# CALIFORNIA INSTITUTE OF TECHNOLOGY

## DYNAMICS LABORATORY

# STATIONARY RANDOM RESPONSE OF MULTIDEGREE-OF-FREEDOM SYSTEMS

by

I-Min Yang

Report No. DYNL-100

Pasadena, California

June 1970

STATIONARY RANDOM RESPONSE OF

MULTIDEGREE-OF-FREEDOM SYSTEMS

Thesis by

I-Min Yang

In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1970

(Submitted March 10, 1970)

## ACKNOWLEDGMENTS

## ABSTRACT

An approximate approach is presented for determining the stationary random response of a general multidegree-of-freedom nonlinear system under stationary Gaussian excitation. This approach relies on defining an equivalent linear system for the nonlinear system. Two particular systems which possess exact solutions have been solved by this approach, and it is concluded that this approach can generate reasonable solutions even for systems with fairly large nonlinearities. The approximate approach has also been applied to two examples for which no exact or approximate solutions were previously available.

Also presented is a matrix algebra approach for determining the stationary random response of a general multidegree-of-freedom linear system. Its derivation involves only matrix algebra and some properties of the instantaneous correlation matrices of a stationary process. It is therefore very direct and straightforward. The application of this matrix algebra approach is in general simpler than that of commonly used approaches.

# TABLE OF CONTENTS

## NOTATIONS

Matrices and vectors will be denoted by capital letters and lower case letters with a bar over them, respectively (exceptions: E, $\Gamma$). Subscripts may be used to designate different matrices or vectors, e.g., $X_1$, $X_{12}$, $R_{\bar{x}}$. Superscripts T and * will denote, respectively, the transpose and the conjugate transpose of a matrix or a vector. The components of a matrix or a vector, say A or $\bar{x}$, will be written as $a_{jk}$ or $x_j$. In the following is a list of frequently used symbols. Others will be defined when they are used.

$c_j$ = $\gamma_{jk}$ when $k = j-1$

C = damping matrix of a linear system

$C^{(o)}$ = damping matrix of the linearized system of a nonlinear system

$\bar{e}$ = difference between a nonlinear system and its equivalent linear system

E [ ] = expectation operator

$f_u$ = ultimate force of a nonlinear spring

$\bar{f}(t)$ = stationary random vector

$\bar{g}$ = internal force vector of a system

G(t) = impulse response function matrix

H($\omega$) = frequency response function matrix

i = $\sqrt{-1}$

I = identity matrix

$k_j$ = $\kappa_{jm}$ when $m = j-1$

$k^{(o)}$ = initial stiffness of a nonlinear spring

| | | |
|---|---|---|
| $K$ | = | stiffness matrix of a linear system |
| $K^{(o)}$ | = | stiffness matrix of the linearized system of a non-linear system |
| $\overline{m}_{\overline{x}}$ | = | mean vector of $\overline{x}$ |
| $M$ | = | mass matrix |
| $O$ | = | null matrix or null vector |
| $p$ | = | Laplace transform parameter or probability density function |
| $q$ | = | transitional probability density |
| $R_{\overline{x}}(\tau)$ | = | correlation function matrix of $\overline{x}$ |
| $R_{\overline{x}}(0)$ | = | instantaneous correlation matrix of $\overline{x}$ |
| $s_j$ | = | $s_{jk}$ when $k = j-1$ |
| $s_{jk}$ | = | force in the nonlinear element connecting the $j^{th}$ and the $k^{th}$ masses of a nonlinear system |
| $\overline{s}(t)$ | = | white noise vector |
| $t$ | = | time |
| $u(\overline{x})$ | = | potential energy of a system |
| $W$ | = | spectral density matrix of a white or a clipped white noise vector |
| $\overline{x}$ | = | displacement vector |
| $y_{jk}$ | = | displacement of the $j^{th}$ mass relative to the $k^{th}$ mass |
| $\overline{y}$ | = | relative displacement vector of a simple n-degree-of-freedom system |
| $\gamma$ | = | proportionality factor between $C$ and $R_{\overline{f}}(\tau)$ or a constant |
| $\gamma_{jk}$ | = | equivalent linear damping constant of $s_{jk}$ |
| $\Gamma(\ )$ | = | Gamma function |
| $\delta(t)$ | = | Dirac delta function |
| $\zeta$ | = | fraction of critical damping |

$\varkappa_{jk}$ = equivalent linear stiffness of $s_{jk}$

$\mu$ = small parameter

$\sigma_y$ = r.m.s. value of y

$\omega_c$ = cutoff frequency of a clipped white noise

$\omega_n$ = natural frequency

## I. INTRODUCTION

In structural dynamics the excitation of a system is often random in nature and hence the response of such a system cannot be accurately predicted by usual deterministic approaches. For example, the ground motion during an earthquake is a random process and it is therefore desirable to apply probabilistic techniques to the analysis of structures subjected to this type of excitation. In recent years the development of rockets and jet engines which give rise to vibration that is essentially random has lead to increased interest in the application of probabilistic techniques to structural dynamics.

The theory of the random process was first successfully applied to dynamic systems by Einstein[1]. He used it to investigate the Brownian motion of a free particle and found that the probability density of the response process was governed by a diffusion equation. Soon his result was generalized to more complicated cases by Fokker[2], Smoluchowski[3], Planck[4], and others. It was found that the transitional probability of a special kind of Markov process was governed by a partial differential equation of parabolic type. This equation is usually called the Fokker-Planck equation. A more general partial differential equation for determining the transitional probability density of a completely general Markov process has also been found[5] and sometimes it is also referred to as the Fokker-Planck equation. In the present study, the general equation will not be used, and hence the more restrictive parabolic equation will be referred to as the Fokker-Planck equation.

In 1919 Ornstein[6] developed another approach which did not require that the response process of a dynamic system be Markovian. It is known that a random process can be completely defined by its moment functions of various orders. For example, a Gaussian process can be specified by its first and second moment functions. Therefore, Ornstein integrated the equation of motion and expressed the random displacement in terms of a stochastic integral. Then he was able to derive all of the required moment functions. This approach will be called the impulse response function approach.

Wiener[7] in 1930, and Khintchine[8] in 1934, found independently that the spectral density and the correlation function of a random process are related by a Fourier cosine transform. Since the spectral density of the stationary response of a linear system can be readily determined by Fourier transform, the Wiener-Khintchine relation furnishes another way to determine the stationary response of a linear system. This approach is usually called the spectral density approach.

All of these approaches were first developed for single-degree-of-freedom linear systems. However, they can be generalized to multidegree-of-freedom linear systems without difficulty[9].

In general, it is just an approximation to consider a real system as linear. Most real systems are nonlinear by their very nature. The results of linear analysis are particularly inadequate for large motions. Hence, the effect of system nonlinearities on the response of structures should not be overlooked.

Unfortunately, only one of the three approaches mentioned above, the Fokker-Planck approach, can be extended to nonlinear problems. However, no exact solution has been found for the Fokker-Planck equation for any second-order nonlinear system. The time-independent or stationary Fokker-Planck equation can be solved, but only under certain rather restrictive conditions. Since an exact solution is available only for limited stationary cases, several approximate approaches have been devised to treat non-linear problems.

In deterministic theory, Krylov and Bogoliubov[10] developed a technique to replace a nonlinear system by an equivalent linear system. By solving this linear system, they obtained an approximate solution to the nonlinear system. The application of this technique to problems of random vibrations was made independently by Booton[11] and Caughey[12]. Later, Caughey further extended this technique to include some special multidegree-of-freedom nonlinear systems[12]. This extension will be called the normal mode approach since it is based on the linear theory concept of normal modes of oscillation.

Another approximate approach which has also been adapted from the deterministic theory is the perturbation approach. Crandall[13] first applied this approach to investigate the random vibration of a nonlinear oscillator. Later, Tung[14] was able to apply it to multidegree-of-freedom nonlinear systems. This approach requires that the nonlinearity of the system be small. Then the

nonlinear problem can be reduced to the solution of several sets of linear differential equations.

In the present study a new approach for multidegree-of-freedom nonlinear systems is presented. Additionally, as a by-product of the nonlinear analysis, a matrix algebra approach is developed which can be applied to find various instantaneous correlation matrices for the stationary random response of a multidegree-of-freedom linear system. The details of the matrix algebra approach are presented in Chapter II. This chapter also contains a brief review of existing approaches for multidegree-of-freedom linear systems and a comparison of the matrix algebra approach and the existing approaches. Two examples are worked out in detail to illustrate the application of the matrix algebra approach.

The new approach for determining the instantaneous correlation matrices of the stationary response of a multidegree-of-freedom nonlinear system is presented in Chapter III along with a brief review of the Fokker-Planck approach, the normal mode approach, and the perturbation approach. This approach is based on the idea of defining an equivalent system by minimizing the difference between the original system and the equivalent system. In the present study the equivalent system is assumed to be linear. Thus, it leads to a generalization of the method of equivalent linearization. The only restrictions on the application of this generalized equivalent linearization approach are that the excitation be stationary and Gaussian.

If the nonlinearities of the system are small, then the response should be close to a Gaussian process. Hence, it is expected that the instantaneous correlation matrices generated by this approach will be a good approximation. For large nonlinearities the response may in general be quite different from a Gaussian process. However, for the examples considered in Chapter III, this approach still gives a reasonable solution.

In Chapter IV, two timely examples which cannot be solved either exactly or approximately by existing techniques are given as illustration of the generalized equivalent linearization approach.

## II. STATIONARY RANDOM RESPONSE OF
## MULTI-DEGREE-OF-FREEDOM
## LINEAR SYSTEMS

### 2.1 Equation of Motion

The general equation of motion in matrix form for an n-degree-of-freedom linear system may be described by

$$M\ddot{\overline{x}} + C\dot{\overline{x}} + K\overline{x} = \overline{f}(t) \qquad (2.1)$$

where

   M   is an nxn generalized mass matrix

   C   is an nxn generalized damping matrix

   K   is an nxn generalized stiffness matrix

   $\overline{x}$   is a generalized displacement vector defining the motion of the system

   $\overline{f}(t)$   is a stationary random vector process specified by its mean vector $\overline{m}_{\overline{f}}$ and its correlation function matrix
$$R_{\overline{f}}(t_1 - t_2) = E\left[\overline{f}(t_1)\overline{f}^T(t_2)\right]$$

Since $\overline{f}(t)$ is a random vector process, Eq. (2.1) is a stochastic differential equation and $\overline{x}$ is also a random vector process. In this study the vector process $\overline{x}$ is assumed to be stationary, continuous, integrable and differentiable to the required order in the sense of mean square. One consequence of these assumptions, which we will use quite often, is that the operations of expectation and mean-square differentiation or integration are commutative, provided, of course, that the expectations in question exist and are continuous at the limit points[12].

## 2.2 Review of Commonly-Used Approaches in Obtaining a Stationary Solution for Multidegree-of-Freedom Linear Systems

Several approaches have been developed to treat the stochastic differential equation of motion (2.1). The most commonly used approaches are the impulsive response function approach, the spectral density approach, and the Fokker-Planck approach. The first two approaches can be used to calculate the mean vector and the correlation function matrix of the response process. However, if the response process is Gaussian, these two quantities completely define the probability densities of any order. In the third approach, the Fokker-Planck equation, which governs the transition probability density of the response process, can be used only when the excitation is a shot or white noise. If the excitation is a Gaussian white noise, and one is interested in the stationary solution, the transition probability alone completely describes the response process. All three approaches can be applied to nonstationary problems as well as stationary problems. In this section we will give a brief description of these three approaches applied to stationary problems.

### 2.2.1 Impulsive Response Function Approach

The first step in this method is to find the impulse response function matrix G(t) of (2.1) which is defined as the solution of the following system:

$$M\ddot{G} + C\dot{G} + KG = I\delta(t) \tag{2.2}$$

with the initial conditions

$$\dot{G}(0) = G(0) = O \tag{2.3}$$

where $\delta(t)$ is the Dirac delta function, I, an nxn identity matrix, and O, an nxn null matrix. The problem specified by (2.2) and (2.3) can be solved in a variety of ways. However, one of the most direct is the Laplace transform method[15].

Taking the Laplace transform of both sides of (2.2) and using the initial conditions (2.3) yields

$$\widetilde{G}(p) = (Mp^2 + Cp + K)^{-1} \tag{2.4}$$

where p is a transform parameter and $\widetilde{G}(p)$ is the Laplace transform of G(t). Let

$$S(p) = Mp^2 + Cp + K \tag{2.5}$$

$$\Delta(p) = \text{the determinant of } S$$

$$R(p) = \text{the adjoint of } S$$

Then G(t) is given by the inversion integral

$$G(t) = \frac{1}{2\pi i} \int_{Br} e^{pt} \frac{R(p)}{\Delta(p)} \, dp \tag{2.6}$$

where Br is the Bromwich contour in the p-plane. For the present case the integral on the right-hand side of (2.6) can be evaluated by calculating the residues at the singularities of the integrand. In this way, G(t) can be expressed in the following form:

$$G(t) = 2 \sum_{k=1}^{n} (X_k \cos\beta_k t - Y_k \sin\beta_k t) e^{-\alpha_k t} \tag{2.7}$$

where

$$X_k + iY_k = \left[ R(p) / \frac{d}{dp} \Delta(p) \right]_{p=\lambda_k}$$

(2.8)

$$\lambda_k = -\alpha_k + i\beta_k \, , \ \beta_k > 0$$

The $\lambda_k$'s are those roots of $\Delta(p) = 0$ which have positive imaginary part, the $\alpha_k$'s are always positive for a stable system.

Differentiating G(t) with respect to t gives

$$\dot{G}(t) = -2 \sum_{k=1}^{n} e^{-\alpha_k t} \left[ \left( X_k \alpha_k + Y_k \beta_k \right) \cos \beta_k t \right.$$

$$\left. + \left( X_k \beta_k - Y_k \alpha_k \right) \sin \beta_k t \right]$$

(2.9)

As $t \to 0^+$, that is, as t approaches zero from the positive side, it can be proved that[16]

$$\lim_{t \to 0^+} G(t) = 0$$

$$\lim_{t \to 0^+} \dot{G}(t) = M^{-1}$$

(2.10)

After having found G(t), the steady-state response to an arbitrary excitation $\overline{f}(t)$ may be evaluated from the superposition integral

$$\overline{x}(t) = \int_{-\infty}^{\infty} G(t-\tau) \overline{f}(\tau) d\tau$$

(2.11)

If $\overline{f}(t)$ is a random process, $\overline{x}(t)$ will also be a random process and the integral in (2.11) becomes a stochastic integral which will exist in mean square iff[17]

$$- \infty < \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(t_1 - \tau_1) R_{\overline{f}}(\tau_1 - \tau_2) G^T(t_2 - \tau_2) d\tau_1 d\tau_2 < \infty \qquad (2.12)$$

for all $t_1$ and $t_2$. When (2.12) is satisfied, the mean vector $\overline{m}_{\overline{x}}$ and correlation function matrix $R_{\overline{x}}(\tau)$ are given by

$$\overline{m}_{\overline{x}} \equiv E\left[\overline{x}(t)\right]$$

$$= \int_{-\infty}^{\infty} G(t - \tau_1) E\left[\overline{f}(\tau_1)\right] d\tau_1$$

$$= \left(\int_{-\infty}^{\infty} G(t - \tau_1) d\tau_1\right) \cdot \overline{m}_{\overline{f}}$$

$$= \left(\int_{-\infty}^{\infty} G(\tau_1) d\tau_1\right) \cdot \overline{m}_{\overline{f}} \qquad (2.13)$$

$$R_{\overline{x}}(\tau) = R_{\overline{x}}(t_1 - t_2) \equiv E\left[\overline{x}(t_1)\overline{x}^T(t_2)\right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(t_1 - \tau_1) E\left[\overline{f}(\tau_1)\overline{f}^T(\tau_2)\right] G^T(t_2 - \tau_2) d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(t_1 - \tau_1) R_{\overline{f}}(\tau_1 - \tau_2) G^T(t_2 - \tau_2) d\tau_1 d\tau_2 \qquad (2.14)$$

where $\tau = t_1 - t_2$.

When $t_1 = t_2 = t$, the correlation function matrix becomes

$$R_{\overline{x}}(0) = E\left[\overline{x}(t)\overline{x}^T(t)\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(t - \tau_1) R_{\overline{f}}(\tau_1 - \tau_2) G^T(t - \tau_2) d\tau_1 d\tau_2$$

$$(2.15)$$

The matrix $R_{\underline{x}}(0)$ will be referred to as the instantaneous correlation matrix of $\underline{x}$.

If the joint moment functions of higher order of the excitation are given, the present approach can also be used to find the joint moment functions of the corresponding order of the response. For example, the third joint moment functions are

$$E\left[x_j(t_1)x_k(t_2)x_m(t_3)\right]$$

$$= \sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{r=1}^{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g_{jp}(t_1-\tau_1)\,g_{kq}(t_2-\tau_2)\,g_{mr}(t_3-\tau_3)$$

$$\cdot E\left[f_p(\tau_1)f_q(\tau_2)f_r(\tau_3)\right]d\tau_1 d\tau_2 d\tau_3$$

$$j,k,m = 1,2,\ldots,n \qquad\qquad (2.16)$$

provided that the right-hand side exists. The function $g_{jk}(t)$ in (2.16) represents the $(j,k)$ element of the matrix $G(t)$.

The mean vector in (2.13) and the correlation function matrix in (2.14) play an important role in application. First, if a process is Gaussian, all moments higher than the second can be computed from these two quantities. Hence, a Gaussian process can be completely specified by its mean vector and correlation function matrix. Second, even when the mean vector and the correlation function matrix cannot specify a random process completely, they still give some important information about that process.

## 2.2.2 Spectral Density Approach

It is well known that the correlation function matrix $R_{\overline{x}}(\tau)$ and the spectral density function matrix $\Phi_{\overline{x}}(\omega)$ of a stationary random vector process $\overline{x}$ form a Fourier transform pair. They are related as follows[*]:

$$\Phi_{\overline{x}}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{\overline{x}}(\tau) \exp(-i\omega\tau)d\tau$$

$$(2.17)$$

$$R_{\overline{x}}(\tau) = \int_{-\infty}^{\infty} \Phi_{\overline{x}}(\omega) \exp(i\omega\tau)d\tau$$

This Fourier transform pair is usually called the Wiener-Khintchine relation. The impulse response function matrix $G(t)$ and the frequency response function matrix $H(\omega)$ also form a Fourier transform pair (or more accurately, $G(t)$ and $H(\omega)/2\pi$)[*]:

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) \exp(i\omega t)d\omega$$

$$(2.18)$$

$$H(\omega) = \int_{-\infty}^{\infty} G(t) \exp(-i\omega t)dt$$

With the aid of (2.14) and the second equations in (2.17) and (2.18), the correlation function matrix becomes

$$R_{\overline{x}}(\tau) = R_{\overline{x}}(t_1 - t_2) \equiv E\left[\overline{x}(t_1)\overline{x}^T(t_2)\right]$$

$$= \iiint_{-\infty}^{\infty} G(t_1 - \tau_1) \Phi_{\overline{f}}(\omega) G^T(t_2 - \tau_2) \exp\left[i\omega(\tau_1 - \tau_2)\right]d\tau_1 d\tau_2 d\omega \qquad (2.19)$$

---

[*]For the conditions under which the integrals in (2.17) and (2.18) exist, see reference (18).

$$= \int_{-\infty}^{\infty} H(\omega) \, \Phi_{\overline{f}}(\omega) \, H^*(\omega) \exp \left[ i\omega(t_1 - t_2) \right] d\omega$$

$$= \int_{-\infty}^{\infty} H(\omega) \, \Phi_{\overline{f}}(\omega) \, H^*(\omega) \exp(i\omega\tau) d\omega \qquad (2.19)$$
$$\text{cont'd}$$

in which $H^*(\omega)$ is the conjugate transpose of $H(\omega)$. Since the Fourier transform of $R_{\overline{x}}(\tau)$ is unique, from (2.19) and the second equation in (2.17) we know that the spectral density function matrix of the response vector $\overline{x}$ is given by

$$\Phi_{\overline{x}}(\omega) = H(\omega) \, \Phi_{\overline{f}}(\omega) \, H^*(\omega) \qquad (2.20)$$

When $\tau = 0$, i.e., $t_1 = t_2$, Eq. (2.19) reduces to

$$R_{\overline{x}}(0) \equiv E\left[ \overline{x}(t) \overline{x}^T(t) \right] = \int_{-\infty}^{\infty} \Phi_{\overline{x}}(\omega) d\omega \qquad (2.21)$$

Thus, the instantaneous correlation matrix is just the sum of the spectral density matrix over all frequencies.

The matrix $H(\omega)$ can be found by taking the Fourier transform of both sides of (2.2), or it can alternatively be found from (2.1) by letting the excitation vector be $\overline{f}_0 e^{i\omega t}$ and $\overline{x} = H(\omega)\overline{f}_0 e^{i\omega t}$ where $\overline{f}_0$ is an arbitrary constant vector. Both approaches lead to

$$H(\omega) = \int_{-\infty}^{\infty} G(t) \exp(-i\omega t) dt = (-\omega^2 M + i\omega C + K)^{-1} \qquad (2.22)$$

If $\omega = 0$, Eq. (2.22) reduces to

$$H(0) = \int_{-\infty}^{\infty} G(t) dt = K^{-1} \qquad (2.23)$$

---

*For the conditions under which the integrals in (2.17) and (2.18) exist, see reference (18).

Hence, the frequency response function matrix at $\omega = 0$ is just equal to the inverse of the stiffness matrix.

Upon substituting (2.23) into (2.13), the mean vector $\overline{m}_{\overline{x}}$ becomes

$$\overline{m}_{\overline{x}} = H(0)\overline{m}_{\overline{f}} = K^{-1}\overline{m}_{\overline{f}} \tag{2.24}$$

### 2.2.3 Fokker-Planck Approach

A random vector process $\overline{z}(t)$ is said to be Markovian if the conditional probability density function $p(\overline{z}_n, t_n | \overline{z}_{n-1}, t_{n-1}; \ldots; \overline{z}_1, t_1)$ where $\overline{z}_n = \overline{z}(t_n), \ldots, \overline{z}_1 = \overline{z}(t_1)$ and $t_n > t_{n-1} > \ldots > t_1$, depends only on the last value $\overline{z}(t_{n-1})$ and not on the preceeding values $\overline{z}(t_{n-2}), \ldots, \overline{z}(t_1)$. Hence, for a Markov vector process, we can write

$$p(\overline{z}_n, t_n | \overline{z}_{n-1}, t_{n-1}; \ldots; \overline{z}_1, t_1) = p(\overline{z}_n, t_n | \overline{z}_{n-1}, t_{n-1}),$$

$$t_n > t_{n-1} > \ldots > t_1 \tag{2.25}$$

The special conditional probability density $p(\overline{z}_n, t_n | \overline{z}_{n-1}, t_{n-1})$ is called the transitional probability density and will be denoted by a special symbol $q(\overline{z}_n, t_n | \overline{z}_{n-1}, t_{n-1})$. From the definition of conditional probability we have

$$p(\overline{z}_1, t_1; \overline{z}_2, t_2; \ldots; \overline{z}_n, t_n) = p(\overline{z}_1, t_1) p(\overline{z}_2, t_2 | \overline{z}_1, t_1) \ldots$$

$$\ldots p(\overline{z}_n, t_n | \overline{z}_{n-1}, t_{n-1}; \ldots; \overline{z}_1, t_1), \ t_n > t_{n-1} > \ldots > t_1 \tag{2.26}$$

Hence, it follows by using (2.25) that in the case of Markov processes,

$$p(\overline{z}_1, t_1; \overline{z}_2, t_2; \ldots; \overline{z}_n, t_n) = p(\overline{z}_1, t_1) \, q(\overline{z}_2, t_2 \,|\, \overline{z}_1, t_1) \ldots$$

$$\ldots q(\overline{z}_n, t_n \,|\, \overline{z}_{n-1}, t_{n-1}), \quad t_n > t_{n-1} > \ldots > t_1 \tag{2.27}$$

Thus, if we know the first probability density and the transitional probability density of $\overline{z}(t)$, we can write its probability density of any order, that is, the first probability density and the transitional probability completely specify a Markov process.

Consider a special Markovian process whose transitional probability density is governed by the Fokker-Planck equation

$$\frac{\partial q}{\partial t} + \sum_{j=1}^{2n} \frac{\partial}{\partial z_j} (a_j q) - \frac{1}{2} \sum_{j=1}^{2n} \sum_{k=1}^{2n} \frac{\partial^2}{\partial z_k \partial z_j} (b_{jk} q) = 0 \tag{2.28}$$

with the initial condition

$$\lim_{t \to t_1} q(\overline{z}, t \,|\, \overline{z}_1, t_1) = \delta(\overline{z} - \overline{z}_1) \tag{2.29}$$

Such a process will be called a continuous Markovian process. Let

$$\Delta z_k = z_k(t + \tau) - z_k(t) \tag{2.30}$$

Then the quantities $a_k$ and $b_{jk}$ in (2.28) are given by

$$a_k = \lim_{\tau \to 0} \frac{E[\Delta z_k]}{\tau}$$

$$\tag{2.31}$$

$$b_{jk} = \lim_{\tau \to 0} \frac{E[\Delta z_k \Delta z_j]}{\tau}$$

provided that all limits exist.

The first probability density of a stationary Markov vector
may be obtained from the transition probability density by letting the
transition time $(t-t_1)$ approach infinity, i.e.,

$$p(\bar{z}, t) = \lim_{(t-t_1) \to \infty} q(\bar{z}, t \,|\, \bar{z}_1, t_1) \qquad (2.32)$$

Hence, a stationary Markov vector is sufficiently defined by its transi-
tion probability density alone.   The first probability density of a
stationary continuous Markovian process can also be found by solving
the time independent or stationary Fokker-Planck equation

$$\sum_{j=1}^{2n} \frac{\partial}{\partial z_j}(a_j p) - \frac{1}{2}\sum_{k=1}^{2n} \sum_{j=1}^{2n} \frac{\partial^2}{\partial z_j \partial z_k}(b_{jk}p)=0 \qquad (2.33)$$

which follows from (2.28) by dropping the first term and replacing
$q(\bar{z}, t_1 +\tau \,|\, \bar{z}_1, t_1)$ by $p(\bar{z}, t)$.

If the vector $\bar{f}$ in (2.1) is a Gaussian process such that

$$E[\bar{f}(t)] = O$$
$$E[\bar{f}(t)\bar{f}^T(t+\tau)] = 2\pi\, W\, \delta(\tau) \qquad (2.34)$$

where W is the spectral density matrix of $\bar{f}(t)$, it may be shown that
the displacement and velocity vectors of the response process form
a 2n continuous Markov vector.   Wang and Uhlenbeck[19] have solved
this problem when M, C, K are symmetric and

$$\pi W = \gamma C, \qquad (2.35)$$

where $\gamma$ is a constant.   They found that the instantaneous correlation
matrices of the stationary response were governed by the following
equations:

$$E\left[\overline{x}\,\dot{\overline{x}}^T\right] = O$$

$$ME\left[\dot{\overline{x}}\,\dot{\overline{x}}^T\right] - K\,E\left[\overline{x}\,\overline{x}^T\right] = O \qquad (2.36)$$

$$CE\left[\dot{\overline{x}}\,\dot{\overline{x}}^T\right]M + ME\left[\dot{\overline{x}}\,\dot{\overline{x}}^T\right]C = 2\gamma C$$

This set of equations can readily be solved to give

$$E\left[\overline{x}\,\dot{\overline{x}}^T\right] = O$$

$$E\left[\overline{x}\,\overline{x}^T\right] = \gamma K^{-1} \qquad (2.37)$$

$$E\left[\dot{\overline{x}}\,\dot{\overline{x}}^T\right] = \nu M^{-1}$$

Wang and Uhlenbeck's approach can be extended to a more general problem where the spectral density matrix W is not related to C and M, C, K are not symmetric. In this case the instantaneous correlation matrices of the stationary response are governed by the following equations (see Appendix A for details)

$$E\left[\overline{x}\,\dot{\overline{x}}^T\right] + E\left[\dot{\overline{x}}\,\overline{x}^T\right] = O$$

$$ME\left[\dot{\overline{x}}\,\dot{\overline{x}}^T\right] - CE\left[\dot{\overline{x}}\,\overline{x}^T\right] - K\,E\left[\overline{x}\,\overline{x}^T\right] = O$$

$$ME\left[\dot{\overline{x}}\,\dot{\overline{x}}^T\right]C^T + CE\left[\dot{\overline{x}}\,\dot{\overline{x}}^T\right]M^T + ME\left[\dot{\overline{x}}\,\overline{x}^T\right]K^T \qquad (2.38)$$

$$+ KE\left[\overline{x}\,\dot{\overline{x}}^T\right]M^T = 2\pi W$$

## 2.3 A Matrix Algebra Approach for Stationary Response

All the approaches discussed in the above section are directed toward finding the correlation function matrices, and the instantaneous correlation matrices are given as special cases when $\tau = 0$. However, in many applications, one needs only the instantaneous

correlation matrices. For example, the first probability density $p(\dot{\bar{x}}, \bar{x})$ of a Gaussian process is completely described by $E\left[\bar{x}\bar{x}^T\right]$, $E\left[\dot{\bar{x}}\dot{\bar{x}}^T\right]$ and $E\left[\bar{x}\dot{\bar{x}}^T\right]$, the probability distribution of peaks and the average frequency of a narrow-band process are all dependent on the instantaneous correlation matrices. If one is only concerned with the instantaneous correlation matrices, then a very straight-forward and direct approach can be used. In this approach, the mean vector and the instantaneous correlation matrices are derived directly from the equation of motion, and only matrix algebra and some special properties of stationary vector processes are involved.

### 2.3.1 Mean Vectors

The mean vector of the response process $\bar{x}$, $\bar{m}_{\bar{x}}$, can be found in the following way.

Taking expectations of both sides of (2.1) gives

$$ME\left[\ddot{\bar{x}}\right] + CE\left[\dot{\bar{x}}\right] + KE\left[\bar{x}\right] = E\left[\bar{f}\right] = \bar{m}_{\bar{f}} \tag{2.39}$$

From the stationarity of $\bar{x}$ it follows that $\bar{m}_{\bar{x}}$ is a constant vector, and

$$E\left[\dot{\bar{x}}\right] = \frac{d}{dt} E\left[\bar{x}\right] = O$$

$$E\left[\ddot{\bar{x}}\right] = \frac{d^2}{dt^2} E\left[\bar{x}\right] = O \tag{2.40}$$

Hence, Eq. (2.39) reduces to

$$E\left[\bar{x}\right] = K^{-1}\bar{m}_{\bar{f}} \tag{2.41}$$

This is the same equation as (2.24).

