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# String Theory on $\mathrm{AdS}_3$

Jan de Boer, Hirosi Ooguri, Harlan Robins and Jonathan Tannenhauser

Department of Physics, University of California at Berkeley,
Berkeley, California 94720

and

Theoretical Physics Group, Mail Stop 50A-5101,
Lawrence Berkeley National Laboratory,
Berkeley, California 94720

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**Abstract**

It was shown by Brown and Henneaux that the classical theory of gravity on $AdS_3$ has an infinite-dimensional symmetry group forming a Virasoro algebra. More recently, Giveon, Kutasov and Seiberg (GKS) constructed the corresponding Virasoro generators in the first-quantized string theory on $AdS_3$. In this paper, we explore various aspects of string theory on $AdS_3$ and study the relation between these two works. We show how semi-classical properties of the string theory reproduce many features of the $AdS/CFT$ duality. Furthermore, we examine how the Virasoro symmetry of Brown and Henneaux is realized in string theory, and show how it leads to the Virasoro Ward identities of the boundary $CFT$. The Virasoro generators of GKS emerge naturally in this analysis. Our work clarifies several aspects of the GKS construction: why the Brown-Henneaux Virasoro algebra can be realized on the first-quantized Hilbert space, to what extent the free-field approximation is valid, and why the Virasoro generators act on the string worldsheet localized near the boundary of $AdS_3$. On the other hand, we find that the way the central charge of the Virasoro algebra is generated is different from the mechanism proposed by GKS.

# 1 Introduction

It was shown by Brown and Henneaux [1] that the semi-classical theory of gravitation on three-dimensional anti-de Sitter space ($AdS_3$) possesses an infinite-dimensional symmetry algebra of Virasoro type. The realization of this Virasoro algebra has recently been clarified in light of the $AdS/CFT$ duality [2–4], according to which string/$M$-theory on a $(p+1)$-dimensional anti-de Sitter space times a compact space is equivalent to a $p$-dimensional conformal field theory ($CFT_p$). The case of $AdS_3$ was studied in more detail in [5–8]. More recently, Giveon, Kutasov and Seiberg (GKS) found that the Brown-Henneaux Virasoro algebra is realized on the first-quantized string theory on $AdS_3$, shedding further light on the duality [9].

The main purpose of this paper is to clarify the relation between the Brown-Henneaux Virasoro algebra and the Virasoro generators constructed by GKS. In the construction of GKS, the Virasoro algebra acts on the first-quantized string Hilbert space. However, as shown in [5–8], the Brown-Henneaux Virasoro operators are creation and annihilation operators of gravitons in $AdS_3$, and as such they are realized on the second-quantized Hilbert space of strings. It was not clear how to reconcile these two points of view. In addition, GKS assume that the string worldsheet is localized near the boundary of $AdS_3$ and winds around the boundary. It is not obvious why we need (and need only) consider such worldsheet configurations. In this paper, we will give answers to these questions, and along the way, we will recover many features of the $AdS/CFT$ duality directly from the worldsheet theory of strings on $AdS_3$.

As by-products of this analysis, we gain new insights into the structure of two-dimensional sigma models with non-compact target spaces such as $AdS_3$. In the case of $AdS_3$ with the Euclidean-signature metric, the sigma model is unitary and its Hilbert space is equipped with a positive-definite inner product. Since $AdS_3$ has an $SO(3,1) \simeq SL(2,C)$ isometry group, the Hilbert space should decompose into a direct sum of unitary representations of $SL(2,C)$. To our surprise, we find that the $AdS/CFT$ duality implies that vertex operators of the sigma model belong to non-unitary representations of $SL(2,C)$. This is not a contradiction, and appears to be a generic phenomenon in non-compact sigma models. It is closely related to the absence of the state-operator correspondence in the Liouville model [10], where it is known that normalizable states make up the Hilbert space, while non-normalizable states correspond to operators. In the $AdS_3$ case, unitary representations of $SL(2,C)$ are realized by normalizable functions on $AdS_3$, whereas non-normalizable functions give non-unitary representations. Thus it is reasonable, by analogy with the Liouville model, that the vertex operators of the sigma model belong to non-unitary representations.

This paper is organized as follows.

In section 2, we briefly summarize the duality between string theory on $AdS_3$ and conformal field theory in two dimensions.

In section 3, we discuss various aspects of the worldsheet theory of strings on Euclidean $AdS_3$, including the $SL(2, C)$ symmetry and the vertex operator construction. We show that the worldsheet vertex operators are closely related to bulk-boundary Green's functions in target space.

In section 4, we perform a semi-classical analysis of correlation functions of primary fields, and show that the vertex operators are subject to the wave function renormalization expected from the $AdS/CFT$ duality and from the holographic identification of the regularizations [11]. The worldsheet stretches to the boundary of $AdS_3$ at the insertion points of the vertex operators, and can be viewed as a thickening of the target-space Feynman diagram involving bulk-boundary and bulk-bulk Green's functions.

