Dynamical Systems and Invariant Manifolds[†]

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Abstract. We review some basic terminology in dynamical systems with the purpose of bridging some of the communication gaps that may exist between mathematicians and engineers at this conference. Recent results on panel flutter and on the existence of horseshoes in the dynamics of a forced beam are briefly sketched to illustrate some of the concepts of interest to both groups.

1. Dynamical Systems on Manifolds. A vector field X on a manifold M is a (smooth) mapping from M to TM, the tangent bundle of M, that assigns to each point $x \in M$ a vector tangent to M at x. Often, M is Euclidean n space $\mathbb{R}^n$, so X is a vector field in the sense of advanced calculus: $X(x^1, \ldots, x^n) = (X^1(x^1, \ldots, x^n), X^2(x^1, \ldots, x^n), \ldots, X^n(x^1, \ldots, x^n))$. A vector field X may be thc of as the right hand side of a system of first order differential e tions in the large, that is, a dynamical system. In $\mathbb{R}^n$, this si corresponds to the system of n-ordinary differential equations

$$\frac{dx^i}{dt} = X^i(x^1(t), \ldots, x^n(t)), \quad i = 1, \ldots,$$

or, abstractly,

$$\dot{x} = X(x(t)).$$

One might ask whether it is worthwhile to engineers to invest i the machinery of dynamical systems on manifolds. This question m answered in the affirmative on two grounds as follows:

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then take $S_\lambda = \bar{S} \cap X \times \{\lambda\}$, an invariant manifold for each λ. One may regard S_λ as implicitly defined in the same way as the function $\phi(x_-, \lambda)$ in the Liapunov-Schmidt procedure is implicitly defined.

Zeros of f near (x_0, λ_0) necessarily lie on $\bar{S}$, so the problem reduces to finding zeros of $f|\bar{S}$, the analogue of the bifurcation equation. For finding fixed points, this is a geometric formulation of the Liapunov-Schmidt procedure. The fact that we are dealing with vector fields entails that the choice of Y_2 (or M in Section 4) is now automatically made; both S_P and M are now replaced by $\bar{S}$.

In order to capture dynamic bifurcations as well as static ones, it is necessary to enlarge $\bar{S}$ to the full center manifold, as is explained in, for example, J. Marsden and M. McCracken, *The Hopf Bifurcation and Its Applications*, Springer Appl. Math. Sciences #19 (1976). (For operators with real eigenvalues, such as potential operators, $\bar{S}$ equals the center manifold.)

The fact that the reduction of a potential operator by the Liapunov-Schmidt procedure results in a potential operator is now clear. In fact, if one uses the space $\bar{S}$, a modification of ϕ is *not necessary*; one needs only to restrict it to $\bar{S}$. This is because of the following obvious fact: the restriction of a gradient vector field to an invariant submanifold is a gradient vector field whose potential is the restriction of the original one; i.e. $(\nabla\phi)|S = \nabla(\phi\,|S)$ if $\nabla\phi$ is tangent to S.

(i) Quite often in practice, the systems that one encounters have state spaces which result from imposing smooth constraints on vector spaces. For instance, in a circuit the dynamics of the capacitor voltages and the inductor currents are constrained along a manifold specified by Kirchoff's laws and the static non-linear resistor characteristics.

(ii) It often helps to identify the state space of a physical object with a more abstract mathematical object with a manifold structure and exploit the convenient geometric intuition of manifolds. For instance in rigid body rotation the state space may be identified with SO(3), the space of all proper orthogonal 3×3 matrices, which is a compact 3-dimensional manifold.

We now state some of the simplest results of dynamical systems culminating in invariant manifold theory (the general reference used here is Abraham-Marsden [1]). First we give some definitions and preliminaries.

Let X be a (time-independent) vector field on M. An <u>integral curve</u> or <u>trajectory</u> of X at $x_0 \in M$ is a curve $x(t)$ in M such that $\dot{x}(t) = X(x(t))$ for each t in an open interval I and $x(0) = x_0$. If X is smooth (or locally Lipschitz will do), then the integral curve of X at x_0 exists and is unique. The vector field X is said to be <u>complete</u> if the domain of each integral curve can be extended to all of $\mathbb{R}$ (i.e. the open interval I can be chosen to be $\mathbb{R}$). If M is compact, any (smooth) vector field X on M is complete; or if the support of a vector field X is compact, it is complete. We assume hence forward that the vector field X is complete, for simplicity. In this case we can associate with X a one-parameter family of diffeomorphism on M called the <u>flow</u> of X denoted F_t and defined by

$$F_0(x_0) = x_0$$

and

$$\frac{d}{dt} F_t(x_0) = X(F_t(x_0))$$

A point x_0 is called a <u>critical point</u> (or <u>singular point</u> or <u>equilibrium point</u>) of X if $X(x_0) = 0$. This is equivalent to x_0

being a fixed point of the flow: $F_t(x_0) = x_0$. The linearization of X at x_0 is the linear map

$$X'(x_0) : T_{x_0}M \to T_{x_0}M \text{ defined by}$$

$$X'(x_0)v = \frac{d}{d\lambda}\left(TF_\lambda(x_0)\cdot v\right)\Big|_{\lambda=0}$$

In $\mathbb{R}^n$, $X'(x_0)$ is the matrix $\partial X^i/\partial x^j$ evaluated at x_0. The eigenvalues of $X'(x_0)$ are called the characteristic exponents of X at x_0 and their exponentials are called the characteristic multipliers of X at x_0.

Of basic interest to engineers is the stability of a critical point, defined as follows: If x_0 is a critical point of λ then x_0 is stable (in the sense of Liapounov) if for any neighbourhood U of x_0, there is a neighbourhood V of x_0, such that $F_t(x) \in U$ for all $x \in V$ and all $t \geq 0$. The point x_0 is said to be asymptotically stable if there exists a neighbourhood V of x_0 such that

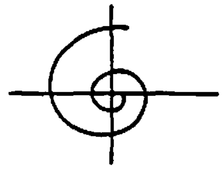
$$F_t(V) \subset F_s(V) \text{ if } t > s \text{ and } \lim_{t\to\infty} F_t(V) = \{x_0\}.$$

An important sufficient condition for checking stability is the following theorem of Liapounov:

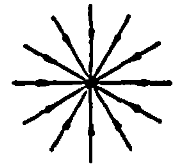
Liapounov's Theorem. Let x_0 be a critical point of X and let the characteristic exponents of X at x_0 have strictly negative real parts. Then x_0 is asymptotically stable (similarly, if the characteristic exponents of X have strictly positive real parts, then x_0 is asymptotically unstable i.e. asymptotically stable as $t \to -\infty$).

A critical point x_0 is said to be hyperbolic if none of its characteristic exponents has zero real part. A result of Hartman shows that near a hyperbolic critical point the flow looks like that of its linearization (i.e. is conjugate to the flow of its linearization). Thus, in the plane we have (upto diffeomorphism) the hyperbolic flows shown in Figure 1.