## 2.3.2 Instantaneous Correlation Matrices

The equations governing the instantaneous correlation matrices $E[\bar{x}\bar{x}^T]$, $E[\bar{x}\dot{\bar{x}}^T]$ and $E[\dot{\bar{x}}\dot{\bar{x}}^T]$ can be obtained as follows.

Postmultiplying through (2.1) respectively by $\bar{x}^T$ and $\dot{\bar{x}}^T$ and then taking expectations of both sides of the resulting equation gives

$$
\left.\begin{aligned}
ME[\ddot{\bar{x}}\bar{x}^T] + CE[\dot{\bar{x}}\bar{x}^T] + KE[\bar{x}\bar{x}^T] &= E[\bar{f}\bar{x}^T] \\
ME[\ddot{\bar{x}}\dot{\bar{x}}^T] + CE[\dot{\bar{x}}\dot{\bar{x}}^T] + KE[\bar{x}\dot{\bar{x}}^T] &= E[\bar{f}\dot{\bar{x}}^T]
\end{aligned}\right\} \quad (2.42)
$$

Postmultiplying the second equation in (2.42) by $M^T$ and adding the resulting equation to its transpose yields

$$
M\left(E[\ddot{\bar{x}}\dot{\bar{x}}^T] - E[\dot{\bar{x}}\ddot{\bar{x}}^T]\right)M^T + CE[\dot{\bar{x}}\dot{\bar{x}}^T]M^T
$$

$$
+ ME[\dot{\bar{x}}\dot{\bar{x}}^T]C^T + KE[\bar{x}\dot{\bar{x}}^T]M^T + ME[\dot{\bar{x}}\bar{x}^T]K^T
$$

$$
= E[\bar{f}\dot{\bar{x}}^T]M^T + ME[\dot{\bar{x}}\bar{f}^T] \quad (2.43)
$$

For a stationary random process which is differentiable, it may be shown that (see Appendix B for details)

$$
\left.\begin{aligned}
E[\bar{x}\dot{\bar{x}}^T] + E[\dot{\bar{x}}\bar{x}^T] &= O \\
E[\bar{x}\ddot{\bar{x}}^T] = E[\ddot{\bar{x}}\bar{x}^T] &= -E[\dot{\bar{x}}\dot{\bar{x}}^T]
\end{aligned}\right\} \quad (2.44)
$$

Thus $E[\bar{x}\dot{\bar{x}}^T]$ is antisymmetric and $E[\bar{x}\ddot{\bar{x}}^T]$ is symmetric. Upon using the relations in (2.44), Eq. (2.43) and the first equation in (2.42) become

$$- ME\left[\dot{\ddot{x}}\dot{x}^T\right] + CE\left[\dot{\ddot{x}}\bar{x}^T\right] + KE\left[\bar{x}\bar{x}^T\right] = E\left[\bar{f}\bar{x}^T\right] \left.\right\}$$

$$ME\left[\dot{\ddot{x}}\dot{\ddot{x}}^T\right]C^T + CE\left[\dot{\ddot{x}}\dot{x}^T\right]M^T + ME\left[\dot{x}\bar{x}^T\right]K^T$$

$$+ KE\left[\bar{x}\dot{x}^T\right]M^T = E\left[\bar{f}\dot{x}^T\right]M^T + ME\left[\dot{x}\bar{f}^T\right] \left.\right\} \qquad (2.45)$$

The matrices $E\left[\bar{x}\bar{x}^T\right]$ and $E\left[\dot{\ddot{x}}\dot{x}^T\right]$ are clearly symmetric and the matrix $E\left[\bar{x}\dot{x}^T\right]$ is antisymmetric, as noted. Suppose that $E\left[\bar{x}\bar{f}^T\right]$ and $E\left[\dot{x}\bar{f}^T\right]$ exist and can be evaluated, then there are $(n \times n) + n(n+1)/2$ unknowns in (2.45). The first matrix equation in (2.45) gives $n \times n$ component equations and because of symmetry, the second one furnishes only $n(n+1)/2$. The number of equations is just equal to the number of unknowns. Let

$$A = \begin{pmatrix} O & -I \\ M^{-1}K & M^{-1}C \end{pmatrix}$$

$$X = \begin{pmatrix} E\left[\bar{x}\bar{x}^T\right] & E\left[\bar{x}\dot{x}^T\right] \\ E\left[\dot{x}\bar{x}^T\right] & E\left[\dot{\ddot{x}}\dot{x}^T\right] \end{pmatrix} \left.\right\} \qquad (2.46)$$

and

$$B = \begin{pmatrix} O & E\left[\bar{x}\bar{f}^T\right]\left(M^{-1}\right)^T \\ M^{-1}E\left[\bar{f}\bar{x}^T\right] & M^{-1}E\left[\bar{f}\dot{x}^T\right] + E\left[\dot{x}\bar{f}^T\right]\left(M^{-1}\right)^T \end{pmatrix}$$

Then (2.45) can be put into the following form

$$AX + XA^T = B \qquad (2.47)$$

It may be shown that X can be uniquely determined from (2.47) iff[20]

$$\lambda_k + \lambda_j \neq 0 \quad k, j = 1, \ldots, 2n \tag{2.48}$$

where $\lambda_k$ are the eigenvalues of A.

The instantaneous correlation matrix for accelerations is given by

$$ME\left[\ddot{x}\ddot{x}^T\right]C^T + CE\left[\ddot{x}\ddot{x}^T\right]M^T = KE\left[\dot{x}\ddot{x}^T\right]C^T + CE\left[\dot{x}\dot{x}^T\right]K^T$$

$$+ E\left[\overline{f}\ddot{x}^T\right]C^T + CE\left[\ddot{x}\overline{f}^T\right] \tag{2.49}$$

which is obtained in the same way as in deriving the second equation in (2.45).

Now we turn our attention to the determination of the matrices $E\left[\overline{x}\,\overline{f}^T\right]$, $E\left[\dot{x}\,\overline{f}^T\right]$ and $E\left[\ddot{x}\,\overline{f}^T\right]$. It follows from (2.11) that

$$
\left.
\begin{aligned}
\overline{x}(t) &= \int_{-\infty}^{\infty} G(t-\tau)\overline{f}(\tau)d\tau = \int_{-\infty}^{\infty} G(\tau)\overline{f}(t-\tau)d\tau \\[2ex]
\dot{\overline{x}}(t) &= \int_{-\infty}^{\infty} \dot{G}(t-\tau)\overline{f}(\tau)d\tau = \int_{-\infty}^{\infty} \dot{G}(\tau)\overline{f}(t-\tau)d\tau \\[2ex]
\ddot{\overline{x}}(t) - M^{-1}\overline{f}(t) &= \int_{-\infty}^{\infty} \ddot{G}(t-\tau)\overline{f}(\tau)d\tau = \int_{-\infty}^{\infty} \ddot{G}(\tau)\overline{f}(t-\tau)d\tau
\end{aligned}
\right\} \tag{2.50}
$$

provided all these stochastic integrals exist. Hence, postmultiplying through (2.50) by $\overline{f}^T(t)$ and taking expectations of both sides of the resulting equations, we obtain

$$E\left[\overline{\dot{x}\dot{f}^T}\right] = \int_0^\infty G(\tau)R_{\overline{f}}(\tau)d\tau$$

$$E\left[\overline{\dot{x}\dot{f}^T}\right] = \int_0^\infty \dot{G}(\tau)R_{\overline{f}}(\tau)d\tau \left.\right\} \qquad (2.51)$$

$$E\left[\overline{\ddot{x}\dot{f}^T}\right] - M^{-1}R_{\overline{f}}(0) = \int_0^\infty \ddot{G}(\tau)R_{\overline{f}}(\tau)d\tau$$

The lower limit of each integral in (2.51) has been changed from $-\infty$ to 0 since $G(\tau)$, $\dot{G}(\tau)$, and $\ddot{G}(t)$ are null matrices for $\tau < 0$. The impulse response function matrices $G(t)$ and $\dot{G}(t)$ are given by (2.7) and (2.9), and $\ddot{G}(t)$ can be found from $\dot{G}(t)$ by direct differentiation.

The integrals in (2.51) are real and can readily be evaluated. Suppose that

$$R_{\overline{f}}(\tau) = A\, b(\tau) \qquad (2.52)$$

where A is a symmetric, constant matrix. Then from (2.7) and (2.8), we know that each integral in (2.51) takes one of the following two forms:

$$\int_0^\infty b(\tau)e^{-\alpha\tau}\cos\beta\tau\, d\tau$$

or $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left.\right\} \qquad (2.53)$

$$\int_0^\infty b(\tau)e^{-\alpha\tau}\sin\beta\tau\, d\tau$$

where $\alpha$ and $\beta$ are positive constants. If $b(\tau)$ is specified, then, usually, these integrals can be found in standard integral tables. After evaluating the integrals in (2.51), the instantaneous correlation matrices can be determined by solving the linear equations

in (2.45). Hence, this method is particularly simple and easily
adapted to digital computation.

### 2.3.3 Instantaneous Covariance Matrices

The instantaneous covariance matrix of $\bar{x}$, $\bar{\Psi}_{\bar{x}}$, is defined
as

$$\bar{\Psi}_{\bar{x}} \equiv E\left[(\bar{x}-\bar{m}_{\bar{x}})(\bar{x}-\bar{m}_{\bar{x}})^T\right] = E\left[\bar{x}\bar{x}^T\right] - \bar{m}_{\bar{x}}\bar{m}_{\bar{x}}^T \qquad (2.54)$$

With the aid of (2.41), Eq. (2.54) becomes

$$\bar{\Psi}_{\bar{x}} = E\left[\bar{x}\bar{x}^T\right] - K^{-1}\bar{m}_{\bar{f}}\,\bar{m}_{\bar{f}}^T(K^{-1})^T \qquad (2.55)$$

Hence, the instantaneous covariance matrix of $\bar{x}$ is identical to its
instantaneous correlation matrix if the mean vector of the excitation
$\bar{m}_{\bar{f}}$ vanishes. However, all the other instantaneous covariance
matrices are the same as their corresponding instantaneous correla-
tion matrices since from (2.40) we know that $\bar{m}_{\dot{x}}$ and $\bar{m}_{\ddot{x}}$ are two null
vectors.

### 2.3.4 Special Cases

White Noise Excitation: If the excitation $\bar{f}(t)$ is a white noise
vector, that is,

$$\bar{m}_{\bar{f}} = O, \quad R_{\bar{f}}(\tau) = 2\pi\,W\,\delta(\tau) \qquad (2.56)$$

in which W represents the spectral density matrix of $\bar{f}(t)$ and $\delta(\tau)$ is
the Dirac delta function, then upon using (2.56) and (2.10), the first
two equations in (2.51) become

$$E\left[\overline{x}\,\overline{f}^T\right] = O$$

$$E\left[\dot{\overline{x}}\,\overline{f}^T\right] = \pi M^{-1} W \qquad (2.57)$$

Hence Eq. (2.45) reduces to

$$ME\left[\dot{\overline{x}}\,\dot{\overline{x}}^T\right] - CE\left[\dot{\overline{x}}\,\overline{x}^T\right] - KE\left[\overline{x}\,\overline{x}^T\right] = O$$

$$ME\left[\dot{\overline{x}}\,\dot{\overline{x}}^T\right]C^T + CE\left[\dot{\overline{x}}\,\dot{\overline{x}}^T\right]M^T + ME\left[\dot{\overline{x}}\,\overline{x}^T\right]K^T$$

$$+ KE\left[\overline{x}\,\dot{\overline{x}}^T\right]M^T = 2\pi W$$

$$(2.58)$$

Eq. (2.58) is the same as Eq. (2.38), but, here, the excitation is not necessary to be Gaussian.

For white noise excitation, the integral in the third equation in (2.51) does not exist because both $\ddot{G}(\tau)$ and $R_{\ddot{f}}(\tau)$ go to infinity at $\tau = 0$. Hence, the instantaneous correlation matrix of the acceleration which depends on the third equation in (2.51), becomes meaningless. However, in some problems, the instantaneous correlation matrix of the absolute acceleration may exist even though the instantaneous correlation matrix of the relative acceleration does not exist. For example, consider an arbitrary system excited at some point by a white noise acceleration input, $\ddot{x}_o(t)$. Let $x_j$ denote the displacement of the $j^{th}$ mass relative to the excitation point. Then it may be shown that $E\left[\ddot{\overline{x}}\,\ddot{\overline{x}}^T\right]$ does not exist. For this example, the absolute acceleration of the $j^{th}$ mass $a_j$ will be

$$a_j(t) = \ddot{x}_j(t) + \ddot{x}_o(t) \qquad (2.59)$$

and the excitation vector $\overline{f}(t)$ in (2.1) becomes

$$\overline{f}(t) = -\begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix} \ddot{x}_o(t) \tag{2.60}$$

Substituting (2.59) and (2.60) into (2.49) and noting that

$$M = \begin{pmatrix} m_1 & 0 & \ldots & 0 \\ 0 & m_2 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & m_n \end{pmatrix} \tag{2.61}$$

we have

$$ME\left[\overline{a}\,\overline{a}^T\right]C^T + CE\left[\overline{a}\,\overline{a}^T\right]M = KE\left[\dot{\overline{x}}\dot{\overline{x}}^T\right]C^T$$

$$+ CE\left[\dot{\overline{x}}\dot{\overline{x}}^T\right]K^T \tag{2.62}$$

It is now obvious that $E\left[\overline{a}\,\overline{a}^T\right]$ exists.

Filtered White Noise Excitation: A white noise has been widely used as an ideal excitation. It not only simplifies the analysis, but also gives very reasonable results for some lightly damped systems. Sometimes, the use of a white noise as an approximation may not be acceptable, but one may still use a filtered white noise. For example, an earthquake may be approximated by passing a white noise through a viscously damped linear oscillator. In the following, a more general filter will be considered.

A random process $r(t)$ is said to be a filtered white noise if it satisfies the stochastic differential equation

$$\sum_{k=0}^{r} a_k \frac{d^k}{dt^k} \eta(t) = s(t) , a_r \neq 0 \tag{2.63}$$

where the $a_k$'s are constants. The excitation $s(t)$ is a white noise whose mean and auto-correlation function are given by

$$\left. \begin{array}{c} E\left[ s(t) \right] = 0 \\[2mm] E\left[ s(t) s(t-\tau) \right] = 2\pi w \, \delta(\tau) \end{array} \right\} \tag{2.64}$$

where $w$ is the spectral density of $s(t)$.

Consider an n-degree-of-freedom system excited at some point by a filtered white noise acceleration input $\eta(t)$. Let $x_j$ denote the displacement of the $j^{th}$ mass relative to the excited point. Then in (2.1)

$$\bar{f}(t) = -\bar{m}\eta(t) = -\begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix} \eta(t) \tag{2.65}$$

Combining (2.1) and (2.63) and using (2.65) yields

$$\sum_{k=0}^{r} a_k \frac{d^k}{dt^k} \left( M\ddot{\bar{x}} + C\dot{\bar{x}} + K\bar{x} \right) = -\bar{m}s(t) \tag{2.66}$$

For simplicity, we rewrite (2.66) as

$$\sum_{k=0}^{r+2} B_k \frac{d^k}{dt^k} \bar{x}(t) = -\bar{m}s(t) , \quad B_{r+2} = a_r M, \quad \frac{d^o x}{dt^o} = x \tag{2.67}$$

where $B_k$'s are constant matrices. Postmultiplying through (2.67) respectively by $\bar{x}^T, \frac{d\bar{x}^T}{dt}, \ldots, \left( d^{r+1}/dt^{r+1} \right)\bar{x}^T$ and taking expectations of both sides of the resulting equations yields

$$\sum_{k=0}^{n+2} B_k X_{kj} = -\overline{m} E\left[ s(t) \frac{d^j}{dt^j} \overline{x}^T \right] \tag{2.68}$$

where

$$X_{kj} = E\left[ \left( \frac{d^k}{dt^k} \overline{x} \right) \left( \frac{d^j}{dt^j} \overline{x} \right)^T \right] \tag{2.69}$$

$$j = 0, 1, \ldots, r+1$$

Analogous to the second order system (2.1), we define the impulse response function matrix G(t) of (2.67) as the solution of the following system

$$\sum_{k=0}^{r+2} B_k \frac{d^k}{dt^k} G(t) = I \delta(t) \tag{2.70}$$

with initial conditions

$$G(0) = \frac{d}{dt} G(0) = \ldots = \frac{d^{r+1}}{dt^{r+1}} G(0) = O \tag{2.71}$$

Then $\dfrac{d^j}{dt^j} G(t)$, $j=1, \ldots, r$, is continuous at $t=0$ and

$$\lim_{t \to 0^+} \frac{d^{r+1}}{dt^{r+1}} G(t) = B_{r+2}^{-1} \tag{2.72}$$

Using (2.71) and (2.72) we obtain

$$E\left[ s(t) \frac{d^k}{dt^k} \overline{x}^T(t) \right] = -\overline{m}^T \int_0^\infty E\left[ s(t) s(t-\tau) \right] \frac{d^k}{dt^k} G^T(\tau) d\tau$$

$$= -\pi w \overline{m}^T \frac{d^k}{dt^k} G^T(0^+) = O; k = 1, \ldots, r \tag{2.73a}$$

$$E\left[s(t)\frac{d^{r+1}}{dt^{r+1}}\overline{x}^T(t)\right] = -\pi\,w\overline{m}^T\frac{d^{r+1}}{dt^{r+1}}G^T(0^+)$$

$$= -\pi\,w\overline{m}^T\left(B_{r+2}^{-1}\right)^T \qquad (2.73b)$$

Thus Eq. (2.68) becomes

$$\sum_{k=0}^{r+1} B_k X_{kj} = B_{r+2}X_{r+1,\,j+1} \qquad j = 0, 1, \ldots, r$$

$$\left. \begin{array}{l} \\ \\ B_{r+2}\sum_{k=0}^{r+1}X_{r+1,\,k}B_k^T + \sum_{k=0}^{r+1}B_k X_{k,\,r+1}B_{r+2}^T = 2\pi\,w\overline{m}\overline{m}^T \end{array} \right\} \qquad (2.74)$$

where the relations

$$X_{gh} = (-)^{g+j}X_{jk}$$

$$X_{gh} = (-)^{h+k}X_{jk}$$

$$g+h = j+k; \ g, h, j, k = 0, 1, \ldots, r+1 \qquad (2.75)$$

have been used (these relations have been proved in Appendix B).
Equations (2.74) and (2.75) can be used to determine all the instan-
taneous correlation matrices provided that they are independent.

## 2.4  Comparison of Various Approaches

In the second section of the present chapter, we have dis-
cussed the impulse function approach, the spectral density approach
and the Fokker-Planck approach. In the third section, a fourth
approach, a matrix algebra approach, was introduced. The first
three approaches can be used to find the correlation function matrices
as well as the instantaneous correlation matrices for the response

process, whereas the last approach can only be used to find the instantaneous correlation matrices. This limitation may not, however, be severe as was mentioned earlier.

We will now give a discussion of comparison of various approaches applied to stationary problems.

(1) The Fokker-Planck equation can be used only if the excitation is a Gaussian white or a filtered Gaussian white noise. In this case, both the Fokker-Planck approach and the matrix algebra approach lead to the same equations for the determination of the instantaneous correlation matrices. However, the matrix algebra approach can also be applied to arbitrary excitations provided that the integrals in (2.39) exist.

(2) In the spectral density approach one needs to evaluate a separate integral for each independent element in each instantaneous correlation matrix. From (2.21) and (2.20) we know that these integrals are in general different and their integrands may be real or complex. Furthermore, the matrix $H(\omega)$ in the integrands must be found analytically from the expression (2.22) and the difficulties involved in the integrations increase with increasing the number of degree-of-freedom of the system. Hence, for complex systems one is generally forced to evaluate these integrals one by one numerically.

In the matrix algebra approach, if the excitation is white, then the instantaneous correlation matrices can be found by solving the system of linear algebraic equations (2.58) without evaluating

any integrals. If the excitation is non-white, it is necessary to evaluate some integrals also, but each of these is real and takes one of the forms in (2.53). Usually, these integrals can be found in standard integral tables. After evaluating the integrals in closed form, again one can find the instantaneous correlation matrices by simply solving a system of linear algebraic equations. Therefore, this approach is very suitable for digital computation.

(3) Both the impulse response function approach and the matrix algebra approach use the idea of impulse response functions. In the first approach the correlation function matrices are expressed in terms of double integrals. In the second approach, one, first, has to evaluate some integral as in (2.53); then, the instantaneous correlation matrices are found by solving a system of linear algebraic equations. Since it is very time consuming to evaluate double integrals numerically, the matrix algebra approach is usually faster in use than the impulse response function approach.

As an illustration, the spectral density approach and the matrix algebra approach were programmed to solve a three-degree-of-freedom system under the excitation of a white noise vector. The program using the spectral density approach is limited to three-degree-of-freedom systems and cannot be generalized to arbitrary systems. For this special excitation, the integrals are evaluated by calculating the residues at the singularities of the integrands. It takes about 200 milliseconds on an IBM 360/75 computer to find the matrix $E\left[x\,x^T\right]$.

The program using the matrix algebra approach can be used for arbitrary systems under white noise excitation. Also, the program itself is simpler than that using the spectral density approach. It takes about 150 milliseconds on the same computer to find the matrices $E\left[\bar{x}\bar{x}^T\right]$, $E\left[\bar{x}\dot{x}^T\right]$, and $E\left[\dot{x}\dot{x}^T\right]$.

## 2.5 Examples

In order to illustrate the application of the matrix algebra approach, we consider two examples below.

In the first example, a single-degree-of-freedom system under clipped white noise excitation is considered. This is a very simple example, but it contains all procedures needed for more complex systems. This problem has also been solved by using the spectral density approach. Both approaches lead to the same solution.

In the second example, we consider a n-degree-of-freedom system under the excitation of a clipped white noise. It is shown that the matrices $E\left[\bar{x}\bar{f}^T\right]$ and $E\left[\dot{x}\bar{f}^T\right]$ can be expressed in terms of the same two types of integrals as in the first example and can therefore be easily evaluated. Only additional computation is the solution of a system of linear algebraic equations. The results of a particular 3-degree-of-freedom system are plotted. From the figures, some expected phenomena can be observed.

## 2.5.1 Single-Degree-of-Freedom Systems Under Clipped White Noise Excitation

Consider a mass-spring-dashpot system governed by the equation of motion

$$\ddot{x} + 2\zeta\omega_n x + \omega_n^2 x = f(t)/m \tag{2.76}$$

The constants $m$, $\omega_n$, $\zeta$ are respectively the mass, the undamped natural frequency, and the ratio of the actual damping to the critical damping of the system. The excitation $f(t)$ is a clipped white noise with cutoff frequency $\omega_c$. It is specified by its mean $m_f$ and its auto-correlation function

$$E\left[f(t)\,f(t-\tau)\right] = \frac{2w}{\tau}\sin\omega_c\tau \tag{2.77}$$

where $w$ is the spectral density in the frequency range $|\omega| < \omega_c$. The impulse response functions $g(t)$ and $\dot{g}(t)$ can be easily found to be

$$\left.\begin{aligned} g(t) &= \frac{1}{\omega_d}\,e^{-\zeta\omega_n t}\sin\omega_d t, \quad \omega_d = \omega_n\sqrt{1-\zeta^2} \\[2mm] \dot{g}(t) &= e^{-\zeta\omega_n t}\left[\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right] \end{aligned}\right\} \tag{2.78}$$

Then from (2.51) one has

$$E\left[xf\right] = \frac{2w}{m^2}\int_0^\infty g(\tau)\,\frac{\sin\omega_c\tau}{\tau}\,d\tau$$

$$= \frac{2w}{m^2\omega_d}\int_0^\infty \frac{e^{-\zeta\omega_n\tau}}{\tau}\sin\omega_c\tau\,\sin\omega_d\tau\,d\tau \tag{2.79}$$

and

$$E[\dot{x}f] = \frac{2w}{m^2} \int_o^\infty \dot{g}(\tau) \frac{\sin \omega_c \tau}{\tau} d\tau$$

$$= \frac{2w}{m^2} \int_o^\infty \frac{e^{-\zeta\omega_n\tau}}{\tau} \sin\omega_c\tau \cos\omega_d\tau \, d\tau - \zeta\omega_n E[xf] \qquad (2.80)$$

By using the results[21]

$$\left. \begin{array}{l} \displaystyle\int_o^\infty \frac{e^{-\zeta\omega_n\tau}}{\tau} \sin\omega_c\tau \cos\omega_d\tau \, d\tau = \frac{1}{2} \tan^{-1} \frac{2\zeta\omega_n\omega_c}{\omega_n^2 - \omega_c^2} \\[20pt] \displaystyle\int_o^\infty \frac{e^{-\zeta\omega_n\tau}}{\tau} \sin\omega_c\tau \sin\omega_d\tau \, d\tau = \frac{1}{4} \ln \frac{(\zeta\omega_n)^2 + (\omega_c + \omega_d)^2}{(\zeta\omega_n)^2 + (\omega_c - \omega_d)^2} \end{array} \right\} \qquad (2.81)$$

Eqs. (2.79) and (2.80) reduce to

$$E[xf] = \frac{w}{2\omega_d m^2} \ln \frac{(\zeta\omega_n)^2 + (\omega_c + \omega_d)^2}{(\zeta\omega_n)^2 + (\omega_c - \omega_d)^2}$$

$$= \frac{w}{2\omega_d m^2} \ln \frac{1 + (\omega_c/\omega_n)^2 + 2\sqrt{1-\zeta^2}(\omega_c/\omega_n)}{1 + (\omega_c/\omega_n)^2 - 2\sqrt{1-\zeta^2}(\omega_c/\omega_n)} \qquad (2.82)$$

$$E[\dot{x}f] = \frac{w}{m^2} \left[ \tan^{-1} \frac{\omega_c + \omega_d}{\zeta\omega_n} + \tan^{-1} \frac{\omega_c - \omega_d}{\zeta\omega_n} \right] - \zeta\omega_n E[xf]$$

$$= \frac{w}{m^2} \tan^{-1} \frac{2\zeta(\omega_c/\omega_n)}{1 - (\omega_c/\omega_n)^2} - \zeta\omega_n E[xf] \qquad (2.83)$$

Upon using $E[x\dot{x}] = 0$, Eq. (2.45) now becomes

$$\left.\begin{array}{c} \omega_n^2 \, E[x^2] - E[\dot{x}^2] = E[xf] \\[2em] 4\zeta\omega_n E[\dot{x}^2] = 2E[\dot{x}f] \end{array}\right\} \qquad (2.84)$$

Thus,

$$E[x^2] = \frac{1}{2\zeta\omega_n^3} \left( 2\zeta\omega_n E[xf] + E[\dot{x}f] \right)$$

$$= \frac{\pi w}{2\zeta\omega_n^3 m^2} \left[ \theta_1\left(\frac{\omega_c}{\omega_n}, \zeta\right) + \theta_2\left(\frac{\omega_c}{\omega_n}, \zeta\right) \right] \qquad (2.85)$$

$$E[\dot{x}^2] = \frac{1}{2\zeta\omega_n} E[\dot{x}f]$$

$$= \frac{\pi w}{2\zeta\omega_n m^2} \left[ \theta_1\left(\frac{\omega_c}{\omega_n}, \zeta\right) - \theta_2\left(\frac{\omega_c}{\omega_n}, \zeta\right) \right] \qquad (2.86)$$

where

$$\left.\begin{array}{c} \theta_1\left(\dfrac{\omega_c}{\omega_n}, \zeta\right) = \dfrac{1}{\pi} \tan^{-1}\left(\dfrac{2\zeta(\omega_c/\omega_n)}{1 - (\omega_c/\omega_n)^2}\right) \\[3em] \theta_2 = \dfrac{\zeta}{2\pi\sqrt{1-\zeta^2}} \, \ell n \, \dfrac{1 + \left(\dfrac{\omega_c}{\omega_n}\right)^2 + 2\sqrt{1-\zeta^2}\left(\dfrac{\omega_c}{\omega_n}\right)}{1 + \left(\dfrac{\omega_c}{\omega_n}\right)^2 - 2\sqrt{1-\zeta^2}\left(\dfrac{\omega_c}{\omega_n}\right)} \end{array}\right\} \qquad (2.87)$$

### 2.5.2 Multidegree-of-Freedom Systems Under Clipped White Noise Excitation

Consider an n-degree-of-freedom system governed by

$$M\ddot{\overline{x}} + C\dot{\overline{x}} + K\overline{x} = \overline{f}(t) \tag{2.88}$$

The excitation $\overline{f}(t)$ is a clipped white noise vector with cutoff frequency $\omega_c$. It is specified by its mean vector $\overline{m}_{\overline{f}}$ and correlation function matrix

$$R_{\overline{f}}(\tau) = 2W \frac{\sin \omega_c \tau}{\tau} \tag{2.89}$$

where W is the spectral density matrix for $|\omega| < \omega_c$. The impulse response function matrices $G(t)$ and $\dot{G}(t)$ are given by (2.7) and (2.9). Thus,

$$E\left[\overline{x}\,\overline{f}^T\right] = \int_0^\infty G(\tau) R_{\overline{f}}(\tau) d\tau$$

$$= 4 \sum_{k=1}^n X_k W \int_0^\infty \frac{e^{-\alpha_k \tau}}{\tau} \sin \omega_c \tau \cos \beta_k \tau d\tau$$

$$- 4 \sum_{k=1}^n Y_k W \int_0^\infty \frac{e^{-\alpha_k \tau}}{\tau} \sin \omega_c \tau \sin \beta_k \tau d\tau \tag{2.90}$$

$$E\left[\dot{\overline{x}}\,\overline{f}^T\right] = \int_0^\infty \dot{G}(\tau) R_{\overline{f}}(\tau) d\tau$$

$$= -4 \sum_{k=1}^n \left(X_k \alpha_k + Y_k \beta_k\right) W \int_0^\infty \frac{e^{-\alpha_k \tau}}{\tau} \cos \beta_k \tau \sin \omega_c \tau d\tau \tag{2.91}$$

$$- 4 \sum_{k=1}^{n} \left( X_k \beta_k - Y_k \alpha_k \right) W \int_{0}^{\infty} \frac{e^{-\alpha_k \tau}}{\tau} \sin \beta_k \tau \sin \omega_c \tau d\tau \quad (2.91)$$

<div align="right">cont'd</div>

where $\alpha_k$, $\beta_k$, $X_k$ and $Y_k$ are defined in (2.8). As mentioned before, here one meets the same two types of integrals as in the single-degree-of-freedom system discussed in Example 1.

