Variational Collision Integrators and Optimal Control

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1 Abstract

This paper presents a methodology for generating locally optimal control policies for mechanical systems that undergo collisions at point contacts. Principles of nonsmooth mechanics for rigid bodies are used in both continuous and discrete time, and provide impact models for a variety of collision behaviors. The **discrete Euler-Lagrange (DEL) equations** that follow from the discrete time analyses yield variational integration schemes for the different impact models. These DEL equations play a pivotal role in the method of **Discrete Mechanics and Optimal Control (DMOC)**, which generates locally optimal control policies as the solution to equality constrained nonlinear optimization problems. The DMOC method is demonstrated on a 4-link planar walking robot model, generating locally optimal periodic walking gaits.

2 Introduction

The problem of optimal control generation for mechanical systems undergoing collisions is challenging due to the complex and nonsmooth nature of impact dynamics. While in reality these dynamics are dependent upon material deformation and elasticity, a common approach is to adopt a rigid body assumption when modelling the impact dynamics [15] [3]. In [5] methods from nonsmooth variational calculus are used to describe elastic impacts for rigid bodies, and a discrete time analysis

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is done to provide a symplectic-momentum integration scheme. The method of Discrete Mechanics and Optimal Control (DMOC) [6] utilizes structured integrators, such as the aforementioned, in a numerical scheme to solve optimal control problems. Combining and extending these ideas, this paper has two objectives: to define variational integrators for a variety of collision models, and to demonstrate the usage of those integrators in the DMOC method.

Variational integration algorithms are generated through discrete variational principles rather than discretized equations of motion. The resulting integrators have been shown to have remarkable structure preserving properties, such as appropriately conserving discrete momenta and a discrete symplectic form. Due to these desirable properties, variational integrator theory remains a topic of interest with ongoing research to provide extensions to a variety of mechanical systems. For example, previous works have developed variational integrators for systems with forcing and constraints [12] [10] [13] , systems on Lie groups [1], and systems with collisions [5]. In this work we treat the case of systems with collisions, but incorporating non-conservative impact forces into the variational principles such that varying post collision behaviors may result (i.e. bouncing or sticking). In some cases, particularly those in which contacts are established or released, the integrators presented utilize two variational principles in sequence in order to handle the changing dimension of the system's constraint distribution.

With a variational integration scheme for non-conservative impacts, we make use of the underlying discrete Euler-Lagrange (DEL) equations in the DMOC method. This extension from a particular variational integrator to DMOC is a natural one, as seen in previous DMOC works [6] [11] [9] [8]. DMOC generates locally optimal control policies as the solution to an equality-constrained nonlinear optimization problem, with the DEL equations serving as constraints. The optimization problem is essentially a standard optimal control problem recast in discrete time, and in this form it can be solved with sequential quadratic programming (SQP) methods.

The structure of this paper is as follows. Section 3 reviews principles of discrete mechanics and variational integration for free systems and holonomically constrained systems. Section 4 extends the material in Section 3 to provide variational collision algorithms for systems with impacts and varying contacts. Section 5 reviews the DMOC method of generating optimal controls and presents results for the case example of a 4-link planar walking robot model.

3 Discrete Mechanics

The central idea in discrete mechanics theory is the derivation of discrete time equations of motion for mechanical systems through discrete variational principles. This is in contrast to the common approach of obtaining discrete time equations of motion with a direct discretization of differential equations (using finite differences, quadrature rules, etc.). Here we review the key details in the theory surrounding free systems and holonomically constrained systems. More in depth derivations can be found in [12].