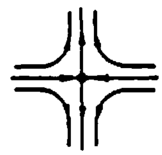
The case when the critical point is not hyperbolic is of obvious interest, for instance in Hamiltonian dynamics, and will be discussed



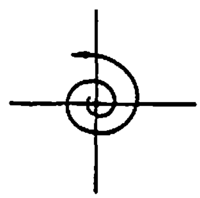
(a) $Re\,\mu_1 = Re\,\mu_2 < 0$
μ_1, μ_2 not real. (stable focus)



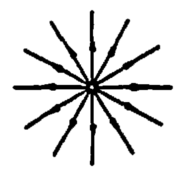
(b) $\mu_1 < 0, \mu_2 < 0$ (stable node)



(c) $\mu_1 > 0, \mu_2 < 0$ (saddle)



(d) $Re\mu_1 = Re\mu_2 > 0$
μ_1, μ_2 not real
(unstable focus)



(e) $\mu_1 > 0, \mu_2 > 0$
(unstable node).

Figure 1 Hyperbolic equilibria with characteristic exponents. (a) $Re\,\mu_1 = Re\,\mu_2 < 0$, with μ_1, μ_2 not real (stable focus). (b) $\mu_1 < 0$, $\mu_2 < 0$ (stable node). (c) $\mu_1 < 0$, $\mu_2 > 0$ (saddle). (d) $Re\,\mu_1 = Re\,\mu_2 > 0$, with μ_1, μ_2 not real (unstable focus). (e) $\mu_1 > 0$, $\mu_2 > 0$ (unstable node).

below.

We next discuss another possible <u>critical element</u> of the vector field X, namely a <u>closed orbit</u>. A <u>periodic point</u> of X is a point $x \in M$ such that for some $\tau > 0$, $F_{t+\tau}(x) = F_t(x)$ for all $t \in \mathbb{R}$, and the <u>period</u> of x is the smallest $\tau > 0$ satisfying this condition. A <u>closed orbit</u> is the orbit of a periodic non-equilibrium point. We have seen how the linearization, $X'(x_0)$ of the vector field X at an equilibrium point x_0 approximates the flow of X near x_0. We now discuss the asymptotic behavior of orbits close to a closed orbit using the Poincaré map on a local-transversal section. This is defined as follows: A <u>local transversal</u> section of X at $x \in M$ is a submanifold $S \subset M$ of codimension one with $x \in S$ and with $X(s)$ not contained in (transversal to) $T_s S$ for all $s \in S$. Then, if γ is a closed orbit of X with period τ and S a local transversal section of X at $x \in \gamma$ then a <u>Poincaré map</u> of γ is a mapping

$\Theta : W_0 \to W_1$ where

(i) W_0, W_1 are open neighbourhoods on S of $x \in S$ and Θ is a diffeomorphism,

(ii) There is a continuous function $\delta : W_0 \to \mathbb{R}$, such that $\Theta(s) = F(s, \tau - \delta(s))$; and

(iii) If $t \in (0, \tau - \delta(s))$ then $F(s,t) \notin W_0$.

This definition is visualized in Figure 2.

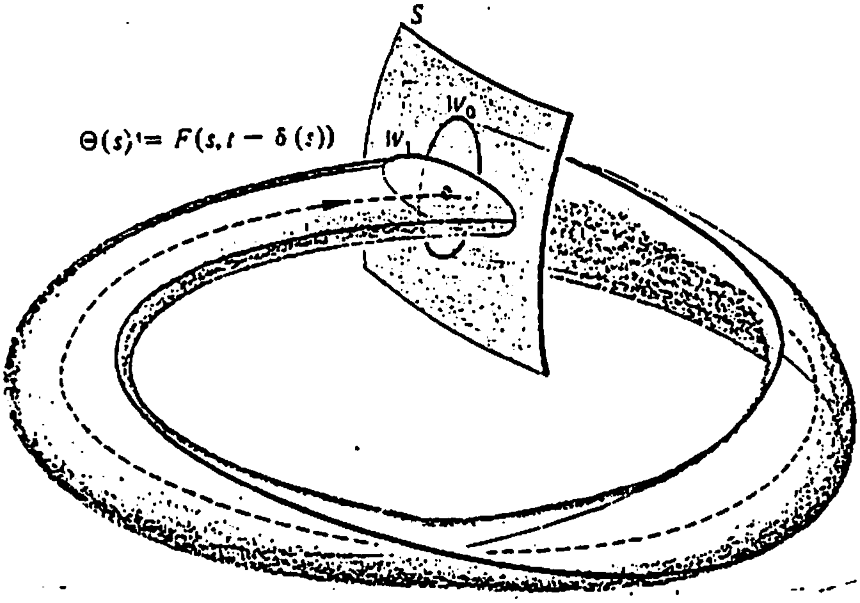


Figure 2. Visualization of the Poincaré map.

It is a basic result that Poincaré maps exist. Also, they are unique upto configuration -- i.e. if S' is another local transverse section at $x' \in \gamma$ with associated Poincaré map Θ' then there are open neighbourhoods $W_2 \subset S$, $W_2' \subset S'$, $W_2 \subset W_0 \cap W_1$, $W_2' \subset W_0' \cap W_1'$ and a diffeomorphism $H : W_2 \to W_2'$ such that the diagram

$$\begin{array}{ccc} \Theta^{-1}(W_2) \cap W_2 & \xrightarrow{\Theta} & W_2 \cap \Theta(W_2) \\ H \downarrow & & \downarrow H \\ W_2' & \xrightarrow{\Theta'} & S' \end{array}$$

commutes.

The linear approximation to Θ at x is $T_x\Theta : T_xS \to T_xS$ and the uniqueness of the Poincaré map upto configuration makes $T_x\Theta'$ similar to $T_x\Theta$ so that the eigenvalues of $T_x\Theta$ are independent of $x \in \gamma$ and the specific transverse section. These eigenvalues of $T_x\Theta$ are referred to as the characteristic multipliers of X at γ. Another linear approximation to the flow near γ is given by $T_xF_\tau\colon T_xM \to T_xM$. It is clear that T_xF_τ has an eigenvalue 1 corresponding to the eigenvector $X(x)$ (since τ is the period of the closed orbit). The remaining eigenvalues are the characteristic multipliers of X at γ.

We now define what we understand by asymptotic stability of a closed orbit. An orbit $F_t(y)$ is said to wind toward γ if for any transversal S to X at $x \in \gamma$ there is a t_0 such $F_{t_0}(y) \in S$ and successive applications of the Poincaré map yield a sequence of points that converge to x. We then have the following condition for this stability.

Proposition. If γ is a closed orbit of X and the characteristic multipliers of γ lie inside the unit circle, then there is a neighbourhood U of γ such that for any $y \in U$, the orbit $F_t(y)$ winds towards γ.

