $\sigma|\mathcal{V}$ induces an injective map

$$\widetilde{\sigma}\colon\ \mathcal{V}\to H_2(X,\partial X;\mathbb{Q}).$$

We pick two elements $\varepsilon'_\infty\in\rho_\infty^{-1}(e(\mathscr{E}(\infty))), \varepsilon'_\alpha\in\rho_\alpha^{-1}(e(\mathscr{E}(\alpha)))$. If $\dim\mathcal{V}>1$, then there exists a nonzero integral element $h\in\widetilde{\sigma}(\mathcal{V})$ such that

$$\langle\varepsilon'_\infty,h\rangle=\langle\varepsilon'_\alpha,h\rangle.$$

Let $H\subset X$ be a proper surface representing h such that $H\cap T=\emptyset$. We claim that this H is what we need. We only need to check 3) since the first two conditions are obvious.

From the Mayer–Vietoris sequence

$$H^1(K(\xi))\longrightarrow H^2(X,\partial X)\xrightarrow{\ \rho_\xi\ }H^2(X(\xi),\partial X(\xi))$$

and the fact that $h\cdot[T]=0$ we conclude that $\langle\varepsilon_\xi,h\rangle$ does not depend on the choice of $\varepsilon_\xi\in\rho_\xi^{-1}(e(\mathscr{E}(\xi)))$. Hence 3) holds. □

Assume the dimension of $\mathcal{V}$ is greater than 1, let H be a surface as in Lemma 4.4, and suppose $H\pitchfork G$. Without loss of generality, we can assume no component of $C=H\cap G$ is nullhomologous in $H_1(G,\partial G)$, and H is efficient in $G\times S^1$, hence we can also assume $H\cap G$ is efficient in G.

Let $p\in G\backslash C$ be a point. Let $\mathcal{S}_m(+C)$ be the set of properly embedded oriented surfaces $S\subset G\times I$, such that $S\cap K=\emptyset$, $\partial_0S=-C\times 0$, $\partial_1S=C\times 1$, and the algebraic intersection number between S and $p\times I$ is m. Similarly, let $\mathcal{S}_m(-C)$ be the set of properly embedded surfaces $S\subset G\times I$, such that $S\cap K=\emptyset$, $\partial_0S=C\times 0$, $\partial_1S=-C\times 1$, and the algebraic intersection number of S with $p\times I$ is m. Since $[C]\cdot i_*([K])=0$, $\mathcal{S}_m(\pm C)\neq\emptyset$.

Suppose $S\subset M$ is a properly embedded surface which is transverse to $\partial G\times 0$. For any component S_0 of S, we define

$$y(S_0)=\max\{\frac{|S_0\cap(\partial G\times 0)|}{2}-\chi(S_0),0\},$$

and let $y(S)$ be the sum of $y(S_i)$ with S_i running over all components of S. Let $y(\mathcal{S}_m(\pm C))$ be the minimal value of $y(S)$ for all $S\in\mathcal{S}_m(\pm C)$.

Lemma 4.5. *When m is sufficiently large, there exist surfaces $S_1\in\mathcal{S}_m(+C)$ and $S_2\in\mathcal{S}_m(-C)$, such that they are taut.*

Proof. Let $x(\cdot)$ be the Thurston norm in $H_2(X,\partial X)$. There exists $N\geq 0$, such that if $k>N$, then $x([H]+(k+1)[G])=x([H]+k[G])+x(G)$. As in the proof of Gabai [2, Theorem 3.13], if $\overline{Q}$ is a Thurston norm minimizing surface in the homology class $[H]+k[G]$, and $\overline{Q}\cap G$ consists of essential curves in G, then Q gives a taut decomposition of M, where Q is obtained from $\overline{Q}$ by cutting open along $\overline{Q}\cap G$. Moreover, we can assume $\overline{Q}$ is efficient in X. Hence $\overline{Q}\cap T=\emptyset$ and for each boundary component δ of $G\times i$, $|\partial Q\cap\delta|=|[\partial Q]\cdot[\delta]|$.