Upon substitution of (2.81) into (2.90) and (2.91), one has finally

$$\left. \begin{aligned} E\left[\overline{x}\,\overline{f}^{T}\right] &= 2 \sum_{k=1}^{n} X_k W \theta_k - \sum_{k=1}^{n} Y_k W \varphi_k \\[2ex] E\left[\dot{\overline{x}}\,\overline{f}^{T}\right] &= -2 \sum_{k=1}^{n} \left( X_k \alpha_k + Y_k \beta_k \right) W \theta_k \\[2ex] &\quad - \sum_{k=1}^{n} \left( X_k \beta_k - Y_k \alpha_k \right) W \varphi_k \end{aligned} \right\} \quad (2.92)$$

where

$$\left. \begin{aligned} \theta_k &= \tan^{-1} \left( \frac{2\omega_c \alpha_k}{\alpha_k^2 + \beta_k^2 - \omega_c^2} \right) \\[3ex] \varphi_k &= \ell n \frac{\alpha_k^2 + (\omega_c + \beta_k)^2}{\alpha_k^2 + (\omega_c - \beta_k)^2} \end{aligned} \right\} \quad (2.93)$$

The instantaneous correlation matrices $E[\bar{x}\bar{x}^T]$, $E[\bar{x}\dot{\bar{x}}^T]$, and $E[\dot{\bar{x}}\dot{\bar{x}}^T]$ can be determined by simply solving the linear algebraic equations in (2.45).

If this problem is solved by the spectral density approach, the matrices $E[\bar{x}\bar{x}^T]$, $E[\bar{x}\dot{\bar{x}}^T]$, and $E[\dot{\bar{x}}\dot{\bar{x}}^T]$ are given by

$$
\left.
\begin{aligned}
E[\bar{x}\bar{x}^T] &= \int_0^{\omega_c} H(\omega)WH^*(\omega)d\omega \\[2mm]
E[\bar{x}\dot{\bar{x}}^T] &= -i\int_0^{\omega_c} H(\omega)WH^*(\omega)\,\omega d\omega \\[2mm]
E[\dot{\bar{x}}\dot{\bar{x}}^T] &= \int_0^{\omega_c} H(\omega)WH^*(\omega)\,\omega^2 d\omega
\end{aligned}
\right\} \qquad (2.94)
$$

where

$$
H(\omega) = (-M\omega^2 + i\,C\omega + K)^{-1} \qquad (2.95)
$$

In the above equation the resultant matrix in the brackets is a function of $\omega$, so its inversion must be done analytically. Even if the matrix $H(\omega)$ has been found, the evaluation of the integrals in (2.94) for a complex system is not trivial. Usually, these integrals are evaluated one by one numerically by computer, but this will generally take considerable computer time.

The matrix algebra approach has been applied to the system shown in Fig. 2.1 with $n = 3$ and $\dot{x}_0$ is a clipped white noise with unit spectral density. The mean square displacements $E[y_1^2]$, $E[y_2^2]$, and

$E\left[y_3^2\right]$, where $y_j$ is the displacement of the $j^{th}$ spring, are plotted against various cut-off frequencies in Figs. 2.2 and 2.3.

In Fig. 2.2 the mean square displacements for the system shown in Fig. 2.1 with C proportional to M is shown. It will be noted from this figure that the contributions from the modes higher than the first one are not negligible. For example, consider $E\left[y_1^2\right]$. The contribution from the first mode is about 5.8, from the second mode is about 2.9, and from the third mode is about 0.4. Hence, the contribution from the second mode is as high as 50% of that from the first mode.

In Fig. 2.3, the mean square relative displacements for the system shown in Fig. 2.1 with C proportional to K is shown. The proportionality constant is chosen so that the system would have the same part of critical damping in the first mode as the system considered in Fig. 2.2. The primary difference between Fig. 2.2 and Fig. 2.3 is that the contributions due to frequencies beyond $w_2$, the second natural frequency, are almost completely damped out in Fig. 2.3. This phenomenon is expected since the modal damping in the first system decreases with increasing modal frequency while that in the second system increases with increasing modal frequency.

## III. STATIONARY RANDOM RESPONSE OF
## MULTIDEGREE-OF-FREEDOM
## NONLINEAR SYSTEMS

### 3.1 Introduction

Since most real physical systems exhibit some kind of non-linearity for sufficiently large motions, it is important that one be able to investigate nonlinear systems as well as linear systems. Of course, the analysis of nonlinear systems is more difficult than that of linear systems since very few nonlinear differential equations can be solved exactly.

One exact method of studying the stationary random response of a nonlinear system is the Fokker-Planck approach. If the excitation is a Gaussian white noise, then the transitional probability density of the response process is governed by the Fokker-Planck equation. As mentioned in Chapter 2, this transitional probability density can completely define the response process. However, no one has succeeded in solving the complete Fokker-Planck equation for any second-order nonlinear system. The first probability density is governed by Eq. (2.33) which can be solved in some cases. From the first probability density, one can deduce all the instantaneous correlation matrices.

Since the exact solution is available only for limited cases, attention has also turned to approximate solutions. If the nonlinearity is small, several approximate methods have been devised. One of them is the normal mode approach in which the approximate solution

can be found by solving several single-degree-of-freedom systems. However, in using this approach, one must impose some conditions on the excitation as well as on the system.

Another method of generating approximate solution is the perturbation approach. In this approach it requires that the non-linear terms of the system must be small compared to the linear terms and the excitation level also has to be sufficiently low.

In this chapter, we will consider a more general approach for multidegree – of – freedom systems. The only restrictions on this approach are that the excitation be stationary and Gaussian. Before introducing this approach, a brief discussion of the Fokker-Planck approach, the perturbation approach, and the normal mode approach is given below.

### 3.2 Fokker-Planck Approach

The equation governing the first probability density for the stationary response process of a nonlinear system has only been solved under the following rather restrictive conditions:

(1) the damping force is proportional to the velocity

(2) the excitation is a Gaussian white noise

(3) the correlation function matrix of the excitation is proportional to the damping matrix of the system

Under the above conditions, the equation of motion may be written as follows:

$$M\ddot{\overline{x}} + C\dot{\overline{x}} + \frac{\partial u(\overline{x})}{\partial \overline{x}} = \overline{f}(t) \qquad (3.1)$$

with

$$\overline{m}_{\overline{f}} = O$$

$$R_{\overline{f}}(\tau) = 2\gamma C \delta(\tau)$$

(3.2a)

where $\gamma$ is a constant, $u(\overline{x})$ is the potential energy of the system and

$$\frac{\partial u(\overline{x})}{\partial \overline{x}} = \begin{pmatrix} \dfrac{\partial u}{\partial x_1} \\ \vdots \\ \dfrac{\partial u}{\partial x_n} \end{pmatrix}$$

(3.2b)

Suppose that there exists an orthogonal matrix A which can simultaneously diagonalize M and C:

$$A^T A = I$$

$$A^T M A = V$$

$$A^T C A = \Lambda$$

(3.3)

where V and $\Lambda$ are two diagonal matrices. Then, upon using the transformation $\overline{x} = A\overline{z}$ and noting that

$$A^T \frac{\partial u(\overline{x})}{\partial \overline{x}} = A^T \frac{\partial \overline{z}}{\partial \overline{x}} \frac{\partial u(\overline{z})}{\partial \overline{z}} = \sum_{j, k=1}^{n} a_{jm} a_{jk} \frac{\partial u(\overline{z})}{\partial z_k} \qquad m = 1, \ldots, n$$

$$= \frac{\partial u(\overline{z})}{\partial z_m}$$

$$= \frac{\partial u(\overline{z})}{\partial \overline{z}}$$

(3.4)

Equation (3.1) becomes

$$V\ddot{\bar{z}} + \Lambda\dot{\bar{z}} + \frac{\partial u(\bar{z})}{\partial\bar{z}} = A^T \bar{f}(t) = \bar{b}(t) \qquad (3.5a)$$

and the correlation function matrix of $\bar{b}(t)$ is given by

$$R_{\bar{b}}(\tau) = 2\gamma\Lambda\,\delta(\tau) \qquad (3.5b)$$

The stationary Fokker-Planck equation associated with (3.5a) is given by

$$\sum_{j=1}^{n} \frac{\partial}{\partial z_j}(\dot{z}_j p) - \sum_{j=1}^{n} \frac{1}{v_j} \frac{\partial}{\partial \dot{z}_j}\left\{\left[\lambda_j\dot{z}_j + \frac{\partial u(\bar{z})}{\partial z_j}\right]p\right\}$$

$$-\sum_{j=1}^{n} \frac{\gamma\lambda_j}{v_j^2} \frac{\partial^2 p}{\partial \dot{z}_j^2} = 0 \qquad (3.6)$$

where $\lambda_j$ and $v_j$ denote the $j^{th}$ diagonal element of $\Lambda$ and $V$, p is the abbreviation for the first probability density of the Markovian vector $\begin{vmatrix}\bar{z}\\\dot{\bar{z}}\end{vmatrix}$ . The solution to (3.6) may be written as follows:

$$p(\bar{z}, \dot{\bar{z}}) = \beta \exp\left\{-\frac{1}{\gamma}\left[\frac{1}{2}\sum_{j=1}^{n} v_j \dot{z}_j^2 + u(\bar{z})\right]\right\} \qquad (3.7)$$

This solution was first obtained by Ariarathan[22] for a two-degree-of-freedom system (n = 2), and it was extended to the above form by Caughey[23]. The constant $\beta$ in (3.7) is a normalizing factor such that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(\bar{z}, \dot{\bar{z}})\, dz_1 \ldots dz_n\, d\dot{z}_1 \ldots d\dot{z}_n = 1 \qquad (3.8)$$

In the original coordinates, Equation (3.7) becomes

$$p(\bar{x}, \dot{\bar{x}}) = \beta \exp\left\{-\frac{1}{\gamma}\left[\frac{1}{2}\dot{\bar{x}}^T M\dot{\bar{x}} + u(\bar{x})\right]\right\} \tag{3.9}$$

It will be noted that the terms in the square brackets are respectively the kinetic energy and the potential energy of the system. Equation (3.9) may also be written as

$$p(\bar{x}, \dot{\bar{x}}) = \beta \exp\left\{-\frac{1}{2\gamma}\dot{\bar{x}} M^T \dot{\bar{x}}\right\} \exp\left\{-\frac{1}{\gamma}u(\bar{x})\right\} \tag{3.10}$$

Hence $\bar{x}$ and $\dot{\bar{x}}$ are linearly independent.

## 3.3 Normal Mode Approach

Consider an n-degree-of-freedom system governed by the equation of motion

$$M\ddot{\bar{x}} + C^{(0)}\dot{\bar{x}} + K^{(0)}\bar{x} + \mu\bar{g}(\dot{\bar{x}}, \bar{x}) = \bar{f}(t), \quad \mu = \text{a small parameter} \tag{3.11}$$

The matrices $C^{(0)}$ and $K^{(0)}$ are respectively the damping matrix and the stiffness matrix of the system due to the linear part of the damping forces and the spring forces, and $\mu\bar{g}(\bar{x}, \dot{\bar{x}})$ represents the nonlinear forces of the system. $\bar{f}(t)$ is a stationary Gaussian random vector. Without loss of generality, assume that $\bar{m}_{\bar{f}} = 0$.

In using this approach, the following two conditions must be satisfied:

(1) the linear system obtained by neglecting the nonlinear term $\bar{g}(\dot{\bar{x}}, \bar{x})$ in (3.11) must possess normal modes

(2) the correlation function matrix $R_{\bar{f}}(\tau)$ must be diagonalized by the same matrix which diagonalizes the matrices $M$, $C^{(o)}$ and $K^{(o)}$.

The second condition is quite restrictive and is seldom realized in real systems.

Assume that the above restrictions can be met, then there exists a matrix A such that

$$
\left.\begin{aligned}
A^T M A &= I \\
A^T K^{(o)} A &= \Omega^{(o)}, \quad w_{kj}^{(o)} = w_k^{(o)}\, \delta_{kj} \\
A^T C^{(o)} A &= \Lambda^{(o)}, \quad \lambda_{kj}^{(o)} = \lambda_k^{(o)}\, \delta_{kj} \\
A^T R_f(\tau) A &= D(\tau), \quad d_{kj} = d_k(\tau)\, \delta_{kj}
\end{aligned}\right\} \tag{3.12}
$$

By using the transformation $\bar{x} = A\bar{z}$, Eq. (3.11) reduces to

$$
\ddot{\bar{z}} + \Lambda^{(o)} \dot{\bar{z}} + \Omega^{(o)} \bar{z} + \mu A^T \bar{g}(\dot{\bar{z}}, \bar{z}) = A^T \bar{f}(\tau) = \bar{b}(t) \tag{3.13a}
$$

where the correlation function matrix of $\bar{b}$ is

$$
R_{\bar{b}}(\tau) = D(\tau) \tag{3.13b}
$$

In component form, Eq. (3.13a) becomes

$$
\ddot{z}_j + \lambda_j^{(o)} \dot{z}_j + (w_j^{(o)})^2 z_j + \sum_{k=1}^{n} a_{kj} g_k(\dot{z}, z) = b_j(t) \tag{3.14a}
$$

and Eq. (3.13b) becomes

$$
E\left[b_k(t) b_j(t+\tau)\right] = d_k(\tau)\, \delta_{kj} \qquad j, k = 1, \ldots, n \tag{3.14b}
$$

The differential equations in (3.14a) may be written as

$$\ddot{z}_j + \lambda_j \dot{z}_j + \omega_j^2 z_j + e_j(\dot{\bar{z}}, \bar{z}) = b_j(t)$$

$$j = 1, \ldots, n \qquad (3.15)$$

where the deficiency term $e_j$ is given by

$$e_j = (\lambda_j^{(o)} - \lambda_j)\dot{z}_j + \left[(\omega_j^{(o)})^2 - \omega_j^2\right] z_j$$

$$+ \mu \sum_{k=1}^{n} a_{kj} g_k(\dot{\bar{z}}, \bar{z}) \quad j = 1, \ldots, n \qquad (3.16)$$

If the quantities $\lambda_j$ and $\omega_j^2$ are chosen in such a way that some measure of the deficiency term is minimized, then it seems reasonable that the statistics of the response of the nonlinear system can be approximated by those of the linear system described by

$$\ddot{z}_j + \lambda_j \dot{z}_j + \omega_j^2 z_j = b_j(t) \quad j = 1, \ldots, n \qquad (3.17)$$

At this stage, the differential equations are uncoupled and the excitation $\bar{b}(t)$ is an uncorrelated vector process. Hence, each uncoupled differential equation can be solved separately.

In order to determine $\lambda_j$ and $\omega_j^2$, Caughey chose them so as to minimize the mean square value of the deficiency term $\bar{e}$. This can be achieved by requiring that

$$\left. \begin{array}{l} \dfrac{\partial}{\partial \lambda_j} E\left[\bar{e}^T \bar{e}\right] = 0 \\[4ex] \dfrac{\partial}{\partial(\omega_j^2)} E\left[\bar{e}^T \bar{e}\right] = 0 \end{array} \right\} \quad j = 1, \ldots, n \qquad (3.18)$$

Substituting (3.16) into (3.18) and interchanging the order of differentiation and expectation, we obtain

$$\lambda_j = \lambda_j^{(o)} + \mu \sum_{k=1}^{n} a_{kj} \, E\left[\dot{z}_j g_k(\dot{\bar{z}}, \bar{z})\right] \Big/ E\left[\dot{z}_j^2\right]$$

$$\left.\begin{array}{c} \\ \\ \end{array}\right\} j = 1, \ldots, n$$

$$\omega_j^2 = (\omega_j^{(o)})^2 + \mu \sum_{k=1}^{n} a_{kj} \, E\left[z_j g_k(\dot{\bar{z}}, \bar{z})\right] \Big/ E\left[z_j^2\right] \quad (3.19)$$

Equations (3.17) and (3.19) can be used to find various mean square values of the response process.

In certain cases, the contribution from the first mode may be dominant. In these cases, we may let $x_j = a_{j1} z_1$ in the above derivation. Then

$$\omega_1^2 = (\omega_1^{(o)})^2 + \mu \sum_{j=1}^{n} a_{j1} \, E\left[z_1 g_j(\dot{z}_1, z_1)\right] \Big/ E\left[z_1^2\right]$$

$$\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \quad (3.20)$$

$$\lambda_1 = \lambda_1^{o} + \mu \sum_{j=1}^{n} a_{j1} \, E\left[\dot{z}_1 g_j(\dot{z}_1, z_1)\right] \Big/ E\left[\dot{z}_1^2\right]$$

This is a rather rough approximation, but it is very simple, and in some cases, it does give reasonable approximate solution as will be demonstrated later.

## 3.4 Perturbation Approach

Consider the same problem defined in the previous section whose equations of motion are

$$M\ddot{\bar{x}} + C^{(o)}\dot{\bar{x}} + K^{(o)}\bar{x} + \mu\bar{g}(\dot{\bar{x}}, \bar{x}) = \bar{f}(t) \quad (3.21)$$

Assume that $\mu$ is so small that the solution of (3.21) can be approximately represented by

$$\bar{x} = \bar{x}_o + \mu\bar{x}_1 \qquad (3.22)$$

Substituting (3.22) into (3.21), neglecting terms involving $\mu^2, \mu^3, \ldots$ and equating corresponding coefficients of $\mu^o$ and $\mu^1$ yields the following sets of linear differential equations:

$$M\ddot{\bar{x}}_o + C^{(o)}\dot{\bar{x}}_o + K^{(o)}\bar{x}_o = \bar{f}(t) \qquad (3.23a)$$

$$M\ddot{\bar{x}}_1 + C^{(o)}\dot{\bar{x}}_1 + K^{(o)}\bar{x}_1 = -\bar{g}(\dot{\bar{x}}_o(t), \bar{x}_o(t)) \qquad (3.23b)$$

Correct to the same order of accuracy, the instantaneous correlation matrix for displacements becomes

$$E\left[\bar{x}\,\bar{x}^T\right] = E\left[\bar{x}_o\bar{x}_o^T\right] + \mu\left\{E\left[\bar{x}_o\bar{x}_1^T\right] + E\left[\bar{x}_1\bar{x}_o^T\right]\right\} \qquad (3.24)$$

Note that

$$E\left[\bar{x}_o\bar{x}_1^T\right] = \left(E\left[\bar{x}_o\bar{x}_1^T\right]\right)^T \qquad (3.25)$$

The matrix $E\left[\bar{x}_o\bar{x}_o^T\right]$ can be found from (3.23a) by the various approaches discussed in the previous chapter, and $E\left[\bar{x}_o\bar{x}_1^T\right]$ may be evaluated as follows. Since (3.23a) and (3.23b) are linear, their stationary solutions are

$$\bar{x}_o = \int_{-\infty}^{\infty} G(t-\tau)\bar{f}(\tau)d\tau \qquad (3.26a)$$

$$\bar{x}_1 = -\int_{-\infty}^{\infty} G(t-\tau)\bar{g}(\tau)d\tau \qquad (3.26b)$$

where $G(t)$ is the common impulse response function matrix of (3.23a) and (3.23b) and $g(\tau)$ is the abbreviation of $\bar{g}(\dot{\bar{x}}_o(\tau), \bar{x}_o(\tau))$.

Thus,

$$E\left[\bar{x}_o \bar{x}_1^T\right] = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(t-\tau_1) E\left[f(\tau_1)\bar{g}^T(\tau_2)\right] G^T(t-\tau_2) d\tau_1 d\tau_2 \qquad (3.27)$$

The matrix $E\left[\bar{f}(\tau_1)\bar{g}^T(\tau_2)\right]$ can be evaluated with the help of properties of Gaussian processes. Therefore, we can find $E\left[\bar{x}_o \bar{x}_1^T\right]$ and hence $E\left[\bar{x}\bar{x}^T\right]$.

Usually, the evaluation of the integrals in (3.27) is not easy, so Tung has developed a different approach to generate $E\left[\bar{x}\bar{x}^T\right]$ from (3.23a) and (3.23b). He applies Foss's[24] method to uncouple (3.23a) and (3.23b) into first order differential equations and then solves the resulting equations to find various instantaneous correlation matrices. For detail, see Reference (14).

This approach will fail if the damping matrix $C^{(o)}$ is a null matrix. In this case, Equation (3.23a) does not have a stationary solution since all of its correlation functions will finally go to infinity. Another limitation of this approach is that not only the nonlinearity of the system has to be small, but also the excitation has to be sufficiently low. This will be demonstrated in Section 3.6.

## 3.5 A Generalized Equivalent Linearization Approach

The normal mode approach is quite powerful if it applies, since it reduces the problem to one of the solution of uncoupled single-degree-of-freedom linear oscillators. However, due to the conditions imposed on the excitation, its application is rather limited. The perturbation approach also has its restrictions. In order to get a

reasonable approximate solution, both the nonlinearity of the system and the excitation have to be sufficiently small.

In this section a more general approach will be introduced. Except that the excitation must be stationary, the only additional restriction to this approach is that the excitation be Gaussian. According to the central limit theorem[25], many random processes in nature can be assumed to be at least approximately Gaussian distributed, so this restriction may not be too severe.

In this approach, we define an auxiliary set of linear differential equations for the original nonlinear system. Some coefficients of the auxiliary set may still be unknown. The solution of the original nonlinear system is approximated by the solution of the auxiliary set and the unknown coefficients are chosen in such a way that some measure of the difference between the two sets of equations is a minimum. With the help of some properties of Gaussian processes and the approach derived in the last chapter, the approximate instantaneous correlation matrices of a nonlinear system can be found from the solution of a set of algebraic equations. Although these algebraic equations are in general nonlinear, they may be solved linearly by a specific iteration scheme.

Consider an n-degree-of-freedom system connected by nonlinear elements. The equation of motion may be written as

$$M\ddot{\bar{x}} + \bar{g}(\dot{\bar{x}}, \bar{x}) = \bar{f}(t) \tag{3.28}$$

where

$x_j$ is the generalized displacement of the $j^{th}$ mass

$g_j(\dot{\bar{x}}, \bar{x})$ is the total force acting on the $j^{th}$ mass by the non-
linear elements .

$\bar{f}(t)$ is a stationary Gaussian random vector representing the

excitation to the system .

It is assumed that the system possesses a stationary solution.

By way of obtaining an approximate solution of (3.28), con-
sider the following linear differential equation

$$M\ddot{\bar{x}} + C\dot{\bar{x}} + K\bar{x} = \bar{f}(t) \tag{3.29}$$

where C and K are two arbitrary matrices. Let the solution to (3.29)
be also the approximate solution to (3.28), then the difference of (3.28)
and (3.29), $\bar{e}$, will be

$$\bar{e} = \bar{g}(\dot{\bar{x}}, \bar{x}) - C\dot{\bar{x}} - K\bar{x} \tag{3.30}$$

Note that here $\bar{x}$ is the solution of (3.29). The matrices C and K,
which are still arbitrary up to this point, will be chosen so as to
make some measure of the vector $\bar{e}$ as small as possible. Then it
is assumed that the solution to the linear system (3.29) will furnish
a good approximate solution to the nonlinear system (3.28). This way
of defining an auxiliary set for (3.28) has been applied by Iwan[26] to
nonlinear systems under deterministic excitation. In that case, the
criterion was that the squared error per cycle was a minimum. But
here we shall use the criterion that the mean square value of $\bar{e}$ is a
minimum, that is,

$$E\left[\bar{e}^T\bar{e}\right] = \text{Minimum} \tag{3.31}$$

The necessary conditions for (3.31) to be true are

$$\frac{\partial E\left[\bar{e}^T \bar{e}\right]}{\partial c_{jk}} = 2E\left[\bar{e}^T \frac{\partial \bar{e}}{\partial c_{jk}}\right] = 2E\left[e_j \dot{x}_k\right] = 0$$

$$\frac{\partial E\left[\bar{e}^T \bar{e}\right]}{\partial k_{jk}} = 2E\left[\bar{e}^T \frac{\partial \bar{e}}{\partial k_{jk}}\right] = 2E\left[e_j x_k\right] = 0$$

(3.32)

Upon using (3.30), they become in the matrix form

$$E\left[\bar{e}\,\dot{\bar{x}}^T\right] = E\left[\bar{g}(\dot{\bar{x}}, \bar{x})\dot{\bar{x}}^T\right] - CE\left[\dot{\bar{x}}\dot{\bar{x}}^T\right] - KE\left[\bar{x}\,\dot{\bar{x}}^T\right] = O$$

$$E\left[\bar{e}\,\bar{x}^T\right] = E\left[\bar{g}(\dot{\bar{x}}, \bar{x})\bar{x}^T\right] - CE\left[\dot{\bar{x}}\bar{x}^T\right] - KE\left[\bar{x}\bar{x}^T\right] = O$$

(3.33)

The conditions in (3.32) will give a true minimum (as opposed to a maximum) if the following inequality holds

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{r=1}^{n} \sum_{s=1}^{n} \left\{ dc_{jk}\, dc_{rs} \frac{\partial^2}{\partial c_{jk}\partial c_{rs}} + dc_{jk}\, dk_{rs} \frac{\partial^2}{\partial c_{jk}\partial k_{rs}} \right.$$

$$\left. + dk_{jk}\, dc_{rs} \frac{\partial^2}{\partial k_{jk}\partial c_{rs}} + dk_{jk}\, dk_{rs} \frac{\partial^2}{\partial k_{jk}\partial k_{rs}} \right\} E\left[\bar{e}^T \bar{e}\right] \geq 0$$

(3.34)

Differentiating (3.32) with respect to $c_{rs}$ and $k_{rs}$, substituting the results into the left-hand side of Inequality (3.34) yields

$$\sum_{j=1}^{n} \begin{pmatrix} dk_{j1} \\ \vdots \\ dk_{jn} \\ dc_{j1} \\ \vdots \\ dc_{jn} \end{pmatrix}^T \begin{pmatrix} E\left[\bar{x}\,\bar{x}^T\right] & E\left[\bar{x}\,\dot{\bar{x}}^T\right] \\ E\left[\dot{\bar{x}}\,\bar{x}^T\right] & E\left[\dot{\bar{x}}\,\dot{\bar{x}}^T\right] \end{pmatrix} \begin{pmatrix} dk_{j1} \\ \vdots \\ dk_{jn} \\ dc_{j1} \\ \vdots \\ dc_{jn} \end{pmatrix} \geq 0 \quad (3.35)$$

Hence (3.31) will be a true minimum if (3.33) and (3.35) hold.

The square matrix in (3.35) is just the instantaneous correlation matrix of the vector process $\begin{pmatrix} \bar{x} \\ \dot{\bar{x}} \end{pmatrix}$. Since the excitation is assumed to be Gaussian, the vector process $\begin{pmatrix} \bar{x} \\ \dot{\bar{x}} \end{pmatrix}$ is also Gaussian. It can be shown[27] that for a Gaussian process, the instantaneous correlation matrix is non-negative definite. Therefore, (3.35) is always true and the conditions in (3.33) do define a minimum for $E\left[\bar{e}^T\bar{e}\right]$.