3.1 Free Systems

Given a mechanical system with a configuration space, Q, and a regular Lagrangian, $L : TQ \to \mathbb{R}$, the system's equations of motion can be derived from *Hamilton's principle*. This principle states that

$$\delta \int_0^T L(q(t), \dot{q}(t))\, dt = 0, \tag{1}$$

for all variations $\delta q(t)$ with $\delta q(0) = \delta q(T) = 0$. In order to discretize this variational principle, the state space TQ is replaced by $Q \times Q$ and a discrete path $q_d : \{0, h, 2h, \ldots, Nh = T\} \to Q$, with $N \in \mathbb{N}$ and $h \in \mathbb{R}^+$, is defined such that $q_k = q_d(kh)$ is considered an approximation to $q(kh)$. Based on this discretization, the action integral in Hamilton's principle is approximated on each time slice $[kh, (k+1)h]$ with the introduction of a *discrete Lagrangian*, $L_d : Q \times Q \to \mathbb{R}$, such that

$$L_d(q_k, q_{k+1}) \approx \int_{kh}^{(k+1)h} L(q(t), \dot{q}(t))\, dt.$$

Typically this approximation is made with simple quadrature rules, such as the midpoint rule. Summing these approximations over the discrete path allows for the definition of a discrete Hamilton's principle

$$\delta \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}) = 0, \tag{2}$$

for all variations $\{\delta q_k\}_{k=0}^{N}$ with $\delta q_0 = \delta q_N = 0$. This is equivalent to the system of *discrete Euler-Lagrange (DEL) equations*

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) = 0, \tag{3}$$

for all $k \in \{1, \ldots, N-1\}$, where the notation D_i indicates differentiation with respect to the i^{th} argument. When used as a variational integrator, (3) implicitly defines a map from (q_{k-1}, q_k) to (q_k, q_{k+1}).

3.2 Systems with Holonomic Constraints

The above derivation can be extended to systems subject to time-independent holonomic constraints. That is, consider that one is given a constraint function, $g : Q \to \mathbb{R}^m$, for which 0 is a regular value, such that system configurations are to be restricted to the constraint submanifold $R = g^{-1}(0)$. One method to obtain the equations of motion for such a system would be to apply Hamilton's principle to paths in R. However, the Lagrange multiplier theorem provides that the following variational principle for paths $(q(t), \lambda(t))$ in $Q \times \mathbb{R}^m$ yields equivalent results:

$$\delta \int_0^T \left[L(q, \dot{q}) - g(q) \cdot \lambda\right] \mathrm{d}t = 0, \tag{4}$$

where $\lambda : \mathbb{R} \to \mathbb{R}^m$ denotes a path of m Lagrange multiplers. In order to discretize this variational principle, in addition to the steps carried out above one must introduce a discrete path of Lagrange multipliers, $\lambda_d : \{0, h, 2h, \ldots, Nh = T\} \to \mathbb{R}^m$, such that $\lambda_k = \lambda_d(kh)$ is considered an approximation to $\lambda(kh)$. Then a *discrete constraint function*, $g_d : Q \to \mathbb{R}$, must be introduced such that the appended terms in the action integral can be approximated as

$$\frac{1}{2} g_d^T(q_k) \cdot \lambda_k + \frac{1}{2} g_d^T(q_{k+1}) \cdot \lambda_{k+1} \approx \int_{kh}^{(k+1)h} g(q) \cdot \lambda \, dt.$$

This expression, which differs slightly from that in [12], is taken from [10] where the particulars of its form are discussed. Now the discrete version of the variational principle above is

$$\delta \sum_{k=0}^{N-1} \left[L_d(q_k, q_{k+1}) - \frac{1}{2} g_d(q_k) \cdot \lambda_k - \frac{1}{2} g_d(q_{k+1}) \cdot \lambda_{k+1} \right] = 0,$$

for all variations $\{\delta q_k, \delta \lambda_k\}_{k=0}^N$ with $\delta q_0 = \delta q_N = 0$. This is equivalent to the system of *constrained discrete Euler-Lagrange equations*

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) - G_d(q_k)^T \cdot \lambda_k = 0, \tag{5}$$

along with $g(q_{k+1}) = 0$, where $G_d(q_k) = Dg_d(q_k)$ denotes the Jacobian of g_d.

4 Variational Collision Algorithms

Using the fundamentals presented in the previous section, we now discuss algorithms for mechanical systems encountering collisions. In the variational approach to modeling collisions we follow the theory in [5].