In section 5, we define the Virasoro generators as the graviton vertex operators corresponding to Brown-Henneaux diffeomorphisms, and explain why these vertex operators do not decouple from the theory.

In section 6, we derive the Virasoro Ward identity of the boundary $CFT$ and show how the Virasoro generators defined by GKS [9] arise.

In section 7, we consider the correlation function of two boundary stress-energy tensors and explain how the central charge appears. A crucial step is to consider disconnected worldsheets, *i.e.*, second-quantized string theory.

We end in section 8 with some conclusions.

## 2 The $AdS_3/CFT_2$ Duality

Following [6], we start with type IIB string theory on $\mathbf{R}^4 \times \mathbf{R}^2 \times M^4$, where $M^4$ is a compact manifold ($M^4 = T^4$ or $K_3$), and consider $Q_1$ fundamental strings on $\mathbf{R}^2$ and $Q_5$ $NS$ fivebranes on $\mathbf{R}^2 \times M^4$. In the near-horizon limit, the target space geometry is $AdS_3 \times S^3 \times M^4$, with a non-zero $NS$-$NS$ 2-form. The curvature radius $l_{AdS}$ of $AdS_3$ is equal to $\sqrt{Q_5} l_s$ where $l_s$ is the string scale. The string coupling constant on $AdS_3$ is

$$g_3^2 = \frac{1}{Q_1\sqrt{Q_5}}. \tag{2.1}$$

Thus we have the following hierarchy of scales:

$$l_{AdS} = \sqrt{Q_5} l_s = 4Q_1 Q_5 l_p, \tag{2.2}$$

where $l_p = \frac{1}{4} g_3^2 l_s$ is the three-dimensional Planck length. When $Q_1 Q_5 \gg 1$, quantum gravity effects are weak. Moreover when $Q_5 \gg 1$, the $\alpha'$-expansion of the worldsheet theory becomes reliable.

According to the $AdS/CFT$ duality, this system is dual to some two-dimensional conformal field theory ($CFT_2$). In the original work of Brown and Henneaux [1], the central charge $c$ of the $CFT_2$ is given in the low-energy gravity approximation by

$$c = \frac{3l_{AdS}}{2l_p} = 6Q_1 Q_5. \tag{2.3}$$

This is consistent with the $S$-dual of the brane-configuration, which is the D1-D5 system, whose field theory limit is a $CFT_2$ with $c = 6Q_1Q_5$ [12].

# 3 Worldsheet Description of Strings on AdS$_3$

## 3.1 Action and Symmetry

In Euclidean $AdS_3$, the bosonic part of the worldsheet Lagrangian is

$$S_E = \frac{Q_5}{2\pi} \int d^2z (\partial\phi\bar{\partial}\phi + e^{2\phi}\partial\bar{\gamma}\bar{\partial}\gamma). \tag{3.1}$$

Here $(\phi, \gamma, \bar{\gamma})$ are the coordinates on $AdS_3$. The coordinate $\phi$ is real, while $\gamma$ and $\bar{\gamma}$ are complex conjugates. The boundary of $AdS_3$ is located at $\phi = \infty$. In this sub-section we will summarize known facts about this action, based on the earlier works [9, 13, 14].

First of all, it is instructive to compare (3.1) with the corresponding action $S_L$ for $AdS_3$ with Lorentzian signature. Lorentzian-signature $AdS_3$ is the group manifold of $SL(2,R)$; the action $S_L$ is the Wess-Zumino-Witten (WZW) action for $SL(2,R)$ with level $Q_5$, and so possesses an affine $SL(2,R) \times SL(2,R)$ symmetry, with independent generators for the left- and right-movers.

Euclidean $AdS_3$ is the coset manifold $SL(2,C)/SU(2)$; the action $S_E$ can be directly obtained from the $SL(2,C)$ WZW action $S_{wzw}(g)$ [13]*. The $SL(2,C)$ WZW model

*This model has been studied in the past, owing to its relation to coset conformal field theories. It was shown in [14] that, when $G$ and $H$ are compact groups, the $G/H$ model is equivalent to the product of the $G$ model and the $H^c/H$ model, where $H^c$ is the complexification of $H$, when some BRST invariance has two independent affine $SL(2,C)$ symmetries, associated with left- and right-movers. The quotient by $SU(2)$ identifies the left and the right affine symmetries by complex conjugation. This can be seen as follows.

