APPENDIX A: SUPPLEMENTAL MATERIAL

This appendix provides supporting calculations for the main text. In the first section, we derive the amplitude equation using perturbation theory. In the second section, we calculate the noise from spontaneous emission.

1. Derivation of amplitude equation

Here we use perturbation theory to calculate how the weak nonlinearities and interactions in Eq. (3) of the main text change the amplitude and phase of the harmonic oscillations on a long time scale. We use the method of averaging with amplitude and phase variables because the noise from spontaneous emission depends on the amplitude.

First we rescale Eq. (3) of the main text with $\tau = \omega_o t$ and $y_n = x_n/\ell$,

$$0 = \frac{d^2}{d\tau^2}y_n + y_n + \alpha y_n^3 - \nu(1-y_n^2)\frac{d}{d\tau}y_n + D[(y_n - y_{n-1}) + (y_n - y_{n+1})] + \zeta_n(\tau) \ , \tag{A1}$$

where $\alpha = \alpha_o \ell^2/\omega_o^2$, $\nu = \mu/\omega_o$, $D = 2k_e e^2/md^3\omega_o^2$, and $\zeta_n(\tau)$ is the noise. Equation (A1) describes a chain of van der Pol-Duffing oscillators.

Let $y_n = r_n'(\tau)\cos[\tau + \theta_n(\tau)]$, where r_n' and θ_n change slowly ($\dot{r}' \ll r'$, $\ddot{r}' \ll \dot{r}'$, $\dot{\theta} \ll 1$, $\ddot{\theta} \ll \dot{\theta}$). Substituting this into Eq. (A1) and keeping the leading terms,

$$\begin{aligned} 0 &= -2\dot{r}_n'\sin(\tau+\theta_n) - 2r_n'\dot{\theta}_n\cos(\tau+\theta_n) + \alpha r_n'^3\cos^3(\tau+\theta_n) + \nu[1 - r_n'^2\cos^2(\tau+\theta_n)]r_n'\sin(\tau+\theta_n) \\ &\quad + D[2r_n'\cos(\tau+\theta_n) - r_{n-1}'\cos(\tau+\theta_{n-1}) - r_{n+1}'\cos(\tau+\theta_{n+1})] + \zeta_n(\tau) \ . \end{aligned} \tag{A2}$$

Then multiply each equation by $\sin(\tau + \theta_n)$ and integrate over the time interval $[\tau, \tau + \Delta\tau]$, where $\Delta\tau$ is a multiple of 2π,

$$\frac{dr_n'}{d\tau} = \frac{\nu}{2}\left(1 - \frac{r_n'^2}{4}\right)r_n' + \frac{D}{2}[r_{n-1}'\sin(\theta_{n-1}-\theta_n) + r_{n+1}'\sin(\theta_{n+1}-\theta_n)] + \xi_n^r(\tau) \ . \tag{A3}$$

Then multiply each equation by $\cos(\tau + \theta_n)$ and integrate similarly,

$$\frac{d\theta_n}{d\tau} = \frac{3}{8}\alpha r_n'^2 + \frac{D}{2}\left[2 - \frac{r_{n-1}'}{r_n'}\cos(\theta_n - \theta_{n-1}) - \frac{r_{n+1}'}{r_n'}\cos(\theta_n - \theta_{n+1})\right] + \xi_n^\theta(\tau) \ . \tag{A4}$$

These equations describe how r_n' and θ_n evolve. The noise functions are,

$$\xi_n^r(\tau) = \frac{1}{\Delta\tau}\int_\tau^{\tau+\Delta\tau} d\tau'\zeta_n(\tau')\sin(\tau'+\theta_n) \tag{A5}$$

$$\xi_n^\theta(\tau) = \frac{1}{r_n'\Delta\tau}\int_\tau^{\tau+\Delta\tau} d\tau'\zeta_n(\tau')\cos(\tau'+\theta_n) \ . \tag{A6}$$