In order to solve (3.33) for K and C, it is first necessary to express $E\left[\bar{g}(\dot{\bar{x}}, \bar{x})\bar{x}^T\right]$ and $E\left[\bar{g}(\dot{\bar{x}}, \bar{x})\dot{\bar{x}}^T\right]$ in terms of $E\left[\bar{x}\,\bar{x}^T\right]$, $E\left[\bar{x}\,\dot{\bar{x}}^T\right]$ and $E\left[\dot{\bar{x}}\,\dot{\bar{x}}^T\right]$. Let $y_{kr}$ denote the displacement of the $k^{th}$ mass relative to the $r^{th}$ mass and let the approximate force acting on the $k^{th}$ mass by the nonlinear element connecting the $k^{th}$ mass and the $r^{th}$ mass be denoted by $s_{kr}(\dot{y}_{kr}, y_{kr})$. Then

$$\left.\begin{aligned}
E\left[g_k(\dot{\bar{x}}, \bar{x})x_j\right] &= \sum_{\substack{r \\ r \neq k}} E\left[s_{kr}(\dot{y}_{kr}, y_{kr})x_j\right] \\
E\left[g_k(\dot{\bar{x}}, \bar{x})\dot{x}_j\right] &= \sum_{\substack{r \\ r \neq k}} E\left[s_{kr}(\dot{y}_{kr}, y_{kr})\dot{x}_j\right]
\end{aligned}\right\} \qquad (3.36)$$

where the sum is taken over all nonlinear elements connected to the $k^{th}$ mass. Since $\bar{x}$ is a Gaussian vector, it follows that the quantities $\dot{y}_{kr}$, $y_{kr}$, $x_j$, and $\dot{x}_j$ will be Gaussian distributed. Hence

$$\left.\begin{aligned}
E\left[s_{kr}(\dot{y}_{kr}, y_{kr})x_j\right] &= E\left[s_{kr}(\dot{y}_{kr}, y_{kr})\dot{y}_{kr}\right]\cdot E\left[\dot{y}_{kr}x_j\right]\Big/E\left[\dot{y}_{kr}^2\right] \\
&\quad + E\left[s_{kr}(\dot{y}_{kr}, y_{kr})y_{kr}\right]\cdot E\left[y_{kr}x_j\right]\Big/E\left[y_{kr}^2\right]
\end{aligned}\right\} \qquad (3.37)$$

$$E\left[s_{kr}(\dot{y}_{kr}, y_{kr})\dot{x}_j\right] = E\left[s_{kr}(\dot{y}_{kr}, y_{kr})\dot{y}_{kr}\right] \cdot E\left[\dot{y}_{kr}\dot{x}_j\right] / E\left[\dot{y}_{kr}^2\right]$$
$$+ E\left[s_{kr}(\dot{y}_{kr}, y_{kr})y_{kr}\right] \cdot E\left[y_{kr}\dot{x}_j\right] / E\left[y_{kr}^2\right] \qquad (3.37)$$

These two results are proved in Appendix C. Let us define

$$\gamma_{kr} = E\left[s_{kr}(\dot{y}_{kr}, y_{kr})\dot{y}_{kr}\right] / E\left[\dot{y}_{kr}^2\right]$$
$$\varkappa_{kr} = E\left[s_{kr}(\dot{y}_{kr}, y_{kr})y_{kr}\right] / E\left[y_{kr}^2\right] \qquad k \neq r \qquad (3.38)$$

Then equation (3.37) reduces to

$$E\left[s_{kr}(\dot{y}_{kr}, y_{kr})x_j\right] = E\left[(\gamma_{kr}\dot{y}_{kr} + \varkappa_{kr}y_{kr})x_j\right]$$
$$E\left[s_{kr}(\dot{y}_{kr}, y_{kr})\dot{x}_j\right] = E\left[(\gamma_{kr}\dot{y}_{kr} + \varkappa_{kr}y_{kr})\dot{x}_j\right] \qquad (3.39)$$

Hence, there exists a linear system with spring constants $\varkappa_{kr}$ and damping coefficients $\gamma_{kr}$ defined by (3.38) such that if the nonlinear system is replaced by this linear system, the expectation values $E\left[\bar{g}(\dot{\bar{x}}, \bar{x})\bar{x}^T\right]$ and $E\left[\bar{g}(\dot{\bar{x}}, \bar{x})\dot{\bar{x}}^T\right]$ will not be changed. Note that up to this point this linear system is not necessary to be a system which minimizes $E\left[\bar{e}^T\bar{e}\right]$

Substituting (3.39) into (3.36) gives

$$E\left[\bar{g}_k(\dot{\bar{x}}, \bar{x})x_j\right] = E\left[\sum_{\substack{r \\ r \neq k}}(\gamma_{kr}\dot{y}_{kr} + \varkappa_{kr}y_{kr})x_j\right]$$
$$E\left[\bar{g}_k(\dot{\bar{x}}, \bar{x})\dot{x}_j\right] = E\left[\sum_{\substack{r \\ r \neq k}}(\gamma_{kr}\dot{y}_{kr} + \varkappa_{kr}y_{kr})\dot{x}_j\right] \qquad (3.40)$$

Let the stiffness matrix and the damping matrix of the linear system defined by (3.38) be denoted by $K^{(e)}$ and $C^{(e)}$. Then (3.40) can also

be written as

$$E\left[g_k(\dot{\bar{x}},\bar{x})x_j\right] = E\left[\sum_{s=1}^{n}(c_{ks}^{(e)}\dot{x}_s + k_{ks}^{(e)}x_s)x_j\right]$$

$$E\left[g_k(\dot{\bar{x}},\bar{x})\dot{x}_j\right] = E\left[\sum_{s=1}^{n}(c_{ks}^{(e)}\dot{x}_s + k_{ks}^{(e)}x_s)\dot{x}_j\right]$$

(3.41)

since the right-hand sides of (3.40) and (3.41) are just two different representations of the total force acting on the $k^{th}$ mass. In matrix form, Eq. (3.41) becomes

$$E\left[\bar{g}(\dot{\bar{x}},\bar{x})\bar{x}^T\right] = C^{(e)}E\left[\dot{\bar{x}}\,\bar{x}^T\right] + K^{(e)}E\left[\bar{x}\,\bar{x}^T\right]$$

$$E\left[\bar{g}(\dot{\bar{x}},\bar{x})\dot{\bar{x}}^T\right] = C^{(e)}E\left[\dot{\bar{x}}\,\dot{\bar{x}}^T\right] + K^{(e)}E\left[\bar{x}\,\dot{\bar{x}}^T\right]$$

(3.42)

Eq. (3.33) may now be solved for the K and C which minimize $E\left[\bar{e}^T\bar{e}\right]$. Substituting (3.42) into (3.33) yields

$$\left(K-K^{(e)}\right)E\left[\bar{x}\,\bar{x}^T\right] + \left(C-C^{(e)}\right)E\left[\dot{\bar{x}}\,\bar{x}^T\right] = O$$

$$\left(K-K^{(e)}\right)E\left[\bar{x}\,\dot{\bar{x}}^T\right] + \left(C-C^{(e)}\right)E\left[\dot{\bar{x}}\,\dot{\bar{x}}^T\right] = O$$

(3.43)

This set of equations may also be written as

$$\begin{pmatrix} E\left[\bar{x}\,\bar{x}^T\right] & E\left[\bar{x}\,\dot{\bar{x}}^T\right] \\ \\ E\left[\dot{\bar{x}}\,\bar{x}^T\right] & E\left[\dot{\bar{x}}\,\dot{\bar{x}}^T\right] \end{pmatrix} \begin{pmatrix} \left(K-K^{(e)}\right)^T \\ \\ \left(C-C^{(e)}\right)^T \end{pmatrix} = O$$

(3.44)

If the square matrix is non-singular, the only solution to (3.44) is

$$K = K^{(e)}$$

$$C = C^{(e)}$$

(3.45)

If the square matrix is singular, (3.45) is not the only solution. But in this case, it may be shown that any solution to (3.44) will lead to the same minimum for $E\left[\bar{e}^T\bar{e}\right]$. Since no criterion is available to determine which of the solutions is the best, for simplicity, we still use the solution in (3.45).

We have thus shown that the linear system formed by replacing each nonlinear element by a linear spring and a linear damper defined by (3.38) will minimize $E\left[\bar{e}^T\bar{e}\right]$ provided that the excitation is Gaussian. This linear system will henceforth be referred to as the equivalent linear system of the nonlinear system (3.28) and the quantities defined in (3.38) will be called the equivalent linear stiffness and equivalent linear damping coefficient of the corresponding nonlinear element.

In the following we will discuss a method of solution of this general equivalent linear system.

### 3.5.1 Method of Solution

The first step in this approach is to find the instantaneous correlation matrices for the equivalent linear system (3.29). This can be done by solving (2.36). Hence, in order to find an approximate solution for the nonlinear system (3.28), one must solve (2.36) and (3.38). These are nonlinear algebraic equations and it is difficult to solve them directly. However, they can in general be solved by the following iteration scheme. Assume a set of values for $\varkappa_{kr}$'s and $\gamma_{kr}$'s. Then (2.36) is reduced to a system of linear algebaic equations and can be solved easily for the instantaneous correlation functions. Substituting the results into (3.38) yields a new set of values for

$\varkappa_{kr}$'s and $\gamma_{km}$'s. This procedure can be repeated until the required accuracy is obtained. This scheme is particularly well suited to digital computation and has been used successfully in solving the examples in Chapter 4.

### 3.5.2 Special Case

Suppose that the force in a nonlinear element, say $s_{jk}(\dot{y}_{jk}, y_{jk})$, may be represented as the sum of separate functions of displacement and velocity, i.e.,

$$s_{jk}(\dot{y}_{jk}, y_{jk}) = s^{(1)}(\dot{y}_{jk}) + s^{(2)}(y_{jk}) \tag{3.46}$$

Since $\dot{y}_{jk}$ and $y_{jk}$ are uncorrelated, Equation (3.38) reduces by the substitution of (3.46) to

$$\left. \begin{array}{l} \varkappa_{jk} = E\left[s^{(2)}(y_{jk})y_{jk}\right] \Big/ E\left[y_{jk}^2\right] \\[2ex] \gamma_{jk} = E\left[s^{(1)}(y_{jk})y_{jk}\right] \Big/ E\left[\dot{y}_{jk}^2\right] \end{array} \right\} \tag{3.47}$$

If $s_{jk}(\dot{y}_{jk}, y_{jk})$ is linear, say,

$$\left. \begin{array}{l} s^{(1)}(\dot{y}_{jk}) = c^{(o)}\dot{y}_{jk} \\[2ex] s^{(2)}(y_{jk}) = k^{(o)}y_{jk} \end{array} \right\} \tag{3.48}$$

where $c^{(o)}$ and $k^{(o)}$ are constants, then it follows from (3.47) that

$$\begin{array}{l} \varkappa_{jk} = k^{(o)} \\[2ex] \gamma_{jk} = c^{(o)} \end{array} \tag{3.49}$$

Hence the equivalent linear system is the original linear system itself.

## 3.6 Accuracy of the Generalized Equivalent Linearization Approach

The accuracy of the generalized linearization approach, of course, depends on the smallness of the nonlinearity. In order to obtain some understanding of range of application, we consider some problems which can be solved by the Fokker-Planck approach as well as by the generalized equivalent linearization approach. It will be shown that for the special problems considered, the generalized equivalent linearization approach gives quite satisfactory results even for rather large nonlinearities. We also give a comparison of the results of the generalized equivalent linearization approach with those of the normal mode approach.

Consider the system shown in Figure 3.1. Its equation of motion may be written as

$$\ddot{\bar{x}} + c\dot{\bar{x}} + \frac{\partial u(\bar{x})}{\partial \bar{x}} = \bar{f}(t) \tag{3.50a}$$

where

$x_i$ is the absolute displacement of the $i^{th}$ mass.

c is a constant.

$u(\bar{x})$ is the potential energy of the system.

Furthermore, let $\bar{f}(t)$ be a Gaussian white noise specified by

$$\left. \begin{array}{l} \bar{m}_{\bar{f}} = O \\[2mm] R_{\bar{f}}(\tau) = 2c\gamma\,\delta(\tau)I \end{array} \right\} \tag{3.50b}$$

where $\gamma$ is a constant. This problem will be solved for several forms of $u(\bar{x})$ first exactly by the Fokker-Planck approach, and then approximately by the generalized equivalent linearization approach.

(1) Exact Solution - Fokker-Planck Approach

It follows from (3.10) that

$$p(\dot{\bar{x}}) = \beta_1 \exp\left\{-\frac{1}{2\gamma}\,\dot{\bar{x}}^T\dot{\bar{x}}\right\}$$

$$p(\bar{x}) = \beta_2 \exp\left\{-\frac{1}{\gamma}\,u(\bar{x})\right\}$$

$$(3.51)$$

where $\beta_1$ and $\beta_2$ are two normalization factors such that

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(\dot{\bar{x}})d\dot{x}_1 \cdots d\dot{x}_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(\bar{x})dx_1 \cdots dx_n = 1 \quad (3.52)$$

Straightforward calculations show that

$$E\left[\dot{\bar{x}}\dot{\bar{x}}^T\right] = \gamma I$$

$$E\left[\bar{x}\dot{\bar{x}}^T\right] = O$$

$$(3.53)$$

and $E\left[\bar{x}\bar{x}^T\right]$ depends on $u(\bar{x})$. Let

$$x = A\bar{y} \qquad (3.54)$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ 1 & 1 & 1 & \ldots & 0 \\ \cdots\cdots\cdots\cdots \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix}$$

and let $y_j$ be the displacement of the $j^{th}$ mass relative to the $(j-1)^{th}$

mass or the base if $j = 1$. Then Equation (3.51) becomes

$$p(\dot{\bar{y}}) = \beta_1 \exp\left\{-\frac{1}{2\gamma}\,\dot{\bar{y}}A^T A\dot{\bar{y}}\right\}$$

$$p(\bar{y}) = \beta_2 \exp\left\{-\frac{1}{\gamma}\sum_{k=1}^{n} u_k(y_k)\right\}$$

$$(3.55)$$

where $u_k(y_k)$ is the potential energy of the $k^{th}$ spring. From (3.55), one immediately finds

$$E\left[\dot{\bar{y}}\,\dot{\bar{y}}^T\right] = \gamma A^{-1}(A^{-1})^T$$

$$E\left[\bar{y}\,\dot{\bar{y}}^T\right] = O$$

and

$$E\left[y_k y_j\right] = \delta_{kj} \int_{-\infty}^{\infty} y_k^2 e^{-\frac{1}{\gamma}u_k(y_k)}dy_k \Big/ \int_{-\infty}^{\infty} e^{-\frac{1}{\gamma}u_k(y_k)}dy_k$$

$$k,j = 1,\ldots,n$$

(3.56)

The first two results are independent of the nonlinear springs, so they will remain unchanged for all kinds of nonlinear springs. The third equation implies that the nonlinear springs are uncorrelated with one another. Two kinds of nonlinear springs will be considered after the discussion of the approximate solution.

(2) Approximate Solution - The Generalized Equivalent Linearization Approach

The general equivalent linear system of (3.50a) is

$$\ddot{\bar{x}} + c\dot{\bar{x}} + K\bar{x} = \bar{f}(t) \tag{3.57}$$

and its correlation function matrices are given by

$$E\left[\dot{\bar{x}}\,\dot{\bar{x}}^T\right] = \gamma I$$

$$E\left[\bar{x}\,\dot{\bar{x}}^T\right] = O$$

and

$$E\left[\bar{x}\,\bar{x}^T\right] = \gamma K^{-1} \text{ or } E\left[\frac{\partial u(\bar{x})}{\partial \bar{x}}\bar{x}^T\right] = \gamma I$$

(3.58)

Under the same transformation defined by (3.54), Equation (3.58) becomes

$$E\left[\overset{\cdot\cdot}{\bar{y}}\,\overset{\cdot\cdot}{\bar{y}}^{T}\right] = \gamma A^{-1}(A^{-1})^{T} \tag{3.59a}$$

$$E\left[\bar{y}\,\overset{\cdot\cdot}{\bar{y}}^{T}\right] = 0 \tag{3.59b}$$

$$E\left[s_{j}(y_{j})y_{j}\right] = \gamma \; ; \quad j=1,\ldots,n \tag{3.59c}$$

where $s_{j}$ is the force in the spring connecting the $j^{th}$ mass and $(j-1)^{th}$ mass. Equations (3.59a) and (3.59b) are the same as those obtained by the Fokker-Planck approach. Equation (3.59c) is different from the third equation in (3.56), but both equations show that the non-linear springs are uncorrelated with one another.

According to the above results, we know that the approximate solution is different from the exact solution only in $E\left[y_{j}^{2}\right]$, $j = 1, \ldots, n$. Moreover, since the equations for the determination of the $E\left[y_{j}^{2}\right]$'s are uncoupled, it is sufficient to consider one equation for each type of nonlinear spring.

In the following, two types of nonlinear springs, a hardening spring and a softening spring, will be considered.

### 3.6.1 Cubic Hardening Spring

Consider a cubic hardening spring having a force-deflection relation

$$s(y) = k^{(o)}(y+\alpha y^{3}); \; k^{(o)}, \; \alpha > 0 \tag{3.60}$$

Using the formulas derived in this section, we have the following results.

## (1) Exact Solution

The first probability density for the displacement is given by

$$p(y) = \exp\left\{-\frac{k^{(o)}}{2\gamma}\left(y^2 + \frac{\alpha}{2}y^4\right)\right\} \Big/ \int_{-\infty}^{\infty}\exp\left\{-\frac{k^{(o)}}{2\gamma}\left(y^2 + \frac{\alpha}{2}y^4\right)\right\}dy \quad (3.61)$$

and the mean square displacement $\sigma_y^2$ is defined as

$$\sigma_y^2 = E\left[y^2\right] = \int_{-\infty}^{\infty}y^2 p(y)dy \quad (3.62)$$

Expanding the exponential functions in p(y) into a Taylor's series

about $y = 0$ and integrating the resulting series term by term yields

$$\sigma_y^2 = \frac{\gamma}{k^{(o)}}\left[1 - 3\left(\frac{\alpha\gamma}{k^{(o)}}\right) + 24\left(\frac{\alpha\gamma}{k^{(o)}}\right)^2 - 297\left(\frac{\alpha\gamma}{k^{(o)}}\right)^3 + \ldots\right] \quad (3.63)$$

If $\frac{\alpha\gamma}{k^{(o)}} \ll 1$, the sum of the first two or three terms will be very close

to the true value of $\sigma_y^2$.

Consider another situation. Let $k^{(o)}$ tend to zero, but let

$k^{(o)}\alpha$ approach some finite value $\epsilon$. In other words, the nonlinearity

parameter $\alpha$ tends to infinity. In this case, one has

$$p(y) = \exp\left\{-\frac{\epsilon}{4\gamma}y^4\right\} \Big/ \int_{-\infty}^{\infty}\exp\left\{-\frac{\epsilon}{4\gamma}y^4\right\}dy \quad (3.64)$$

and

$$\sigma_y^2 = \int_{-\infty}^{\infty}y^2\exp\left\{-\frac{\epsilon}{4\gamma}y^4\right\}dy \Big/ \int_{-\infty}^{\infty}\exp\left\{-\frac{\epsilon}{4\gamma}y^4\right\}dy \quad (3.65)$$

Upon using the result[28]

$$\int_{o}^{\infty}y^a e^{-(ry)^b}dy = \frac{1}{br^{a+1}}\Gamma\left(\frac{a+1}{b}\right), \quad a+1, b, r > 0 \quad (3.66)$$

where $\Gamma(\ )$ is the Gamma function, Eqns. (3.64) and (3.65) reduce to

$$p(y) = \left(\frac{\varepsilon}{\gamma}\right)^{1/4} \exp\left\{-\frac{\varepsilon}{4\gamma} y^4\right\} / \frac{1}{4} \Gamma\left(\frac{1}{4}\right) \qquad (3.67)$$

and

$$\sigma_y^2 = \frac{2}{3} \frac{\Gamma(7/4)}{\Gamma(5/4)} \left(\frac{\gamma}{\varepsilon}\right)^{1/2} = 0.6760 \left(\frac{\gamma}{\varepsilon}\right)^{1/2}$$

or

$$\sigma_y = 0.8222 \left(\frac{\gamma}{\varepsilon}\right)^{1/4} \qquad (3.68)$$

(2) Approximate Solution

The response process is of course Gaussian and the mean square displacement $E\left[y^2\right]$ is governed by

$$k^{(o)}\left(E\left[y^2\right] + \alpha E\left[y^4\right]\right) = \gamma \qquad (3.69)$$

Noting that $E\left[y^4\right] = 3E\left[y^2\right]$ for a Gaussian process, one can solve (3.69) for $E\left[y^2\right]$:

$$\sigma_y^2 = E\left[y^2\right] = -\frac{1}{6\alpha}\left(1 - \sqrt{1 + 12\frac{\alpha\gamma}{k^{(o)}}}\right) \qquad (3.70)$$

If $\frac{\alpha\gamma}{k^{(o)}} < \frac{1}{12}$, the second term in the brackets can be expanded into the following power series

$$\sqrt{1 + 12\frac{\alpha\gamma}{k^{(o)}}} = 1 + \frac{1}{2}\left(12\frac{\alpha\gamma}{k^{(o)}}\right) - \frac{1}{8}\left(12\frac{\alpha\gamma}{k^{(o)}}\right)^2 + \frac{1}{16}\left(12\frac{\alpha\gamma}{k^{(o)}}\right)^3 \ldots$$

$$= 1 + 6\frac{\alpha\gamma}{k^{(o)}} - 18\left(\frac{\alpha\gamma}{k^{(o)}}\right)^2 + 108\left(\frac{\alpha\gamma}{k^{(o)}}\right)^3 \ldots \qquad (3.71)$$

Thus

$$\sigma_y^2 = \frac{\gamma}{k^{(o)}}\left[1 - 3\left(\frac{\alpha\gamma}{k^{(o)}}\right) + 18\left(\frac{\alpha\gamma}{k^{(o)}}\right)^2 \cdots \right] \tag{3.72}$$

Again let $k^{(o)}$ tend to zero and let $k^{(o)}\alpha$ approach $\epsilon$. Then Eq. (3.69)
reduces to

$$\sigma_y = \left(\frac{\gamma}{3\epsilon}\right)^{1/4} = 0.7600\left(\frac{\gamma}{\epsilon}\right)^{1/4} \tag{3.73}$$

From the above calculations, we have the following results:
When $\frac{\alpha\gamma}{k^{(o)}} \ll 1$, we know from (3.63) and (3.72) that the approximate
solution for $\sigma_y$ shows good agreement with the exact one. When the
nonlinearity of the spring is very large, from (3.68) and (3.73) we
find that the error for the approximate $\sigma_y$ is about $(0.822 - 0.760)/0.822$
$= 7.5\%$. For an arbitrary nonlinearity, one might expect that the error
will be less than 7.5%.

For comparison, the exact and approximate values for $\frac{\sigma_y}{\sigma_o}$
have been plotted against $\frac{\alpha\gamma}{k^{(o)}}$ in Fig. 3.2. $\sigma_o$ is the linear solution
for $\sigma_y$ and is equal to $\gamma/k^{(o)}$. Both the exact and approximate $\sigma_y$ decrease
as the nonlinearity parameter $\alpha$ increases. The approximate solution
is always less than or equal to the exact solution, but the error is
less than 7.5%.

Also plotted in Fig. 3.2 is the first order approximate solu-
tion found by the perturbation approach. It is clear that this approach
is valid only if both the nonlinearity of the system and the excitation
are sufficiently small.

### 3.6.2 <u>Softening Spring</u>

In deterministic theory, a softening spring can also be represented by (3.60) with $\alpha < 0$. But in probabilistic theory this representation often leads to unbounded solutions. In order to avoid this difficulty, here, the following representation will be used.

$$s(y) = \frac{2}{\pi} f_u \tan^{-1} \left( \frac{\pi k^{(o)} y}{2 f_u} \right) \tag{3.74}$$

in which $f_u$ is the ultimate force of the spring and $k^{(o)}$ is the initial slope of the load-displacement curve. Fig. 3.3 shows some general features of this type of springs.

### (1) <u>Exact Solution</u>

The potential energy of a spring specified by (3.74) is given by

$$u(y) = \frac{2}{\pi} f_u \left\{ y \tan^{-1} \left( \frac{\pi k^{(o)} y}{2 f_u} \right) - \frac{f_u}{\pi k^{(o)}} \ell n \left[ 1 + \left( \frac{\pi}{2} \frac{k^{(o)} y}{f_u} \right)^2 \right] \right\} \tag{3.75}$$

The first probability density and the mean square displacement are given by

$$p(y) = \exp\left\{ -\frac{1}{\gamma} u(y) \right\} \Big/ \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{\gamma} u(y) \right\} dy \tag{3.76}$$

$$\sigma_y^2 = E\left[ y^2 \right] = \int_{-\infty}^{\infty} y^2 p(y) dy \tag{3.77}$$

Now consider a special case. Let $k^{(o)}/f_u \to \infty$. Then Eq. (3.74) becomes

$$s(y) = \begin{cases} f_u & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ -f_u & \text{if } y < 0 \end{cases} \qquad (3.78a)$$

and the potential energy $u(y)$ reduces to

$$u(y) = \lim_{k^{(o)}/f_u \to \infty} \frac{2}{\pi} f_u \left\{ y \tan^{-1}\left(\frac{\pi}{2} \frac{k^{(o)}y}{f_u}\right) - \frac{f_u}{\pi k^{(o)}} \ln\left[1 + \left(\frac{\pi k^{(o)}y}{2f_u}\right)^2\right] \right\} = f_u |y| \quad (3.78b)$$

where $|y|$ denotes the absolute value of $y$. The first probability density now becomes

$$p(y) = \exp\left\{-\frac{f_u}{\gamma}|y|\right\} \Big/ \int_{-\infty}^{\infty} \exp\left\{-\frac{f_u}{\gamma}|y|\right\} dy$$

$$= \frac{1}{2} \exp\left\{-\frac{f_u}{\gamma}|y|\right\} \Big/ \int_{o}^{\infty} \exp\left\{-\frac{f_u}{\gamma}y\right\} dy$$

$$= \frac{f_u}{2\gamma} \exp\left\{-\frac{f_u}{\gamma}|y|\right\} \qquad (3.79)$$

and the mean square displacement is given by

$$\sigma_y^2 = \int_{-\infty}^{\infty} \frac{f_u}{2\gamma} y^2 \exp\left\{-\frac{f_u}{\gamma}|y|\right\} dy$$

$$= \frac{2\gamma^2}{f_u^2} \qquad (3.80)$$

## (2) Approximate Solution

First, we shall evaluate $E\big[s(y)y\big]$:

$$E[s(y)y] = \frac{1}{\sqrt{2\pi}\ \sigma_y} \int_{-\infty}^{\infty} \frac{2f_u}{\pi} ye^{-\frac{y^2}{2\sigma_y^2}} \tan^{-1}\left(\frac{\pi k^{(o)}}{2f_u}y\right)dy$$

$$= \left(\frac{2}{\pi}\right)^{3/2} \sigma_y f_u \alpha^2 \int_{-\infty}^{\infty} re^{-\alpha^2 r^2} \tan^{-1} r\, dr \qquad (3.81)$$

where we have set $\alpha = \sqrt{2}\, f_u/\pi k^{(o)}\sigma_y$ and used the change of variable $r = \pi k^{(o)}y/2f_u$. Integration by parts and using the result[29]

$$\int_{o}^{\infty} \frac{e^{-\alpha^2 r^2}}{1+r^2} dr = \frac{\pi}{2}e^{\alpha^2} \text{erfc}(\alpha) \qquad (3.82)$$

where erfc ( ) is the complementary error function, yields

$$\int_{-\infty}^{\infty} re^{-\alpha^2 r^2}\tan^{-1} r\, dr = -\frac{e^{-\alpha^2 r^2}}{2\alpha^2}\tan^{-1} r \bigg|_{-\infty}^{\infty} + \frac{1}{2\alpha^2}\int_{-\infty}^{\infty} \frac{e^{-\alpha^2 r^2}}{1+r^2}dr$$

$$= \frac{\pi}{2\alpha^2} e^{\alpha^2} \text{erfc}(\alpha) \qquad (3.83)$$

Substituting (3.83) into (3.81) gives

$$E[s(y)y] = \sqrt{\frac{2}{\pi}}\ \sigma_y f_u e^{\alpha^2}\text{erfc}(\alpha) \qquad (3.84)$$

Then it follows from (3.59c) that

$$\sigma_y e^{\alpha^2}\text{erfc}(\alpha) = \sqrt{\frac{\pi}{2}}\ \frac{\gamma}{f_u} \qquad (3.85)$$

As $k^{(o)}/f_u \to \infty$, this becomes

$$\sigma_y = \sqrt{\frac{\pi}{2}}\ \frac{\gamma}{f_u} \qquad (3.86)$$

From (3.80) and (3.86) we know that the error of the approximate $\sigma_y$ for $k^{(o)}/f_u \to \infty$ is about $(1-\sqrt{\pi}/2) = 11.4\%$.

For comparison, Equations (3.77) and (3.85) have been solved numerically and the results are plotted in Figure 3.4. The approximate solution is always bigger than the exact solution, but the error is less than 11.4%.

We have considered two types of nonlinear springs for the problem defined by (3.50) using the exact approach and the generalized equivalent linearization approach. Both approaches can diagonalize the matrix $E\left[\overline{y}\,\overline{y}^T\right]$ and therefore we consider only one element of this matrix. If the same system is solved by the normal mode approach, the matrix $E\left[\overline{y}\,\overline{y}^T\right]$ is not diagonal, so we must consider a particular system. The system as shown in Figure 3.1 when $n = 3$ will be used. The parameters of this three-degree-of-freedom system are as follows:

$$\left.\begin{aligned} & m_1 = m_2 = m_3 = 1 \\[6pt] & s_j(y_j) = k_j^{(o)}(y_j + \alpha y_j^3) \quad, \; j = 1, 2, 3 \\[6pt] & k_1^{(o)} = 3, \; k_2^{(o)} = 2, \; k_3^{(o)} = 1 \\[6pt] & c = 0.1 \\[6pt] & R_{\overline{f}}(\tau) = 0.2\gamma\,\delta(\tau)I \;, \quad \overline{m}_f = O \end{aligned}\right\} \tag{3.87}$$

where $\alpha$ and $\gamma$ are constants. Again, this approach yields

$$E\left[\dot{\overline{x}}\dot{\overline{x}}^T\right] = \gamma I$$

$$E\left[\overline{x}\dot{\overline{x}}^T\right] = O \tag{3.88}$$

Hence for these two matrices both approximate approaches give the exact solution. This is because the exact velocity process is Gaussian and both approximate approaches recognize that the velocity process is uncorrelated with the displacement process.

The mean square values for the displacements $\sigma_{y_j}$ from the exact approach and two approximate approaches are plotted against the spectral density of the white noise excitation in Figs. 3.5 to 3.8 for different $\alpha$'s. From these figures one sees that both approximate approaches give reasonable results. As far as the percentage of error is concerned, the three percentages of error for $\sigma_{y_1}$, $\sigma_{y_2}$, and $\sigma_{y_3}$ given by the generalized equivalent linearization approach are more uniform than those for $\sigma_{y_1}$, $\sigma_{y_2}$, and $\sigma_{y_3}$ given by the normal mode approach.

In the above example, the damping matrix of the system is proportional to the spectral density of the excitation. This problem can be solved exactly as well as approximately. Here we are going to consider another example in which the damping matrix is proportional to the stiffness matrix formed by the linear part of the nonlinear springs. This problem cannot be solved exactly, but can be solved by both approximate approaches. The results are shown in Fig. 9. Also plotted in Fig. 9 are the results from the one mode approximation discussed at the end of section 3.3. The system considered here is the same one used in Fig. 3.6 except that the damping matrix is now proportional to the stiffness matrix formed by the linear part of the nonlinear springs. The proportionality constant is chosen in such a way that both systems used in Figs. 3.6 and 3.9 have the same amount of damping in the first mode. Since in this case the modal damping in-

creases with increasing modal frequency, contributions from higher

modes are nearly damped out. That is the reason why the results

in Fig. 3.9 are much lower than those in Fig. 3.6. By the same

reasoning the results given by the one mode approximation are not

too bad.

## IV. EXAMPLES

In this chapter we shall consider two examples which can be treated by the generalized equivalent linearization approach described in the previous chapter.

It is well known that damping plays an important role in structural dynamics. In order to simplify analysis, it is usually assumed that the damping force is directly proportional to velocity. However, in some systems, this force does not obey such a simple linear law. For example, the damping force in the landing system of certain aircraft is found to be proportional to the square of velocity[30]. The vibration of submerged structures furnishes another example in which the external damping force, the resistance to the surrounding water, can often be considered to be proportional to the square of velocity. Hence in the first example, we consider a multidegree-of-freedom system connected by linear springs, but having dampers whose resisting force is proportional to the square of velocity. The system is excited by a base acceleration which is a white noise. This problem cannot be solved exactly by the Fokker-Planck approach because the conditions listed in Section 3.2 are not satisfied. The perturbation approach fails because the damping matrix $C^{(o)}$ vanishes. An equivalent linear system which possesses normal modes may be constructed, but the modal exictation process is correlated. Therefore, the normal mode approach is also not applicable.

Many structures can be considered as linear in analysis without significant error if the excitation is sufficiently small.