We may regard the coset $SL(2,C)/SU(2)$ as the space of $2\times 2$ hermitian complex matrices $h$ with unit determinant. To compare with the action (3.1), we parametrize an $SL(2,C)/SU(2)$ matrix $h$ as

$$h = \begin{pmatrix} e^{-\phi} + \gamma\bar{\gamma}e^{\phi} & e^{\phi}\gamma \\ e^{\phi}\bar{\gamma} & e^{\phi} \end{pmatrix}. \tag{3.2}$$

The string action (3.1) is simply the $SL(2,C)$ WZW action $S_{wzw}(h)$, with $h$ restricted to the form (3.2). By construction, the WZW action $S_{wzw}(g)$ is invariant for arbitrary $g \in SL(2,C)$ under the left and the right $SL(2,C)$ symmetries

$$g \to U(z)gV^{\dagger}(\bar{z}), \quad U, V \in SL(2,C). \tag{3.3}$$

However, $S_E = S_{wzw}(g=h)$ is invariant only under the diagonal action

$$h \to U(z)hU^{\dagger}(\bar{z}), \quad U \in SL(2,C), \tag{3.4}$$

since $h$ is constrained to be hermitian. The matrix $U$ is an arbitrary holomorphic function of $z$; consequently, by Noether's theorem, the corresponding currents $J^a$ $(a=\pm,3)$ are holomorphically conserved,

$$\bar{\partial}J^a = 0. \tag{3.5}$$

So far we have discussed the classical symmetry of the action $S_E$. The currents $J^a$ could receive quantum corrections, but we expect that the conservation law (3.5) still holds. There are two instances in which quantum effects can be perturbatively treated.

(A) When $Q_5$ is large, worldsheet quantum effects are suppressed by $1/Q_5$.

(B) If the functional integral is dominated by contributions at large $\phi$, we can use the action

$$S' = \frac{1}{4\pi}\int d^2z(\partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \beta\bar{\beta}e^{-2\phi/\alpha_+} - \frac{2}{\alpha_+}\phi\sqrt{g}R), \tag{3.6}$$

where $\alpha_+ = \sqrt{2Q_5 - 4}$ and $R$ is the curvature of the worldsheet. The theory defined by the action $S'$ can be shown to be equivalent to the original one, upon integrating out $(\beta, \bar{\beta})$,

is imposed on the product theory. It is interesting to note that, if we take $G = H = SU(2)$, we find that the topological $SU(2)/SU(2)$ model is equivalent to the product of the $SL(2,C)/SU(2) = AdS_3$ model and the $SU(2) = S^3$ model (with the BRST invariance). Before imposing the BRST invariance, the product model is nothing but the worldsheet theory of strings on $AdS_3 \times S^3$. The $SU(2)/SU(2)$ model may be useful to study some topological aspects of the string theory in question.

taking into account effects on the measure of the functional integral, and rescaling the scalar fields by $\phi \to \phi\alpha_+$, $\gamma \to \sqrt{2Q_5}\gamma$ [16]. For large $\phi$, the interaction term $\beta\bar{\beta}e^{-2\phi/\alpha_+}$ is suppressed and the free-field approximation to the fields $(\beta, \gamma)$ becomes reliable. The $SL(2,C)$ currents in this notation are given by

$$\begin{aligned} J^- &= \frac{1}{2}\beta \\ J^3 &= \frac{1}{2}(\beta\gamma - \alpha_+\partial\phi) \\ J^+ &= \frac{1}{2}(\beta\gamma^2 - 2\alpha_+\gamma\partial\phi - \alpha_+^2\partial\gamma). \end{aligned} \tag{3.7}$$

Moreover, because of the coupling of $\phi$ to the worldsheet curvature $R$ in (3.6), the effective string coupling constant depends on the coordinate $\phi$ (the linear dilaton background). For $\phi \to \infty$, the string coupling constant vanishes asymptotically. Thus the spacetime theory as well as the worldsheet theory is weakly coupled for $\phi \to \infty$ in this picture [9][†].

### 3.2 Vertex Operators

According to the $AdS/CFT$ duality, correlation functions of $CFT$ correspond to string amplitudes on $AdS$ [3, 4]. It is therefore useful to study vertex operators of the $AdS_3$ string. Generally speaking, if a $CFT$ has a global affine $G$ symmetry, its vertex operators $V(z,\bar{z})$ take values in vector spaces representing the $G$ symmetry. In the case of $AdS_3$, since the group $SL(2,C)$ is non-compact, we are led to consider infinite-dimensional representations as well as finite-dimensional ones. Teschner [15] introduced auxiliary coordinates $(x,\bar{x})$ to organize these representations. Because $SL(2,C)$ acts on the matrix $h$ as $h \to UhU^\dagger$, it is natural to consider the combination

$$(1, -x)h\begin{pmatrix} 1 \\ -\bar{x} \end{pmatrix} = e^{\phi/\alpha_+}(\gamma - x)(\bar{\gamma} - \bar{x}) + e^{-\phi/\alpha_+}, \tag{3.8}$$