Now we can apply Gabai [3, Lemma 0.6] to get a new taut surface Q' such that $\partial_0 Q'=-C\times 0, \partial_1 Q'=C\times 1$. When m is sufficiently large, let S_1 be the surface obtained by doing oriented cut-and-pastes of Q' with $(m-Q'\cdot(p\times I))$ copies of G, then $S_1\in\mathcal{S}_m(+C)$ is the surface we need. Similarly, we can find the surface $S_2\in\mathcal{S}_m(-C)$. □

Correction 4.6. In Ni [9], after the statement of Proposition 3.4, the author claims that there exists a circle or arc $C\subset G$ such that $[C]\cdot i_*[K]=0$. This claim is not true. The correct statement should be there exists an essential efficient curve C in G such that $[C]\cdot i_*[K]=0$. The proof only needs slight changes: one can make use of the above Lemma 4.5 to find taut surfaces. □

The following result is Ni [9, Lemma 3.6], whose proof uses the assumption that (M,γ) contains no essential product disks or product annuli and an argument from Gabai [4].

Lemma 4.7. *For any positive integers p,q,*

$$y(\mathcal{S}_p(+C))+y(\mathcal{S}_q(-C))>(p+q)y(G).$$

□

Proof of Proposition 4.3. By Lemma 4.5, when m is large there exist taut surfaces $S_1\in\mathcal{S}_m(+C)$, $S_2\in\mathcal{S}_m(-C)$. By Theorem 2.9, S_i remains taut in $M(\xi_i)$ for some $\xi_i\in\{\infty,\alpha\}$, $i=1,2$. Let $\mathscr{F}_i$ be a taut foliation of $X(\xi_i)$ induced by S_i.

Let $\overline{S_1},\overline{S_2}\subset X$ be the surfaces obtained from S_1,S_2 by gluing $C\times 0$ to $C\times 1$. We have

$$[\overline{S_1}]=[H]+m[G],\quad [\overline{S_2}]=-[H]+m[G]$$

in $H_2(X,\partial X)$ and $H_2(X(\xi),\partial X(\xi))$.

We have

$$\chi(\overline{S_i})=\chi(S_i)-|\partial_0 S_i|=-y(S_i),$$

and by Proposition 2.12

$$\begin{aligned}\chi(\overline{S_1}) &= \langle e(\mathscr{F}_1),[\overline{S_1}]\rangle\\ &= \langle e(\mathscr{E}(\xi_1)),[H]+m[G]\rangle,\\ \chi(\overline{S_2}) &= \langle e(\mathscr{F}_2),[\overline{S_2}]\rangle\\ &= \langle e(\mathscr{E}(\xi_2)),-[H]+m[G]\rangle.\end{aligned}$$

By Lemma 4.4, $\langle e(\mathscr{E}(\xi_1)),[H]\rangle=\langle e(\mathscr{E}(\xi_2)),[H]\rangle$. So

$$\begin{aligned}\chi(\overline{S_1})+\chi(\overline{S_2}) &= \langle\mathscr{E}(\xi_1),m[G]\rangle+\langle\mathscr{E}(\xi_2),m[G]\rangle\\ &= 2m\chi(G),\end{aligned}$$

which contradicts Lemma 4.7. □

Proof of Proposition 4.1. By Proposition 4.3, $b_1(G) \leq 2$, so G is an annulus, a pair of pants or a genus-one surface with one boundary component. We only need to show that the last case is not possible.

Suppose $g(G) = 1$ and $|\partial G| = 1$. Let $C \subset G$ be a simple closed curve such that $[C] \cdot i_*[K] = 0$, then there exists a closed taut surface $H \subset X$ such that $[H] = [C \times S^1]$ and $H \cap T = \emptyset$. Since M does not contain any product annuli, H is not a torus, hence H is not taut in $X(\infty)$. By Theorem 2.9, H is taut in $X(\alpha)$.

Consider the monodromy φ of $X(\alpha)$, the surface $H \subset X(\alpha)$ forces $\varphi_*[C] = [C]$. Since G is a once-punctured torus, $\varphi(C)$ is isotopic to C. Thus there exists a torus $R \subset X(\alpha)$ such that $R \cap G = C = H \cap G$, which implies that $[H] = [R] + m[G]$ for some integer m. Since H is closed, $m = 0$. This contradicts the facts that H is taut in $X(\alpha)$ and that H is not a torus. □

5 Knots in pants$\times I$

In this section, we study the case where G is a pair of pants.

The following elementary observation is stated without proof.