To put Eqs. (A3) and (A4) in simpler form, rescale time ($\bar{t} = \nu\tau/2$) and amplitude ($r_n = r_n'/2$),

$$\frac{dr_n}{d\bar{t}} = (1 - r_n^2)r_n + b[r_{n-1}\sin(\theta_{n-1}-\theta_n) + r_{n+1}\sin(\theta_{n+1}-\theta_n)] + \psi_n^r(\bar{t}) \tag{A7}$$

$$\frac{d\theta_n}{d\bar{t}} = cr_n^2 + b\left[2 - \frac{r_{n-1}}{r_n}\cos(\theta_n-\theta_{n-1}) - \frac{r_{n+1}}{r_n}\cos(\theta_n-\theta_{n+1})\right] + \psi_n^\theta(\bar{t}) \ . \tag{A8}$$

where $b = D/\nu$, $c = 3\alpha/\nu$, and ψ_n^r and ψ_n^θ are the rescaled noise functions. Then write everything in terms of a complex amplitude $A_n = r_n e^{-i\theta_n}$, so that $y_n(\tau) = 2\mathrm{Re}[A_n(\tau)e^{-i\tau}]$. (We use $e^{-i\theta_n}$ instead of $e^{i\theta_n}$ in order to match up with the sign convention in Ref. [1].) Then $A_n(\bar{t})$ evolves according to,

$$\frac{dA_n}{d\bar{t}} = A_n + ib(-2A_n + A_{n-1} + A_{n+1}) - (1+ic)|A_n|^2A_n + \psi_n^A(\bar{t}, A_n) \ , \tag{A9}$$

where ψ_n^A is the complex-valued noise function.

2. Noise from spontaneous emission

Here we calculate the expected noise from spontaneous emission. When an ion absorbs a photon from a laser, it gets a momentum kick in the direction of the laser, and when it spontaneously emits the photon, it gets a momentum kick in a random direction. Spontaneous emission is the inherent source of noise in our scheme, so we explain how to represent it with the noise term ψ^A in Eq. (A9).

There are two factors that must be taken into account. First, for the experimental conditions assumed in the text, an ion scatters on the order of one photon per oscillation cycle. Thus, the noise is a sequence of occasional impulses happening at random times. Second, the noise is position dependent due to the intensity gradient of the red beams.

We just consider a single ion, since the noise for each ion is independent and identically distributed. Each scattering event happens at a random time, and the spontaneous emission of a photon causes a momentum kick $\hbar k$ in a random direction. Suppose the ion scatters photons at times t_n. Then the noise in Eq. (3) of the main text is

$$\chi(t) \;=\; \frac{\hbar k}{m}\sum_n \delta(t-t_n)q_n \;, \tag{A10}$$

where q_n is a random variable (with variance σ_q) for the projection of a momentum kick along the trap axis. Each kick is independent ($\langle q_j q_k\rangle = \delta_{jk}$). For simplicity, we assume that the emission is isotropic ($\sigma_q^2 = 1/3$), although there is a slight anisotropy relative to the laser direction [2].

With the assumptions on experimental parameters given in the text, the scattering rate Γ may be calculated rigorously from the Optical Bloch Equations [3, 4],

$$\Gamma(x) \;=\; \frac{\gamma^3}{I_s}\left[\frac{I_R(x)}{\gamma^2+4\Delta\omega_R^2}+\frac{I_B}{\gamma^2+4\Delta\omega_B^2}\right] \;. \tag{A11}$$

The first and second terms correspond to scattering by the red and blue beams, respectively. (Remember that we assume counter-propagating beams for mathematical convenience, but one would use single beams in practice.) Note that Γ depends on position and is independent of velocity to first order.

After rescaling ($\tau = \omega_o t$ and $y = x/\ell$) to get Eq. (A1), the noise is

$$\zeta(\tau) \;=\; \frac{\hbar k}{m\omega_o\ell}\sum_n \delta(\tau-\tau_n)q_n \;, \tag{A12}$$

and the scattering rate becomes,

$$\tilde\Gamma(y) \;=\; \tilde\Gamma_R(y)+\tilde\Gamma_B \tag{A13}$$

$$\tilde\Gamma_R(y) \;=\; \frac{1}{\omega_o}\left(\frac{y}{\cos\phi}\right)^2\frac{I_B}{I_s}\frac{\gamma^3}{\gamma^2+4\Delta\omega^2} \tag{A14}$$

$$\tilde\Gamma_B \;=\; \frac{1}{\omega_o}\frac{I_B}{I_s}\frac{\gamma^3}{\gamma^2+4\Delta\omega^2} \;, \tag{A15}$$

where we have used the intensity relation given in Eq. (2) of the main text.