2. Invariant Manifolds. The motivation for invariant manifolds comes from the study of critical elements of linear differential equations of the form

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n.$$

Let W_s, W_c and W_u be the (generalized) real eigenspaces of A associated with eigenvalues of A lying the open left half plane, the imaginary axes and open right half plane respectively. Each of these spaces is invariant under the flow of $\dot{x} = Ax$ and represents respectively a stable, center and unstable manifold.

To return to the non-linear case, a submanifold $S \subset M$ is said to be invariant under the flow of X if for $x \in S$, $F_t(x) \in S$ for small $t > 0$; i.e. X is tangent to S. Invariant manifolds are, then, "non-linear eigenspaces". We may define invariant manifolds S of a critical element γ to be stable or unstable depending on whether they are comprised of orbits in S that wind toward γ with increasing time, or wind toward γ with decreasing time. In a neighbourhood of x in the critical element γ, the tangent spaces to the stable and unstable manifolds are provided by the eigenspaces in T_xM of characteristic multipliers of modulus <1 and modulus >1 respectively. The eigenspace corresponding to eigenvalues of modulus $= 1$ (not including $T_x\gamma$) is tangent to the center manifold of γ. We state this result formally as the local center stable manifold (for proof, see for example Kelley's appendix in Abraham-Robbin [2]).

Local-Center Stable Manifold Theorem. If $\gamma \subset M$ is a critical element of X, there exist submanifolds S_γ, CS_γ, C_γ, CUS_γ, US_γ of M (also denoted $W^s(\gamma)$, $W^{cs}(\gamma)$, $W^c(\gamma)$, $W^{cu}(\gamma)$ and $W^u(\gamma)$) such that

(i) Each sub-manifold is locally invariant under X and contains γ

(ii) For $x \in \gamma, T_x(S_\gamma)$ [resp. $T_x(CS_\gamma)$, $T_x(C_\gamma)$, $T_x(CUS_\gamma)$, $T_x(US_\gamma)$] is the sum of the eigenspace in T_xM of characteristic multipliers of modulus <1 [resp. ≤ 1, $= 1$, ≥ 1, >1] and the subspace $T_x\gamma$.

(iii) If $x \in S_\gamma$, then $\bigcap_{n \in \mathbb{Z}} F_{(n,\infty)}(x) = \gamma$; and if $x \in US_\gamma$, then $\bigcap_{n \in \mathbb{Z}} F_{(-\infty,n)}(x) = \gamma$.

(iv) S_γ and US_γ are (locally) unique.

Comments: (i) The configuration of these manifolds is slightly different in the cases covered $\gamma = \{x\}$, a critical point in which case $T_x\gamma = \{0\}$ or γ is a closed orbit in which case $T_x\gamma$ is the subspace generated by $X(x)$. These two cases are shown in Figure 3.

(ii) The stable and unstable manifolds are unique; but the center manifold is not unique (see Kelley's article cited above, Marsden and McCracken [10] and Wan [17]).

(iii) The theorem says in addition that if γ is hyperbolic then locally, the orbits behave qualitatively (actually, up to diffeomorphism) like the linear case.

(iv) If γ is hyperbolic we only have the locally unique manifolds S_γ and US_γ. These can be extended to globally unique, immersed submanifolds by means of the integral of X. This is the global stable manifold theorem of Smale. (There is also a global center manifold theorem due to Fenichel).

(v) From the stability (instability) of the center manifold C we can conclude the stability (instability) of the center stable CS (center-unstable CUS) manifold. This is an important theorem of Pliss and

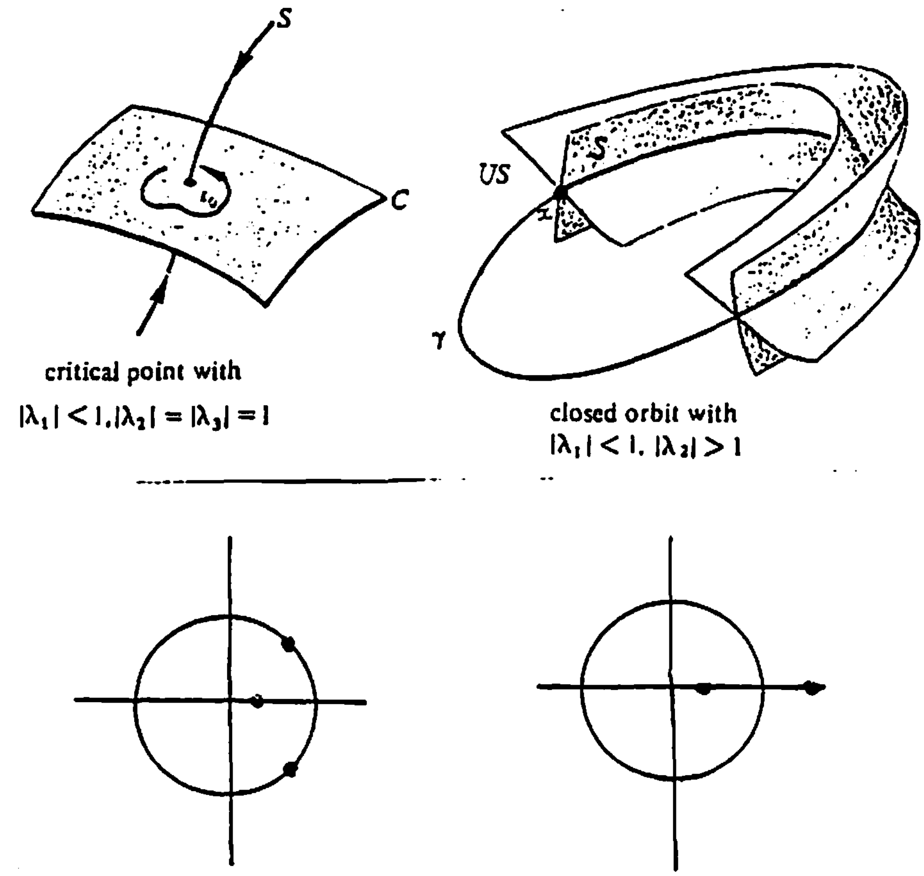


Figure 3. Invariant Manifolds.

Kelley.

It may be mentioned here in conclusion that the theory of invariant manifolds can be generalized in two important directions: (i) to maps rather than to dynamical systems and (ii) to arbitrary non-wandering sets of the flow rather than elementary critical elements. We make a few comments on (i) and (ii).

(i) Mappings rather than flows arise in at least 3 basic ways:

(a) Many systems are directly described by discrete dynamics: $x_{n+1} = f(x_n)$. For example, many population problems are best understood this way. Delay and difference equations can be viewed in this category as well.

(b) The Poincaré map of a closed orbit has already been mentioned.

(c) Suppose we are interested in non-autonomous systems of the form $\dot{x} = f(x,t)$ where f is T-periodic in t. Then the map P that advances solutions by time T, also called the Poincaré map, is very basic to a qualitative study of the orbits. (See Figure 4). This map is often used in forced oscillations as we shall see in Section 5.