Lemma 5.1. *Suppose $C_1, C_2 \subset G$ are two efficient curves consisting of essential arcs. If they are homologous in $H_1(G, \partial G)$, then they are isotopic.* □

Let a, b, c be the three boundary components of G, u, v, w be three mutually disjoint oriented arcs in G such that u connects b to c, v connects c to a, w connects a to b. Then

$$[u] + [v] + [w] = 0 \in H_1(G, \partial G). \qquad (6)$$

Lemma 5.2. *None of u, v, w has zero intersection number with $i_*[K]$.*

Proof. The argument is similar to the once-punctured torus case of Proposition 4.1. Assume that $[u] \cdot i_*[K] = 0$, then there exists a closed taut surface $H \subset X$ such that $[H] = [u \times S^1]$. We may assume that H is efficient in X, hence H has two boundary components and $H \cap T = \emptyset$. By Lemma 5.1, we may assume that $H \cap G = u$.

Since M does not contain any product disks, H is not an annulus, hence H is not taut in $X(\infty) = G \times S^1$. By Theorem 2.9, H is taut in $X(\alpha)$. Let φ be the monodromy of $X(\alpha)$, then H forces $\varphi(u)$ to be homologous hence isotopic to u by Lemma 5.1. Thus there exists an annulus $A \subset X(\alpha)$ such that $A \cap G = u = H \cap G$, which implies that $[H] = [A] + m[G]$ for some integers. Since H has only two boundary components, $m = 0$. This contradicts the facts that H is taut in $X(\alpha)$ and H is not an annulus. □

Lemma 5.3. *The intersection number of $i_*[K]$ with each of u, v, w is ± 1 or ± 2.*

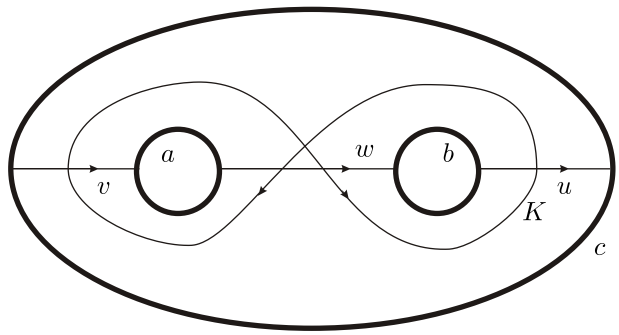

Figure 3: The homology class of K in $G \times I$

Proof. Capping off a with a disk, we get an annulus G_a. Now $K \subset G_a \times I$ and the α–surgery on K does not change the homeomorphism type of the pair $(G_a \times I, (\partial G_a) \times I)$. By the previous lemma, K is nontrivial in $G_a \times I$. By Lemma 3.1, K is the core or the $(2, \pm 1)$–cable in $G_a \times I$, so $i_*[K] \cdot [u]$ is ± 1 or ± 2. The same argument applies to v and w. □

Using the previous two lemmas and (6), we may assume

$$[u] \cdot i_*[K] = [v] \cdot i_*[K] = 1, \quad [w] \cdot i_*[K] = -2, \tag{7}$$

after reversing the orientation of K and renumbering a, b, c, u, v, w if necessary. We give a, b, c the boundary orientation induced from G, then

$$[v] \cdot [a] = [w] \cdot [b] = -[u] \cdot [b] = [u] \cdot [c] = 1. \tag{8}$$

See Figure 3 for the homology class of K.

Let τ_a, τ_b, τ_c be the right-hand Dehn twists along (parallel copies of) a, b, c. The mapping class group $\mathcal{MCG}(G, \partial G)$ of G is generated by τ_a, τ_b, τ_c. (See, for example, Farb–Margalit [1] for preliminaries on the mapping class groups of surfaces with boundary.) Since a, b, c are disjoint, $\mathcal{MCG}(G, \partial G) \cong \mathbb{Z}^3$.

Lemma 5.4. *If K is the $(2, 1)$–cable in $G_c \times I$, then the map induced by the α–surgery is*

$$\varphi_\alpha = \tau_a^2 \tau_b^2 \tau_c^{-1}.$$

If K is the $(2, -1)$–cable in $G_c \times I$, then

$$\varphi_\alpha = \tau_a^{-2} \tau_b^{-2} \tau_c.$$

Proof. Capping off a, b with two disks, G becomes a disk G_{ab}. K has a canonical frame λ, which is null-homologous in $(G_{ab} \times I) \backslash K$. Hence λ is homologous to $l[a] + m[b]$ in M for some integers l, m. By (7), (8) we conclude that λ is homologous to $a - b$ in M. Hence λ is also the canonical frame in $G_c \times I$, where G_c is obtained from G by capping off c with a disk.