4.1 Elastic Collisions

A main result in [5] is an algorithm for perfectly elastic impacts. The derivation of that algorithm begins with a mechanical system with Q and L as above, and some submanifold with boundary, $C \subset Q$, defining admissible configurations. The boundary of C, denoted ∂C, defines the set of contact configurations.[1] With the contact configurations defined, the same Hamilton's principle (1) is used, but with a more complex path space which allows for nonsmooth trajectories in $q(t)$ and reparameterizations of time. For a precise definition of this path space, and the apparent advantages of its usage, the reader is referred to [5]. On a time interval $[0, T]$ in which it is assumed the system encounters only one collision, the continuous

[1]Note that this definition of the set contact configurations implies that it is of codimension 1 relative to the set admissible configurations. This is in agreement with a point contact assumption. Cases of higher codimension, such as the case when multiple contacts are established simultaneously, are excluded from consideration.

time variational principle (1) on the nonsmooth path space yields that the *extended Euler-Lagrange equations* hold everywhere away from impact. That is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0, \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} E = 0, \tag{7}$$

in $\big[0,\tilde{t}\big) \cup \big(\tilde{t},T\big]$, where $E = (\frac{\partial L}{\partial \dot{q}}\dot{q} - L)$ is the system's total energy and $\tilde{t}$ denotes the impact time. At impact, the principle yields the transition equations

$$\left.\frac{\partial L}{\partial \dot{q}}\right|_{\tilde{t}^-}^{\tilde{t}^+} \cdot \delta q = 0, \tag{8}$$

$$\left. E \right|_{\tilde{t}^-}^{\tilde{t}^+} = 0, \tag{9}$$

for any $\delta q \in T_{q(\tilde{t})}\partial C$. This implies conservation of momentum tangent to the contact surface and conservation of energy through the impact. The transition equations provide no information about the system's momentum normal to the contact surface, as the nonsmooth path space does not permit variations in that direction.

In order to obtain a description of the dynamics above in discrete time, we again use a discrete variational principle. As taking variations with respect to the collision time is essential in deriving the continuous time dynamics above, we must ensure that our discrete time arguments incorporate such variations. In order to do this, we first assume knowledge of the time step $[ih,(i+1)h]$ in which the system will encounter an impact at a time $\tilde{t}$ in a configuration $\tilde{q}$. We parameterize the impact time in that step as $\tilde{t} = (i+\alpha)h$ using $\alpha \in [0,1]$. Also, in the remainder of our derivations we consider the discrete Lagrangian to be a function of both configuration and time. That is, $L_d : \mathbb{R} \times Q \times \mathbb{R} \times Q \to \mathbb{R}$ provides the same approximation as previously discussed

$$L_d(t_k, q_k, t_{k+1}, q_{k+1}) \approx \int_{t_k}^{t_{k+1}} L(q(t), \dot{q}(t))\, dt,$$

where we have introduced the notation $t_k = kh$. In the arguments that follow, we assume that L_d is readily adaptable to time slices that are not of length h, in particular $[t_i, \tilde{t}\,]$ preceding the collision and $[\tilde{t}, t_{i+1}]$ following the collision. For brevity, we will begin using the notation $\bar{q}_k = (t_k, q_k)$ and $\tilde{\bar{q}} = (\tilde{t}, \tilde{q})$, however it is important to keep in mind that L_d is a function of four arguments. Now, the discrete Hamilton's principle takes a form slightly different from that of (2), as

$$\delta\left[\sum_{k=0}^{i-1} L_d(\bar{q}_k, \bar{q}_{k+1}) + L_d(\bar{q}_i, \tilde{\bar{q}}) + L_d(\tilde{\bar{q}}, \bar{q}_{i+1}) + \sum_{k=i+1}^{N-1} L_d(\bar{q}_k, \bar{q}_{k+1})\right] = 0, \tag{10}$$

