THE EFFECTIVE COMPRESSIBILITY OF A HOLLOW SPHERE

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INTRODUCTION

The problem of determining the effective compressibility of a hollow sphere is practically important because it relates to the problem of compacting a real material containing voids. For example, the void content in propellants is such that the initial slope of the P-V curve is three-to-ten times lower than the limiting slope obtained after all the voids have been collapsed. The phenomenon of compaction-decompaction is extremely important in determining the mechanical properties of propellants since it accounts for much of the batch-to-batch variability and also is a function of the stress-time-temperature history of the material. In this memo we consider only a very special type of compaction, but it is believed that this will pave the way for further analysis along these lines.

DEFINITION OF THE PROBLEM

A real material with void defects usually entrains up to a few-tenths per cent of such pinholes, ranging from one to several hundred microns in diameter. This scale of dimensions is particularly true in composite propellants, where a void may be the occasion of a missing oxidizer particle. Because of the relatively low number of such holes, it is assumed that the stress fields do not interact. Thus the problem of how much a voided material compacts as a result of external forces reduced to that of the effective compressibility of a hollow sphere, internal radius a, external radius b. For simplicity, we assume point-symmetric loading. The more general problem is extremely difficult. We denote the fractional cavity volume by the symbol $\delta \equiv a^3/b^3$; and we assume that the sphere is subjected to hydrostatic pressure P at $r = b$, and stress-free at $r = a$. The problem is to calculate the effective P-V curve, and the initial effective bulk modulus.

We denote coordinates in the deformed body by capital symbols, in the undeformed body, by small symbols, and look upon the undeformed coordinates as functions of the deformed coordinates, or, because of symmetry,

$$r = r(R) \tag{1}$$

It is easily established that the four metric tensors (cf. Green and Zerna's Theoretical Elasticity) are given by:

$$g_{ij} = \begin{pmatrix} r'^2 & & \\ & r^2\cos^2\phi & \\ & & r^2 \end{pmatrix} \quad (2)^*$$

$$G_{ij} = \begin{pmatrix} 1 & & \\ & R^2\cos^2\phi & \\ & & R^2 \end{pmatrix} \quad (4)$$

$$g^{ij} = \begin{pmatrix} \frac{1}{r'^2} & & \\ & \frac{1}{r^2\cos^2\phi} & \\ & & \frac{1}{r^2} \end{pmatrix} \quad (3)$$

$$G^{ij} = \begin{pmatrix} 1 & & \\ & \frac{1}{R^2\cos^2\phi} & \\ & & \frac{1}{R^2} \end{pmatrix} \quad (5)$$

The mixed stretch tensor is defined by:

$$M^{i}_{\cdot k} \equiv g^{ij} G_{jk} = \begin{pmatrix} \frac{1}{r'^2} & & \\ & \frac{R^2}{r^2} & \\ & & \frac{R^2}{r^2} \end{pmatrix} \quad (6)$$

The invariants of $M^{i}_{\cdot k}$ are important:

$$I_m = \frac{1}{r'^2} + \frac{2R^2}{r^2} \quad (7)$$

$$II_m = \frac{2R^2}{r^2 r'^2} + \frac{R^4}{r^4} \quad (8)$$

$$\sqrt{III_m} \equiv J_3 = \frac{R^2}{r' r^2} = \frac{V}{V_0} \quad (9)$$

For a constitutive law we choose a form which is both mathematically convenient and in excellent agreement with Bridgman's hydrostatic data [1] on rubberlike materials:

$$\frac{\tau^{il}}{\mu} = \frac{g^{il}}{J_3} + \left(C - 1 + \frac{C}{J_3^{1/3}}\right) G^{il} \quad (10)$$

where $C = 3K/\mu$. This value for C is needed to guarantee that (10) reduces to Hooke's law for small strains.

Since the stress tensor is diagonal and the coordinates in the deformed body orthogonal, we can establish equilibrium by computing the physical stresses and substituting in the equation:

$$\frac{\partial \sigma_{RR}}{\partial R} + \frac{2(\sigma_{RR} - \sigma_{\theta\theta})}{R} = 0 \quad (11)$$

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* Primes denote differentiation re R; zeros have been omitted.

where

$$\sigma_{ij} \equiv \tau^{ij} \sqrt{\frac{G_{jj}}{G^{ii}}} \tag{12}$$

Making the appropriate substitution, we obtain:

$$2r - 2Rr' - \frac{r^2 r''}{r'^2} + \frac{CR^2 J_3'}{3 J_3^{4/3}} = 0 \tag{13}$$