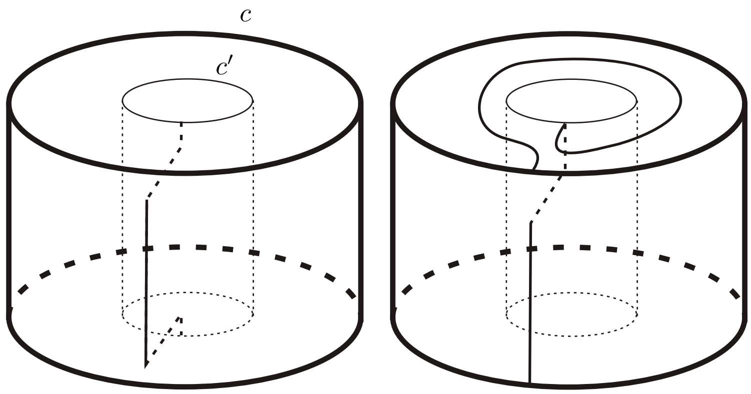

Figure 4: Local pictures of $X(\alpha)$ near $c \times S^1$

Suppose $\varphi_\alpha = \tau_a^p \tau_b^q \tau_c^r$. If K is the $(2,1)$–cable in $G_c \times I$, then by Lemma 3.1, the slope α is 1 with respect to λ.

There is a natural map

$$q_a\colon\ \mathcal{MCG}(G, \partial G) \to \mathcal{MCG}(G_a, \partial G_a),$$

where $\mathcal{MCG}(G_a, \partial G_a)$ is generated by τ_b. Since K is the core in $G_a \times I$ and the slope α is 1, $q_a(\varphi_\alpha)$ must be τ_b by Lemma 3.2. The map q_a sends both τ_b and τ_c to τ_b, and sends τ_a to 1. So $q_a(\varphi_\alpha) = \tau_b^{q+r}$, thus $q + r = 1$. The same argument shows that $p + r = 1$.

Now consider the natural map

$$q_c\colon\ \mathcal{MCG}(G, \partial G) \to \mathcal{MCG}(G_c, \partial G_c) = \langle \tau_a \rangle.$$

By Lemma 3.2, $q_c(\varphi_\alpha) = \tau_a^4$. Hence $p + q = 4$. So we conclude that $p = q = 2, r = -1$. The same argument works when K is the $(2,-1)$–cable in $G_c \times I$. □

Proposition 1.4 follows from the above lemma.

The manifold $G \times S^1$ has a unique product structure. Let $\omega, \omega_\alpha \subset c \times S^1$ be S^1–fibres with respect to the product structures on $X(\infty)$ and $X(\alpha)$, respectively.

Lemma 5.5. *If K is the $(2,1)$–cable in $G_c \times I$, then*

$$[\omega_\alpha] = [\omega] + [c].$$

Proof. The manifold $X(\infty)$ is obtained from $G \times I$ by identifying $(x, 0)$ with $(x, 1)$ for each $x \in G$. By Lemma 1.3, $X(\alpha)$ is obtained from $G \times I$ by identifying $(x, 0)$ with $(\varphi_\alpha(x), 1)$ for each $x \in G$. Choose parallel copies of a, b, c in G, denoted a', b', c'. Let φ_α be supported in the three annuli bounded by $a - a'$, $b - b'$ and $c - c'$. Pick points $p \in c, p' \in c'$, then $p' \times S^1$ is an S^1–fibre of the product structures on both $X(\infty)$ and $X(\alpha)$, while $p \times S^1$ is an S^1–fibre of the product structure on $X(\infty)$.

In $X(\alpha)$, we isotope $p' \times S^1$ such that it becomes a curve $\mathcal{S}$ which is the union of four segments $J, J_\epsilon, J_{1-\epsilon}, J'$, where J is a vertical segment in the interior of $c \times I$, $J_\epsilon \subset G \times \epsilon, J_{1-\epsilon} \subset G \times (1-\epsilon)$, J' is a vertical segment in $c' \times S^1$. See the left hand side of Figure 4.