for all variations $\delta\alpha, \delta\tilde{q}, \{\delta q_k\}_{k=0}^{N}$ with $\delta q_0 = \delta q_N = 0$. Taking variations implies that the DEL equations hold away from impact

$$D_4 L_d(\bar{q}_{k-1}, \bar{q}_k) + D_2 L_d(\bar{q}_k, \bar{q}_{k+1}) = 0, \tag{11}$$

for all $k \neq i-1, i, i+1$. Also, the principle describes more complex conditions surrounding the impact

$$D_4 L_d(\bar{q}_{i-1}, \bar{q}_i) + D_2 L_d(\bar{q}_i, \tilde{\bar{q}}) = 0, \tag{12}$$

$$i^* \Big(D_4 L_d(\bar{q}_i, \tilde{\bar{q}}) + D_2 L_d(\tilde{\bar{q}}, \bar{q}_{i+1}) \Big) = 0, \tag{13}$$

$$D_3 L_d(\bar{q}_i, \tilde{\bar{q}}) + D_1 L_d(\tilde{\bar{q}}, \bar{q}_{i+1}) = 0, \tag{14}$$

$$D_4 L_d(\tilde{\bar{q}}, \bar{q}_{i+1}) + D_2 L_d(\bar{q}_{i+1}, \bar{q}_{i+2}) = 0, \tag{15}$$

where $i^* : T^*Q \to T^*\partial C$ is the cotangent lift of the embedding $i : \partial C \to Q$. One should notice that equations (13) and (14) represent, in discrete time, the conservation of momentum and energy specified by (8) and (9).

When using the above equations as a variational collision integration algorithm, one would initially iterate solving (11) as an implicit map from (q_{k-1}, q_k) to (q_k, q_{k+1}). This is continued until coming across $q_{k+1} \notin C$. Discarding that inadmissible configuration, one solves (12) along with the condition $\tilde{q} \in \partial C$ for $(\tilde{t}, \tilde{q})$. This is followed by solving (13) and (14) to determine q_{i+1}. Lastly, (15) is solved for q_{i+2}, after which one can proceed in using (11) as prior.

4.2 Lossful Collisions

In this section we extend the results of [5] above, in order to model lossful impacts which may perform work on and reduce the total energy of a given mechanical system. This is carried out by defining a *contact force field*, $f^{con} \in T^*(\mathbb{R} \times \partial C)$, in order to incorporate virtual work done by the impact into the variational principle (1). The force field has a "time component", denoted $f_t^{con} \in T^*\mathbb{R}$, that may not have an intuitive physical meaning at first. Further developments and discussion will clarify this point. With this forcing, the system's dynamics are now derived from a *Lagrange-d'Alembert principle* of the form

$$\delta \int_0^T L(q(t), \dot{q}(t))\, dt + f^{con} \cdot \delta\tilde{\bar{q}} = 0. \tag{16}$$

The implications of this variational principle leave equations (6) and (7) unchanged, but yield modified transition equations of the form

$$\frac{\partial L}{\partial \dot{q}}\bigg|_{\tilde{t}^-}^{\tilde{t}^+} \cdot \delta q + f_q^{con} \cdot \delta q = 0, \tag{17}$$

$$E\bigg|_{\tilde{t}^-}^{\tilde{t}^+} - f_t^{con} = 0, \tag{18}$$

where again $\delta q \in T_{q(\tilde{t})}\partial C$. These equations capture the standard jump conditions for inelastic impacts with friction. In (17), f_q^{con}, the portion of f^{con} in $T^*\partial C$, determines the change in momentum tangent to ∂C due to friction. In (18), f_t^{con} defines the energy dissipated through the impact. Jointly, these equations implicitly define the momentum change normal to ∂C.