To calculate the amplitude noise ξ^r, plug Eq. (A12) into Eq. (A5),

$$\xi^r(\tau) \;=\; \frac{\hbar k}{\Delta\tau m\omega_o\ell}\sum_{\tau<\tau_n<\tau+\Delta\tau} q_n\sin(\tau_n+\theta) \;, \tag{A16}$$

where τ_n is the time of a scattering event. Since the damping is weak, the ion scatters on the order of one photon in an oscillation cycle ($\Gamma \sim \omega_o/2\pi$), so there is significant time between scattering events. This means that the phase of oscillation at which a scattering event occurs is approximately uncorrelated with the phase of the next event. Each scattering event has a random projection and phase. Thus, the sum in Eq. (A16) is over independent samples of the random variable $w_n = q_n u_n$, where $u_n = \sin\tau_n$ (ignoring the unimportant phase offset θ for now).

Now we find w_n's distribution ρ_w. The intensity gradient of the red beams causes them to scatter more at certain phases within a cycle, while the blue beams scatter uniformly. Thus, ρ_w is actually a weighted average of red and blue components. First we find the distribution of scattering times τ_n (mod 2π) from the intensity profiles,

$$\rho_\tau(\tau) \;=\; \begin{cases} \frac{1}{\pi}\cos^2\tau & \text{red} \\ \frac{1}{2\pi} & \text{blue} \end{cases} \;, \tag{A17}$$

since $y = r' \cos\tau$. Thus the distribution of $u_n = \sin\tau_n$ is

$$\rho_u(u) = \rho_\tau \left| \frac{d\tau}{du} \right| \tag{A18}$$

$$= \begin{cases} \frac{2}{\pi}\sqrt{1-u^2} & \text{red} \\ \frac{1}{\pi}\frac{1}{\sqrt{1-u^2}} & \text{blue} \end{cases} , \tag{A19}$$

for $|u| \leq 1$. Since we assume isotropic spontaneous emission, the distribution of the projection q_n is $\rho_q(q) = 1/2$ for $|q| \leq 1$. Then the distribution of $w_n = q_n u_n$ is

$$\rho_w(w) = \int_{-1}^{1} du \int_{-1}^{1} dq \; \rho_u(u)\rho_q(q)\delta(w - uq) \tag{A20}$$

$$= \begin{cases} \frac{2}{\pi}\left[-\sqrt{1-w^2} + \log\frac{1+\sqrt{1-w^2}}{|w|}\right] & \text{red} \\ \frac{1}{\pi}\log\frac{1+\sqrt{1-w^2}}{|w|} & \text{blue} \end{cases} , \tag{A21}$$

for $|w| \leq 1$. The variance of w_n is

$$\sigma_w^2 = \begin{cases} \frac{1}{12} & \text{red} \\ \frac{1}{6} & \text{blue} \end{cases} . \tag{A22}$$

To find the phase noise ξ^θ, plug Eq. (A12) into Eq. (A6),

$$\xi^\theta(\tau) = \frac{\hbar k}{r'\Delta\tau m\omega_o \ell} \sum_{\tau<\tau_n<\tau+\Delta\tau} q_n \cos(\tau_n + \theta) , \tag{A23}$$

and go through the same process to find the variance of $v_n = q_n \cos\tau_n$,

$$\sigma_v^2 = \begin{cases} \frac{1}{4} & \text{red} \\ \frac{1}{6} & \text{blue} \end{cases} . \tag{A24}$$

Although w_n and v_n come from the same scattering event, they are statistically uncorrelated because $\langle \sin\tau_n \cos\tau_n \rangle = 0$.