[†]It may appear that, in the opposite limit $\phi \to -\infty$, the effective string coupling constant diverges, and the spacetime theory is strongly coupled. This, however, is an artifact of the description in terms of the action $S'$. In the limit $\phi \to -\infty$, the transformation relating $S$ to $S'$ breaks down, because the factor $e^{2\phi}$ multiplying the kinetic term for $\gamma$ in (3.1) vanishes. In fact, this transformation is an intermediate step in the T-duality transformation along the isometry generated by a constant shift of $(\gamma, \bar{\gamma})$. (It becomes T-duality if we write $\beta = \partial\tilde{\gamma}$ [17].) The T-duality transformation is subtle when there is a fixed point in the isometry. After T-duality, the dilaton diverges at the fixed point, but this is an artifact, if the original theory is well-defined at that point. This is the case for the string on $AdS_3$ since $\phi = -\infty$ is a regular boundary point on $AdS_3$ and the string coupling is constant, $g_s^{-2} = Q_1\sqrt{Q_5}$ in the original picture. In this paper, we will use $S'$ only when we analyze the behavior of the functional integral for large $\phi$.

and to define the vertex operator $V_j$ by*

$$V_j(z,\bar{z};x,\bar{x}) = \left((\gamma - x)(\bar{\gamma} - \bar{x})e^{\phi/\alpha_+} + e^{-\phi/\alpha_+}\right)^{2j}. \tag{3.9}$$

In the free-field approximation, it is straightforward to show that this vertex operator gives the correct operator product expansion with the $SL(2,C)$ currents,

$$J^a(z)V_j(w,\bar{w};x,\bar{x}) \sim \frac{1}{z-w}D^aV_j(w,\bar{w};x,\bar{x}), \tag{3.10}$$

where $a = 3, \pm$, and

$$D^- = \frac{\partial}{\partial x}, \quad D^3 = x\frac{\partial}{\partial x} - j, \quad D^+ = x^2\frac{\partial}{\partial x} - 2jx. \tag{3.11}$$

As we will show in section 6, in evaluating the operator product expansion of $J^a$ with $V_j$, we can take $\phi$ to be arbitrarily large. Therefore the computation in (3.10) belongs to the case (B) discussed in section 3.1, and justifies the use of the free-field approximation.

The global $SL(2,C)$ symmetry of $AdS_3$ corresponds to the global conformal symmetry of the boundary $CFT_2$ generated by $L_0$ and $L_{\pm 1}$ [2, 6]. One can then relate the highest weight $j$ of $SL(2,C)$ to the Virasoro highest weight $h$ of the boundary $CFT$ by

$$h = -j. \tag{3.12}$$

In [15], Teschner considered the case $j \in -1/2 + \sqrt{-1}\mathbf{R}$, corresponding to principal representations of $SL(2,C)$. These are unitary representations and therefore appear in the Hilbert space of the sigma model.

In this paper, we are interested in the situation when $h = -j$ is real, since $h$ is a conformal weight of the boundary $CFT_2$. In this case, the $SL(2,C)$ representation is non-unitary, and the corresponding supergravity mode is non-normalizable. For $h > 1/2$, because of the identity

$$\delta^{(2)}(z) = \frac{n-1}{\pi}\lim_{\epsilon\to 0}\frac{\epsilon^{2n-2}}{(\epsilon^2 + |z|^2)^n}, \tag{3.13}$$

the vertex operator $V_j$ behaves as

$$V_{j=-h} \sim e^{2(h-1)\phi/\alpha_+}\delta^{(2)}(\gamma - x), \tag{3.14}$$

near the boundary ($\phi \to \infty$) of $AdS_3$. That is, the vertex operator $V_j$ has the same structure as the bulk-boundary Green's function used in the supergravity computation

*The vertex operators of [9] correspond to the leading large $\phi$ part of the coefficients of the $x, \bar{x}$ expansion of $V_j$.

of $CFT$ correlation functions [3, 4]. Of course, this is not a coincidence. In the semi-classical approximation, if a vertex operator $V_j$ is expressed as a function of $(\phi, \gamma, \bar{\gamma})$, the operator product expansion (3.10) with the $SL(2, C)$ currents implies that $V_j$ solves the supergravity wave equation

$$(\Delta + j(j + 1))V_j = 0, \tag{3.15}$$

where $\Delta$ is the Laplacian on $AdS_3$, expressed in the coordinates $(\phi, \gamma, \bar{\gamma})$. The identification of vertex operators and bulk-boundary Green's functions motivates us to interpret $(x, \bar{x})$ as coordinates for the boundary $CFT_2$.

When $j$ is real, the vertex operator $V_j$ carries the $SL(2, C)$ weights $h = \bar{h} = -j$ and corresponds to a scalar field on $AdS_3$, such as a Kaluza-Klein excitation (on $S^3 \times M^4$) of the dilaton field. To construct a vertex operator with $h \neq \bar{h}$, corresponding to tensor fields on $AdS_3$, we must include derivatives of the fields $(\phi, \gamma, \bar{\gamma})$. Indeed, we will see in section 5 that the graviton vertex operator corresponding to the energy-momentum tensor $T(x)$ of $CFT_2$ is of this form.