In discrete time we must redefine the contact force as $f^{con} \in T^*([0,1] \times \partial C)$. Using this form we can construct a discrete Lagrange-d'Alembert principle of the form

$$\delta\left[\sum_{k=0}^{i-1} L_d(\bar{q}_k, \bar{q}_{k+1}) + L_d(\bar{q}_i, \tilde{\bar{q}}) + L_d(\tilde{\bar{q}}, \bar{q}_{i+1}) + \sum_{k=i+1}^{N-1} L_d(\bar{q}_k, \bar{q}_{k+1})\right] + f^{con} \cdot \delta(\alpha, \tilde{q}) = 0. \tag{19}$$

The implications of this variational principle leave equations (11), (12), and (15) unchanged, but (17) and (18) have additional terms such that

$$i^*\left(D_4 L_d(\bar{q}_i, \tilde{\bar{q}}) + D_2 L_d(\tilde{\bar{q}}, \bar{q}_{i+1}) + f_q^{con}\right) = 0, \tag{20}$$

$$D_3 L_d(\bar{q}_i, \tilde{\bar{q}}) + D_1 L_d(\tilde{\bar{q}}, \bar{q}_{i+1}) + \frac{1}{h} f_t^{con} = 0, \tag{21}$$

where in this case $f_t^{con} \in T^*([0,1])$. Just as (13) and (14) served as the discrete time analogy of (8) and (9) in the case of elastic impacts, here we see (20) and (21) are analogous to (17) and (18) in the case of lossful impacts.

4.3 Plastic Collisions

The previous theory regarding lossful impacts naturally leads one to consider plastic impacts. That is, what happens in the case in which an impact removes enough energy from the system that it remains on the contact manifold? The work presented in [2] aids in answering this question by contrasting elastic and plastic impacts in a geometric framework. This is done by characterizing collisions in terms of distributions on TQ to which the system must belong before and after impact. The previous two sections handled cases in which the pre-impact distribution, $\mathcal{D}^-$, as well as the post-impact distribution, $\mathcal{D}^+$, were TC. This section handles the case in which a given system is initially free and then is subject to a holonomic constraint after impact, meaning $\mathcal{D}^- \neq \mathcal{D}^+$. As previously, we consider constraints described by a constraint function[2] g, and now also have the conditions for the corresponding constraint submanifold R that $R \in \partial C$ and $\mathcal{D}^+ = TR$. As in the elastic case, $\mathcal{D}^- = TC$ and thus the pre-impact equations on $[0, \tilde{t})$ keep the form of (6) and (7). The transition equations maintain the form of (17) and (18) in the last section, with the added condition that the post impact phase lies in $\mathcal{D}^+$. This condition, which is of the form $(q(\tilde{t}^+), \dot{q}(\tilde{t}^+)) \in TR$, is essentially a constraint on the allowable force fields f^{con}. Following the impact the system will obey the following *constrained Euler-Lagrange equations* in $(\tilde{t}, T]$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + G(q)^T \cdot \lambda = 0, \tag{22}$$

together with $g(q) = 0$, where $G(q) = Dg(q)$ denotes the Jacobian of g.

[2] In some cases the precise definition of g cannot be specified prior to a collision occurring. Nonetheless, knowing the type of the constraint g (i.e. sticking or sliding contact) will allow one to fill in these details upon contact with ∂C

One should note that in describing plastic collisions we made use of two variational princples. The results from (16), a principle for unconstrained lossful collisions, and the results from (4), a principle for smooth constrained systems, were joined at $\tilde{t}^+$. While obtaining the perfectly plastic impact equations from a single variational principle remains a subject of interest, it seems unlikely that this is possible. The main difficulty is the lack of smoothness in a path space that describes trajectories subject to constraint distributions of varying dimension. Our technique of bootstrapping variational results is also utilized in the next section.

To define an algorithm for the integration of trajectories through plastic collisions we follow arguments analogous to those above for the continuous time case. The pre-impact equations match those of the case of elastic collisions in discrete time, meaning one would integrate according to (11) until determining some $q_{k+1} \notin C$. Discarding this q_{k+1}, the equations (12), (20), and (21), along with the added condition $(\tilde{q}, q_{i+1}) \in (R \times R)$,[3] are used determine $\tilde{t}, \tilde{q}$, and q_{i+1}. Finally, after impact one would integrate according to (5) for all $k \geq i+1$.