As on the right hand side of Figure 4, we push the previous curve $\mathcal{S}$ down in distance ϵ to get a new curve $\mathcal{S}_-$, then J_ϵ becomes an arc on $G \times 1$. Using Lemma 1.3, this new arc is $\varphi_\alpha(J_\epsilon) = \tau_c^{-1}(J_\epsilon)$. $\mathcal{S}_-$ is a fibre of $X(\alpha)$, and it is homologous to $[p \times S^1] + [c \times 1]$. Hence our conclusion holds. □

Lemma 5.6. *Let $C = v - u$. Pick a point $p \in c \backslash (\partial C)$, we can then define $\mathcal{S}_m(\pm C)$ as in Section 4. Then there exists a connected surface $S \in S_1(C)$ such that $y(S) = 1$. Moreover, Let $S' \subset G \times [0,1]$ be the surface obtained from $-C \times I$ and $G \times 0$ by oriented cut-and-pastes, then S is isotopic to S' in $G \times [0,1]$.*

Proof. For any homology class $h \in H_2(X, (\partial G) \times S^1)$, let $x(h), x_\infty(h), x_\alpha(h)$ denote its Thurston norm in $X, X(\infty), X(\alpha)$, respectively.

Let $U = -[u \times S^1], V = -[v \times S^1] \in H_2(G \times S^1, (\partial G) \times S^1)$. Since $(V - U) \cdot [K] = 0$, $V - U$ also represents an element in $H_2(X, (\partial G) \times S^1)$. Note that the Thurston norm of $h \in H_2(G \times S^1, (\partial G) \times S^1)$ is the absolute value of its algebraic intersection number with the S^1–fibre. Consider $V - U + m[G]$ for $m \geq 0$, using Lemma 5.5, we can compute

$$x_\infty(V - U + m[G]) = m,$$

$$x_\alpha(V - U + m[G]) = (V - U + m[G]) \cdot ([\omega] + [c]) = |m - 2|.$$

Since $x_\infty(V - U + [G]) = x_\alpha(V - U + [G]) = 1$, Theorem 2.9 implies that $x(V - U + [G]) = 1$. Let $\overline{S} \subset X$ be a taut surface in this homology class such that $\overline{S}$ is efficient in X. Then $\overline{S}$ is disjoint from T. Isotope $\overline{S}$ so that it is transverse to G and its intersection with G contains no trivial loops. Now $\overline{S} \cap G$ is homologous to C. Moreover, $\overline{S} \cap G$ can be made efficient in G. So $\overline{S} \cap G$ is isotopic to C by Lemma 5.1. Without loss of generality, we can assume

$$\overline{S} \cap G = C \quad \text{and} \quad \overline{S} \cap ((\partial C) \times S^1) \subset G.$$

Cutting $\overline{S}$ open along C, we get a surface $S \in \mathcal{S}_1(+C)$ such that $y(S) = 1$. After an isotopy of S, we can assume the two surfaces $S, C \times [0,1] \subset G \times [0,1]$ are transverse. Since $S \cap ((\partial C) \times (0,1)) = \emptyset$, $S \cap (C \times (0,1))$ consists of closed curves which bounds disks in $C \times (0,1)$. Since S is incompressible and $G \times [0,1]$ is irreducible, we can isotope S such that $S \cap (C \times (0,1)) = \emptyset$, hence $S \cap (C \times [0,1]) = C \times \{0,1\}$. Now we glue S and $C \times [0,1]$ together along $C \times \{0,1\}$ and perturb the resulting surface slightly, then we get a connected surface G' with $x(G') = 1$ and $\partial G'$ is parallel to $(\partial G) \times 0$ in $(\partial G) \times [0,1]$. Hence G' is parallel to $G \times 0$ in $G \times [0,1]$. It follows that S is isotopic to S' in $G \times [0,1]$. □

Lemma 5.7. *Let S be the surface obtained in Lemma 5.6. Let*

$$G \times I \overset{S}{\rightsquigarrow} (M_1(\infty), \gamma_1)$$

be the sutured manifold decomposition associated with S, then $(M_1(\infty), \gamma_1)$ is a product manifold, and there is an ambient isotopy of $M_1(\infty)$ which takes K to a curve in $R_+(\gamma_1)$ such that the frame of K specified by $R_+(\gamma_1)$ is α.