We set:

$$\lambda = \frac{R}{r} \tag{14}$$

and get:

$$\frac{CRJ_3'}{3J_3^{4/3}} + \frac{\frac{4R\lambda'}{\lambda^2} - \frac{6R^2\lambda'^2}{\lambda^3} + \frac{2R^3\lambda'^3}{\lambda^4} + \frac{R^2\lambda''}{\lambda^2}}{\left(1 - \frac{R\lambda'}{\lambda}\right)^2} = 0 \tag{15}$$

Now set:

$$R = e^s \tag{16}$$

to get:

$$\frac{C\frac{dJ_3}{ds}}{3J_3^{4/3}} + \frac{\frac{3}{\lambda^2}\frac{d\lambda}{ds} - \frac{6}{\lambda^3}\left(\frac{d\lambda}{ds}\right)^2 + \frac{2}{\lambda^4}\left(\frac{d\lambda}{ds}\right)^3 + \frac{1}{\lambda^2}\frac{d^2\lambda}{ds^2}}{\left(1 - \frac{1}{\lambda}\frac{d\lambda}{ds}\right)^2} = 0 \tag{17}$$

Equation (9) becomes:

$$J_3 = \frac{\lambda^3}{1 - \frac{1}{\lambda}\frac{d\lambda}{ds}} \tag{18}$$

or

$$\frac{d\lambda}{ds} = \lambda - \frac{\lambda^4}{J_3} \tag{19}$$

$$\frac{d^2\lambda}{ds^2} = \left(1 - \frac{4\lambda^3}{J_3} + \frac{\lambda^4}{J_3^2}\frac{dJ_3}{d\lambda}\right)\left(\lambda - \frac{\lambda^4}{J_3}\right) \tag{20}$$

Substitution of (19) and (20) into (17) yields:

$$-\frac{2J_3}{\lambda^4}\left(1 - \frac{\lambda^3}{J_3}\right)^2 + \frac{1}{\lambda^3}\left(1 - \frac{\lambda^3}{J_3}\right)\frac{dJ_3}{d\lambda} + \frac{C\frac{dJ_3}{ds}}{3J_3^{4/3}} = 0 \tag{21}$$

Division of (21) by (19) eliminates ds:

$$\frac{dJ_3}{d\lambda} = \frac{2\,(J_3 - \lambda^3)}{\lambda\left(1 + \frac{C\lambda^4}{3J_3^{4/3}}\right)} \tag{22}$$

We now set $\quad J_3 = \lambda^3/w^3$ (23)

whereupon (22) reduces to:

$$d\ln\lambda = \frac{Cw^4 + 3}{Cw^4 + 2w^3 + 1}\,\frac{dw}{w} \tag{24}$$

Now we note that $w \sim \lambda$, since J_3 is of the order of unity, i.e., unless pressure become extremely high, the material itself will not compress significantly. Furthermore, most of the initial effective compression is provided by the collapse of the void, as $\lambda \to 0$. We shall assume that, within the range of pressures ordinarily encountered $Cw \gg 1$. Since for most real materials, K/μ is of the order of 1000, this is equivalent to stating that w or λ never becomes less than 1 . Thus we gain an important algebraic simplification as long as we do not decrease the radius of the cavity by more than one-hundred fold. Now (24) becomes, to the first order in ($1/C$):

$$d\ln\lambda = \frac{dw}{w} - \frac{2}{C}\left(\frac{dw}{w} - \frac{dw}{w^4}\right) + \ \cdots \tag{25}$$

which integrates to:

$$\lambda = Aw\exp\left[\frac{1}{C}\left(\frac{1}{w} - \frac{1}{2w^4}\right)\right] \tag{26}$$

$$J_3 = A^3\exp\left[\frac{3}{C}\left(\frac{1}{w} - \frac{1}{2w^4}\right)\right] \tag{27}$$

with A an integration constant.

$$\text{If} \quad |\ln w| \gg \left|\frac{1}{C}\left(\frac{1}{w} - \frac{1}{2w^4}\right)\right| \tag{28}$$

the (26) and (27) reduce to:

$$\lambda = Aw \tag{29}$$

$$J_3 = A^3 \tag{30}$$

Actually, assumption (28) is valid only for $w > 10\%$, so it restricts our area of analysis somewhat further, but affords an even more useful algebraic simplification. From here on, we treat J_3 as a constant.