However, for severe excitation, the linear treatment are often not accurate enough. Hence, the systems must be considered as non-linear. For example, buildings subjected to severe excitations often behave like lightly damped softening systems, that is, the stiffness of the system decreases as the displacement increases. In addition, the effective damping in the first few modes is often only a few percent of the critical damping. As a second example we consider a multidegree-of-freedom system of this type. The springs in this system are nonlinear and follow the arctangent law discussed in Chapter III. The dampers are linear and arranged in such a manner that the equivalent linear system has normal modes and the equivalent damping in each mode is known. This system is excited by a base acceleration which is a Gaussian white noise. This problem cannot be solved exactly by the Fokker-Planck approach since the conditions listed in Section 3.2 are not satisfied. The perturbation approach is not applicable because the damping matrix $C^{(o)}$ is not known. Although the equivalent linear system is assumed to possess normal modes, the normal mode approach also fails since the modal displacement process is correlated.

## 4.1 Example 1

Consider the system shown in Figure 4.1. n equal masses are connected by n identical springs and n identical dampers. The springs are assumed to be linear. The dampers are nonlinear and their damping force is proportional to the square of the relative velocity. The base acceleration $a_o(t)$ is a Gaussian white noise

which is specified by its spectral density w. Let $y_j$ denote the displacement of the $j^{th}$ mass relative to the $(j-1)^{th}$ mass or the base if $j=1$. Then the equations of motion of the system will be

$$M\ddot{\overline{y}} + K\overline{y} + \overline{g}(\dot{\overline{y}}) = \overline{f}(t) \tag{4.1}$$

where

$$f_j(t) = -ma_o(t) , \quad j=1, \ldots, n \tag{4.2}$$

The $g_j(\dot{\overline{y}})$ are given by

$$g_j(\dot{\overline{y}}) = \gamma\left[\dot{y}_j^2 sgn(\dot{y}_j) - \dot{y}_{j+1}^2 sgn(\dot{y}_{j+1})\right]$$
$$j=1, \ldots, n \quad \dot{y}_{n+1}=0 \tag{4.3}$$

where $\gamma$ is the damping coefficient of the dampers and

$$sgn(\dot{y}_j) = \begin{cases} -1 & \text{if} \quad \dot{y}_j < 0 \\ +1 & \text{if} \quad \dot{y}_j > 0 \end{cases} \tag{4.4}$$

The equivalent linear system of (4.2) can be written as

$$M\ddot{\overline{y}} + C\dot{\overline{y}} + K\overline{y} = \overline{f}(t) \tag{4.5}$$

Let $c_j$ be the equivalent linear damping coefficient of the damper between the $j^{th}$ mass and the $(j-1)^{th}$ mass. Then from (3.38), we obtain

$$c_j = \gamma E\left[\dot{y}_j^3 sgn(\dot{y}_j)\right] \Big/ E\left[\dot{y}_j^2\right] \tag{4.6}$$

Since $a_o(t)$ is Gaussian, $\dot{y}_j$ will also be Gaussian. Hence

$$E\left[\dot{y}_j^3 sgn(\dot{y}_j)\right] = \frac{1}{\sqrt{2\pi E\left[\dot{y}_j^2\right]}} \int_{-\infty}^{\infty} \dot{y}_j^3 sgn(\dot{y}_j) exp\left\{\frac{\dot{y}_j^2}{2E\left[\dot{y}_j^2\right]}\right\} d\dot{y}_j \tag{4.7}$$

$$= \frac{2}{\sqrt{2\pi E\left[\dot{y}_j^2\right]}} \int_0^\infty \dot{y}_j^3 \exp\left\{-\frac{\dot{y}_j^2}{2E\left[\dot{y}_j^2\right]}\right\} d\dot{y}_j$$

$$= \sqrt{\frac{8}{\pi}} \left(E\left[\dot{y}_j^2\right]\right)^{3/2} \tag{4.7}$$

<div align="right">cont'd</div>

Therefore,

$$c_j = \left(8\gamma^2 E\left[\dot{y}_j^2\right]/\pi\right)^{1/2} \tag{4.8}$$

The instantaneous correlation matrices of the equivalent linear system are given by (2.58). In order to reduce the number of unknowns in computation, the matrix $E\left[\overline{x}\dot{\overline{x}}^T\right]$, which is known to be antisymmetric, will be eliminated from (2.58) to give

$$ME\left[\dot{\overline{x}}\dot{\overline{x}}^T\right]C^T + CE\left[\dot{\overline{x}}\dot{\overline{x}}^T\right]M^T - CE\left[\overline{x}\overline{x}^T\right]K^T - KE\left[\overline{x}\overline{x}^T\right]C^T = O$$

$$ME\left[\dot{\overline{x}}\dot{\overline{x}}^T\right]C^T + CE\left[\dot{\overline{x}}\dot{\overline{x}}^T\right]M^T + MC^{-1}M\,E\left[\dot{\overline{x}}\dot{\overline{x}}^T\right]K^T + KE\left[\dot{\overline{x}}\dot{\overline{x}}^T\right]\left(MC^{-1}M\right)^T$$

$$- KE\left[\overline{x}\overline{x}^T\right]\left(MC^{-1}K\right)^T - MC^{-1}KE\left[\overline{x}\overline{x}^T\right]K^T = 2\pi W \tag{4.9}$$

where W is the spectral density matrix of $\overline{f}(t)$. If the base excitation is a clipped Gaussian white noise, Equation (4.8) is still valid, but Equation (2.58) has to be replaced by (2.45).

By using the iteration scheme discussed in Chapter III, Equations (4.9) and (4.8) can be solved numerically. The results of a three-degree-of-freedom system are plotted in Figures 4.2 to 4.10.

Figures 4.2 and 4.3 show the general behavior of the non-linear system. As in a linear system, the mean square displacements increase with increasing excitation level w and decrease with increas-

ing damping coefficient $\gamma$. But they do not increase linearly as in a linear system when the excitation level increases and they decrease faster than those in a linear system when the damping coefficient $\gamma$ increases.

The base excitation used in obtaining Figure 4.4 is a clipped Gaussian white noise with cutoff frequency $\omega_c$. Here the mean square displacement is plotted against $\omega_c$. When $\omega_c$ is not close to the first modal frequency of the equivalent linear system $\omega_1$, the curves are very flat but they rise sharply from nearly zero to values corresponding to solutions for a Gaussian white noise excitation in the neighborhood of $\omega_1$. Hence it is apparent that for a Gaussian white noise excitation the greatest contribution to the mean square displacement comes from the first mode of the equivalent linear system and contributions from higher modes are nearly damped out. This indicates that the equivalent linear damping in the higher modes is much greater than that in the first mode. The slope field of this figure represents approximately (exactly for linear systems) the spectral density for the displacement. Hence from Figure 4.4 we also observe that the spectral density has its most significant peak in the vicinity of $\omega_1$. We therefore conclude that the displacement $y_i$ will be essentially narrow band processes.

In order to obtain more insight into the damping behavior of the nonlinear system, it is instructive to consider a linear system which has the same $M, K, \bar{f}(t)$, and instantaneous correlation matrices as the nonlinear system (4.3). Assume that this system possesses

normal modes. Then the damping ratio $\zeta_i$ in each mode of this linear system can be determined without difficulty. This will give a measure of the damping in the nonlinear system. As expected, the damping ratio for the second and third modes is much higher than that for the first mode (the damping ratios are roughly proportional to the modal frequencies). The damping ratio $\zeta_1$ for three different $\gamma$'s is plotted against the excitation level w in Figure 4.5. It increases as either the damping coefficient or the excitation level increases. Since, for a linear system, damping is not a function of w, the $\zeta_1$ curve will be just a horizontal line. Thus in Figure 4.5 we can compare the nonlinear system with a linear system which is different from the nonlinear system only in the dampers used. For example, a nonlinear system with $\gamma = 1$ will be more effective than a linear system with $\zeta_1 = 0.15$ in reducing mean square displacements if the excitation level w is greater than $0.2/2\pi$. However, if w is less than this value, the linear system will be better.

On the basis of the above observations, the following conclusions can be made.

(1) Since the response is essentially a narrow band process, only those components of the spectral density of the excitation which are in the neighborhood of $\pm w_1$ are important. Thus, if the spectral density is slowly varying in these neighborhoods, then the excitation can be approximated by a white noise whose spectral density is just equal to that of the original excitation at $w = w_1$. In this way the analysis of the equivalent linear system can be much simplified because the stationary response of a linear system subjected to a

white noise excitation can be found by just solving a system of linear algebraic equations.

(2) The dampers used in this example are effective in reducing the mean square displacements only when the excitation level is high. Hence, if the excitation of a system is usually very severe, then the use of this kind of dissipation could be beneficial. However, if the excitation level of a system is in general rather low, then the need for additional linear viscous dampers is indicated.

Although these conclusions are drawn from a three-degree-of-freedom system, it is not difficult to see that they are also valid for n-degree-of-freedom systems.

## 4.2  Example 2

Consider the system shown in Figure 4.6. n equal masses are connected by n identical nonlinear springs and linear dampers (not shown in the figure). The linear dampers are arranged in such a way that the equivalent linear system for the nonlinear system has normal modes and the damping in each mode is 5% of critical damping. The spring force $s_j(y_j)$ is governed by following law

$$s_j(y_j) = \frac{2f_u}{\pi} \tan^{-1} \frac{\pi k^{(o)} y_j}{2f_u} \tag{4.10}$$

where $f_u$ and $k^{(o)}$ are the ultimate force and the initial spring constant of the nonlinear springs. The base acceleration $a_o(t)$ is assumed to be a Gaussian white noise with spectral density w. The equations of motion of this system will be

$$M\ddot{\overline{y}} + C\dot{\overline{y}} + \overline{g}(\overline{y}) = \overline{f}(t) \tag{4.11}$$

where

$$g_j(y) = s_{j+1}(y_{j+1}) - s_j(y_j) \quad , \quad y_{n+1} = 0$$

$$(j=1,\dots,n) \tag{4.12}$$

$$f_j(t) = -m a_o(t)$$

and $C$ is an unknown matrix. The equivalent linear system of (4.11) may be written as

$$M\ddot{\overline{y}} + C\dot{\overline{y}} + K\overline{y} = \overline{f}(t) \tag{4.13}$$

Since it is assumed that this system has normal modes, there exists a matrix $A$ such that

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a_{jr} m_{jk} a_{ks} = \delta_{rs} \quad (r, s = 1, 2, \dots, n) \tag{4.14}$$

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a_{jr} k_{jk} a_{ks} = \omega_r^2 \delta_{rs} \quad (r, s = 1, 2, \dots, n) \tag{4.15}$$

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a_{jr} c_{jk} a_{ks} = 2\zeta_r \omega_r \delta_{rs} \quad (r, s=1, 2, \dots, n) \tag{4.16}$$

where $\zeta_r$ and $\omega_r$ are respectively, the modal damping ratios and modal frequencies of the equivalent linear system (4.13). Equations (4.16) can be solved for $c_{jk}$:

$$c_{jk} = 2 \sum_{q=1}^{n} \sum_{r=1}^{n} \sum_{s=1}^{n} a_{qr} m_{qj} \zeta_r \omega_r a_{sr} m_{sk} \tag{4.17}$$

Since $\zeta_r$ are known, $C$ can be determined from this equation once $K$ is known.

Let $k_j$ denote the equivalent linear spring constant of $s_j(y_j)$. Then it is found that

$$k_j = \frac{E\left[s_j(y_j)y_j\right]}{E\left[y_j^2\right]} = \sqrt{\frac{2}{\pi E\left[y_j^2\right]}} \, f_u e^{\alpha} \text{erfc}(\alpha) \qquad (4.18)$$

where

$$\alpha = \sqrt{\frac{2}{E\left[y_j^2\right]}} \, \frac{f_u}{\pi k^{(o)}} \qquad (4.19)$$

Following the same procedure as used in Example 1, we can solve (4.18) and (4.13) numerically to find all the instantaneous correlation functions. The results of a three-degree-of-freedom system are plotted in Figures 4.7 to 4.10.

Figure 4.7 shows that, as the excitation level increases, the mean square displacement of the first spring increases much faster than that of the second and the third springs. Hence, if this nonlinear system is subjected to severe excitation, the mean square displacement of the first spring will be dominant.

In Figure 4.8 the mean square displacement is plotted against the nonlinear factor $k^{(o)}/f_u$ of the springs. Note that when $k^{(o)}/f_u$ approaches zero, Equation (4.10) reduces to

$$s_j(y_j) = k^{(o)}y_j \quad , \quad k^{(o)} \neq 0 \qquad (4.20)$$

Hence the system becomes linear. Figure 4.8 shows that the mean square displacement of the second and the third springs does not differ too much from the linear solution $(k^{(o)}/f_u = 0)$. As the nonlinear factor $k^{(o)}/f_u$ increases, the mean square displacement of

the third spring actually decreases monotonously, while that of the

second spring increases slowly at first, then also decreases when

$k^{(o)}/f_u$ is greater than 2. However, the mean square displacement

of the first spring differs considerably from the linear solution

when the nonlinear factor is large. For example, at $k^{(o)}/f_u = 1.5$

the mean square value, $\sigma_{y_1}^2$, of the nonlinear system is about four

times as large as that of the linear system.

In Figures 4.9 and 4.10 the excitation is a clipped white

noise. These figures show respectively the mean square displace-

ments of the first and the third springs as a function of $\omega_c$. They

also indicate the contributions from different modes of the equiva-

lent linear system when the excitation is a white noise. For the

first spring the contribution from the first mode is the largest mean

square displacement in that mode while the contributions from the

other modes are considerably smaller. Hence in Figure 4.9 one

sees that the contributions to the first spring from the higher modes

is negligible compared to that from the first mode. However, for

other than the first spring the mean square displacement in the first

mode may be comparable with those in the other modes. Therefore,

in Figure 4.10 one sees that for the third spring the contribution

from the second mode may be of the same order of magnitude as

that from the first mode. Thus the response $y_1$ will be essentially

a narrow band process while the response $y_3$ (also the response $y_2$)

may in general be much less so.

From the above results the following conclusions can be made.

(1) Since only the response of the first spring can be accurately considered as a narrow band process, the use of a white noise to replace an actual excitation may be not a good approximation in this case and the actual excitation should be used if possible. This will, of course, be true for n-degree-of-freedom system.

(2) For the second and third springs the mean square displacements obtained from the linear analysis and the equivalent linearization approach are quite close. However, for the first spring the two solutions will be close only if the excitation is sufficiently low and the nonlinear solution will be much larger than the linear solution if the excitation is large. Therefore, the usual linear analysis furnishes a good approximation only for low excitation. If the excitation is severe, the linear analysis is unconservative for design purposes and the system should be considered as nonlinear. For large systems (n>3) the big difference between the two solutions may not be limited to the first spring.

## V. SUMMARY AND CONCLUSIONS

A matrix algebra approach for determining the mean vector and the instantaneous correlation matrices of the stationary random response of a multidegree-of-freedom linear system is presented in Chapter II. Its derivation is quite straightforward and involves only some simple matrix algebra. Its application consists of two steps: evaluating some simple integrals and solving a system of linear algebraic equations. The first step sometimes becomes very trivial. For example, if the excitation is white, then the integrals can be readily evaluated since their integrand contains a Dirac delta function. The second step can be done by a digital computer and a single program will be valid for any kind of multidegree-of-freedom linear systems. From the comparison of this approach and the existing approaches, it seems that if one is only interested in instantaneous correlation matrices, this approach will be simpler than other approaches.

Several approaches are available for determining the stationary response of a multidegree-of-freedom nonlinear system, but each of them has certain limitations. In all these approaches the excitation is assumed to be stationary and Gaussian. The Fokker-Planck approach is the only approach available which can generate an exact solution for some nonlinear systems, but its applicability is rather limited because of the following restrictions:

(1) the damping force must be proportional to the velocity;

(2) the correlation function matrix of the excitation must be proportional to the damping matrix of the system;

(3) the excitation must be a Gaussian white noise.

The perturbation approach may be used only when the solution of a nonlinear system is close to its linearized solution. Hence it requires that

(1) the nonlinear system possess a linearized solution, that is, $C^{(o)}$ is not a null matrix;

(2) both the system nonlinearities and the excitation be sufficiently small.

Under these conditions this approach reduces the nonlinear differential equations of motion to several sets of linear differential equations.

The normal mode approach is perhaps the simplest one among the existing approaches. It reduces an n-degree-of-freedom nonlinear system to n single-degree-of-freedom linear systems. However, the restrictions imposed on this approach are rather severe. It requires that the linearized system have normal modes and that the correlation function matrix of the excitation be diagonalized by the same matrix that uncouples the linearized system. The first condition may not be too serious, but the second condition on the excitation makes the application of this approach quite limited.

In the present study a generalized equivalent linearization approach for determining the instantaneous correlation matrices of the stationary random response of a multidegree-of-freedom nonlinear system has been presented. The only restrictions on its

application are that the excitation is stationary and Gaussian. The equivalent linear system is formed by replacing each nonlinear element in the original system by a linear spring and a linear damper. After applying the matrix algebra approach to the equivalent linear system and the special iteration scheme described in Chapter III, one is led to the repeated solution of a system of linear algebraic equations.

It is implicitly assumed that in order to obtain a good approximate solution the nonlinearities of the system must be small. However, this approach can also be used to generate approximate solution for systems with large nonlinearities. Two examples which can be solved exactly by the Fokker-Planck approach were also worked out by the generalized equivalent linearization approach. A comparison of the results shows that for a cubic hardening system the error in the root mean square displacement is always within 7.5% of the exact solution and that for an arctangent softening system the largest error is 11.4%. Therefore, it seems that even for large nonlinearities, this approximate approach still gives very reasonable results.

Two more examples which can be solved by the new approach were given in Chapter IV. The first example, linear springs and dampers whose damping force is proportional to the square of the velocity were used. The results show that the dampers used in this example are particularly suitable for systems which are subjected to severe excitation. In the second example, softening springs which follow an arctangent law, were used. The linear dampers were arranged in such a way that the equivalent linear system of the nonlinear

system has normal modes and the damping in each mode is specified. The results show that the usual linearized analysis is unconservative for design purpose.

## APPENDIX A

### Multidegree-of-Freedom Systems Under Gaussian White Noise Excitation

Under certain conditions (see Section 2.2.3) Wang and Uhlenbeck[19] have solved the system (2.1) by the Fokker-Planck approach. However, their approach can be extended to more general problems.

Consider the following equation

$$M\ddot{\overline{x}} + C\dot{\overline{x}} + K\overline{x} = \overline{s}(t) \tag{A.1}$$

where $\overline{s}(t)$ is a Gaussian white noise vector. Without loss of generality, assume that

$$\left.\begin{array}{l} E[\overline{s}(t)] = O \\[2mm] E\left[\overline{s}(t)\overline{s}^T(t-\tau)\right] = 2\pi W \delta(\tau) \end{array}\right\} \tag{A.2}$$

where $W$ is the spectral density matrix of $\overline{s}(t)$. The stationary response $\overline{x}, \dot{\overline{x}}$ will form a $2n$ dimensional continuous Markovian process $\overline{z} = (\overline{x}, \dot{\overline{x}})$ whose first probability density $p(\overline{z})$ is governed by the stationary Fokker-Planck equation

$$\sum_{j=1}^{2n} \frac{\partial}{\partial z_j}(a_j p) - \frac{1}{2}\sum_{k=1}^{2n}\sum_{j=1}^{2n} \frac{\partial^2}{\partial z_k \partial z_j}(b_{kj}p) = 0 \tag{A.3}$$

From (2.31) one easily finds that

$$B = \begin{pmatrix} O & O \\[2mm] O & 2\pi M^{-1}W(M^{-1})^T \end{pmatrix} \tag{A.4}$$

and

$$\bar{a} = D\bar{z} \quad \text{where} \quad D = \begin{pmatrix} O & I \\ -M^{-1}K & -M^{-1}C \end{pmatrix} \tag{A.5}$$

Instead of solving (A.3) directly we make first the linear transformation

$$\bar{z} = A\bar{u} \tag{A.6}$$

The matrix A is the matrix which diagonalizes D, so that:

$$A^{-1}DA = \Lambda \tag{A.7}$$

where $\Lambda$ is a diagonal matrix which consists of the eigenvalues of D. Since it is assumed the system (A.1) possesses a stationary response, hence all eigenvalues will have a negative real part. After applying the linear transformation (A.6), Equation (A.3) takes the form:

$$\sum_{j=1}^{2n} \lambda_j \frac{\partial}{\partial u_j}(u_j p) - \frac{1}{2} \sum_{k=1}^{2n} \sum_{j=1}^{2n} v_{kj} \frac{\partial^2 p}{\partial u_k \partial u_j} = 0 \tag{A.8}$$

where

$$V = A^{-1}B(A^{-1})^T \tag{A.9}$$

Let $h(\bar{\xi})$ denote the characteristic function of $\bar{u}$ which is just the Fourier transform of $p(\bar{u})$:

$$h(\bar{\xi}) = \int_{-\infty}^{\infty} \ldots \int p(\bar{u}) \exp\left\{-i\sum_{j=1}^{2n} \xi_j u_j\right\} du_1 \ldots du_{2n} \tag{A.10}$$

Then in terms of $h(\bar{\xi})$, Equation (A.8) becomes

$$\sum_{j=1}^{2n} \lambda_j \xi_j \frac{\partial h(\xi)}{\partial \xi_j} - \frac{1}{2} h(\bar{\xi}) \sum_{k=1}^{2n} \sum_{j=1}^{2n} v_{kj} \xi_k \xi_j = 0 \tag{A.11}$$

By direct substitution it can be easily shown that

$$h(\xi) = \exp\left\{\frac{1}{2}\sum_{j=1}^{2n}\sum_{k=1}^{2n}\frac{v_{jk}}{\lambda_j+\lambda_k}\xi_j\xi_k\right\} \tag{A.12}$$

From the definition of characteristic function and the properties of a Gaussian distribution, one knows that $\bar{u}$ is Gaussian distributed with zero means and its variances and covariances are given by

$$E[u_ju_k] = -\frac{v_{jk}}{\lambda_j+\lambda_k} \tag{A.13}$$

In the matrix form Equation (A.13) becomes

$$\Lambda E\left[\bar{u}\,\bar{u}^T\right] + E\left[\bar{u}\,\bar{u}^T\right]\Lambda = -V \tag{A.14}$$

Substituting (A.6) into (A.14), using (A.7) and (A.1), one finds that

$$D E\left[\bar{z}\,\bar{z}^T\right] + E\left[\bar{z}\,\bar{z}^T\right]D^T = -B \tag{A.15}$$

Substituting (A.4) and (A.5) into (A.15) and noting that

$$E\left[\bar{z}\,\bar{z}^T\right] = \begin{pmatrix} E\left[\bar{x}\,\bar{x}^T\right] & E\left[\bar{x}\,\dot{\bar{x}}^T\right] \\ E\left[\dot{\bar{x}}\,\bar{x}^T\right] & E\left[\dot{\bar{x}}\,\dot{\bar{x}}^T\right] \end{pmatrix} \tag{A.16}$$

one obtains

$$E\left[\bar{x}\,\dot{\bar{x}}^T\right] + E\left[\dot{\bar{x}}\,\bar{x}^T\right] = O$$

$$ME\left[\dot{\bar{x}}\,\dot{\bar{x}}^T\right] - CE\left[\dot{\bar{x}}\,\bar{x}^T\right] - KE\left[\bar{x}\,\bar{x}^T\right] = O$$

$$\tag{A.17}$$

$$ME\left[\dot{\bar{x}}\,\dot{\bar{x}}^T\right]C^T + ME\left[\dot{\bar{x}}\,\bar{x}^T\right]K^T$$

$$+ CE\left[\dot{\bar{x}}\,\dot{\bar{x}}^T\right]M^T + KE\left[\bar{x}\,\dot{\bar{x}}^T\right]M^T = 2\pi W$$

If M, C, K are symmetric and

$$\pi W = \gamma C$$

then by direct substitution it can be shown that the following solution

found by Wang and Uhlenbeck

$$E\left[\bar{x}\,\dot{\bar{x}}^T\right] = 0$$

$$E\left[\bar{x}\,\bar{x}^T\right] = \gamma K^{-1} \qquad\qquad (A.18)$$

$$E\left[\dot{\bar{x}}\,\dot{\bar{x}}^T\right] = \gamma M^{-1}$$

also satisfies (A.17).

## APPENDIX B

### Some Properties of the Correlation Functions of a Stationary Random Vector Process

Let x(t) be a stationary random vector process which is assumed to be differentiable in mean square to the required order and $R_{\bar{x}}(\tau)$ be its cross-correlation function matrix. Then we have

$$R_{\bar{x}}(\tau) = E\left[\bar{x}(t)\bar{x}^T(t+\tau)\right] = E\left[\bar{x}(t-\tau)\bar{x}^T(t)\right] \qquad (B.1)$$

Differentiating (B.1) with respect to $\tau$ gives

$$\frac{d}{d\tau} R_{\bar{x}}(\tau) = E\left[\bar{x}(t)\dot{\bar{x}}^T(t+\tau)\right] = -E\left[\dot{\bar{x}}(t-\tau)\bar{x}^T(t)\right] \qquad (B.2)$$

The stationarity of $\bar{x}$ implies that (B.2) can be rewritten as

$$\frac{d}{d\tau} R_{\bar{x}}(\tau) = E\left[\bar{x}(t-\tau)\dot{\bar{x}}^T(t)\right] = -E\left[\dot{\bar{x}}(t)\bar{x}^T(t+\tau)\right] \qquad (B.3)$$

Differentiating (B.2) and (B.3) with respect to $\tau$ again yields

$$\frac{d^2}{d\tau^2} R_{\bar{x}}(\tau) = E\left[\bar{x}(t)\ddot{\bar{x}}^T(t+\tau)\right] = E\left[\ddot{\bar{x}}(t-\tau)\bar{x}^T(t)\right]$$

$$= -E\left[\dot{\bar{x}}(t-\tau)\dot{\bar{x}}^T(t)\right] = -E\left[\dot{\bar{x}}(t)\dot{\bar{x}}^T(t+\tau)\right] \qquad (B.4)$$

Finally, setting $\tau = 0$ in (B.2) and (B.4) gives

$$E\left[\bar{x}\dot{\bar{x}}^T\right] + E\left[\dot{\bar{x}}\bar{x}^T\right] = 0$$

$$E\left[\bar{x}\ddot{\bar{x}}^T\right] = E\left[\ddot{\bar{x}}\bar{x}^T\right] = -E\left[\dot{\bar{x}}\dot{\bar{x}}^T\right] \qquad (B.5)$$

where the common argument t has been omitted. Thus $E\left[\bar{x}\dot{\bar{x}}^T\right]$ is antisymmetric and $E\left[\bar{x}\ddot{\bar{x}}^T\right]$ is symmetric.

Further, the above results can be generalized to give

$$E\left[\left(\frac{d^g}{dt^g}\,\overline{x}\right)\left(\frac{d^h}{dt^h}\,\overline{x}^T\right)\right] = (-)^s E\left[\left(\frac{d^j}{dt^j}\,\overline{x}\right)\left(\frac{d^k}{dt^k}\,\overline{x}^T\right)\right] \qquad (B.6)$$

where

$$g + h = j + k, \quad s = g + j \ \text{ or } \ s = h + k$$

$$g, h, j, k = 0, 1, \ldots, r$$

and $d^r/dt^r\,\overline{x}$ is the highest derivative of $\overline{x}$ that exists.

## APPENDIX C $\quad E\left[x_1 f(x_2, x_3)\right]$

Suppose that $x_1$, $x_2$ and $x_3$ are Gaussian distributed. Then

$$E\left[x_1 f(x_2, x_3)\right] = \frac{1}{(2\pi)^{3/2}\sqrt{\det(A)}} \int\!\!\!\int\!\!\!\int_{-\infty}^{\infty} x_1 f(x_2, x_3) \exp\left\{-\frac{1}{2}\bar{x}^T A^{-1}\bar{x}\right\} dx_1\, dx_2\, dx_3$$

$$(C.1)$$

where

$$a_{kj} = E\left[x_k x_j\right] \qquad k, j = 1, 2, 3$$

$$\det(A) = \text{the determinant of } A$$

We may also write (C.1) as follows:

$$E\left[x_1 f(x_2, x_3)\right] = \frac{1}{(2\pi)^{3/2}\sqrt{\det(A)}} \int\!\!\!\int\!\!\!\int_{-\infty}^{\infty} f(x_2, x_3) \exp\left\{-\frac{1}{2}(\alpha_{22}x_2^2 + \alpha_{33}x_3^2 + 2\alpha_{23}x_2 x_3)\right\}$$

$$x_1 \exp\left\{-\frac{\alpha_{11}}{2}\left[x_1^2 + 2x_1\left(\frac{\alpha_{12}}{\alpha_{11}}x_2 + \frac{\alpha_{13}}{\alpha_{11}}x_3\right)\right]\right\} dx_1\, dx_2\, dx_3$$

$$= \frac{1}{(2\pi)^{3/2}\sqrt{\det(A)}} \int\!\!\!\int_{-\infty}^{\infty} f(x_2, x_3) \exp\left\{-\frac{1}{2\alpha_{11}}\left[\left(\alpha_{11}\alpha_{22} - \alpha_{12}^2\right)x_2^2 + \right.\right.$$

$$\left.\left. + \left(\alpha_{33}\alpha_{11} - \alpha_{13}^2\right)x_3^2 + 2\left(\alpha_{23}\alpha_{11} - \alpha_{12}\alpha_{13}\right)x_2 x_3\right]\right\}$$

$$\left\{\int_{-\infty}^{\infty} x_1 \exp\left[-\frac{1}{2\alpha_{11}}\left(\alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3\right)^2\right] dx_1\right\} dx_2\, dx_3$$

$$(C.2)$$

where $\alpha_{kj}$ = the (k, j) element of $A^{-1}$ = the co-factor of $a_{jk}$ in A.