We have found that the $AdS/CFT$ duality implies that the vertex operators $V_j$ belong to non-unitary representations of $SL(2, C)$, even though both the two-dimensional sigma model for Euclidean $AdS_3$ and the boundary $CFT_2$ are expected to be unitary theories, with Hilbert spaces of positive-definite inner product. Therefore there is no state-operator correspondence in the sigma model[†]. This phenomenon is well-known in the Liouville model. In the Liouville model, normalizable states make up the Hilbert space and non-normalizable states correspond to operators [10]. In the $AdS_3$ case, unitary representations of $SL(2, C)$ are realized by normalizable functions on $AdS_3$, whereas non-normalizable functions give non-unitary representations. Thus it is in fact reasonable, by the analogy with the Liouville model, that the vertex operators of the sigma model belong to non-unitary representations.

[†]A generalized version of the correspondence may hold if we suitably extend the notion of states and allow for analytic continuation of the quantum number $j$ in (3.9) [15, 18].

## 4 Semi-classical Analysis

In this section* we will analyze correlation functions of the vertex operators (3.9) semi-classically. We propose the correspondence†

$$\langle\prod_i\int d^2z_i V_{j_i}(z_i,\bar z_i;x_i,\bar x_i)\rangle_{\rm worldsheet}=\langle\prod_i V_{j_i}(x_i,\bar x_i)\rangle_{\rm boundary\ CFT}. \tag{4.1}$$

(In this expression, factors coming from the $S^3\times M^4$ part of the target space are suppressed.) Two facts directly support this proposal. First, there should be a one-to-one correspondence between vertex operators of the boundary $CFT$ inserted at specific boundary points and vertex operators of the worldsheet theory. Second, according to (3.10), the worldsheet $SL(2,C)$ currents generate the standard $SL(2,C)$ action on the boundary coordinates $x,\bar x$.

We obtain further insight in the structure of the correlation functions (4.1) by studying the worldsheets that contribute to it in the semi-classical approximation. The general solution to the equations of motion of (3.1) in the absence of sources is

$$\begin{aligned}\phi&=\log(1+b(z)\bar b(\bar z))+\rho(z)+\bar\rho(\bar z)\\ \gamma&=a(z)+e^{-2\rho(z)}\bar b(\bar z)(1+b(z)\bar b(\bar z))^{-1}\\ \bar\gamma&=\bar a(\bar z)+e^{-2\bar\rho(\bar z)}b(z)(1+b(z)\bar b(\bar z))^{-1},\end{aligned} \tag{4.2}$$

for arbitrary holomorphic functions $a,b,\rho$. The case with sources can be dealt with by allowing poles in $a,b,\rho$. To find these functions in the presence of arbitrary vertex operators is rather complicated (it is the analogue of the uniformization problem in Liouville theory [19]). We will therefore only consider the behavior of the worldsheet near a single vertex operator

$$V=\left((\gamma-x)(\bar\gamma-\bar x)e^{\phi}+e^{-\phi}\right)^{2j}(z_0) \tag{4.3}$$

at the point $z=z_0$. The relevant equations of motion read

$$\frac{1}{2\pi}\partial\bar\partial\phi-\frac{1}{2\pi}e^{2\phi}\partial\bar\gamma\bar\partial\gamma+2j\frac{(\gamma-x)(\bar\gamma-\bar x)e^{\phi}-e^{-\phi}}{(\gamma-x)(\bar\gamma-\bar x)e^{\phi}+e^{-\phi}}\delta^{(2)}(z-z_0)=0 \tag{4.4}$$

$$\frac{1}{4\pi}\partial(e^{2\phi}\bar\partial\gamma)+2j\frac{(\gamma-x)e^{\phi}}{(\gamma-x)(\bar\gamma-\bar x)e^{\phi}+e^{-\phi}}\delta^{(2)}(z-z_0)=0 \tag{4.5}$$

$$\frac{1}{4\pi}\bar\partial(e^{2\phi}\partial\bar\gamma)+2j\frac{(\bar\gamma-\bar x)e^{\phi}}{(\gamma-x)(\bar\gamma-\bar x)e^{\phi}+e^{-\phi}}\delta^{(2)}(z-z_0)=0. \tag{4.6}$$

*From now on we will work with the original variables as they appear in (3.1). Furthermore, we will suppress the $Q_5$ dependence until the discussion of the central charge after equation (7.6).

†This proposal is not complete as it stands; see section 7 for a more precise statement.