Proof. By Lemma 5.6, S is obtained from $-C \times I$ and $G \times 0$ by oriented cut-and-pastes. So $(M_1(\infty), \gamma_1)$ is a product sutured manifold and $R_+(\gamma_1)$ is an annulus.

Let $(M_1(\alpha), \gamma_1)$ be the sutured manifold obtained from $M(\alpha)$ by decomposing along S. Then $M_1(\alpha)$ can also be obtained from $M_1(\infty)$ by α–surgery on K.

We claim that $M_1(\alpha)$ is not taut. In fact, let S'' be the surface obtained from S and $G \times 0$ by oriented cut-and-pastes. Let $\overline{S''} \subset X$ be the surface obtained from S'' by gluing $\partial_0 S''$ to $\partial_1 S''$. Then $x(\overline{S''}) = 2$ and $\overline{S''}$ represents $V - U + 2[G]$. We already computed

$$x_\infty(V - U + 2[G]) = 2 > x_\alpha(V - U + 2[G]) = 0,$$

so $\overline{S''}$ is not taut in $X(\alpha)$. Let $M''(\alpha)$ be the non-taut sutured manifold obtained by decomposing $X(\alpha)$ along $\overline{S''}$.

Since S'' is obtained from S and $G \times 0$ by oriented cut-and-pastes, and $S \cap (G \times 0) = -C \times 0$ consists of two arcs, there exist two product disks in $M''(\alpha)$ such that the result of decomposing $M''(\alpha)$ along these two disks is $(M_1(\alpha), \gamma_1)$. See the proof of Gabai [2, Theorem 3.13] for an explanation of this fact. So $(M_1(\alpha), \gamma_1)$ is not taut by Gabai [3, Lemma 0.4].

Now Theorem 1.5 implies our conclusion. □

Proof of Theorem 1.1. By the results in Sections 3 and 4, we only need to consider the case $F = G$ is a pair of pants. By Lemma 5.7, K lies on $R_+(\gamma_1)$, and the frame specified by $R_+(\gamma_1)$ is α.

Since $R_+(\gamma_1)$ is an annulus, the only essential curve on it is its core. As in Figure 5, $R_+(\gamma_1)$ can be constructed in the following way. Cut $G \times \{0, 1\}$ open along $(v - u) \times \{0, 1\}$, we get two octagons P_0, P_1. There are two edges of P_0 which are copies of $v \times 0$ with different orientations. We call these two edges $v \times 0, -v \times 0$. Similarly, there are edges $\pm u \times 0, \pm v \times 1, \pm u \times 1$. Now we glue two product disks to P_0, P_1, such that one product disk connects $v \times 0$ to $-v \times 1$ and the other connects $-u \times 0$ to $u \times 1$. The annulus we get is isotopic to $R_+(\gamma_1)$. The core of this annulus is clearly a one-crossing knot in $G \times I$. The result about the frame also follows since the vertical projection $p\colon R_+(\gamma_1) \to G$ is an immersion. □

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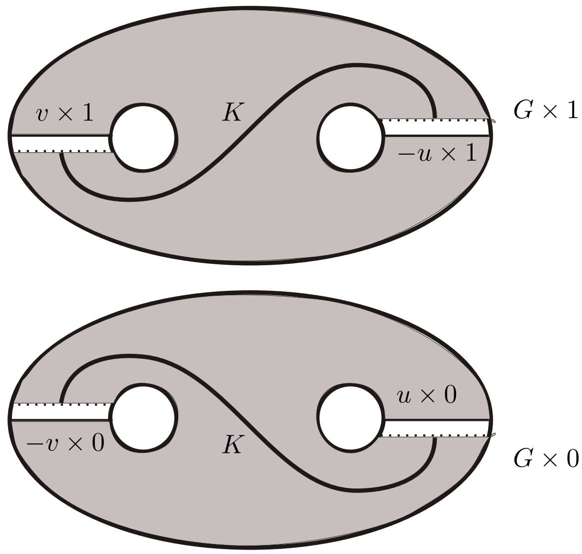

Figure 5: The surface $R_+(\gamma_1)$ containing the knot K

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