Now consider the integral

$$b = \int_{-\infty}^{\infty} x_1 \exp\left[-\frac{1}{2\alpha_{11}}\left(\alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3\right)^2\right] dx_1 \qquad (C.3)$$

which reduces by the change of variable $z^2 = \dfrac{1}{2\alpha_{11}} \left( \displaystyle\sum_{j=1}^{3} \alpha_{1j} x_j \right)^2$ to the form

$$b = \int_{-\infty}^{\infty} \left( \sqrt{\frac{2}{\alpha_{11}}}\, z - \frac{\alpha_{12}}{\alpha_{11}} x_2 - \frac{\alpha_{13}}{\alpha_{11}} x_3 \right) e^{-z^2} \frac{2\, dz}{\sqrt{2\alpha_{11}}} \qquad (C.4)$$

By making use of results

$$\int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi} \quad , \quad \int_{-\infty}^{\infty} z e^{-z^2}\, dz = 0 \qquad (C.5)$$

(C.4) reduces to

$$b = -\sqrt{\frac{2\pi}{\alpha_{11}}} \left( \frac{\alpha_{12}}{\alpha_{11}} x_2 + \frac{\alpha_{13}}{\alpha_{11}} x_3 \right) \qquad (C.6)$$

Hence (C.2) becomes

$$E[x_1 f(x_2, x_3)] = \frac{-1}{(2\pi)^{3/2} \sqrt{\det(A)}} \sqrt{\frac{2\pi}{\alpha_{11}}} \int\!\!\int_{-\infty}^{\infty} \left( \frac{\alpha_{12}}{\alpha_{11}} x_2 + \frac{\alpha_{13}}{\alpha_{11}} x_3 \right) \cdot f(x_2, x_3)$$

$$\cdot \exp\left\{ -\frac{1}{2} \binom{x_2}{x_3}^T A_{11}^{-1} \binom{x_2}{x_3} \right\} dx_2\, dx_3$$

$$= -\left\{ \frac{\alpha_{12}}{\alpha_{11}} E[x_2 f(x_2, x_3)] + \frac{\alpha_{13}}{\alpha_{11}} E[x_3 f(x_2, x_3)] \right\} \qquad (C.7)$$

where

$$A_{11} = \text{the minor of } a_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \qquad (C.8)$$

and the following relation has been used

$$\frac{\alpha_{11}\alpha_{kj} - \alpha_{1k}\alpha_{1j}}{\alpha_{11}} = (-1)^{k+j} \frac{a_{rs} \cdot \det(A^{-1})}{\det(A_{11})/\det(A)} = (-1)^{k+j} \frac{a_{rs}}{\det(A_{11})} \quad (C.9)$$

$$k, j, r, s = 2, 3 \qquad r = s \neq k \quad \text{if} \quad k = j$$

$$r = k, \ s = j \quad \text{if} \quad k \neq j$$

With the aid of the definition of the cofactor, we have finally,

$$E[x_1 f(x_2, x_3)] = \frac{a_{21}a_{33} - a_{23}a_{31}}{a_{22}a_{33} - a_{23}a_{32}} E[x_2 f(x_2, x_3)]$$

$$- \frac{a_{21}a_{32} - a_{31}a_{22}}{a_{22}a_{33} - a_{23}a_{32}} E[x_3 f(x_2, x_3)] \qquad (C.10)$$

If $x_2$ and $x_3$ are uncorrelated, for example, $x_3 = \dot{x}_2$, then $a_{23} = a_{32}$ $= E[x_2, x_3)] = 0$. Thus,

$$E[x_1 f(x_2, x_3)] = \frac{a_{21}}{a_{22}} E[x_2 f(x_2, x_3)] + \frac{a_{31}}{a_{33}} E[x_3 f(x_2, x_3)] \qquad (C.11)$$

Replacing $a_{kj}$ by $E[x_k x_j]$, we obtain

$$E[x_1 f(x_2, x_3)] = \frac{E[x_2 x_1]}{E[x_2^2]} E[x_2, x_3)] + \frac{E[x_3 x_1]}{E[x_3^2]} E[x_3 f(x_2, x_3)] \quad (C.12)$$
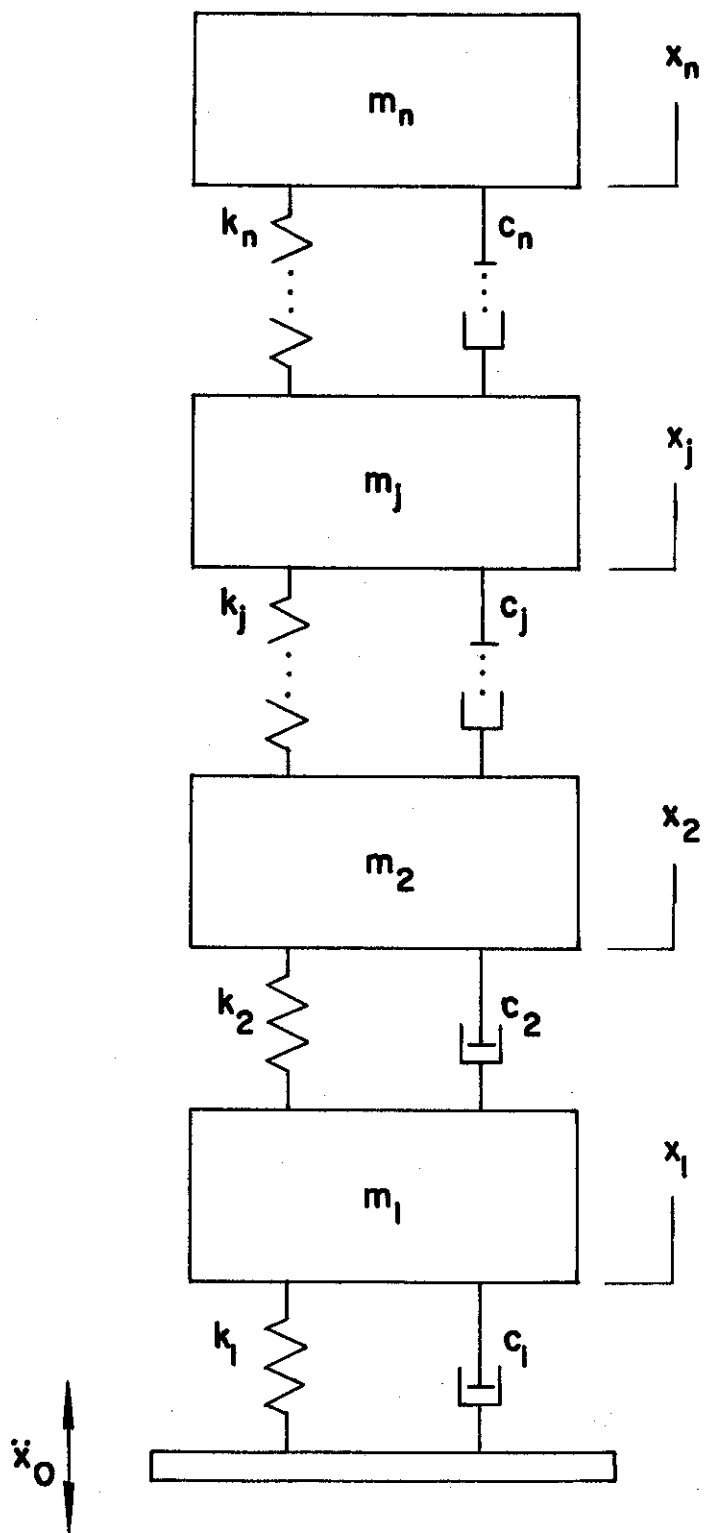
Figure 2.1: An n-degree-of-freedom linear system
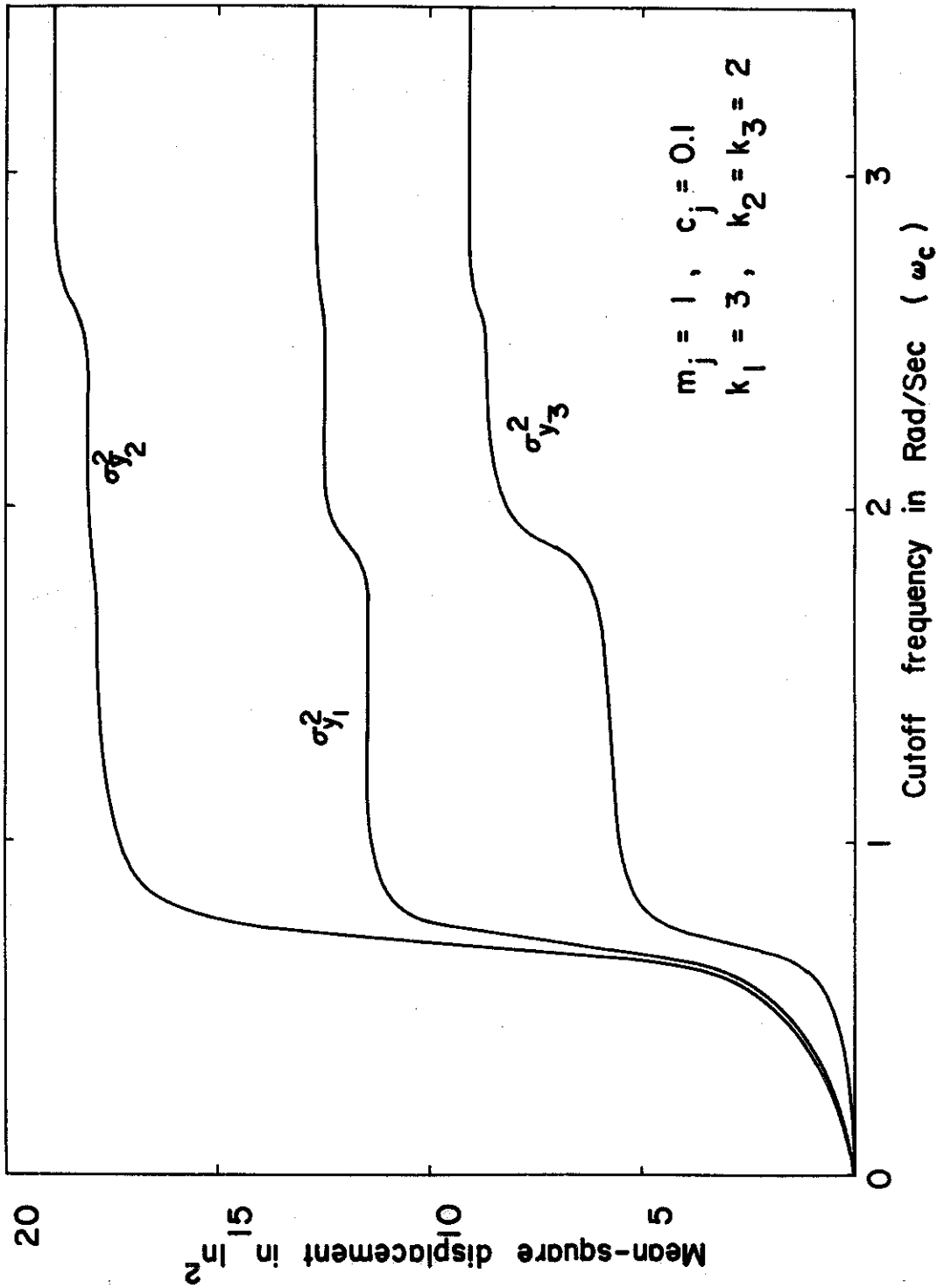
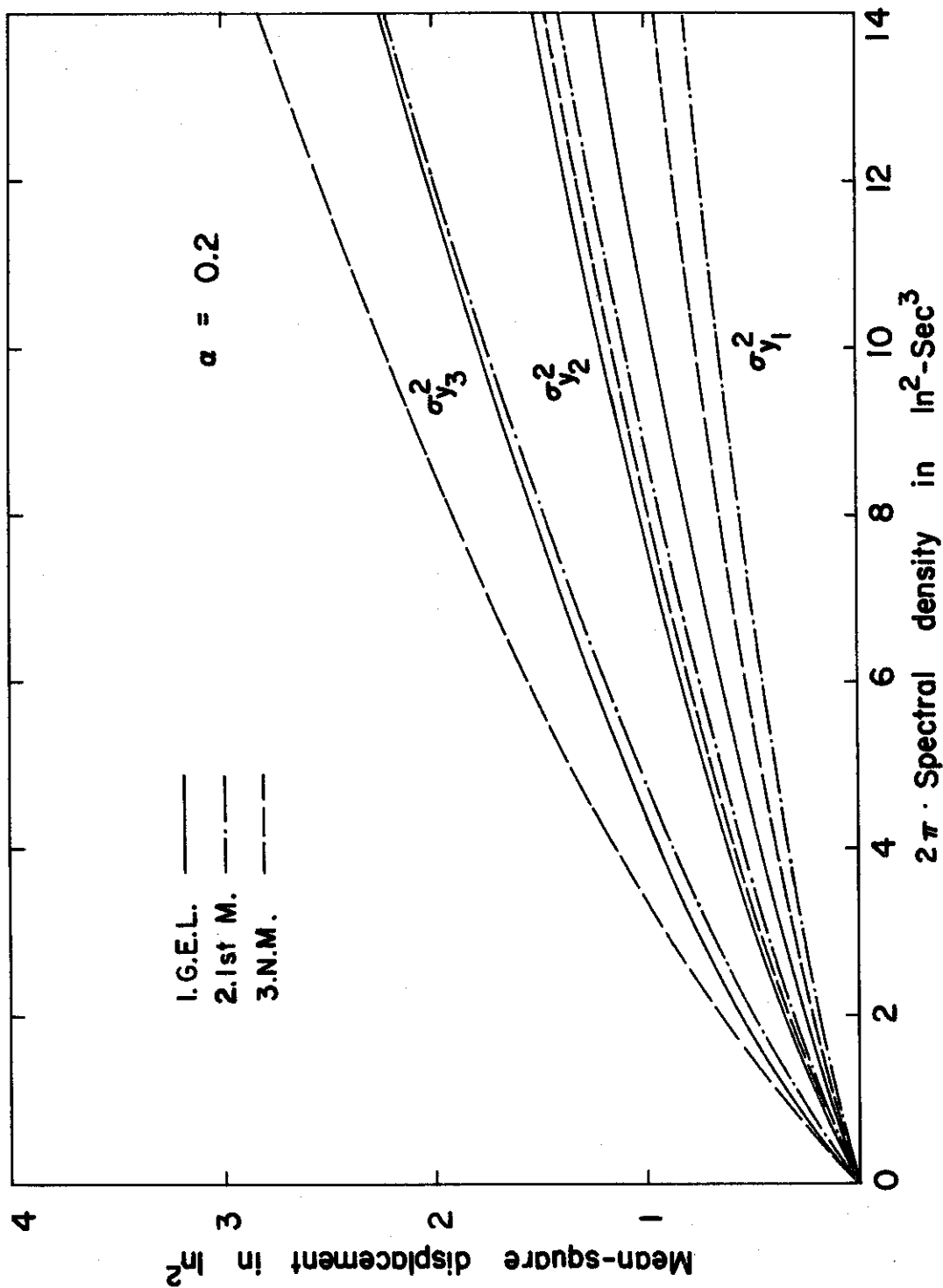Figure 2.2: Stationary mean-square response of a 3-degree-of-freedom system
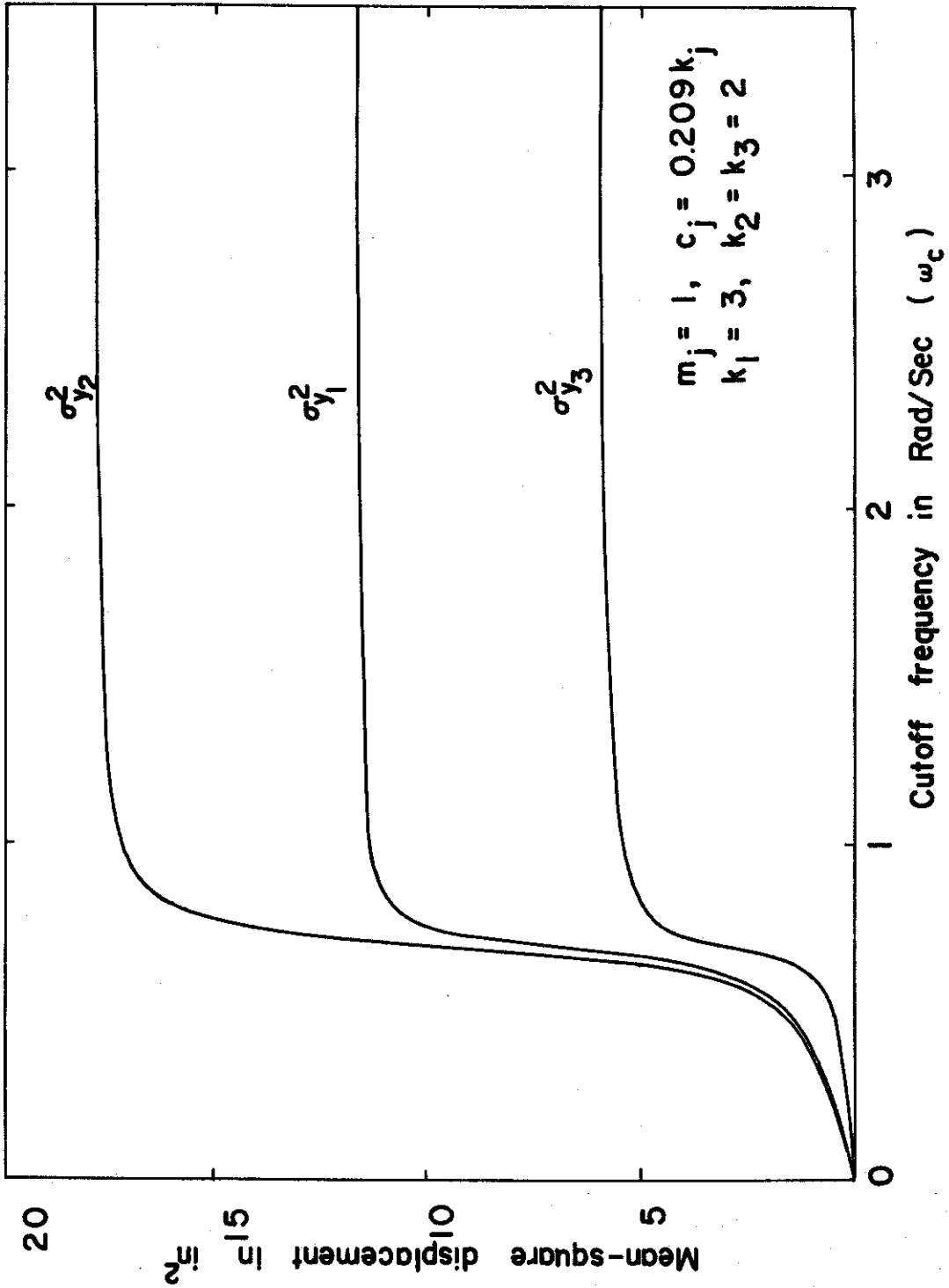
Figure 2.3: Stationary mean-square response of a 3-degree-of-freedom system
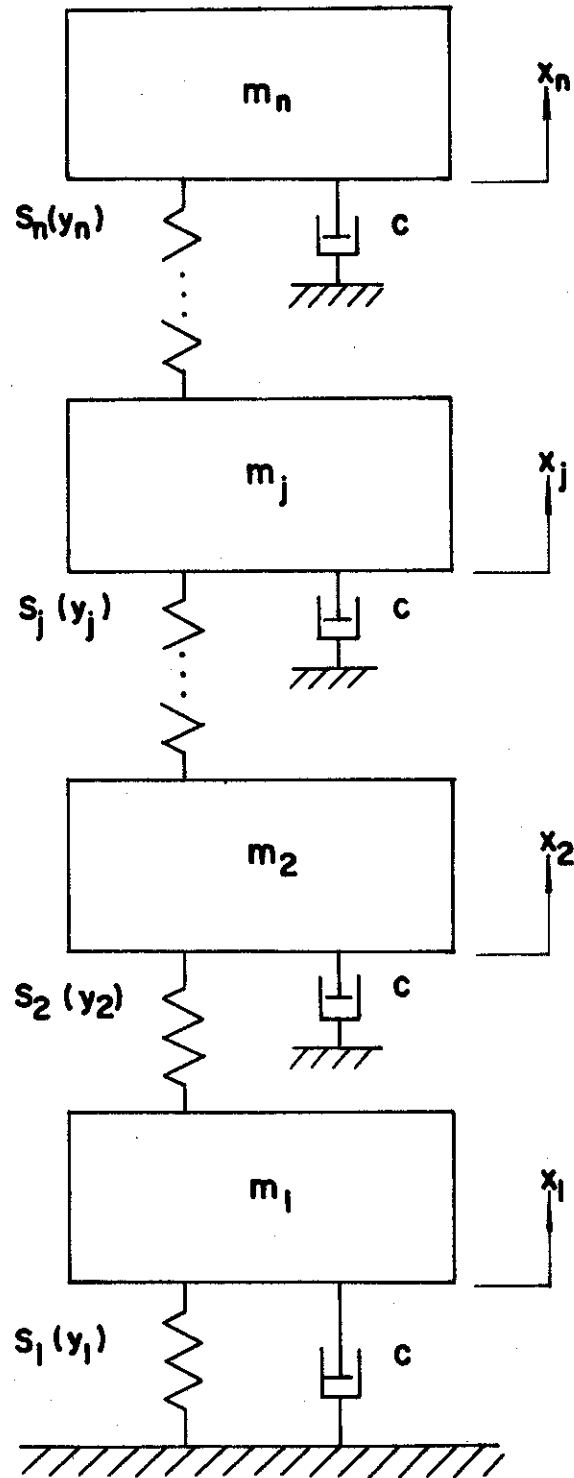
Figure 3.1: An n-degree-of-freedom nonlinear system
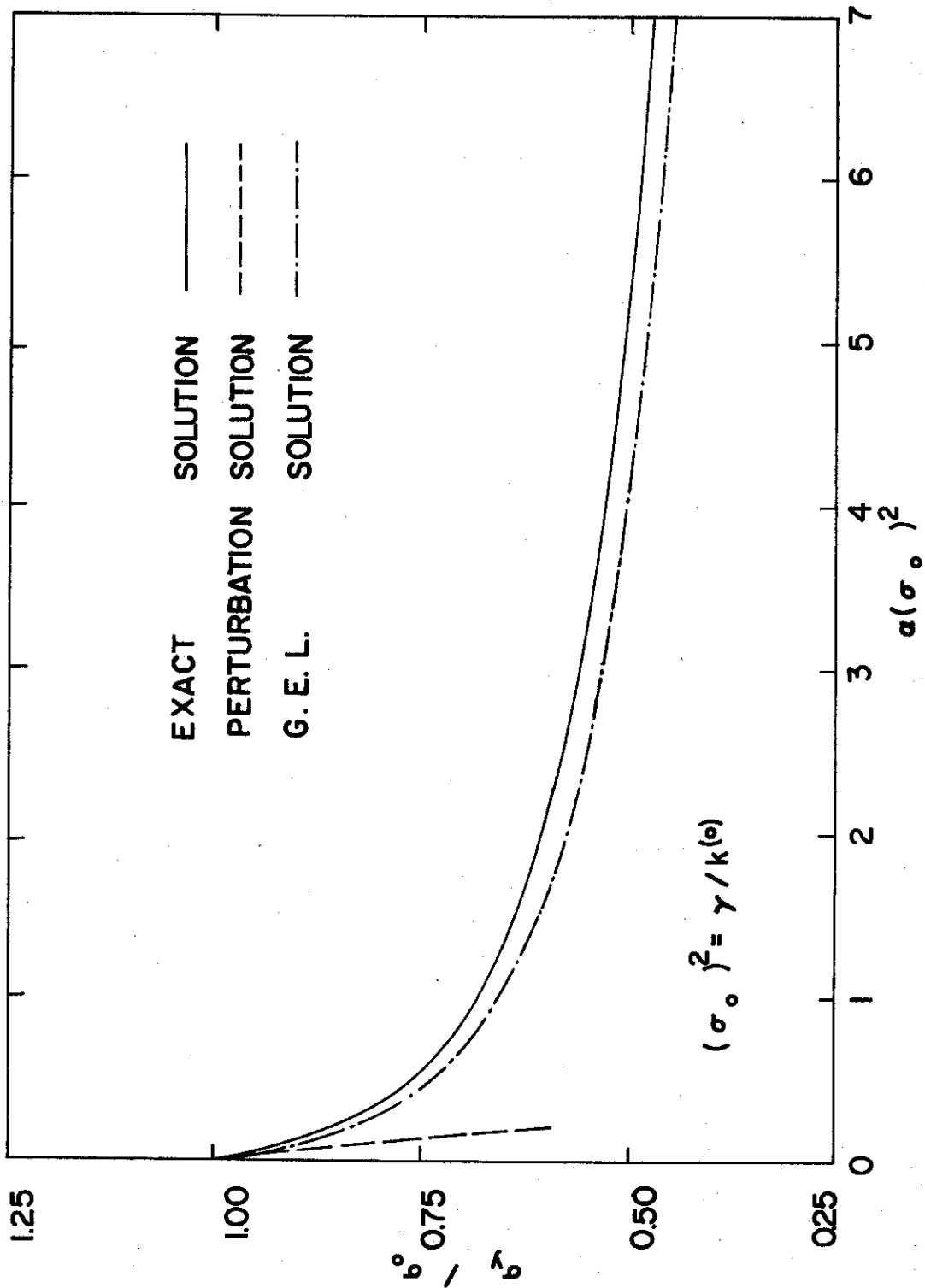
Figure 3.2: Root mean-square displacement of a hardening system