This system has the solution

$$\begin{aligned}\phi &= 2j\log|z-z_0|^2+b+c(z-z_0)+\bar{c}(\bar{z}-\bar{z}_0)+\dots\\ \gamma &= x+a(z-z_0)^{-4j}(\bar{z}-\bar{z}_0)^{1-4j}-2ac(z-z_0)^{1-4j}(\bar{z}-\bar{z}_0)^{1-4j}+\dots\\ \bar{\gamma} &= \bar{x}+\bar{a}(z-z_0)^{1-4j}(\bar{z}-\bar{z}_0)^{-4j}-2\bar{a}\bar{c}(z-z_0)^{1-4j}(\bar{z}-\bar{z}_0)^{1-4j}+\dots,\end{aligned} \tag{4.7}$$

where $a, b, c$ are some arbitrary constants and the dots indicate higher-order regular terms. The corresponding functions in (4.2) are

$$a(z)=x, \qquad b(z)=ae^b(z-z_0)^{1-4j}, \qquad \rho(z)=2j\log(z-z_0)+\frac{b}{2}+c(z-z_0). \tag{4.8}$$

Since we consider only vertex operators with $j \leq -1/2$, corresponding to boundary conformal weight $h \geq 1/2$, the worldsheet coordinates at $z_0$ are $(\phi, \gamma, \bar{\gamma})(z_0) = (\infty, x, \bar{x})$. Thus the worldsheet develops an infinite tube that attaches to the point $(x, \bar{x})$ at the boundary of $AdS_3$. In the field theory limit, the worldsheet degenerates, and we recover the picture of [4], where boundary correlation functions are expressed in terms of Feynman diagrams consisting of bulk-bulk and bulk-boundary propagators. This is further evidence for the identification (4.1). The structure of the worldsheet is illustrated in figure 1.

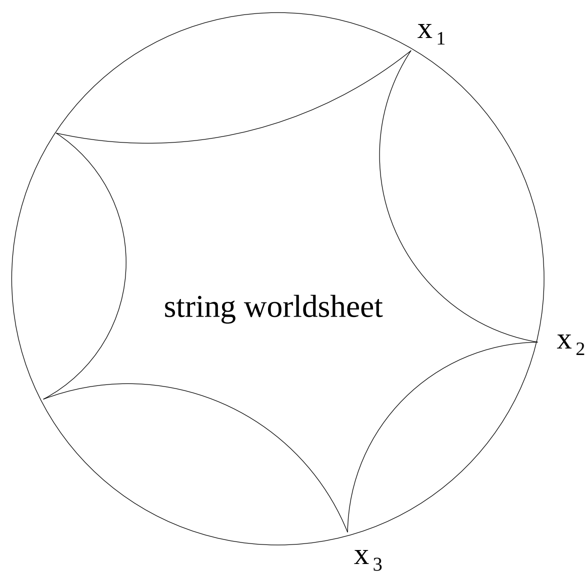


Figure 1: Semi-classical worldsheet in the presence of vertex operators

When we evaluate the semi-classical contribution to the correlation function (4.1), we encounter a divergence arising from the stretching of the worldsheet to the boundary at infinity of $AdS_3$. To regularize this divergence, we introduce a worldsheet UV cutoff $\epsilon$, and multiply the correlation function by a suitable power of $\epsilon$ before taking the limit $\epsilon \to 0$. The appropriate power is easily determined (see [10] for a similar analysis for Liouville theory) and corresponds to a wave function renormalization for each vertex operator $V_j$,

$$V_j^{ren} = \epsilon^{8j^2} V_j. \tag{4.9}$$

A similar renormalization has also been found in [16], where correlation functions of $V_j$ with $j > 0$ were studied. In that situation, one consequence of the renormalization was that the $e^{-\phi}$ in $V_j$ could be dropped, leading to an exact free-field representation of the correlation functions. It should be possible to find similar exact free-field representations of the correlation functions of $V_j$ with $j < 0$, because $SL(2, C)$ representations with spins $j$ and $-1 - j$ are equivalent. We shall not pursue this further here; nevertheless, we will find that the wave function renormalization brings about many simplifications. In particular, it explains why the free-field approximation is valid, and plays a crucial role in proving the Virasoro Ward identities of the boundary *CFT*.

Besides $\epsilon$, there are two other cutoffs in the problem, the IR cutoff of the bulk theory and the UV cutoff of the boundary *CFT*. All three cutoffs are related. According to (4.7), the bulk IR cutoff $U_0$ in $U = e^{\phi}$ is

$$U_0 = \epsilon^{4j}, \tag{4.10}$$

and depends on which vertex operator is inserted. The UV cutoff $\tilde{\epsilon}$ of the boundary *CFT* is related to $U_0$ by [11]

$$\tilde{\epsilon} = U_0^{-1}. \tag{4.11}$$

With this identification of the cutoff parameters, (4.9) may be expressed in terms of the boundary *CFT* cutoff $\tilde{\epsilon}$ as

$$V_j^{ren} = \tilde{\epsilon}^{2h} V_j. \tag{4.12}$$

The factor $\tilde{\epsilon}^{2h}$ matches the scaling behavior of the primary field of the boundary *CFT* corresponding to the worldsheet vertex operator $V_j$. This fits well with the *AdS*/*CFT* duality[‡]. The relation between $U_0$ and $\epsilon$ is illustrated in figure 2.