4.4 Transfer of Contact

The final behavior we will describe is that of an instantaneous transfer of contacts. That is, a mechanical system with one established contact, corresponding to a constraint manifold $R_1 = g_1^{-1}(0)$, undergoes a plastic impact which establishes a new contact on a constraint manifold $R_2 = g_2^{-1}(0)$ and causes contact release from R_1. That is, $\mathcal{D}^- = TR_1$ and $\mathcal{D}^+ = TR_2$. For simplicity we assume that on a specified interval of time $[0, T]$ a transfer of contact occurs at some $\tilde{t}$ in the interval, and no other impacts occur. Under this assumption, prior to impact the system will obey (22) with g_1 inserted as the constraint function. The transition equations keep the form of (17) and (18) with the condition $(q(\tilde{t}^+), \dot{q}(\tilde{t}^+)) \in TR_2$. These transition equations do not guarantee release from R_1 following the collision, and in fact conditions for that release are not trivial. In certain circumstances, the system could encounter contacts representative of the paradox of Painleve. Incorporation of the resolution to this paradox, which involves measure differential inclusions and tangential impact forces [15], remains a topic of future work. Currently, we simply assume the impact causes the system to release from R_1, and verify this assumption with the condition $q(t) \in C$ for $t > \tilde{t}$. Upon release from R_1 the system obeys (22) with g_2 inserted as the constraint function.

The discrete time algorithm for integration of a trajectory through transfer of contacts is as follows. One integrates according to (5) on R_1 until $q_{k+1} \notin C$. Discarding this inadmissible configuration, the following equations are used to solve for $\lambda_i, \tilde{t}$ and $\tilde{q}$,

$$D_4 L_d(\bar{q}_{i-1}, \bar{q}_i) + D_2 L_d(\bar{q}_i, \tilde{\bar{q}}) - G_{1_d}(q_i)^T \cdot \lambda_i = 0, \tag{23}$$

along with $g_1(\tilde{q}) = 0$ and $\tilde{q} \in R_2$. Following this, one integrates the system forward

[3]This condition is analogous to the requirement in continuous time that $(q(\tilde{t}^+), \dot{q}(\tilde{t}^+)) \in TR$

off of R_1 and onto R_2 by calculating $\tilde{\lambda}, \dot{\tilde{q}}$, and q_{i+1} that satisfy

$$\begin{aligned} i^*\Big(D_4L_d(\bar{q}_i,\tilde{\bar{q}}) + D_2L_d(\tilde{\bar{q}},\bar{q}_{i+1}) - \frac{1}{2}G_{1_d}(\tilde{q})^T\cdot\tilde{\lambda} + f_q^{con}\Big) &= 0,\\ D_3L_d(\bar{q}_i,\tilde{\bar{q}}) + D_1L_d(\tilde{\bar{q}},\bar{q}_{i+1}) + \frac{1}{h}f_t^{con} &= 0,\\ D_4L_d(\bar{q}_i,\tilde{\bar{q}}) - \frac{1}{2}G_{1_d}(\tilde{q})^T\cdot\tilde{\lambda} - D_2L(\tilde{q},\dot{\tilde{q}}) &= 0,\\ G_{1_d}(\tilde{q})\cdot\dot{\tilde{q}} &= 0, \end{aligned} \tag{24}$$

along with $(\tilde{q}, q_{i+1}) \in (R_2 \times R_2)$. Upon solving these equations, release from R_1 is verified by checking that $q_{i+1} \in C$, meaning the released contact has accelerated away from the contact surface. Assuming this condition is met, one can proceed in integrating with (5) on R_2.