$$f / f_u = (2/\pi) \tan^{-1} (\pi k^{(o)} y / 2 f_u)$$
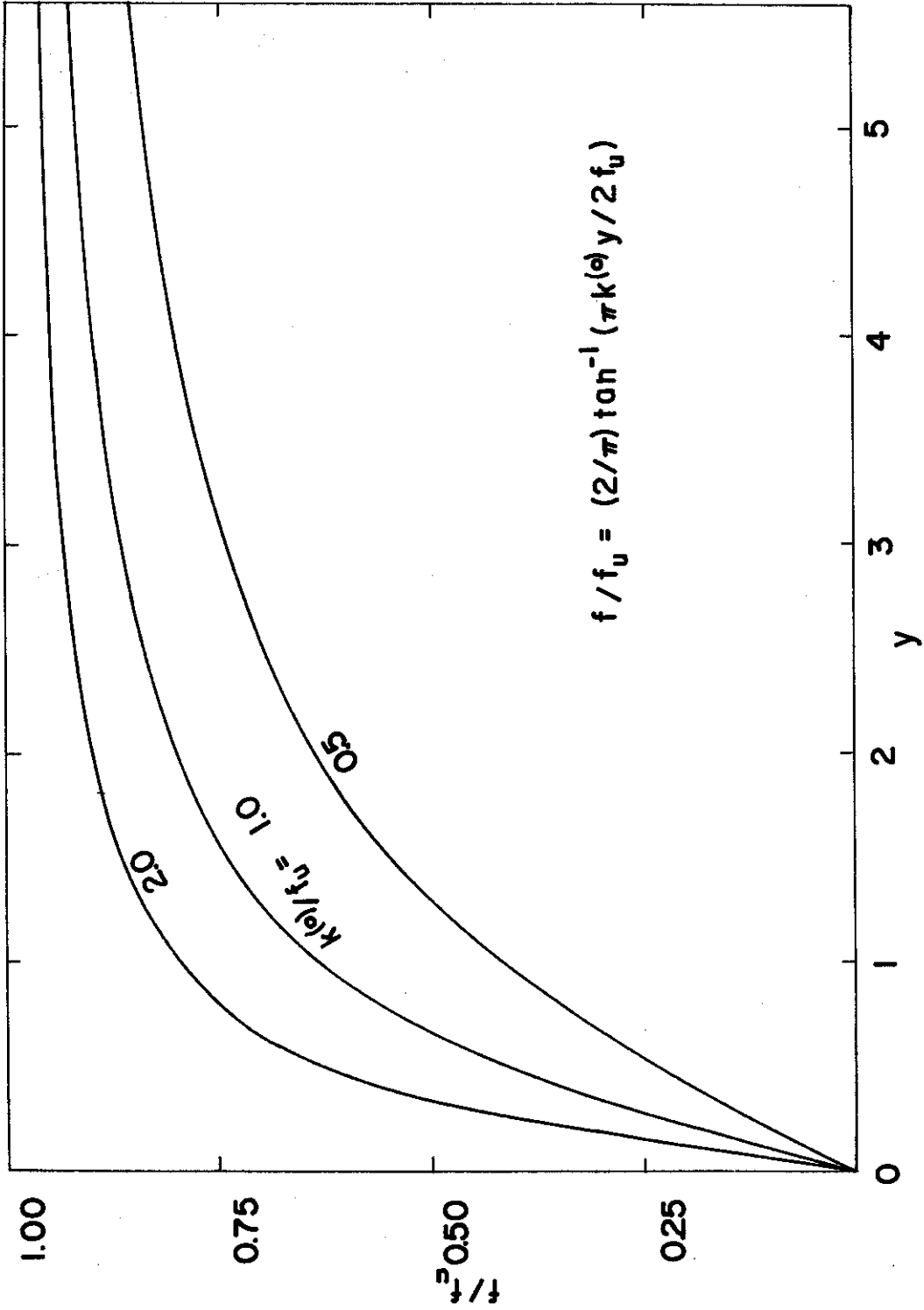
Figure 3.3: Typical force displacement curves for an arctangent softening spring
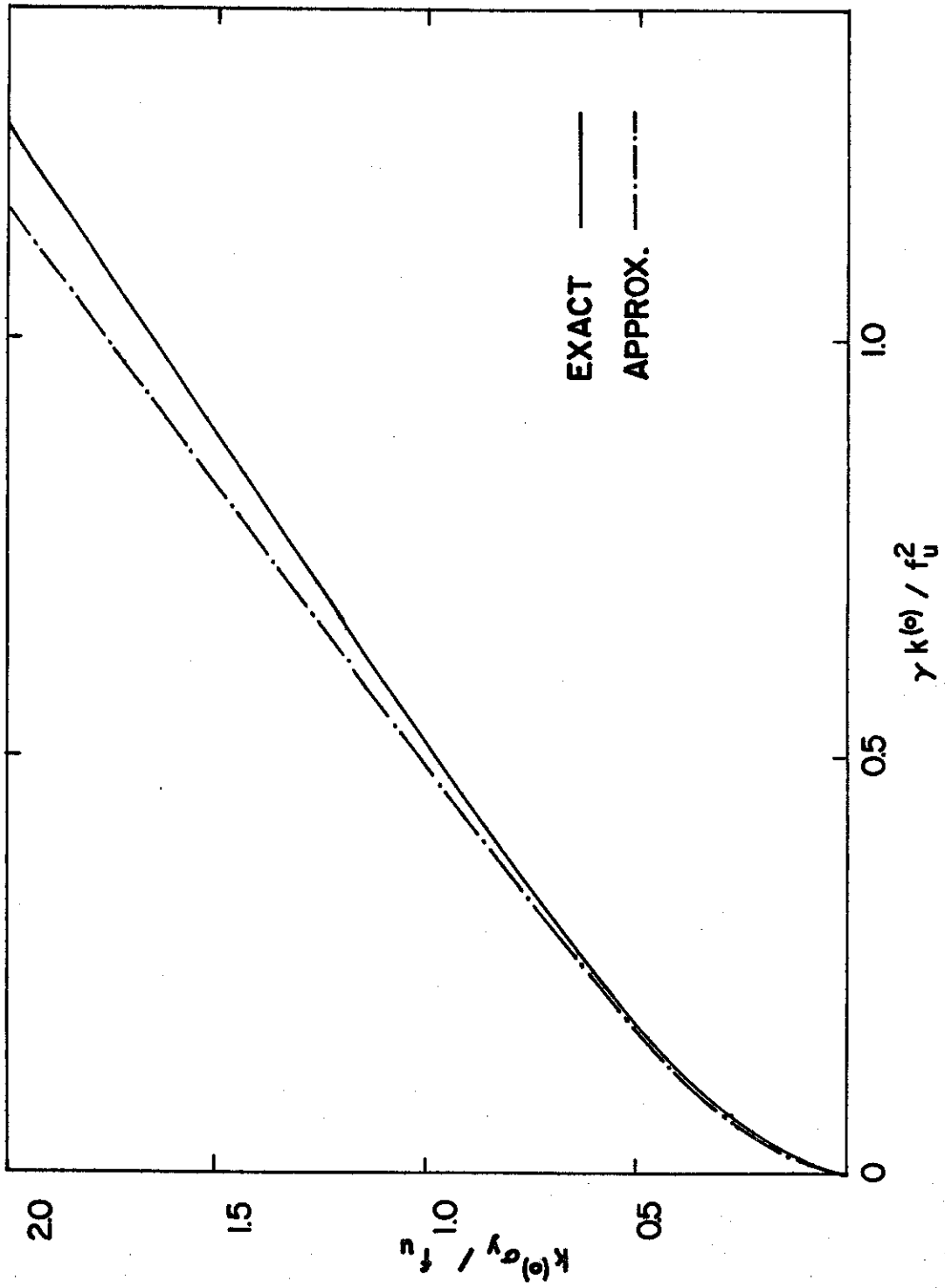
Figure 3.4: Stationary root mean-square response of an arctangent softening system
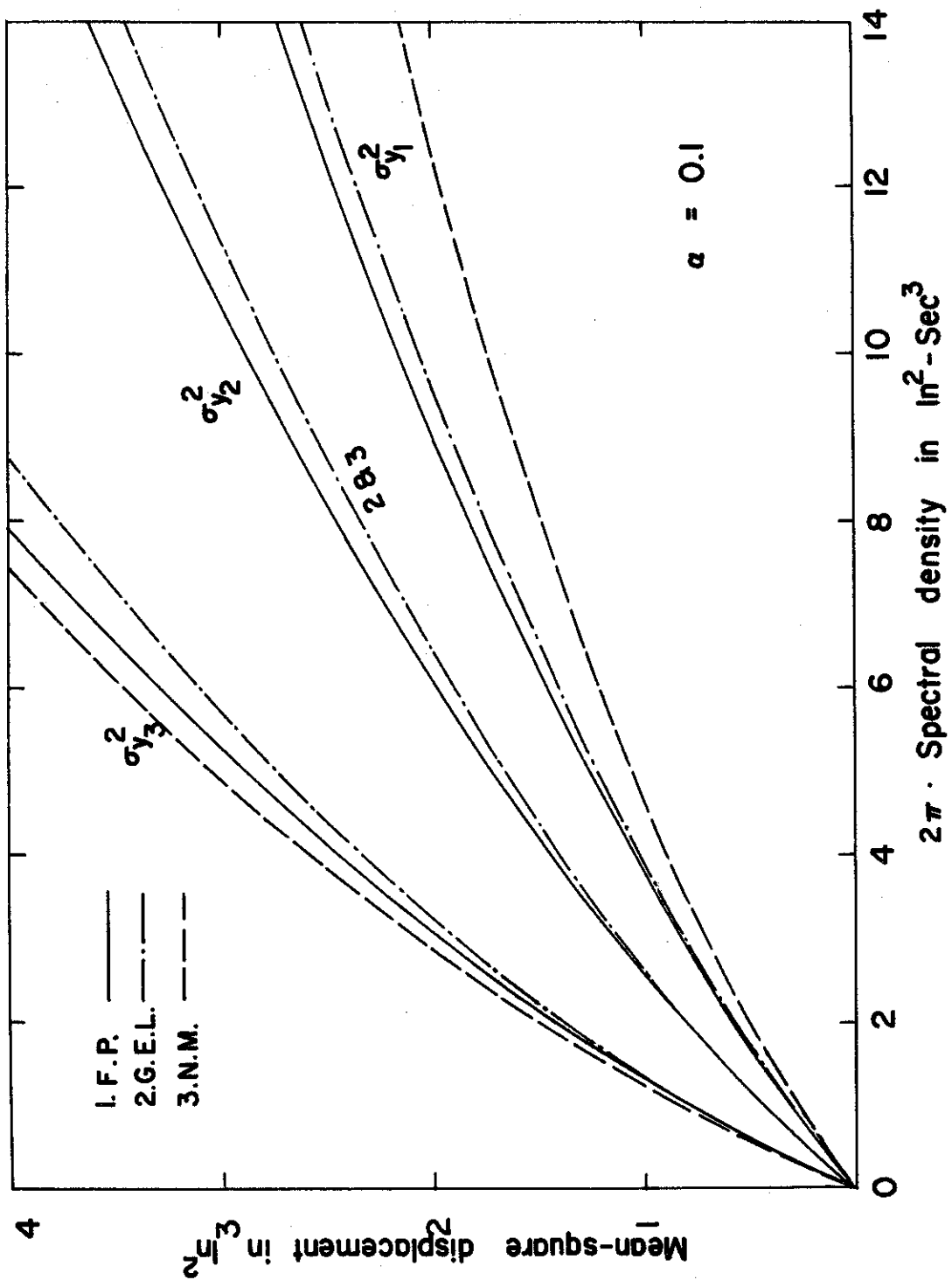
Figure 3.5: Stationary mean-square response of a 3-degree-of-freedom nonlinear system
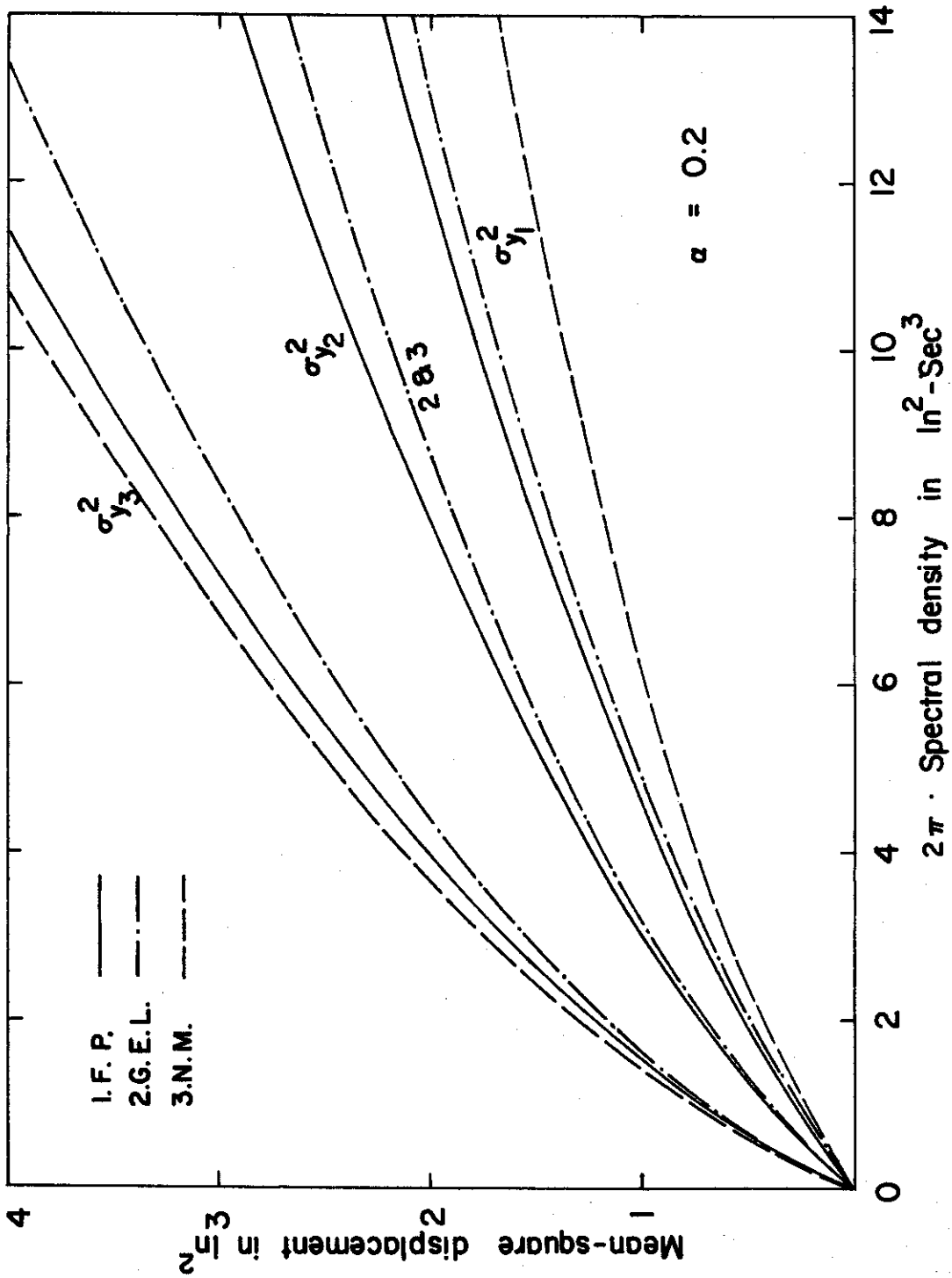
Figure 3.6: Stationary mean-square response of a 3-degree-of-freedom nonlinear system
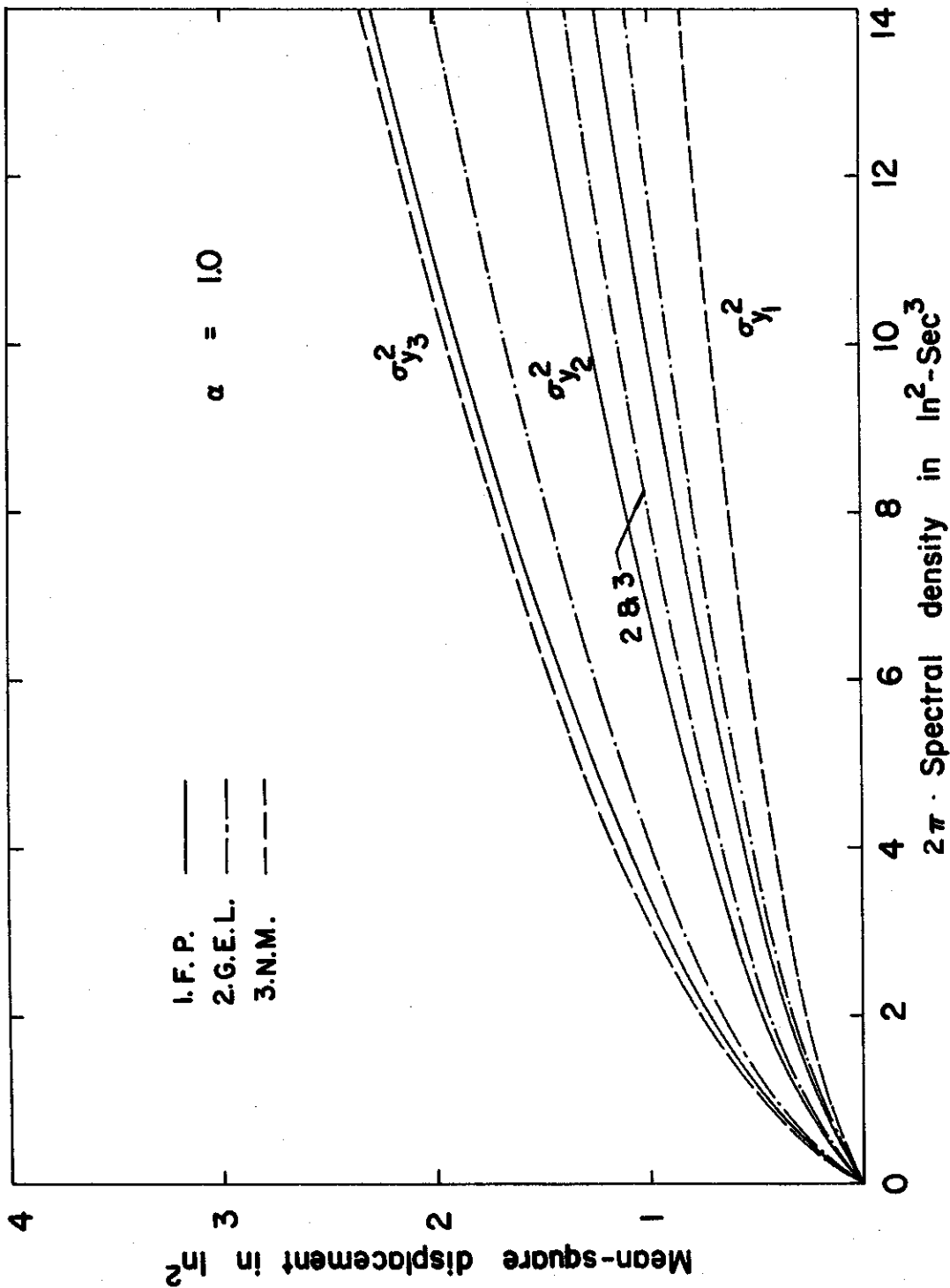
Figure 3.7: Stationary mean-square response of a 3-degree-of-freedom nonlinear system
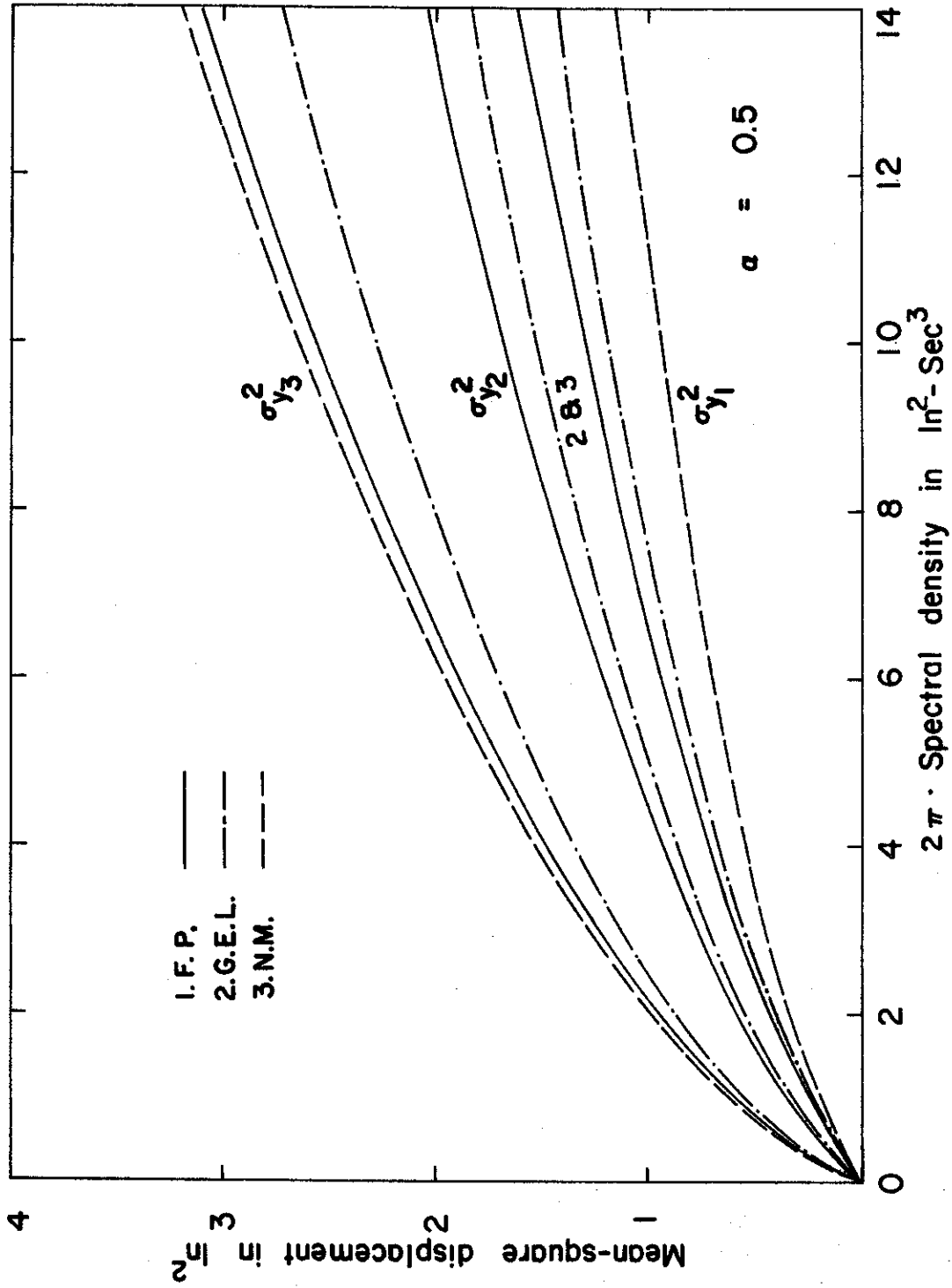
Figure 3.8: Stationary mean-square response of a 3-degree-of-freedom nonlinear system