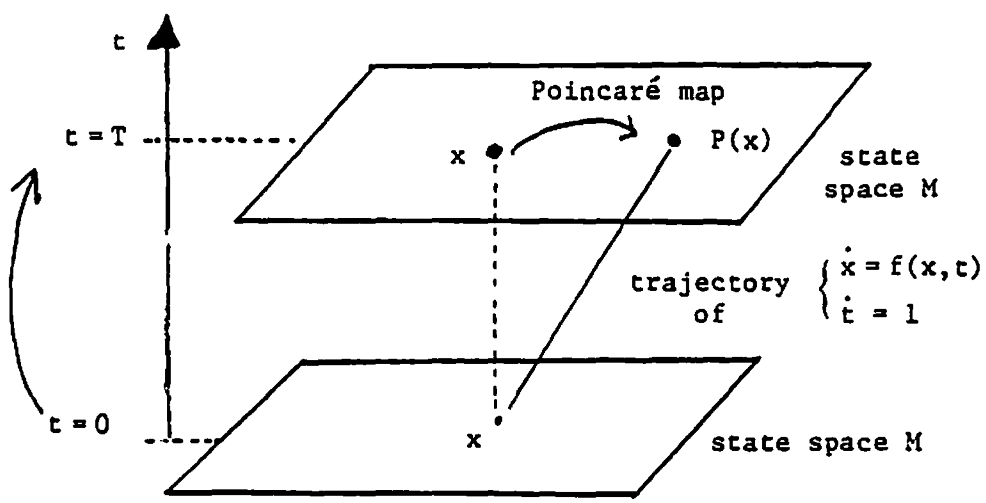


Figure 4. The Poincaré map for forced oscillations.

(ii) The generalization of the invariant manifold theory to arbitrary nonwandering sets is significant; see for instance, Hirsch, Pugh and Shub [4]). The main problem here is the lack of a spectrum (characteristic multipliers) to be able to define a hyperbolic property. The definitions are rather in terms of contractions and expansions under the flow of the norm of the tangent to the flow (with the norm induced by an appropriate Riemannian metric). In the study of chaotic dynamics such complex invariant sets can arise in very concrete systems as many of the other lectures will demonstrate. In Section 5 we shall briefly sketch an example of a complex invariant set, namely the horseshoe.

3. Invariant Manifold Theory for Partial Differential Equations. In applications to partial differential equations a useful assumption is that the semi-flow F_t defined by the equations be smooth for each $t \geq 0$, i.e. $F_t : Z \to Z$ is smooth, where Z is a suitably chosen Banach space of functions and F_t takes Cauchy data at $t = 0$ to the solution

at time t. This enables the invariant manifold machinery to go through along with bifurcation theorems (for an example of applications to the Hopf bifurcation, see ([10]). General conditions for checking smoothness are technical, but the following special instances are easily understood:

(i) F_t is smooth for semilinear p.d.e.s.

$$\frac{\partial u}{\partial t} = \Delta u + f(u) \quad \text{with} \quad u \in H_0^2(\Omega) = Z$$

or more abstractly $\dot{x} = Ax + F(x)$

where A is a generator and F is a smooth function of Z to Z. This is due to Segal [15].

(ii) F_t is smooth for the Navier-Stokes equation

$$\dot{u} = +(u\cdot\nabla)u = \text{grad } p; \quad \text{div } u = 0; \quad Z = H_0^2(\Omega)$$

See [10]. This is essentially due to Kato and Fujita.

(iii) F_t is smooth for the Euler-equations for a fluid in Lagrangian coordinates. This case is due to Ebin and Marsden; see [9]; (F_t is not, however, smooth in Eulerian coordinates).

(iv) F_t is not smooth for the Korteweg de Vries equation (but is in a kind of Lagrangian coordinates); cf. Ratiu [14].

(v) F_t seems not to be smooth for quasilinear hyperbolic equations, for instance in 3 dimensional (conservative) elasticity.

One now assumes that the spectra of $DF_t(x_0)$ or $D\Theta(x)$ split into 3 pieces, one inside the unit circle (at a non zero distance from the unit circle), the second on the unit circle and the third outside the unit circle (at a non-zero distance from it). Then there are corresponding invariant manifolds. The idea is to apply the invariant manifold theorems for smooth maps to each F_t separately; since $F_t \circ F_s = F_{t+s} = F_s \circ F_t$, these manifolds can be chosen common to all the F_t.

4. Applications of Invariant Manifold Theory to Bifurcations. Invariant manifolds are useful in qualitative investigations and in bifurcation theory. To give a specific example we will show the application of the center-manifold theory to reduce the dimension of a bifurcation problem; this method is due to Ruelle and Takens (for details see Marsden and McCracken [10]). Let F_t be a flow on a Banach space Z depending on a bifurcation parameter $\lambda \in \mathbb{R}^p$. The idea is to apply

the invariant manifold theorems to the suspended flow

$$F_t : Z \times \mathbb{R}^p \to Z \times \mathbb{R}^p$$
$$(x,\lambda) \to (F_t(x),\lambda)$$

The invariant manifold theorem shows that if the spectrum of the linearlization of $DF_t(z_0,\lambda_0)$ at a fixed point (z_0,λ_0) splits into $\sigma_s \cup \sigma_c$ where σ_s lies inside the unit disc (at a non-zero distance) and σ_c is on the unit disc, then the flow F_t leaves invariant manifolds S and C tangent to the eigenspaces corresponding to σ_s and σ_c respectively; S is the stable and C is the center manifold respectively. For suspended systems we always have $1 \in \sigma_c$. We now state the center manifold theorem for flows in this context:

Center Manifold Theorem for Suspended Flows. Let Z be a Banach space (or manifold) and let ψ be the time one map of the suspended flow defined in a neighbourhood of (z_0,λ_0). Assume that $\psi(z_0,\lambda_0) = (z_0,\lambda_0)$, that ψ has k continuous derivatives, that $d\psi(z_0,\lambda_0)$ has spectral radius 1 and that the spectrum of $d\psi(z_0,\lambda_0)$ splits into a part on the unit circle and a part at a nonzero distance from the unit circle. Let Y denote the generalized eigenspace of $d\psi(z_0,\lambda_0)$ belonging to the part of the spectrum on the unit circle and that Y has dimension $d < \infty$. Then, there exists a submanifold M defined in a neighbourhood V of (z_0,λ_0) in $Z \times \mathbb{R}^p$ passing through (z_0,λ_0) and tangent to Y at (z_0,λ_0) such that

(i) If $x \in M$ and $\psi(x) \in V$, then $\psi(x) \in M$.

(ii) If $\psi^n(x) \in V$ for $n = 0,1,2,$ then as $n \to \infty$, $\psi^n(x) \to M$.