5 DMOC Method and Results

5.1 The DMOC Method

The standard continuous time optimal control problem that we consider seeks to find a local minima of a cost function, J, while moving a mechanical system from a specified initial phase, $(q^0, \dot{q}^0)$, to a specified final phase, $(q^T, \dot{q}^T)$, in time T. It is assumed that J is the integral of a performance metric, F, and thus the problem can be formally stated

$$\text{Minimize } J(q(t), u(t)) = \int_0^T F(q(t), \dot{q}(t), u(t)),$$

subject to $(q(0), \dot{q}(0)) = (q^0, \dot{q}^0)$, $(q(T), \dot{q}(T)) = (q^T, \dot{q}^T)$, and the appropriate forced Euler-Lagrange equations of motion. In the case example to come, we replace the notion of specified boundary conditions with a periodicity relation $P : TQ \to TQ$ such that $(q(T), \dot{q}(T)) = P(q(0), \dot{q}(0))$.

The DMOC method moves all aspects of the problem above to the discrete time setting. That is, a discrete path q_d is introduced as previously discussed, as well as *left and right discrete forces* u_k^- and u_k^+ at each time node. As described in [12], these discrete forces approximate virtual work on each slice of time in order to incorporate the influence of nonconservative forcing into discrete variational principles. That is, u_k^- and u_k^+ provide the approximation

$$u_k^- \cdot \delta q_k + u_k^+ \cdot \delta q_{k+1} \approx \int_{kh}^{(k+1)h} u(t) \cdot \delta q(t)\, dt.$$

By appending a sum of these approximations to any of the variational principles we've introduced, one can determine forced DEL equations for the various collision behaviors we've described with little change to the variational arguments. To provide a discrete time approximation of J, a discrete cost function, J_d, is introduced as the sum of a discrete performance metric, F_d, which is to be evaluated at each

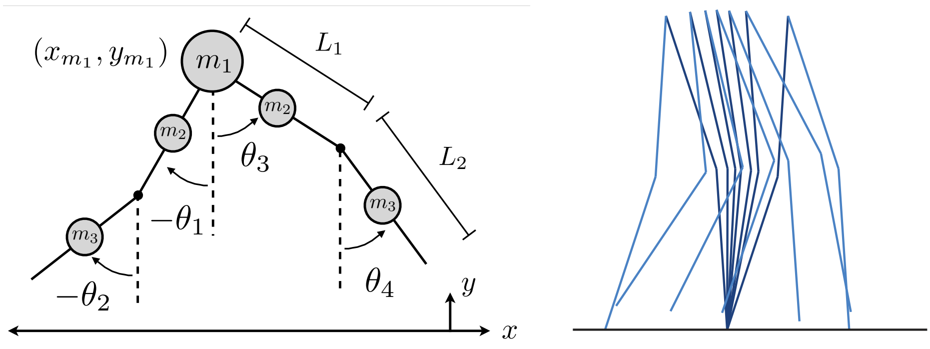

Figure 1. *Model and gait snapshots for the 4-Link Planar Biped. The snapshots depict one step of the periodic gait, in which the stance (dark blue) leg remains in contact with the ground and the swing (light blue) leg travels from contact release to the next impact. Snapshots are 0.15 sec. apart.*

time slice. Finally, the periodic boundary conditions are recast with a discrete periodicity relation, $P_d : Q \times Q \to Q \times Q$. With all of these components, the problem which DMOC considers is stated as

$$\text{Minimize } J_d(q_d, u_d) = \sum_{k=0}^{N-1} F_d(q_k, q_{k+1}, u_k^-, u_k^+),$$

subject to $(q_{N-1}, q_N) = P_d(q_0, q_1)$, and the appropriate forced DEL equations of motion. This problem, which is an equality constrained nonlinear optimization problem, can be solved using sequential quadratic programming (SQP) methods.