Figure 3.9: Stationary mean-square response of a 3-degree-of-freedom nonlinear system
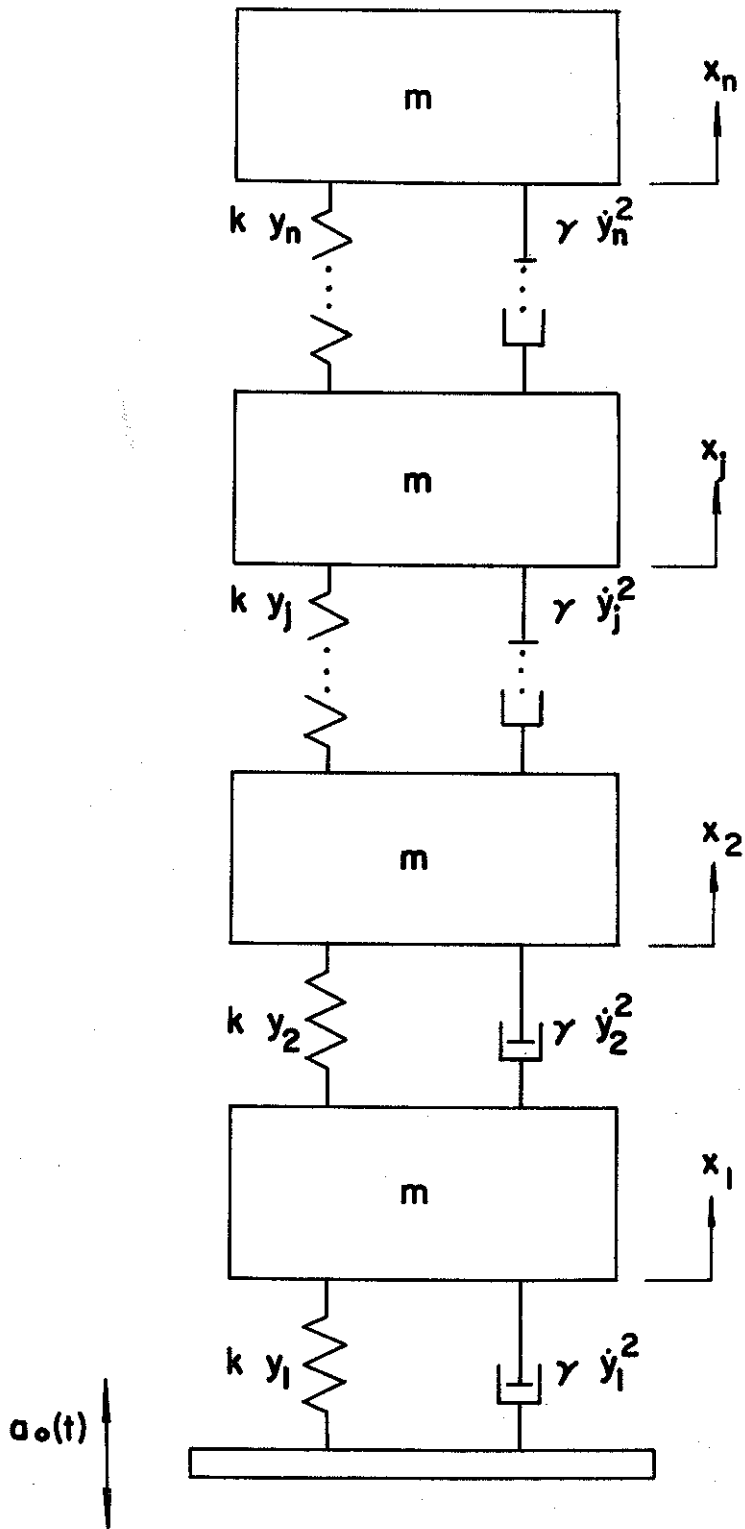
Figure 4.1: An n-degree-of-freedom nonlinear system
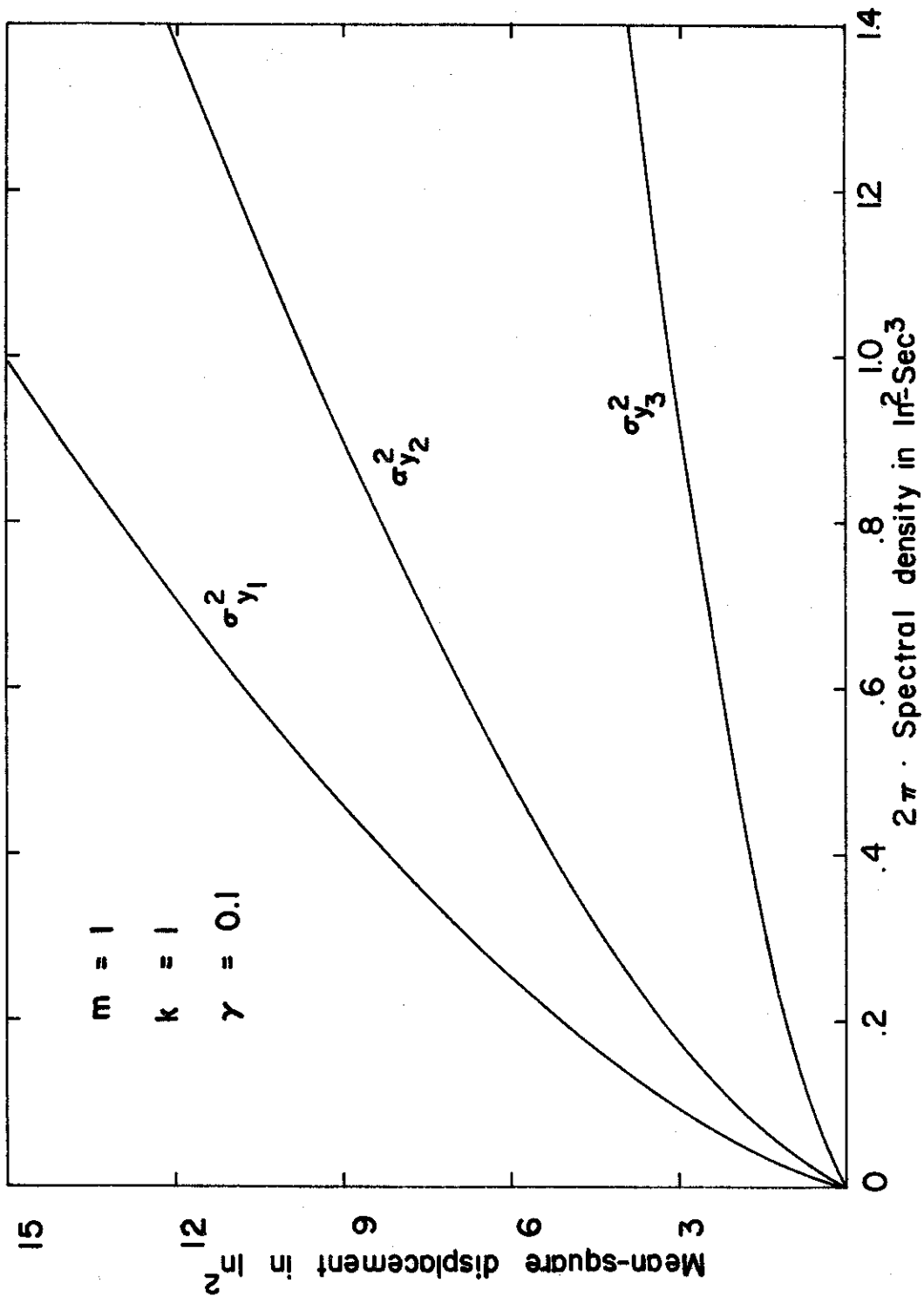
Figure 4.2: Mean-square response as a function of spectral density of base excitation
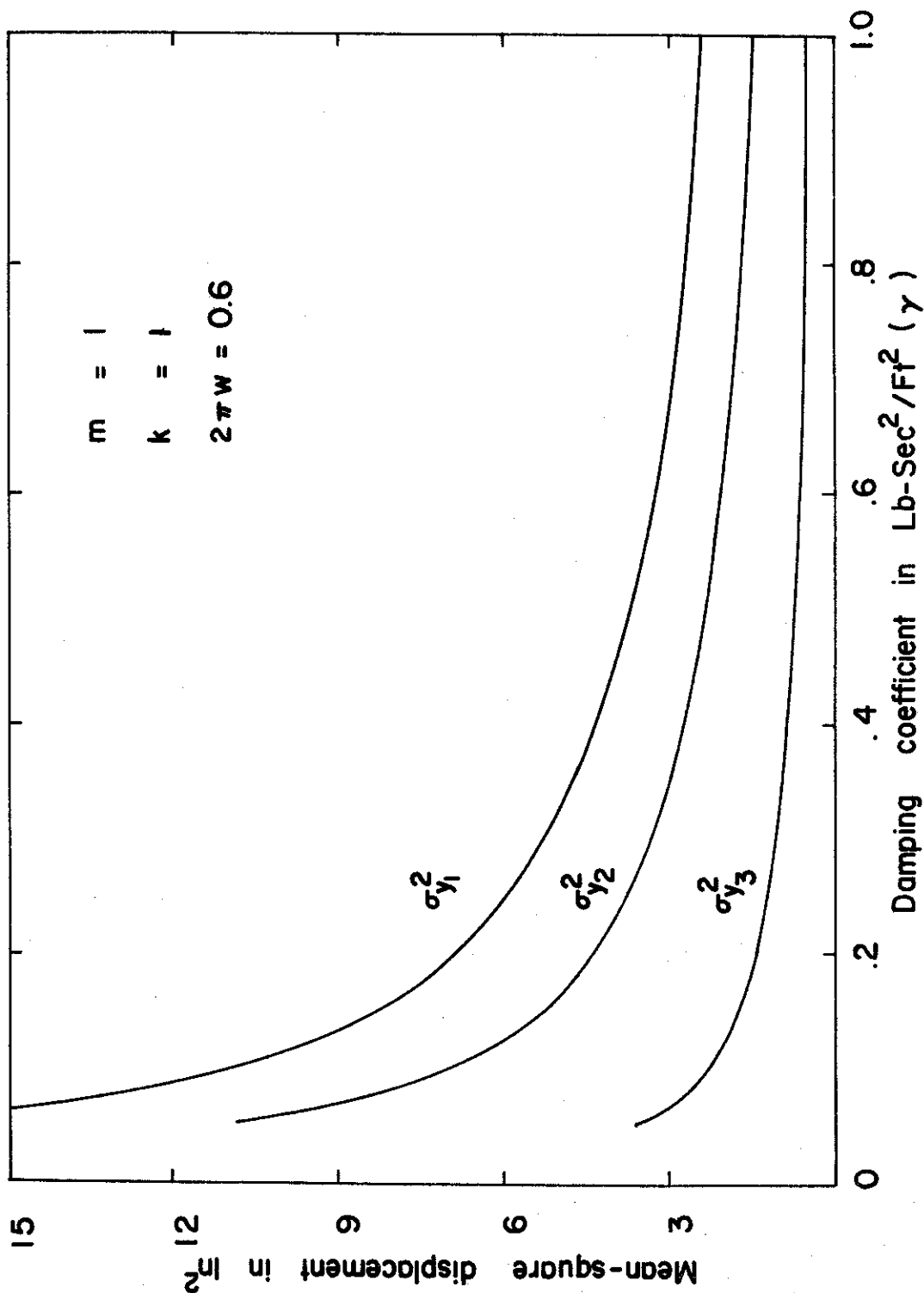
Figure 4.3: Mean-Square response as a function of damping coefficient
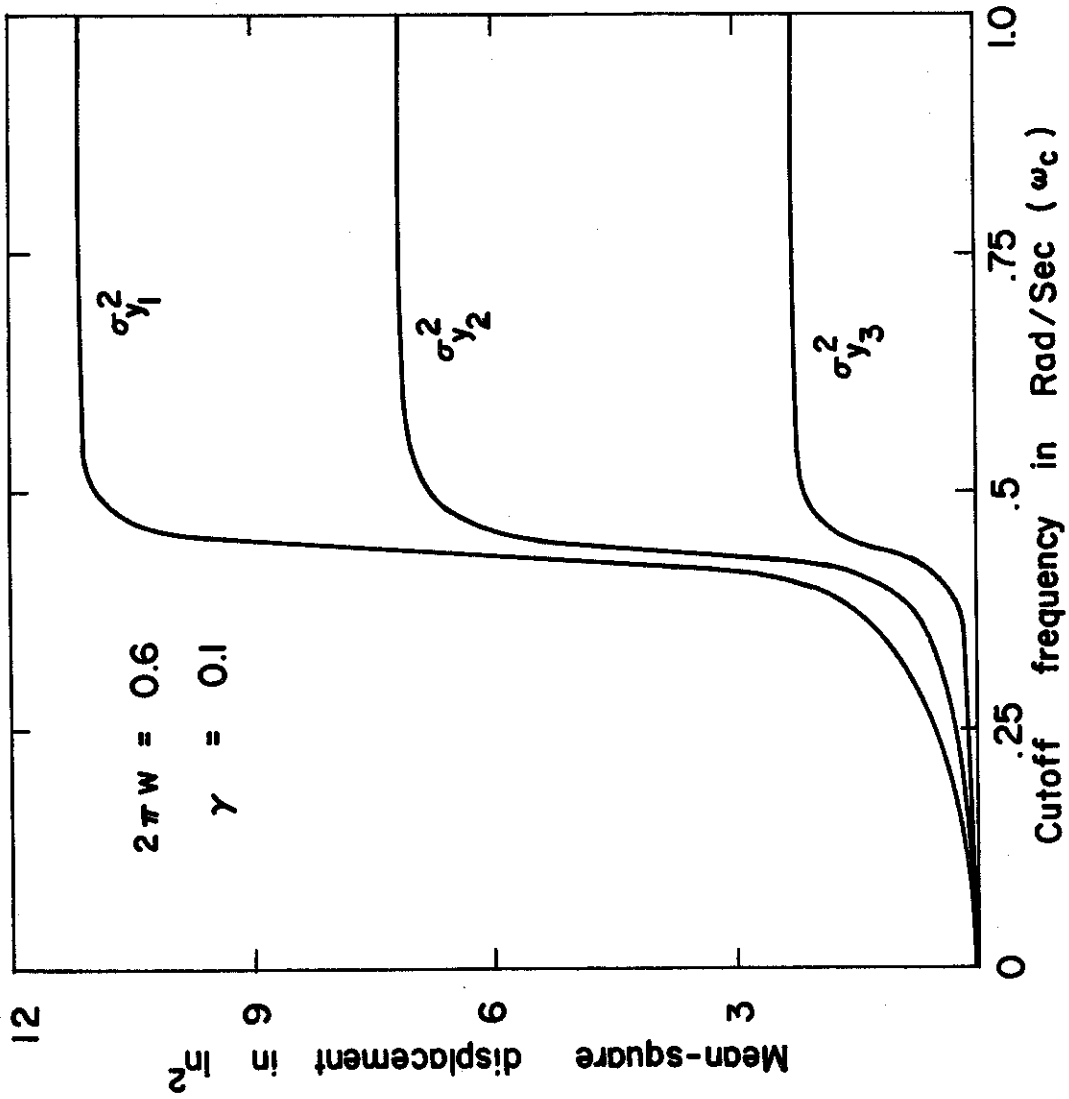
Figure 4.4: Mean-square response as a function of cutoff frequency of base excitation
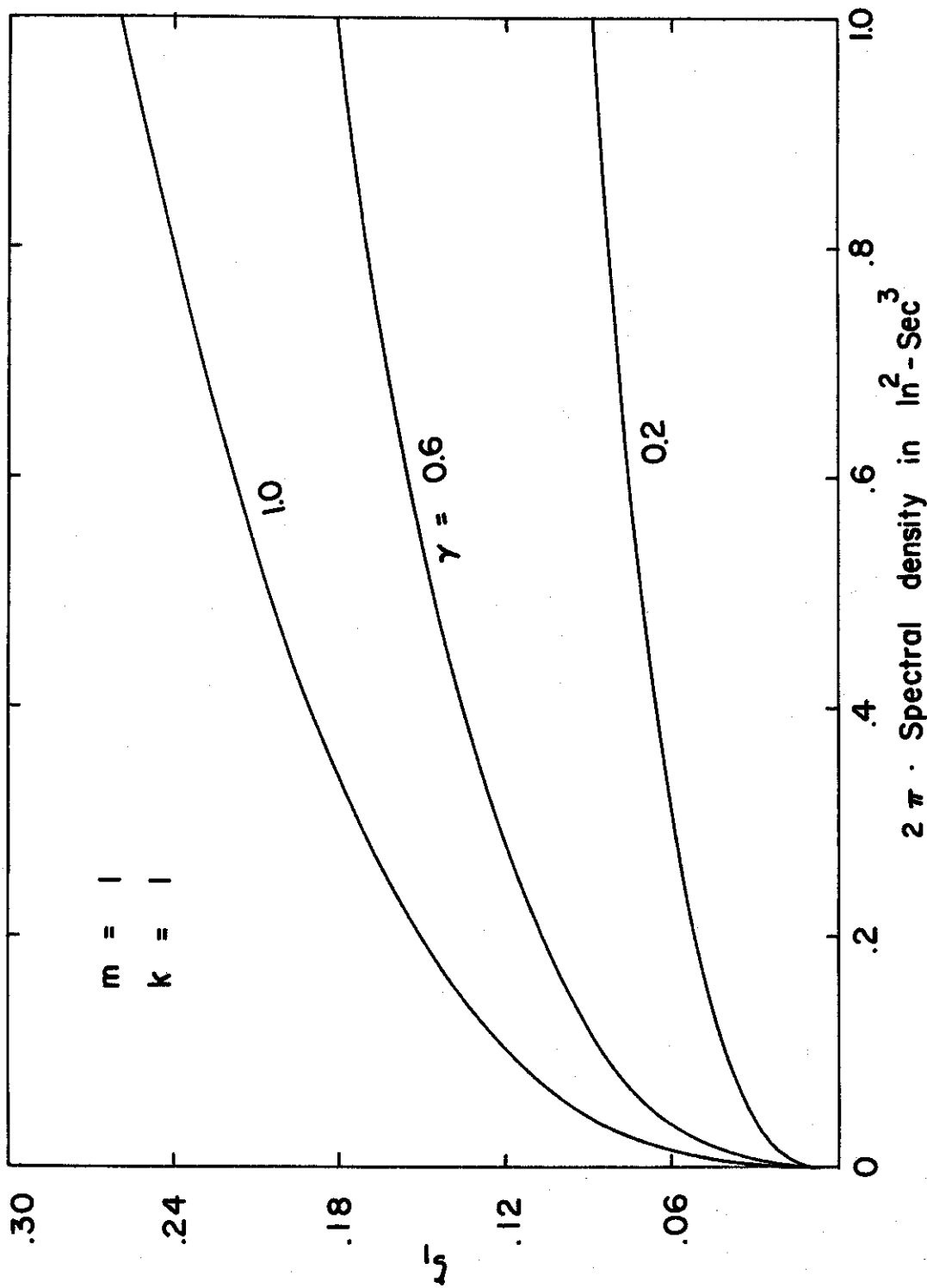
Figure 4.5: $\zeta_1$ as a function of spectral density of base excitation
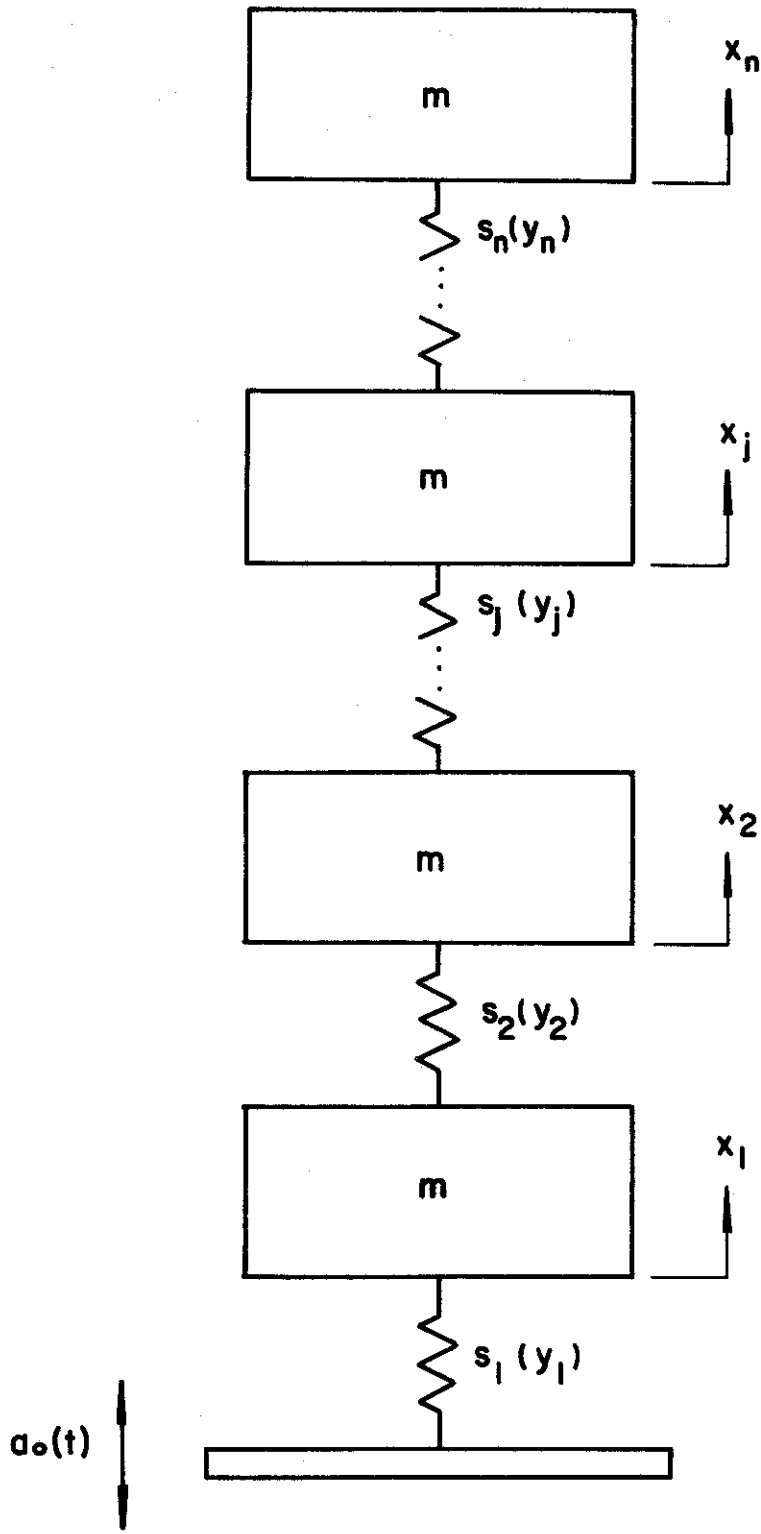
Figure 4.6: An n-degree-of-freedom nonlinear
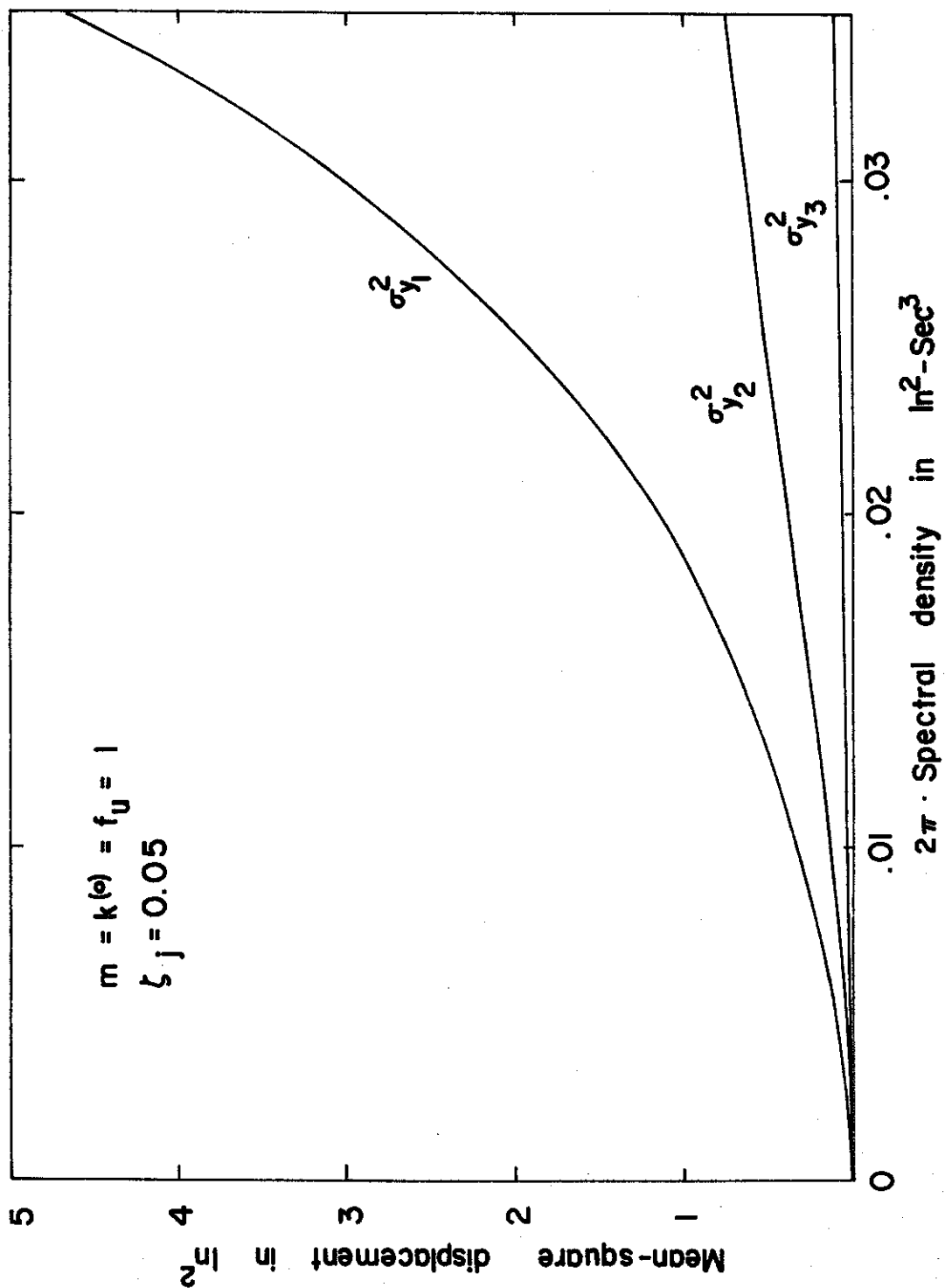system (linear dampers are not shown)

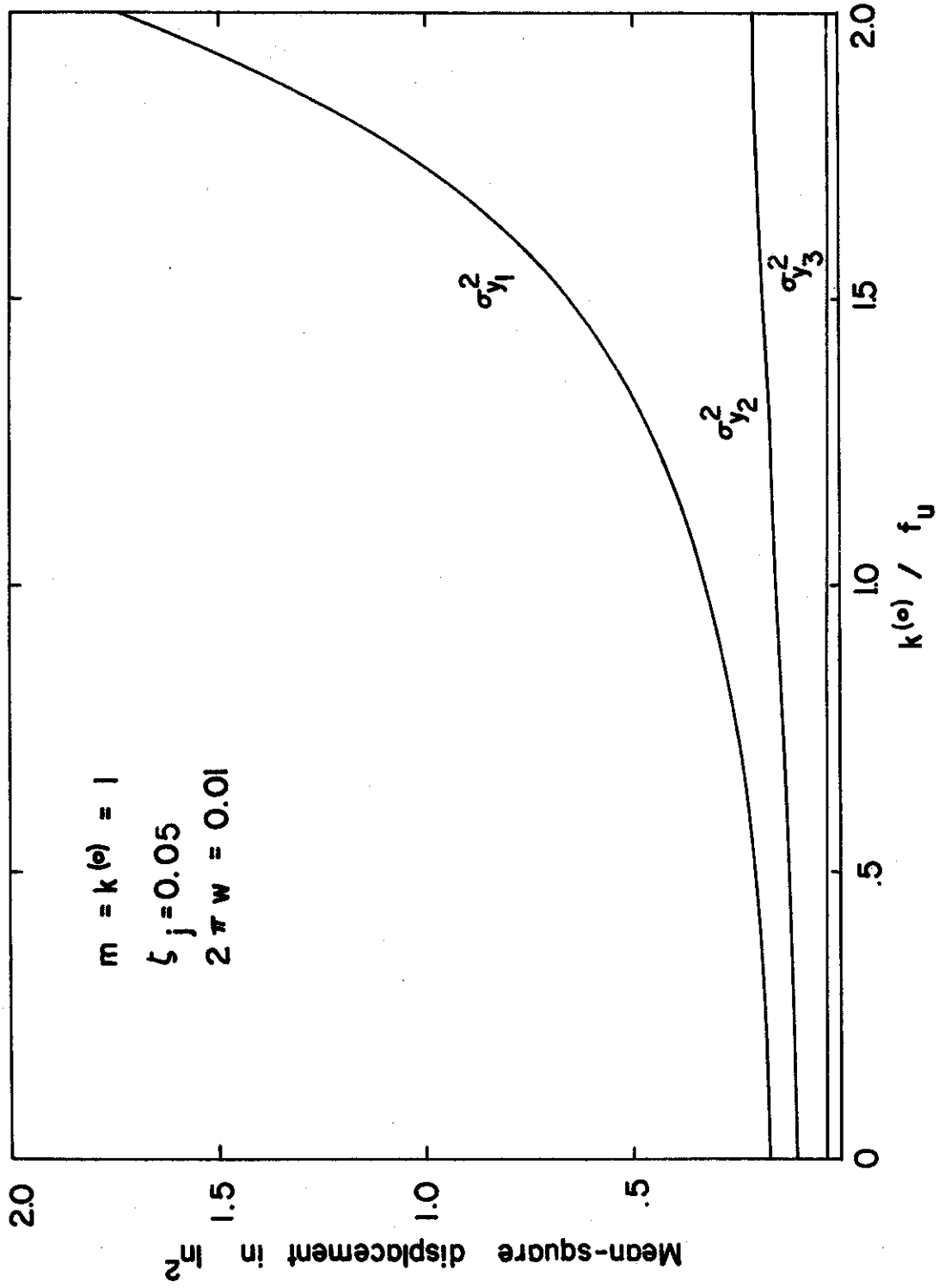Figure 4.7: Mean-square displacement as a function of spectral density of base excitation

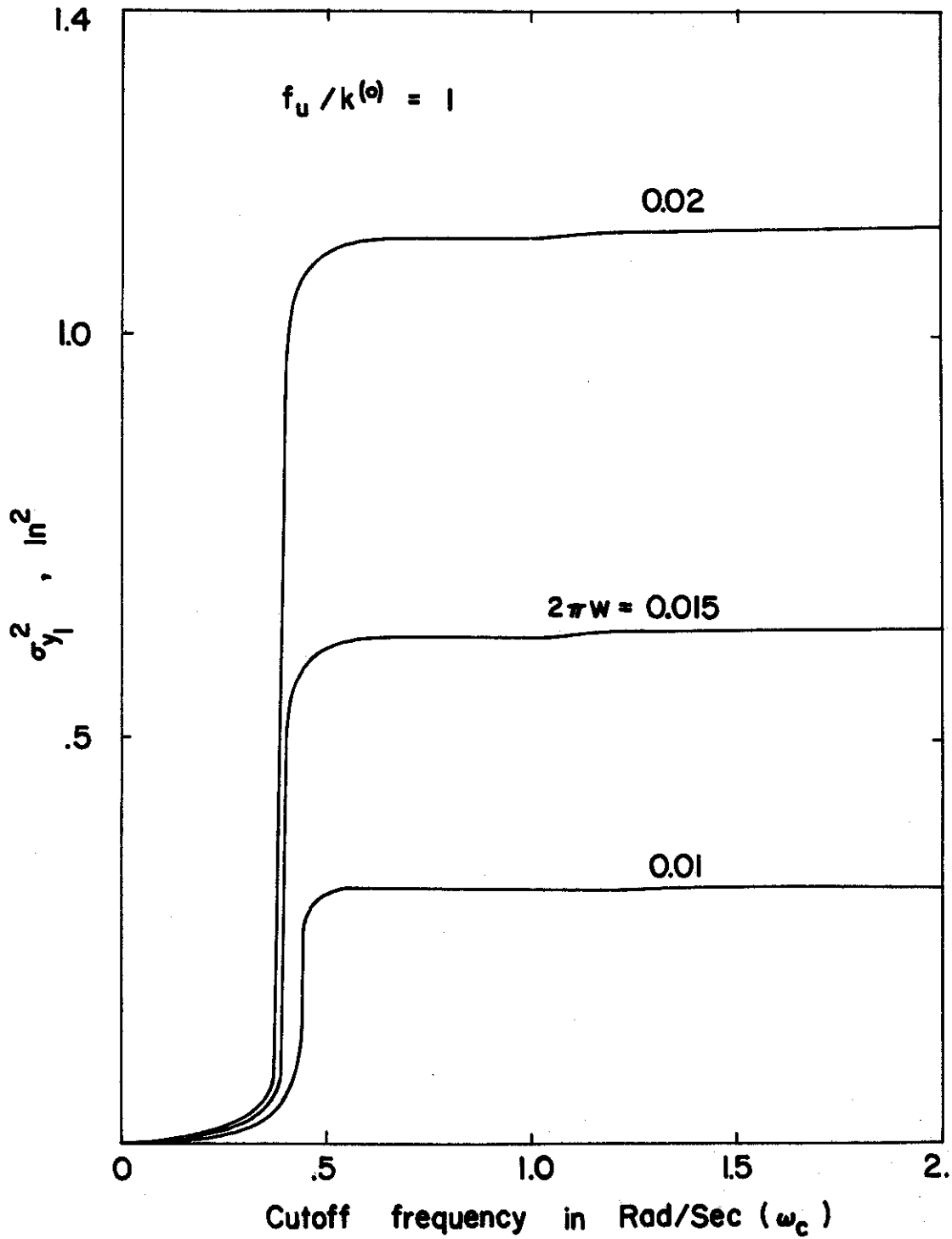Figure 4.8: Mean-square displacement as a function of $k^{(o)}/f_u$

Figure 4.9: $\sigma^2_{y_1}$ as a function of $\omega_c$
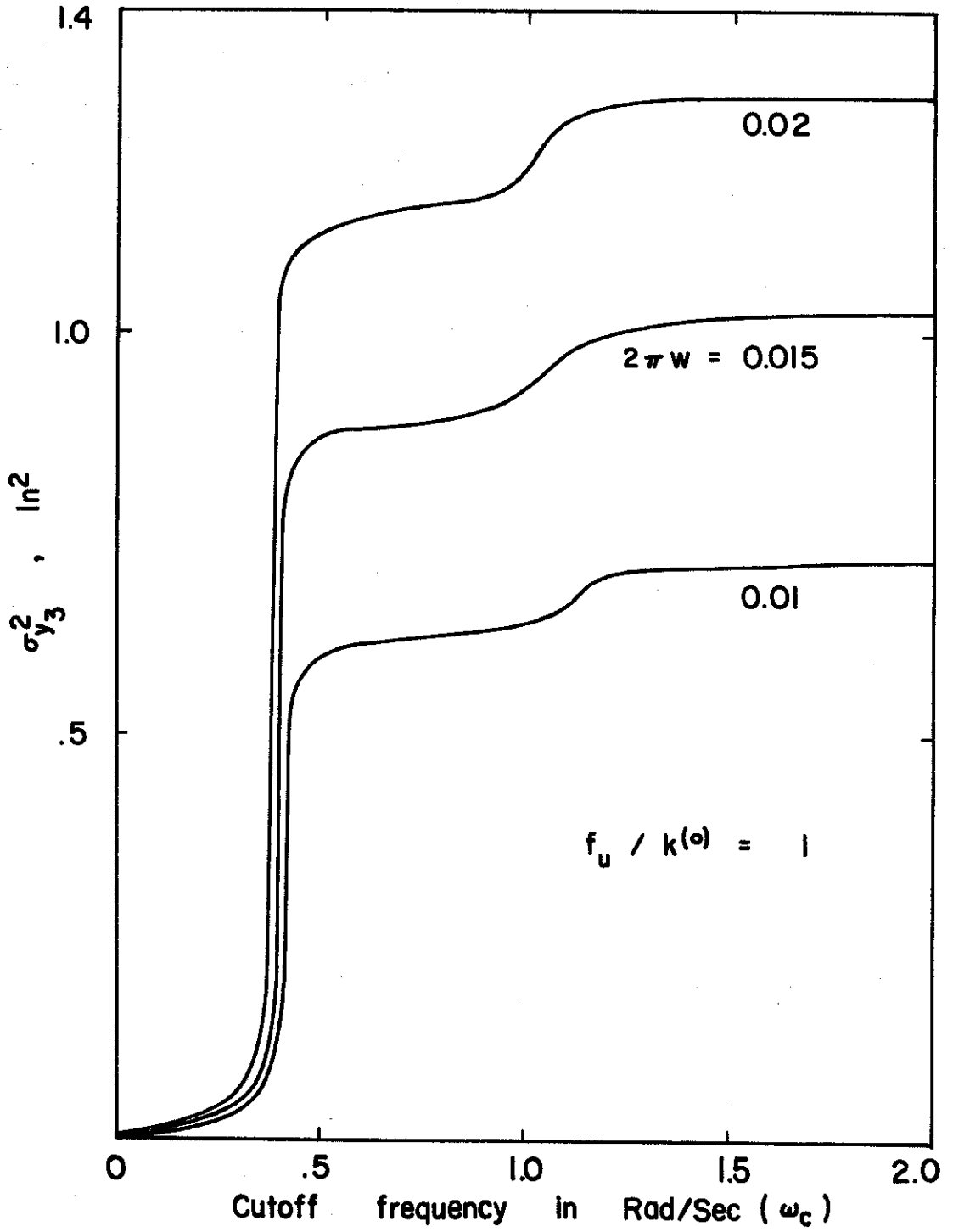
Figure 4.10: $\sigma^2_{y_3}$ as a function of $\omega_c$

# REFERENCES

1. Einstein, A., "Investigations on the Theory of the Brownian Movement", Dover Publication (1956).

2. Fokker, A. D., "Die mittlere Energie rotierender elektrischer Dipole im Strahlungsfeld", Annalen der Physik, Vol. 43, pp. 810-820 (1914).

3. Smoluchowski, M., "Über Brownsche Molekularbewegung unter Einwirkung äusserer Kräfte und deren Zusammenhang mit der verallgemeinerten Diffusionsgleichung", Annalen der Physik, Vol. 48, 1103-1112 (1915).

4. Planck, M., "Über einen Satz der statistischen Dynamik und seine Erweiterung in der Quantentheorie", Akademie der Wissenschaften, Sitzungsberichte, Berlin, pp. 324-341 (1917).

5. Stratonovich, R. L., "Topics in the Theory of Random Noise", translated from the Russian by R. A. Silverman, Vol. I, Gordon and Breach (1963).

6. Ornstein, L. S., "On the Brownian Motion", Koninklijke Akademie van Wetenschappen, Amsterdam, Vol. 21, pp. 96-108 (1919).

7. Wiener, N., "Generalized Harmonic Analysis", Acta Mathematica, Vol. 55, pp. 117-258 (1930).

8. Khintchine, A., "Korrelationstheorie der stationären stochastischen Prozesse", Mathematische Annalen, Vol. 109, pp. 604-615 (1934).

9. Lin, Y. K., "Probabilistic Theory of Structural Dynamics", McGraw-Hill, Chapters 5-6 (1967).

10. Minorsky, N., "Introduction to Nonlinear Mechanics", Edwards Brothers (1947).

11. Booton Jr., R. C., "The Analysis of Nonlinear Control Systems with Random Inputs", Proceedings of the Symposium on Nonlinear Circuit Analysis, Vol. II, pp. 369-391 (1953).

12. Caughey, T. K., "Equivalent Linearization Technique", Journal of the Acoustic Society of America, Vol. 35, No. 11, pp. 1706-1711 (1963).

13. Crandall, S. T., "Perturbation Techniques for Random Vibration of Nonlinear Systems", _Journal of the Acoustic Society of America_, Vol. 35, No. 11, pp. 1700-1705 (1963).

14. Tung, C. C., "The Effects of Runway Roughness on the Dynamic Response of Airplanes", _Journal of Sound and Vibration_, Vol. 5, pp. 164-172 (1967).

15. Carslaw, H. S. and Jaeger, J. C., "_Operational Methods in Applied Mathematics_", Dover Publications (1963).

16. Carslaw, H. S. and Jaeger, J. C., "_Operational Methods in Applied Mathematics_", Dover Publications, pp. 237-239 (1963).

17. Lin, Y. K., "_Probabilistic Theory of Structural Dynamics_", McGraw-Hill, Chapter 3 (1967).

18. Sneddon, I. N., "_Fourier Transforms_", McGraw-Hill, Chapter 1 (1951).

19. Wang, M. C. and Uhlenbeck, G. E., "On the Theory of the Brownian Motion II", _Reviews of Modern Physics_, Vol. 17, Nos. 2-3, pp. 323-342 (1945).

20. Bellman, R. E., "_Introduction to Matrix Analysis_", McGraw-Hill, Chapter 12 (1960).

21. Dwight, H. B., "_Tables of Integrals and Other Mathematical Data_", Dover Publications, pp. 235-236 (1964).

22. Ariarathan, S. T., "Random Vibration of Nonlinear Suspensions", _Journal of Mechanical Engineering Science_, Vol. 2, pp. 195-201 (1960).

23. Caughey, T. K., "Derivation and Application of the Fokker-Planck Equation to Discrete Nonlinear Systems Subjected to White Random Excitation", _Journal of Acoustic Society of America_, Vol. 35, pp. 1683-1692 (1963).

24. Foss, K. A., "Coordinates Which Uncouple the Equations of Motion of Damped Linear Systems", _Journal of Applied Mechanics_, Vol. 25, pp. 361-364 (1958).

25. Feller, W., "_An Introduction to Probability Theory and Its Applications_", John Wiley and Sons, Vol. II, Chapter 8 (1966).

26. Iwan, W. D., "On Defining Equivalent Systems for Certain Ordinary Nonlinear Differential Equations", Accepted for publication in International Journal of Nonlinear Mechanics.

27. Cramer, H., "Mathematical Methods of Statistics", Princeton University Press, Chapter 21 (1946).

28. Gradshteyn, I. S., and Ryzhik, I. M., "Table of Integrals, Series, and Products", Academic Press, p. 342 (1965).

29. Abramowitz, M., and Stegun, I. A., editors, "Handbook of Mathematical Functions", National Bureau of Standards, p. 302 (1964).

30. Milwitzky, B., and Cook, F. E., "Analysis of Landing-Gear Behavior", NACA, TR1154 (1953).