We let the time interval $\Delta\tau$ be large enough to include many scattering events but smaller than the characteristic time scales in Eqs. (A3) and (A4). We average the scattering rate $\tilde{\Gamma}(y)$ in Eq. (A13) over $\Delta\tau$ to find the time-averaged scattering rates of the red beams ($\bar{\Gamma}_R$) and blue beams ($\bar{\Gamma}_B$),

$$\bar{\Gamma}_R(r') = \frac{1}{2\omega_o}\left(\frac{r'}{\cos\phi}\right)^2 \frac{I_B}{I_s}\frac{\gamma^3}{\gamma^2 + 4\Delta\omega^2} \tag{A25}$$

$$\bar{\Gamma}_B = \frac{1}{\omega_o}\frac{I_B}{I_s}\frac{\gamma^3}{\gamma^2 + 4\Delta\omega^2} . \tag{A26}$$

$\bar{\Gamma}_R$ depends on r' due to the intensity gradient of the red beam. Then the amplitude and phase noises are Gaussian and described by,

$$\langle \xi^r(\tau)\xi^r(\tau') \rangle = \left(\frac{\hbar k}{m\omega_o \ell}\right)^2 \left(\frac{1}{12}\bar{\Gamma}_R + \frac{1}{6}\bar{\Gamma}_B\right)\delta(\tau - \tau') \tag{A27}$$

$$\langle \xi^\theta(\tau)\xi^\theta(\tau') \rangle = \frac{1}{r'^2}\left(\frac{\hbar k}{m\omega_o \ell}\right)^2 \left(\frac{1}{4}\bar{\Gamma}_R + \frac{1}{6}\bar{\Gamma}_B\right)\delta(\tau - \tau') . \tag{A28}$$

They are uncorrelated with each other: $\langle \xi^r(\tau)\xi^\theta(\tau') \rangle = 0$.

After rescaling ($\bar{t} = \nu\tau/2$, $r = r'/2$) to get Eqs. (A7) and (A8), the noises become,

$$\langle \psi^r(\bar{t})\psi^r(\bar{t}')\rangle = \frac{1}{2\nu}\left(\frac{\hbar k}{m\omega_o \ell}\right)^2 \left(\frac{1}{12}\bar{\Gamma}_R + \frac{1}{6}\bar{\Gamma}_B\right)\delta(\bar{t}-\bar{t}') \tag{A29}$$

$$\langle \psi^\theta(\bar{t})\psi^\theta(\bar{t}')\rangle = \frac{1}{2\nu r^2}\left(\frac{\hbar k}{m\omega_o \ell}\right)^2 \left(\frac{1}{4}\bar{\Gamma}_R + \frac{1}{6}\bar{\Gamma}_B\right)\delta(\bar{t}-\bar{t}') \,. \tag{A30}$$

Again, $\langle \psi^r(\bar{t})\psi^\theta(\bar{t}')\rangle = 0$. Finally, the complex-valued noise in Eq. (A9) is,

$$\psi^A(\bar{t}, A) = [\eta^R(\bar{t}) + i\sigma^R(\bar{t})]A + [\eta^B(\bar{t}) + i\sigma^B(\bar{t})] \tag{A31}$$

$$\langle \eta^R(\bar{t})\eta^R(\bar{t}')\rangle = \frac{H}{\cos^2\phi}\delta(\bar{t}-\bar{t}') \tag{A32}$$

$$\langle \sigma^R(\bar{t})\sigma^R(\bar{t}')\rangle = \frac{3H}{\cos^2\phi}\delta(\bar{t}-\bar{t}') \tag{A33}$$

$$\langle \eta^B(\bar{t})\eta^B(\bar{t}')\rangle = \langle \sigma^B(\bar{t})\sigma^B(\bar{t}')\rangle = H\delta(\bar{t}-\bar{t}') \tag{A34}$$

$$H = \frac{\hbar(\gamma^2 + 4\Delta\omega^2)}{96 m\omega_o^2 \ell^2 \Delta\omega} \,, \tag{A35}$$

where H is a measure of the noise, and we have simplified using Eq. (4) of the main text. The noise functions for the red beams (η^R,σ^R) and blue beams (η^B,σ^B) are all uncorrelated with each other. The noise from the red beams increases with amplitude and causes more phase noise than amplitude noise.

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