For dynamical bifurcations the center manifold theorem plays the same role as the Lyapunov-Schmidt procedure for static bifurcation — namely, it reduces the bifurcation problem to a finite dimensional one. In the instance of the Hopf bifurcation with a single parameter ($\lambda \in \mathbb{R}$), we obtain as center manifold for the suspended flow a 3-manifold tangent to the eigenspace of the two simple, purely imaginary eigenvalues crossing the imaginary axis at $\lambda = \lambda_0$ and tangent to the λ-axis at $\lambda = \lambda_0$. By looking at λ = constant sections, the problem is now reduced to that of a vector-field in two dimensions.

This general method has been useful in a number of specific problems.

We illustrate briefly by sketching one used by Holmes and Marsden [6] for the two parameter panel flutter problem (see Dowell [3] for background).

We consider the one dimensional thin panel shown in Figure 5 and are interested in bifurcations near the zero solution

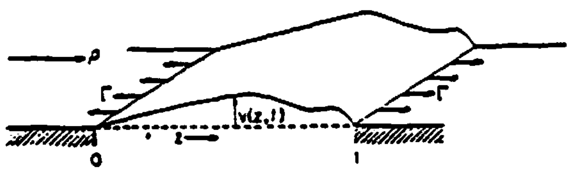


FIG. 5. The panel flutter problem.

written as

$$\alpha \dot{v}'''' + v'''' - \{\Gamma + \kappa \int_0^1 (v'(z))^2\, dz + \sigma \int_0^1 (v'(z)\cdot\dot{v}'(z))\, dz\} v''$$

$$+ \rho v' + \sqrt{\rho}\,\delta \dot{v} + \ddot{v} = 0 \qquad (4.1)$$

Here $\cdot \equiv \partial/\partial t$, $' \equiv \partial/\partial z$; viscoelastic structural damping terms are α, σ; aerodynamic damping terms are $\sqrt{\rho}\,\delta$; κ is the non-linear (membrane) stiffness; ρ the dynamic pressure and Γ an in-plane tensile load. All quantities are assumed non-dimensionalized and boundary conditions at $z = 0,1$ are typically simply supported $(v = v'' = 0)$ or clamped $(v = v' = 0)$. The control parameter is $\lambda = (\rho,\Gamma)$, $\rho > 0$. We redefine (4.1) as an ordinary differential equation on a Banach space $Z = H_0^2([0,1]) \times L^2([0,1])$ where H_0^2 denotes H^2 functions on $[0,1]$ vanishing at 0 and 1. Define the norm on Z by $\|(v,\dot{v})\|_Z = (|\dot{v}|^2 + |v''|^2)^{1/2}$ with $|\cdot|$ denoting the L^2 norm and define the linear operator.

$$A_\mu = \begin{pmatrix} 0 & I \\ C_\mu & D_\mu \end{pmatrix}; \qquad \begin{aligned} C_\mu v &= -v'''' + \Gamma v'' - \rho v' \\ D_\mu v &= \alpha v'''' - \sqrt{\rho}\,\delta v \end{aligned}$$

The basic domain of A_μ consists of $\{v,\dot{v}\} \in Z$ such that $\dot{v} \in H_0^2$ and $v + \alpha\dot{v} \in H^4$; specific boundary conditions necessitate further restrictions.

Also, define the nonlinear operator

$$B(v,\dot{v}) = \begin{pmatrix} 0 \\ \\ [\kappa|v'|^2 + \sigma\langle v',\dot{v}'\rangle]\, v'' \end{pmatrix}$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product. Then we can write (4.1) in the form

$$\dot{x} = A_\mu x + B(x) \quad \text{with} \quad x = \{v,\dot{v}\} \in \text{the domain of definition of } A_\mu .$$

Using ideas of Segal [15] and Parks [13] it can be shown that (4.2) does define (globally) a smooth flow on the domain of $A_\mu \subset Z$, and for $\alpha = 0.005$ and $\delta = 0.1$ the bifurcations take place in the vicinity of $\rho = 108$ and $\Gamma = -22$ when a double zero eigenvalue occurs. The center manifold theorem again reduces this to a 2-dimensional problem. This leads to the bifurcation diagram of Figure 6 (which is the Andronov-Takens normal form for the two-dimensional flow $\ddot{x} = -\nu_2 x - \nu_1\dot{x} - x^3 - x^2\dot{x}$ with parameters ν_1, ν_2).

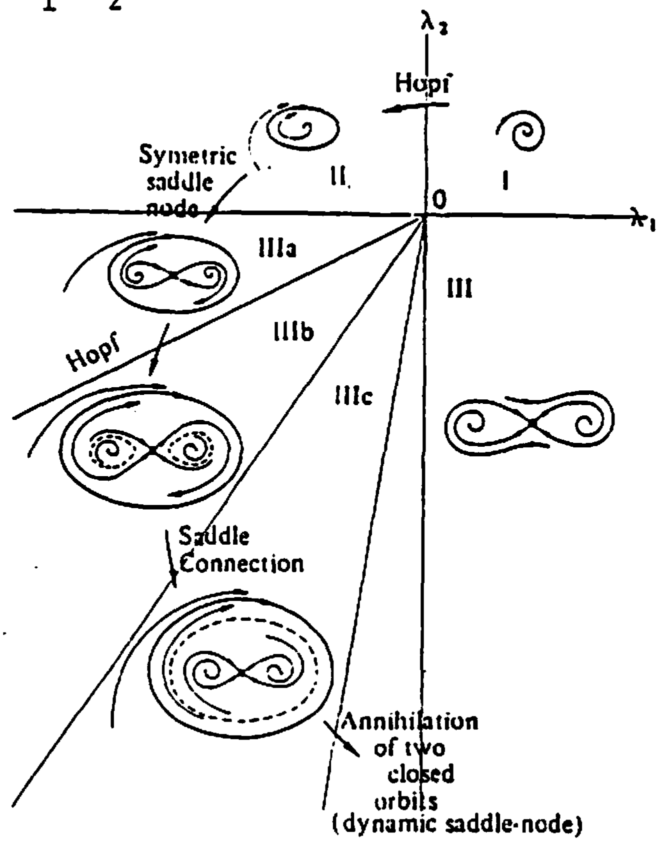


Figure 6 Takens' (2.—) normal form showing the local phase portrait in each region of parameter space.

In particular, there is a supercritical Hopf bifurcation on crossing from region I to region II, and a symmetric saddle node bifurcation on crossing from region I to region III. On crossing from region II to region IIIa the stable closed orbit persists and the unstable critical point bifurcates to a saddle and two unstable fixed points. This bifurcation diagram is actually a structurally stable (symmetric) bifurcation in the sense that it persists under any symmetric perturbation.

The transition from region IIIb to IIIc is especially interesting because a homoclinic loop occurs at the instant of bifurcation; i.e. there is an orbit from the saddle point to itself.

As we shall see in the next section, small (unsymmetric) perturbations of homoclinic loops generally invite horeshoes and "chaos". In fact, this may provide an explanation for the "chaos" discussed in Dowell's lecture.

5. A Horseshoe in the Dynamics of a Forced Beam. In this section we describe a situation in which complex dynamics arises by perturbing a Hamiltonian system with forcing and damping. Several other lectures will be on similar themes; in particular we refer to those of Hale and Levi.

