A FORMULA FOR INSERTING POINT MASSES

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ABSTRACT. Let $d\mu$ be a probability measure on the unit circle and $d\nu$ be the measure formed by adding a pure point to $d\mu$. We give a formula for the Verblunsky coefficients of $d\nu$, based on a result of Simon.

1. Introduction

Suppose we have a probability measure $d\mu$ on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. We define the inner product associated with $d\mu$ and the norm on $L^2(\partial\mathbb{D}, d\mu)$ respectively by

$$\langle f, g \rangle = \int_{\partial\mathbb{D}} \overline{f(e^{i\theta})} g(e^{i\theta}) d\mu(\theta) \tag{1.1}$$

$$\|f\|_{d\mu} = \left(\int_{\partial\mathbb{D}} |f(e^{i\theta})|^2 d\mu(\theta) \right)^{1/2} \tag{1.2}$$

The family of monic orthogonal polynomials associated with the measure $d\mu$ is denoted as $(\Phi_n(z, d\mu))_{n=0}^{\infty}$, while the normalized family is denoted as $(\varphi_n(z, d\mu))_{n=0}^{\infty}$.

Let $\Phi_n^*(z) = z^n \overline{\Phi_n(1/\overline{z})}$ and $\varphi_n^*(z) = \Phi_n^*(z)/\|\Phi_n\|$ be the reversed polynomials. Orthogonal polynomials obey the Szegő recursion relation

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha_n}\Phi_n^*(z) \tag{1.3}$$

α_n is called the n^{th} Verblunsky coefficient. It is well known that there is a one-to-one correspondence between $d\mu$ and $(\alpha_j(d\mu))_{j=0}^{\infty}$ and that

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the Verblunsky coefficients carry much information about the family of orthogonal polynomials. For example,

$$\|\Phi_n\|^2 = \prod_{j=0}^{n-1}(1 - |\alpha_j|^2) \tag{1.4}$$

For a comprehensive introduction to the theory of orthogonal polynomials on the unit circle, the reader should refer to [4, 5], or the classic reference [6].

The result that we would like to present is the following

Theorem 1.1. *Suppose $d\mu$ is a probability measure on the unit circle and $0 < \gamma < 1$. Let $d\nu$ be the probability measure formed by adding a point mass $\zeta = e^{i\omega} \in \partial\mathbb{D}$ to $d\mu$ in the following manner*

$$d\nu = (1-\gamma)d\mu + \gamma\delta_\omega \tag{1.5}$$

Then the Verblunsky coefficients of $d\nu$ are given by

$$\alpha_n(d\nu) = \alpha_n + \frac{(1-|\alpha_n|^2)^{1/2}}{(1-\gamma)\gamma^{-1} + K_n(\zeta)}\overline{\varphi_{n+1}(\zeta)}\varphi_n^*(\zeta) \tag{1.6}$$

where

$$K_n(\zeta) = \sum_{j=0}^{n} |\varphi_j(\zeta)|^2 \tag{1.7}$$

and all objects without the label $(d\nu)$ are associated with the measure $d\mu$.

The proof is based on a result obtained by Simon in the proof of Theorem 10.13.7 in [5] (See Theorem 2.1 below).

In fact, the following formula had been found by Geronimus [2]

$$\Phi_n(z, d\nu) = \Phi_n(z) - \frac{\Phi_n(\zeta)K_{n-1}(z,\zeta)}{(1-\gamma)\gamma^{-1} + K_{n-1}(\zeta,\zeta)} \tag{1.8}$$

The formula for the real case was rediscovered by Nevai [3], Later, the same formula for the unit circle case was rediscovered by Cachafeiro-Marcellan [1]. Unaware of Geronimus' result and the fact that Nevai's result also applies to the unit circle, Simon reconsidered this problem and proved formula (2.7) independently using a totally different method.

For applications of formula (1.6), the reader may refer to [7] and [8].

2. The Proof

First, we will prove a few lemmas.