We next turn to the fluctuations around the semi-classical worldsheet. If we denote the semi-classical worldsheet by $(\phi_0(z), \gamma_0(z), \bar{\gamma}_0(z))$ and quantum fluctuations by $(\phi_q(z), \gamma_q(z), \bar{\gamma}_q(z))$, we see from (4.7) that the dominant contribution to the kinetic term of the quantum fields near $z = z_0$ is

$$\int d^2z |z - z_0|^{-2} (\phi_q(z)^2 + |z - z_0|^{8j} |\gamma_q(z)|^2). \tag{4.13}$$

[‡]These relations among the cutoff parameters hold even when we restore the $Q_5$ dependence.

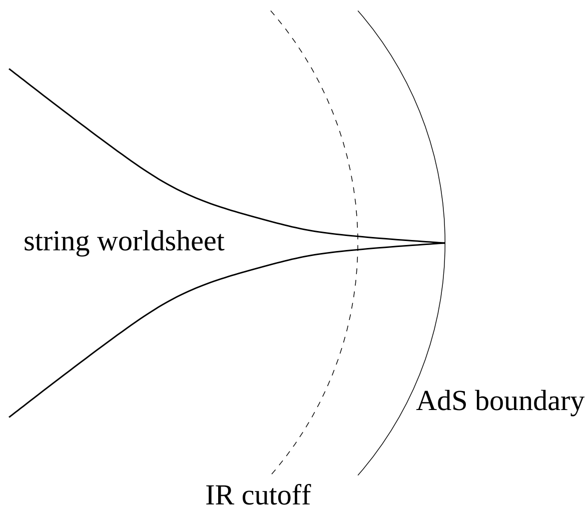


Figure 2: Bulk IR cutoff versus worldsheet UV cutoff

For the action to be finite, we need

$$\phi_q(z_0) \sim \epsilon^{\frac{1}{2}}, \qquad \gamma_q(z_0) \sim \epsilon^{\frac{1}{2}-4j}. \tag{4.14}$$

In particular, the fluctuations of the worldsheet vanish near the boundary as we take $\epsilon \to 0$. Furthermore, no quantum terms in the background field expansion of the vertex operators $V_j$ contribute to the correlation function (4.1). Thus the one-loop worldsheet correction to the correlation function consists only of the determinant of the kinetic term of the quantum fields $(\phi_q(z), \gamma_q(z), \bar{\gamma}_q(z))$.

# 5 The Virasoro Algebra

So far we have discussed the primary fields of the boundary *CFT*. We now turn our attention to the boundary Virasoro algebra. Let us briefly recall how the Virasoro algebra arises in [1]. First, we define spaces that are asymptotically anti-de Sitter by imposing on the metric the boundary conditions

$$G_{\phi\phi} = 1 + \mathcal{O}(e^{-2\phi}), \qquad G_{\phi\gamma} = G_{\phi\bar{\gamma}} = \mathcal{O}(e^{-2\phi}) \tag{5.1}$$

$$G_{\gamma\gamma} = G_{\bar{\gamma}\bar{\gamma}} = \mathcal{O}(1), \qquad G_{\gamma\bar{\gamma}} = \frac{1}{2}e^{2\phi} + \mathcal{O}(1). \tag{5.2}$$

Next, we consider the group $G$ of diffeomorphisms that preserve these boundary conditions. To each of these one can associate an ADM-type charge that vanishes identically for a subgroup $H$ of diffeomorphisms that decay sufficiently fast at infinity. The algebra of the quotient $G/H$ is the Virasoro algebra. The infinitesimal diffeomorphisms corresponding to the generators $L_n$ are

$$\begin{aligned} \xi^{\gamma} &= -\gamma^{n+1} + \mathcal{O}(e^{-4\phi}) \\ \xi^{\bar{\gamma}} &= \frac{1}{2} n(n+1)\gamma^{n-1} e^{-2\phi} + \mathcal{O}(e^{-4\phi}) \\ \xi^{\phi} &= \frac{1}{2}(n+1)\gamma^{n} + \mathcal{O}(e^{-2\phi}). \end{aligned} \tag{5.3}$$

We have given only the holomorphic part of the Virasoro algebra—the full Virasoro algebra consists of the sum of these generators and their complex conjugates. Moreover, our choice of generators is not unique—we could equally well replace $\gamma$ by $\gamma - \gamma_0$ everywhere.