A physical model will help motivate the analysis. One considers a beam that is buckled by an external load Γ so there are two stable equilibrium states and one unstable. (See Figure 7). The whole structure is then shaken with a transverse periodic displacement $f \cos \omega t$ and the beam moves due to its inertia. In a (related) experiment one observes periodic motion about the two stable equilibria for small f but as f is increased, the motion becomes aperiodic or "chaotic". (See Moon and Holmes [12]). The mathematical problem is to develop theorems to explain this bifurcation.

There are a number of specific models that can be used to describe the beam in Figure 7. One such model is the following p.d.e. for the deflection $w(z,t)$ of the center line of the beam:

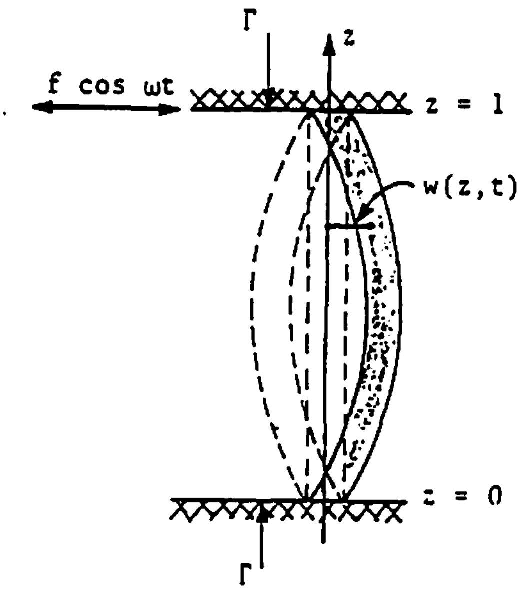


Figure 7. A buckled beam undergoing periodic forcing.

$$\ddot{w} + w'''' + \Gamma w'' - \kappa\left(\int_0^1 [w']^2 d\zeta\right) w'' = \varepsilon(f \cos \omega t - \delta\dot{w}) \qquad (5.1)$$

where $\cdot = \partial/\partial t$, $' = \partial/\partial z$, Γ = external load, κ = stiffness due to "membrane" effects, δ = damping, and ε is a small parameter used to measure the relative size of f and δ. We use hinged boundary conditions: $w = w'' = 0$ at $z = 0, 1$. We assume the beam is in its first buckled state: $\pi^2 < \Gamma < 4\pi^2$.

A simpler model is obtained by looking for "lowest mode" solutions of the form $w(z,t) = x(t) \sin(\pi z)$. Substituting into (5.1), one finds the following Duffing type equation for x:

$$\ddot{x} - \beta x + \alpha x^3 = \varepsilon(\gamma \cos \omega t - \delta\dot{x}), \qquad (5.2)$$

where

$$\beta = \pi^2(\Gamma - \pi^2) > 0, \quad \alpha = \kappa\pi^4/2, \ \gamma = 4f/\pi.$$

The methods used are inspired by Melnikov [11]; see Holmes [5] for an account. We shall set it up in an abstract fashion that applies to the above p.d.e.

We consider an evolution equation in a Banach space X of the form

$$\dot{x} = f_0(x) + \varepsilon f_1(x,t), \tag{5.3}$$

where f_1 is periodic of period T in t. Our hypothesis on (5.3) are as follows:

1. (a) Assume $f_0(x) = Ax + B(x)$ where A is an (unbounded) linear operator that generates a C^0 one parameter group of transformations on X and where $B: X \to X$ is C^∞, and has bounded derivatives on bounded sets.

(b) Assume $f_1: X \times S^1 \to X$ is C^∞ and has bounded derivatives on bounded sets, where $S^1 = \mathbb{R}/(T)$, the circle of length T.

(c) Assume that F_t^ε is defined for all $t \in \mathbb{R}$ for $\varepsilon > 0$ sufficiently small and F_t^ε maps bounded sets in $X \times S^1$ to bounded sets in $X \times S^1$ uniformly for small $\varepsilon \geq 0$ and t in bounded time-intervals.

Assumption 1 implies that the associated suspended autonomous system on $X \times S^1$,

$$\begin{cases} \dot{x} = f_0(x) + \varepsilon f_1(x,\theta), \\ \dot{\theta} = 1, \end{cases} \tag{5.4}$$

has a smooth local flow, F_t^ε, which can be extended globally in time, i.e. solutions do not escape to infinity in finite time. Energy estimates suffice to prove this for equation (5.1).

2. (a) Assume that the system $\dot{x} = f_0(x)$ (the unperturbed system) is Hamiltonian with energy $H_0: X \to \mathbb{R}$.

(b) Assume there is a symplectic 2-manifold $\Sigma \subset X$ invariant under the flow F_t^0 and that on Σ there is a fixed point p_0 and a homoclinic orbit $x_0(t)$, i.e.,

$$f_0(p_0) = 0, \; \dot{x}_0(t) = f_0(x_0(t))$$

and

$$p_0 = \lim_{t \to +\infty} x_0(t) = \lim_{t \to -\infty} x_0(t)$$

This means that X carries a skew symmetric continuous bilinear map $\Omega: X \times X \to \mathbb{R}$ that is weakly non-degenerate (i.e., $\Omega(u,v) = 0$ for all v implies $u = 0$) called the symplectic form and there is a smooth function $H_0 : X \to \mathbb{R}$ such that

$$\Omega(f_0(x), u) = dH_0(x) \cdot u$$

for all x in D_A, the domain of A. Consult Abraham and Marsden [1] for details about Hamiltonian systems.

The next assumption states that the homoclinic orbit through p_0 arises from a hyperbolic saddle.

3. Assume that $\sigma(Df_0(p_0))$, the spectrum of $Df_0(p_0)$, consists of two nonzero real eigenvalues $\pm\lambda$, with the remainder of the spectrum on the imaginary axis, strictly bounded away from 0. Assume that $\sigma(e^{tDf_0(p_0)})$, the spectrum of $e^{tDf_0(p_0)}$, equals the closure of $e^{t\sigma(Df_0(p_0))}$.

Consider the suspended system (5.4) with its flow $F_t^\varepsilon : X \times S^1 \to X \times S^1$. Let $P : X \to X$ be defined by

$$P^\varepsilon(x) = \pi_1 \cdot (F_T^\varepsilon(x, 0))$$

where $\pi_1 : X \times S^1 \to X$ is the projection onto the first factor. The map P^ε is the Poincaré map for the flow F_t^ε. Note that $P^0(p_0) = p_0$, and that fixed points of P^ε correspond to periodic orbits of F_t^ε. One can prove that for $\varepsilon > 0$ small, there is a unique fixed point p_ε for P^ε near p_0; moreover p_ε is a smooth function of ε.

Our final hypothesis means in effect that the perturbation $f_1(x,t)$ is Hamiltonian plus damping. Using an assumption like 3, above, this condition can be stated either in terms of the spectrum of the linearization of equation (5.4) or in terms of the Poincaré map.

