Constant-Temperature Magneto-Gasdynamic Channel Flow

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Introduction

In the course of investigating boundary-layer flow in continuous plasma accelerators with crossed electric and magnetic fields, it was found advantageous to have at hand simple closed-form solutions for the magneto-gasdynamic flow in the duct which could serve as free-stream conditions for the boundary layers. Nontrivial solutions of this sort are not available at present, and in fact, as in the work of Resler and Sears,[1] the variation of conditions along the flow axis must be obtained through numerical integration.

Consequently, some simple solutions of magneto-gasdynamic channel flow were sought, possessing sufficient algebraic simplicity to serve as free-stream boundary conditions for analytic investigations of the boundary layer in a physically reasonable accelerator. In particular, since the cooling of the accelerator tube is likely to be an important physical problem because of the high gas temperatures required to provide sufficient gaseous conductivity, channel flow with constant temperature appears interesting. Some simple algebraic solutions for the case of a constant temperature plasma are developed in the following paragraphs.

Analysis

If heat conduction and viscous effects are neglected, magneto-gasdynamic channel flow may be described by the following equations:

energy, $$\rho u C_p T' = up' + (j^2/\sigma) \qquad (1)$$

momentum, $$\rho u u' = -p' + jB \qquad (2)$$

continuity, $$\rho u A = C \qquad (3)$$

state, $$p = \rho RT \qquad (4)$$

Here, ρ is the density, u is the x directed velocity, C_p is the specific heat at constant pressure, T is the temperature, j is the current density which is normal to the flow direction, and B is the magnetic field which is normal to both current and flow direction. σ is the conductivity, which is assumed scalar, so that the current and effective electric field are connected by Ohm's law:

$$j = \sigma(E - uB) \qquad (5)$$

The magnetic Reynolds Number is assumed sufficiently small that B may be regarded as prescribed. The applied electric field, the temperature, and the conductivity will be assumed constant.

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Then, combining Eqs. (1) and (2) and using Eq. (4) to eliminate B, we find

$$p' = -(j^2/u\sigma) = RTE(j/u^2u')' \qquad (6)$$

Let $\zeta = j/u^2u'$, then

$$\zeta'/\zeta^2 = -[u^3(u')^2/\sigma RTE]$$

and, integrating from the origin,

$$(1/\zeta) - [1/\zeta(0)] = (1/\sigma RTE)\int_0^x u^3(u')^2 dx \qquad (7)$$

If the behavior of u is specified, that of j is readily calculated from Eq. (7), and that of B from Eq. (5).

Particularly simple results are obtained when u varies as a power of x. Suppose that

$$u = \alpha x^n \qquad (8)$$

then, since the temperature is constant, the Mach Number, M, is given by $M = \alpha x^n/(\gamma RT)^{1/2}$. From Eq. (7),

$$1/\zeta = (\alpha^5 n^2/\sigma RTE)[x^{5n-1}/(5n-1)] \qquad (9)$$

and, from the definition of ζ,

$$j = (\sigma RTE/\alpha^2 n)(5n-1)x^{-2n} = \sigma E(5n-1)/n\gamma M^2 \qquad (10)$$

The pressure may be computed from Eqs. (1), (2), and (4):

$$p = [\sigma(RTE)^2/\alpha^5][(5n-1)/n^2]x^{1-5n} \qquad (11)$$

and the flow area from Eq. (3):

$$A = (C\alpha^4 n^2/\sigma E^2 RT)[x^{4n-1}/(5n-1)] \qquad (12)$$

C is the constant value of $\rho u A$. Finally, from Eq. (5),

$$B = (E/u)\{1 - [(5n-1)/n\gamma M^2]\} \qquad (13)$$

In the special case of $n = 1/5$, these results are modified slightly, since then the integral in Eq. (7) is logarithmic.

All positive values of n greater than 1/5 yield flows in which continuous acceleration from zero velocity to an arbitrarily high velocity is possible, in a channel of continuously changing area. The area and velocity variations are such that the conditions given by Resler and Sears[1] for smooth passage through the sonic velocity are satisfied for all such values of n. For $n = 1/4$, the area is constant, and for this case, the solution passes through the singular point at $M = 1$, $u = [(\gamma - 1)/\gamma](E/B)$. For n less than 1/4, the area decreases with increasing velocity, and for n greater than 1/4, it increases with increasing velocity. In view of the very rapid variation of both pressure and area with x for n very different from 1/4, values of n near 1/4 are likely to be the most interesting for plasma accelerators.

References

[1] Resler, E. L., and Sears, W. R., *The Prospects for Magneto-Aerodynamics*, Journal of the Aero/Space Sciences, Vol. 25, No. 4, pp. 235–245, April, 1958.
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[3] Wilson, T. A., *Remarks on Rocket and Aerodynamic Applications of Magnetohydrodynamic Channel Flow*, Cornell University, TN-58-1058, ASTIA 207 228, December, 1958.

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