Combining (14) and (19), we obtain

$$d\ln r^3 = \frac{d(\lambda^3)}{J_3 - \lambda^3} = -\,d\ln(J_3 - \lambda^3) \tag{31}$$

or

$$r^3 = \frac{B}{J_3 - \lambda^3} \tag{32}$$

with B another integration constant. The radial component of the physical stresses assumes the form, using (3), (10), and (12):

$$\frac{\sigma_{RR}}{\mu} = \frac{J_3}{\lambda^4} + C - 1 - \frac{C}{J_3^{1/3}} \tag{33}$$

Equations (32) and (33), evaluated at $r=a$, and $r=b$, form a set of four simultaneous algebraic equations in the unknowns: $\{J_3, B, \lambda_a, \text{and } \lambda_b\}$. We can eliminate B immediately by using the definition for the fractional void volume δ, whereupon we obtain:

$$\frac{a^3}{b^3} \equiv \delta = \frac{J_3 - \lambda_b^3}{J_3 - \lambda_a^3} \tag{34}$$

$$-\frac{P}{\mu} = \frac{J_3}{\lambda_b^4} + C - 1 - \frac{C}{J_3^{1/3}} \tag{35}$$

$$0 = \frac{J_3}{\lambda_a^4} + C - 1 - \frac{C}{J_3^{1/3}} \tag{36}$$

In order to relate the pressure P to the volume of the sphere including cavity, we note that

$$\frac{V}{V_o} = \lambda_b^3 \tag{37}$$

so that it is merely necessary to eliminate J_3 and λ_a from the set (34, 35, 36). This, of course, involves the solution of a fourth order algebraic equation, which is best done by plotting. Since it is not our interest to perform data analysis, we shall not pursue this point further.

We can however establish the analytic flavor of the effective bulk modulus by setting

$$\lambda_a \equiv 1 + \epsilon_a \tag{38}$$

$$\lambda_b \equiv 1 + \epsilon_b \tag{39}$$

$$\frac{\Delta V}{V_o} \equiv 1 - \frac{V}{V_o} \tag{40}$$

to obtain the following <u>linear</u> approximations:

$$J_3 = 1 - \frac{P}{K} \tag{41}$$

$$\epsilon_a = -\frac{P(1-\delta)}{4\mu} \tag{42}$$

$$\epsilon_b = -P(1-\delta)\left(\frac{\delta}{4\mu} + \frac{1}{3K}\right) \tag{43}$$

$$\frac{\Delta V}{V_o} = P(1-\delta)\left(\frac{3\delta}{4\mu} + \frac{1}{K}\right) \equiv \frac{P(1-\delta)}{K^*} \tag{44}$$

where K^* is the effective bulk modulus.

Equation (41) states the dilatation of the material alone is never significantly different from unity, in agreement with our original assumption, as long as $P \ll K$. Since K for real materials is of the order of 10^6 psi, (41) is probably good up to 10, 000 psi, much higher than is encountered in rocket motors. The tangential strain at the surface of the cavity, given by (42), should not exceed 10% in order for the linear approximation to be valid. For a value of shear modulus of 500 psi, this severely limits the range of pressure to no higher than 50 psi, within which (41) - (44) are useful. Thus for higher pressure it is necessary to return to (34) - (37) and plot. The hoop strain at the outer surface of the sphere is of the order of 10^{-6} P, and is practically negligible. Similarly the volume change of sphere plus cavity is practically negligible. Equation (44) does display, however, the fact that the effective bulk modulus depends on the shear modulus and fractional volume of the void.

In conclusion, it is interesting to note the form of the collapse radius of the void, i.e. we set λ_a equal to a small quantity and obtain:

$$\lambda_b = \left(\frac{V}{V_o}\right)^{\frac{1}{3}} = \frac{(1-\delta)^{\frac{1}{3}}}{1 + \frac{P}{3K}} \tag{45}$$

$$\lambda_a^4 = \frac{1}{\left(1 + \frac{P}{\mu}\right)\left(1 + \frac{P}{3K}\right)^3} \Rightarrow \frac{27 K^3 \mu}{P^4} \tag{46}$$

$$\text{or} \quad \lambda_a \Rightarrow \frac{\sqrt[4]{27 K^3 \mu}}{P} \tag{47}$$

$$J_3 = \frac{1}{\left(1 + \frac{P}{K}\right)^3} \Rightarrow \frac{27 K^3}{P^3} \tag{48}$$

Thus (47) shows that the collapse radius goes as one over the pressure. On the other hand (48) shows that the dilatation is far from constant at large pressure. Therefore, it is better to seek an asymptotic solution to (22), which is:

$$J_3 = D\lambda^2 \tag{49}$$

Equation (49) shows that the order of dependence of J_3 on P is double that of λ on P, whereas (47) and (48) show that the order of dependence of J_3 is triple that of λ on P. The exact ratio is probably somewhere between two and three, but cannot be determined without numerical integration.