Lemma 2.1. *Let* $\beta_{jk} = \langle \Phi_j(d\mu), \Phi_k(d\mu)\rangle_{d\nu}$. *Then*

$$\Phi_n(d\nu)(z) = \frac{1}{D^{(n-1)}} \begin{vmatrix} \beta_{00} & \beta_{0\,1} & \dots & \beta_{0\,n} \\ \vdots & & & \vdots \\ \beta_{n-1\,0} & \beta_{n-1\,1} & \dots & \beta_{n-1\,n} \\ \Phi_0(d\mu) & \dots & \dots & \Phi_n(d\mu) \end{vmatrix} \tag{2.1}$$

where

$$D^{(n-1)} = \begin{vmatrix} \beta_{0\,0} & \beta_{0\,1} & \dots & \beta_{0\,n-1} \\ \vdots & & & \vdots \\ \beta_{n-1\,0} & \beta_{n-1\,1} & \dots & \beta_{n-1\,n-1} \end{vmatrix} \tag{2.2}$$

Proof. Let $\tilde{\Phi}_n(d\nu)$ be the right hand side of (2.1). We observe that the inner product $\langle \Phi_j(d\mu), \tilde{\Phi}_n(d\nu)\rangle_{d\nu}$ is zero for $j = 0, 1, \dots, n-1$ as the last row and the j^{th} row of the determinant are the same. By expanding in minors, we see that the leading coefficient of $\tilde{\Phi}_n(d\nu)$ in (2.1) is one. In other words, $\tilde{\Phi}_n(d\nu)$ is an n^{th} degree monic polynomial which is orthogonal to $1, z, \dots, z^{n-1}$ with respect to $\langle\ ,\ \rangle_{d\nu}$, hence $\tilde{\Phi}_n(d\nu)$ equals $\Phi_n(d\nu)$. □

Lemma 2.2. *Let* C *be the following* $(n+1)\times(n+1)$ *matrix*

$$\begin{pmatrix} A & v \\ w & \beta \end{pmatrix} \tag{2.3}$$

where A *is an* $n\times n$ *matrix,* β *is in* $\mathbb{C}$, v *is the column vector* $(v_0, v_1, \dots, v_{n-1})^T$ *and* w *is the row vector* $(w_0, w_1, \dots, w_{n-1})$. *If* $det(A) \neq 0$, *we have*

$$\det(C) = \det(A)\left(\beta - \sum_{0\le j,k\le n-1} w_k v_j (A^{-1})_{jk}\right) \tag{2.4}$$

Proof. We expand in minors, starting from the bottom row to get

$$\det(C) = \beta det(A) + \sum_{0\le j,k\le n-1} w_k v_j (-1)^{j+k+1} \det(\tilde{A}_{jk}) \tag{2.5}$$

where $\tilde{A}_{jk}$ is the matrix A with the j^{th} row and k^{th} column removed.

By Cramer's rule, since $\det(A) \neq 0$,

$$\tilde{A}_{jk} = (-1)^{j+k} \det(A)(A^{-1})_{jk} \tag{2.6}$$

proving Lemma 2.2. □

Next, we are going to prove the following formula by Simon [5]:

Theorem 2.1. *The Verblunsky coefficient of $d\nu$ (as defined in (1.5)) is given by*

$$\alpha_n(d\nu) = \alpha_n - q_n^{-1}\gamma\overline{\varphi_{n+1}(\zeta)}\left(\sum_{j=0}^{n} \alpha_{j-1}\frac{\|\Phi_{n+1}\|}{\|\Phi_j\|}\varphi_j(\zeta)\right) \tag{2.7}$$

where

$$K_n(\zeta) = \sum_{j=0}^{n} |\varphi_j(\zeta)|^2 \tag{2.8}$$

$$q_n = (1-\gamma) + \gamma K_n(\zeta) \tag{2.9}$$

$$\alpha_{-1} = -1 \tag{2.10}$$

and all objects without the label $(d\nu)$ are associated with the measure $d\mu$.

Proof. Since $\alpha_{n-1}(d\nu) = -\overline{\Phi_n(0, d\nu)}$ and $\overline{\beta_{jk}} = \beta_{kj}$, by Lemma 2.1,

$$\alpha_{n-1}(d\nu) = \frac{1}{D^{(n-1)}} \begin{vmatrix} \beta_{0\,0} & \beta_{1\,0} & \dots & \beta_{n\,0} \\ \vdots & & & \vdots \\ \beta_{0\,n-1} & \beta_{1\,n-1} & \dots & \beta_{n\,n-1} \\ -1 & \alpha_0 & \dots & \alpha_{n-1} \end{vmatrix} \tag{2.11}$$

Let C be the matrix with entries as in the determinant in (2.11) above. It could be expressed as follows

$$C = \begin{pmatrix} A & v \\ w & \alpha_{n-1} \end{pmatrix} \tag{2.12}$$

where A is the $n \times n$ matrix with entries $A_{jk} = \beta_{kj}$, v is the column vector $(\beta_{n0}, \dots, \beta_{nn-1})^T$ and w is the row vector $(-1, \alpha_0, \dots, \alpha_{n-2})$. Note that $\det(A) = D^{(n-1)}$ and it is real as A is Hermitian.