4. Assume that for $\varepsilon > 0$ the spectrum of $DP^\varepsilon(p_\varepsilon)$ lies strictly inside the unit circle with the exception of a single real eigenvalue $e^{T\lambda_\varepsilon^+} > 1$.

We remark that the fixed point p_0 perturbs to another fixed point p_ε for the perturbed system. The same is true for the local invariant manifolds [4] of the map P^ε, $W_\varepsilon^{ss}(p_\varepsilon)$ and $W_\varepsilon^u(p_\varepsilon)$, which remains C^r close to the unperturbed manifolds $W_0^s(p_0)$ and $W_0^u(p_0)$.

Here $W_\varepsilon^{ss}(p_\varepsilon) \subset W_\varepsilon^s(p_\varepsilon)$ and the superscript ss denotes the strong stable manifold Assumptions 3 and 4 guarantee that the center-stable manifold $W_0^{sc}(p_0)$ of the unperturbed system and the perturbed stable manifold $W_\varepsilon^s(p_\varepsilon)$ are codimension one, while the unstable manifolds are one dimensional. The flow in $X \times S^1$ similarly has a periodic orbit γ_ε, C^r close to $\{p_0\} \times S^1$, with invariant manifolds close to $W_0^s(p_0) \times S^1$, etc. One now proceeds to calculate the separation of the perturbed manifolds $W_\varepsilon^s(p_\varepsilon)$ and $W_\varepsilon^u(p_\varepsilon)$, by calculating the $O(\varepsilon)$ components of perturbed solution curves of equation (5.3) from the first variation equation of (5.3):

$$\frac{d}{dt} x_1^s(t,t_0) = Df_0(x_0(t-t_0))\, x_1^s(t,t_0) + f_1(x_0(t-t_0),t) \qquad (5.5)$$

Here we have expanded solution curves in $W_\varepsilon^s(\gamma_\varepsilon)$; a similar expression holds for those in $W_\varepsilon^u(\gamma_\varepsilon)$. Points in $W_\varepsilon^s(p_\varepsilon)$ are obtained by intersecting $W_\varepsilon^s(\gamma_\varepsilon)$ with the section $X \times \{0\}$. This can also be done on general sections $X \times \{t_0\}$ and equation (5.5) contains t_0 as an initial starting time.

It is then possible to compute a function $M(t_0)$ which acts as a measure of the separation of the perturbed manifolds $W_\varepsilon^s(p_\varepsilon)$, $W_\varepsilon^u(p_\varepsilon)$ on different Poincaré sections $X \times \{t_0\}$.

<u>Theorem</u>. <u>Let hypotheses 1-4 hold. Let</u>

$$M(t_0) = \int_{-\infty}^{\infty} \Omega(f_0(x_0(t-t_0)),\, f_1(x_0(t-t_0),t))\, dt \qquad (5.6)$$

<u>Suppose that</u> $M(t_0)$ <u>has a simple zero as a function of</u> t_0. <u>Then for</u> $\varepsilon > 0$ <u>sufficiently small, the stable manifold</u> $W_\varepsilon^s(p_\varepsilon(t_0))$ <u>of</u> p_ε <u>for</u> $P_{t_0}^\varepsilon$ <u>and the unstable manifold</u> $W_\varepsilon^u(p_\varepsilon(t_0))$ <u>intersect transversally</u>.

We refer to Holmes and Marsden [7] for the proof. We also have:

<u>Theorem</u>. <u>If the diffeomorphism</u> $P_{t_0}^\varepsilon : X \to X$ <u>possesses a hyperbolic saddle point</u> p_ε <u>and an associated transverse homoclinic point</u> $q \in W_\varepsilon^u(p_\varepsilon) \pitchfork W_\varepsilon^s(p_\varepsilon)$, <u>with</u> $W_\varepsilon^u(p_\varepsilon)$ <u>of dimension 1 and</u> $W_\varepsilon^s(p_\varepsilon)$ <u>of codimension 1, then some power of</u> $P_{t_0}^\varepsilon$ <u>possesses an invariant zero</u>

dimensional hyperbolic set Λ homeomorphic to a Cantor set on which a power of $P^{\varepsilon}_{t_0}$ is conjugate to a shift on two symbols.

This implies the following:

Corollary. A power of $P^{\varepsilon}_{t_0}$ restricted to Λ possesses a dense set of periodic points, there are points of arbitrarily high period and there is a non-periodic orbit dense in Λ.

We now briefly sketch some intuition behind this result.

If the hypotheses above hold, we end up with a Poincaré map $P^{\varepsilon}:X \to X$ that has a hyperbolic saddle point x'_{ε} which has a 1 dimensional unstable manifold intersecting a codimension 1 stable manifold transversally. For $X = \mathbb{R}^2$, this situation implies that the dynamics contains a Smale horseshoe. Figure 8 shows the situation in $\mathbb{R}^2$. The rectangle R gets squashed horizontally, stretched vertically and laid down as shown. A little thought shows that $\bigcap_{-\infty}^{\infty} (P^{\varepsilon})^N(R) = \Lambda$ is locally an interval $\times$ a cantor set. This structure is responsible for the complex dynamics. The account given in Smale [16] is very readable.