5.2 4-Link Biped Results

As a case example we have applied DMOC to determine a periodic walking gait for a 4-link planar bipedal robot. We seek trajectories that are locally optimal with respect to its specific cost of transport, a standard metric in locomotion problems [4]. The biped and snapshots of its locally optimal gait are shown in Fig. 1. Fig. 2 plots the configuration and controls for two steps of the gait depicted in Fig. 1. For precise correspondence between the figures, one could view the step portrayed in the gait snapshots as taking place between the dotted lines of Fig. 2 that mark the impact times, and in this interval consider that θ_3 and θ_4 indicate the configuration of the stance leg. For the biped model we prescribed $m_1 = 10$ kg, $m_2 = 3$ kg, $m_3 = 2$ kg, $L_1 = L_2 = 0.5$ m, and $T = 0.9$ s. Further we prescribed that control inputs u_1, u_2, and u_3 had the form of torques about the θ_2 knee joint, the hip, and the θ_4 knee joint respectively. The locally optimal value of the specific cost of transport for the gait is 0.034, roughly one-tenth of that for average human walking. We hypothesize that this figure could be improved with actuation at the ground, in the form of a control torque at the contact point or the addition of feet to the

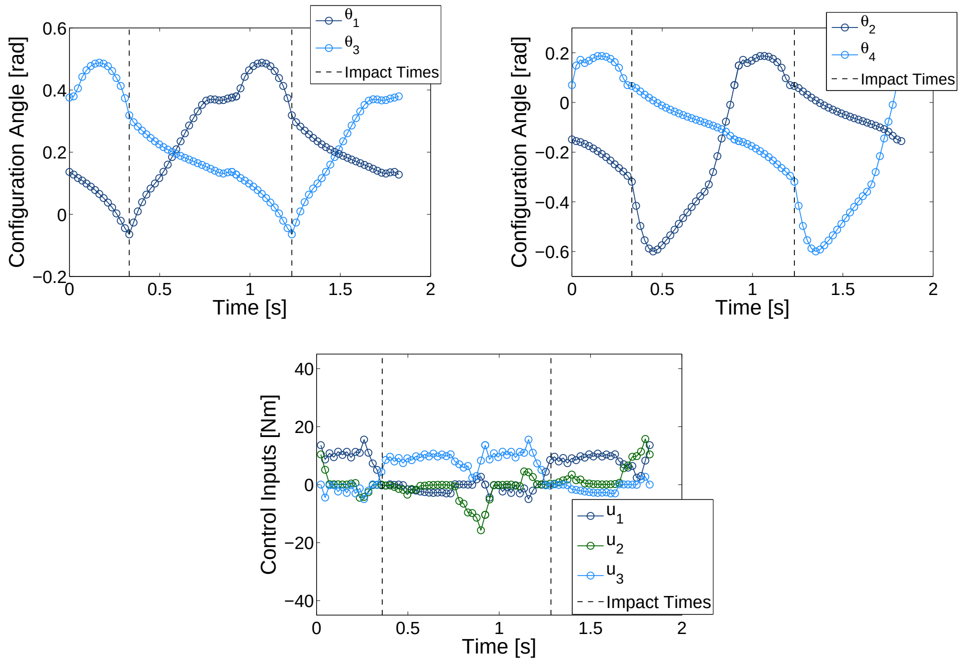

Figure 2. *Locally optimal discrete paths of configuration and control input provided by the DMOC method. Two steps of the biped's gait are plotted, in order to view periodicity.*

model. As is, the robot is underactuated, in the sense that it has more degrees of freedom than control inputs, and thus may have to make inefficient movements to maintain forward momentum.

6 Conclusions and Future Directions

We have used variational principles to derive governing equations for a variety of impact behaviors. Furthermore, we have described how the discrete time equations of motion provide variational integration algorithms and act as constraints in the DMOC method for determining optimal controls. As a case example, we have determined controls for a 4-link planar biped that produce a periodic gait which is locally optimal with respect to the specific cost of transport. One extension to this work that is currently underway is the incorporation of the discrete null space method [10] into our variational description of impacts. Another extension incorporates our implementation of DMOC into a "Design of Dynamics" scheme that uses tools from trend optimization [14] to optimize system designs as well as trajectories and controls. Beyond this, future work may involve more detailed descriptions of contact release conditions, tools from the measure differential inclusion formulation of contact mechanics, or impact models involving multiple contacts.

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