Now we use Lemma 2.2 to compute $det(C)$. To do that, we need to find out what A^{-1} is.

By the definition of ν,

$$A_{jk} = (1-\gamma)\|\Phi_k\|^2\delta_{kj} + \gamma\overline{\Phi_k(\zeta)}\Phi_j(\zeta) = \|\Phi_k\|\|\Phi_j\|M_{jk} \tag{2.13}$$

where

$$M_{jk} = (1-\gamma)\delta_{kj} + \gamma\overline{\varphi_k(\zeta)}\varphi_j(\zeta) \tag{2.14}$$

Observe that for any column vector $x = (x_0, x_1, \ldots, x_{n-1})^T$,

$$Mx = (1-\gamma)x + \gamma\left(\sum_{j=0}^{n-1}\varphi_j(\zeta)x_j\right)(\varphi_0(\zeta), \varphi_1(\zeta), \ldots, \varphi_0(\zeta))^T \tag{2.15}$$

Therefore, if P_φ denotes the orthogonal projection onto the space spanned by the vector $\varphi = (\varphi_0(\zeta), \varphi_1(\zeta), \ldots, \varphi_0(\zeta))$, we can write

$$M = (1-\gamma)\mathbf{1} + \gamma K_{n-1}P_\varphi \tag{2.16}$$

Hence, the inverse of M is

$$M^{-1} = (1-\gamma)^{-1}(\mathbf{1} - P_\varphi) + ((1-\gamma) + \gamma K_{n-1})^{-1}P_\varphi \tag{2.17}$$

and the inverse of A is

$$A^{-1} = D^{-1}M^{-1}D^{-1} \tag{2.18}$$

where $D_{ij} = \|\Phi_i\|\delta_{ij}$.

Recall that $v = (\beta_{n0}, \beta_{n1}, \ldots, \beta_{nn-1})^T$, which is a multiple of φ. Therefore,

$$(A^{-1}v)_j = ((1-\gamma) + \gamma K_{n-1})^{-1}\,\gamma\,\overline{\Phi_n(\zeta)}\,\|\Phi_j\|^{-1}\varphi_j(\zeta) \tag{2.19}$$

(2.19), (2.11) and Lemma 2.2 then imply

$$\alpha_{n-1}(d\nu) = \alpha_{n-1} - ((1-\gamma) + \gamma K_{n-1})^{-1}\,\gamma\,\overline{\varphi_n(\zeta)}\left(\sum_{j=0}^{n-1}\alpha_{j-1}\frac{\|\Phi_n\|}{\|\Phi_j\|}\varphi_j(z_0)\right) \tag{2.20}$$

□

This concludes the proof of Theorem 2.1.

Now we are going to prove Theorem 1.1.

Proof. First, observe that $\alpha_{j-1} = -\overline{\Phi_j(0)}$. Therefore, $\alpha_{j-1}/\|\Phi_j\| = -\overline{\varphi_j(0)}$. Second, observe that $\|\Phi_{n+1}\|$ is independent of j so it could be taken out from the summation. As a result, (2.7) in Theorem 2.1 becomes

$$\alpha_n(d\nu) = \alpha_n(d\mu) + q_n^{-1}\gamma\ \overline{\varphi_{n+1}(\zeta)}\ \|\Phi_{n+1}\| \left(\sum_{j=0}^{n} \overline{\varphi_j(0)}\varphi_j(\zeta)\right) \tag{2.21}$$

Then we use the Christoffel-Darboux formula, which states that for $x, y \in \mathbb{C}$ with $x\bar{y} \neq 1$,

$$(1 - \overline{x}y)\left(\sum_{j=0}^{n} \overline{\varphi_j(x)}\varphi_j(y)\right) = \overline{\varphi_n^*(x)}\varphi_n^*(y) - \overline{x}y\overline{\varphi_n(x)}\varphi_n(y) \tag{2.22}$$

Moreover, note that $q_n^{-1}\gamma = ((1-\gamma)\gamma^{-1} + K_n(\zeta))^{-1}$ Therefore, (2.21) could be simplified as follows

$$\alpha_n(d\nu) = \alpha_n + \frac{\overline{\varphi_{n+1}(\zeta)}\varphi_n^*(0)\varphi_n^*(\zeta)}{(1-\gamma)\gamma^{-1} + K_n(\zeta)}\|\Phi_{n+1}\| \tag{2.23}$$

Finally, observe that $\varphi_n^*(0) = \|\Phi_n\|^{-1}$ and that by (1.4), $\|\Phi_{n+1}\|/\|\Phi_n\| = (1 - |\alpha_n|^2)^{1/2}$. This completes the proof. □

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