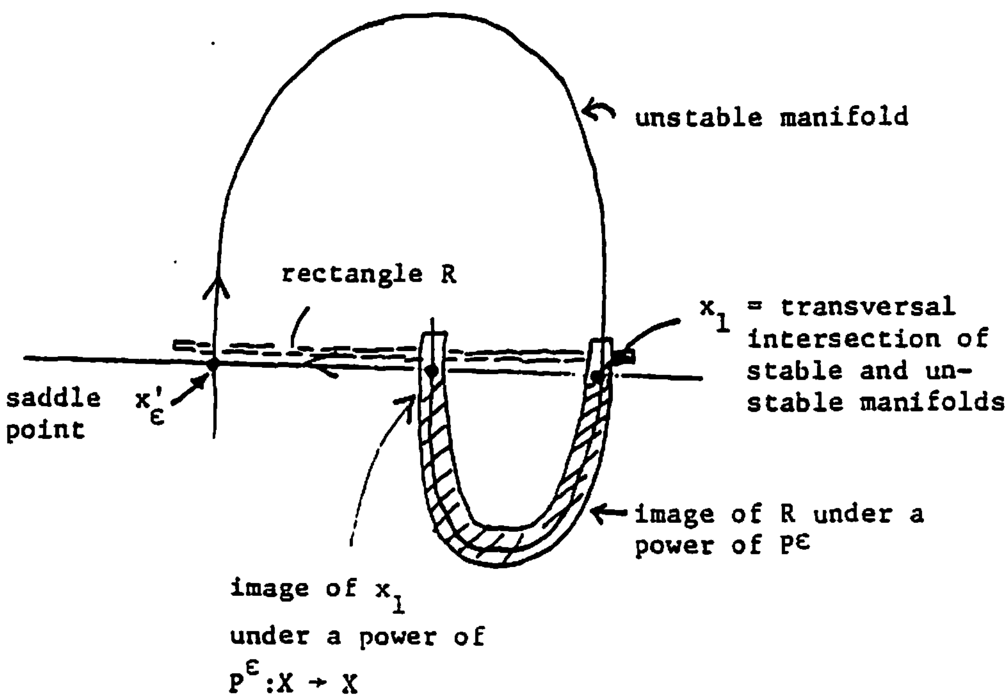


Figure 8. A Smale horseshoe

One can use these results to show that the beam equation (5.1) possesses horseshoes if the force, f, exceeds a certain critical level, dependent upon the damping, δ. The basic space is $X = H_0^2 \times L^2$ where H_0^2 denotes the set of all H^2 functions on $[0,1]$ satisfying the boundary condition $w = 0$ at $z = 0,1$. For $x = (w,\dot{w}) \in X$, the X-norm is the "energy" norm $\| x \|^2 = |w''|^2 + |w|^2$ where $|\cdot|$ denotes the L_2 norm. We write (5.1) in the form (5.3):
where

$$f_0(x) = Ax + B(x) \quad \text{and} \quad f_1(x,t) = \begin{pmatrix} 0 \\ f\cos\omega t - \delta\dot{w} \end{pmatrix}.$$

Here A is the linear operator

$$A\begin{pmatrix} w \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \dot{w} \\ -w'''' - \Gamma w'' \end{pmatrix},$$

with domain

$$D(A) = \{(w,\dot{w}) \in H^4 \times H^2 \,|\, w = w'' = 0 \text{ and } \dot{w} = 0 \text{ at } z = 0,1\}$$

and B is the nonlinear mapping of X to X given by

$$B(x) = \begin{pmatrix} 0 \\ \kappa\left(\int_0^1 [w']^2\, d\zeta \dot{w}''\right) \end{pmatrix}.$$

The theorems of Holmes and Marsden [6] show that A is a generator and that B and f_1 are smooth maps. This, together with energy estimates shows that the equations generate a global flow $F_t^\varepsilon: X \times S^1 \to X \times S^1$ consisting of C^∞ maps for each ε and t. If x_0 lies in the domain of the (unbounded) operator A, then $F_t^\varepsilon(x_0,s)$ is t-differentiable and equation (5.1) is literally satisfied.

For $\varepsilon = 0$ the equation is Hamiltonian using the sympletic form

$$\Omega((w_1,\dot{w}_1), (w_2,\dot{w}_2)) = \int_0^1 \{\dot{w}_2(z)w_1(z) - \dot{w}_1(z)w_2(z)\}\, dz$$

and

$$H(w,\dot{w}) = \frac{1}{2}\,|\dot{w}|^2 = \frac{\Gamma}{2}\,|w'|^2 + \frac{1}{2}|w''|^2 + \frac{\kappa}{4}\,|w'|^4$$

The invariant symplectic 2 manifold Σ is the plane in X spanned by the functions $(a \sin \pi z,\ b \sin \pi z)$ and the homoclinic loop is given by

$$w_0(z,t) = \frac{2}{\pi}\sqrt{\frac{\Gamma-\pi^2}{\kappa}}\,\sin(\pi z)\,\operatorname{sech}(t\pi\sqrt{\Gamma-\pi^2})$$

For $\varepsilon = 0$ one finds by direct calculation that the spectrum of $Df_0(p_0)$, where $p_0 = (0,0)$, is discrete with two real eigenvalues

$$\pm\lambda = \pm\pi\sqrt{\Gamma-\pi^2}$$

and the remainder pure imaginary (since $\Gamma < 4\pi^2$) at

$$\lambda_n = \pm\, n\pi\sqrt{\Gamma-n^2\pi^2}, \quad n = 2,3, \ldots$$

The Melnikov function is given by

$$M(t_0) = \int_{\infty}^{\infty} \Omega\begin{bmatrix} \dot{w} & , & 0 \\ -w'''' + \kappa|w'|^2 w'' - \Gamma w' & & f\cos\omega t - \delta\dot{w} \end{bmatrix} dt$$

$$= \int_{\infty}^{\infty}\left(\int_0^1 f\cos\omega t\ \dot{w}(z,t-t_0) - \delta\dot{w}(z,t-t_0)\dot{w}(z,t-t_0)\,dz\right) dt$$

Substituting the expressions for w, $\dot{w}$ along the homoclinic orbit, the integral can be evaluated by contour methods to give

$$-M(t_0) = \frac{4\omega}{\pi}\sqrt{\frac{\Gamma-\pi^2}{\kappa}}\, f\,\frac{\sin(\omega t_0)}{\cosh\left(\dfrac{\omega}{2\sqrt{\Gamma-\pi^2}}\right)} + \frac{4\delta(\Gamma-\pi^2)^{3/2}}{3\pi\kappa}$$

If

$$|f| > \frac{4\delta(\Gamma-\pi^2)}{3\omega\delta\sqrt{\kappa}} \left\{ \cosh\left(\frac{\omega}{2\sqrt{\Gamma-\pi^2}} \right) \right\}$$

Then $M(t_0)$ has simple zeros, so the stable and unstable manifolds intereact transversally. Note that in [5] the integral was given incorrectly.

This shows that there is a complicated invariant hyperbolic Cantor set Λ embedded in the Poincaré map of equation 5.1 for a calculable open set of parameter values. Although the dynamics near Λ is complex, we do not assert that Λ is a strange attractor i.e. that Λ is a structurally stable attractor. In fact, Λ is unstable in the sense that its generalized unstable manifold (or outset), $W^u(\Lambda)$ is non-empty (it is one dimensional and thus points starting near Λ may wander, remaining near Λ for a relatively long time, but eventually leaving a neighborhood of Λ and approaching an attractor. This kind of behavior has been referred to as transient chaos (or pre-turbulence). In two dimensions, Λ can coexist with two simple sinks of period one or with a strange attractor, depending on the parameter values. There is experimental evidence for transient chaos in the magnetic cantilever problem (Moon and Holmes [12]).

We close with a comment on the bifurcations in which the transversal intersections are created. Since the Melnikov function $M(t_0)$ has nondegenerate maxima and minima, it can be shown that, near the parameter values at which $M(t_0) = M'(t_0) = 0$, but $M''(t_0) \neq 0$, the stable and unstable manifolds $W^s_\varepsilon(p_\varepsilon(t_0))$. $W^u_\varepsilon(p_\varepsilon(t_0))$ have quadratic tangencies. This "Newhouse" mechanism implies that $P^\varepsilon_{t_0}$ can have infinitely many stable periodic orbits of arbitrarily high periods near the bifurcation point, at least in the finite dimensional examples. In practice it may be difficult to distinguish these long period stable periodic points from transient chaos and from "true" chaos itself. In fact it is not yet understood what role the Newhouse sinks play in the experimental and computer generated chaotic motions.

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