If we perform one of the infinitesimal diffeomorphisms (5.3) in the worldsheet theory, the result is the insertion of a combined vertex operator for the graviton and the *NS*-*NS* two-form field. This vertex operator is given by

$$L_n = \delta S_n = \int d^2 z \left( \frac{1}{2}(n+1)n\gamma^{n-1}(\partial\gamma\bar{\partial}\phi - \bar{\partial}\gamma\partial\phi) + \frac{1}{2}(n+1)n(n-1)\gamma^{n-2}\partial\gamma\bar{\partial}\gamma \right). \tag{5.4}$$

We have neglected subleading terms in (5.3).

Normally, the graviton vertex operator corresponding to a diffeomorphism is on-shell BRST exact and decouples from the theory, as it corresponds to an unphysical graviton. Alternatively, the graviton vertex operator is the sum of a total derivative and equation of motion terms, and the latter can be dropped by the canceled propagator argument [20].

In the case of $AdS_3$, however, something special happens. Although we can formally write $\delta S_n$ as $\{Q_{BRST}, X\}$, $X$ is not normalizable, and therefore $\delta S_n$ is a non-trivial element of the BRST cohomology. Alternatively, as we will show below, the total derivative terms cannot be dropped: in fact, these terms give rise to the contour integral representation of the Virasoro generators of [9]. From either perspective, then, the vertex operators $\delta S_n$ are physical states of the theory. Since there are no propagating gravitons in three dimensions, they correspond to degrees of freedom living purely on the boundary of $AdS_3$ (*i.e.*, singletons).

Altogether we are led to identify an insertion of the boundary stress-energy tensor $T(x)$ in a boundary correlation function with the insertion of the vertex operator $T(\phi, \gamma, \bar{\gamma}; x)$ in the worldsheet correlation function given by

$$T(x) = \sum_{n=-2}^{-\infty} L_n x^{-n-2}$$

$$= \int d^2z \left( \frac{1}{(\gamma - x)^3} (\partial\gamma\bar{\partial}\phi - \bar{\partial}\gamma\partial\phi) - \frac{3}{(\gamma - x)^4} \partial\gamma\bar{\partial}\gamma \right). \qquad (5.5)$$

We saw previously in (3.14) that, for large $\phi$, vertex operators behave like bulk-boundary Green's functions, and in particular that they become localized at single points. The same is true for the stress-energy tensor, although this is less obvious from (5.5). Consider for definiteness the second term in (5.5). For large $\phi$, this term seems to be subleading compared to the term $e^{2\phi}\partial\bar{\gamma}\bar{\partial}\gamma$. However, we must be careful, because $(\gamma - x)^{-4}$ blows up near $\gamma = x$. Up to terms subleading in $e^{-2\phi}$, the second term in (5.5) can be rewritten as*

$$-3 \int d^2z e^{2\phi} \left( \frac{(\bar{\gamma} - \bar{x})^2}{(\gamma - x)^2} \frac{e^{-2\phi}}{(|\gamma - x|^2 + e^{-2\phi})^2} \right) \partial\gamma\bar{\partial}\gamma. \qquad (5.6)$$

Since the Brown-Henneaux diffeomorphisms are defined up to subleading terms only, the same is true for $T$, and we might as well have used (5.6) in our definition of $T$. For large $\phi$, (5.6) behaves as

$$-3 \int d^2z e^{2\phi} \left( \frac{(\bar{\gamma} - \bar{x})^2}{(\gamma - x)^2} \delta^{(2)}(\gamma - x) \right) \partial\gamma\bar{\partial}\gamma. \qquad (5.7)$$

This is the analogue of (3.14) for the stress tensor. As in (3.14), it behaves like a bulk-boundary Green's function, and is localized on the boundary of $AdS_3$.

## 6 Boundary Ward Identity

As a first application of the definition (5.5), we will show that it correctly reproduces the Virasoro Ward identities of the boundary *CFT*. We first discuss the case of a single insertion of the stress-energy tensor and an arbitrary number of primary fields. The case with more than one stress tensor insertion is more complicated and will be discussed later.

Our strategy for proving the Virasoro Ward identities is to perform a change of variables in the path integral corresponding to a Brown-Henneaux diffeomorphism. The diffeomorphism corresponding to $T(x)$ is

$$\xi^{\gamma} = -\frac{1}{\gamma - x} + \mathcal{O}(e^{-4\phi})$$
$$\xi^{\bar{\gamma}} = \frac{1}{(\gamma - x)^3} e^{-2\phi} + \mathcal{O}(e^{-4\phi})$$

*For example, one can choose the representative

$$\xi^{\bar{\gamma}} = \frac{\bar{\gamma} - \bar{x}}{(\gamma - x)^2} \frac{e^{-2\phi}}{|\gamma - x|^2 + e^{-2\phi}}$$

in place of (6.1) to find the expression (5.6) for large $\phi$.

$$\xi^\phi = \frac{-1}{2(\gamma - x)^2} + \mathcal{O}(e^{-2\phi}). \tag{6.1}$$

Let us perform this change of variables on the correlation function

$$\langle \prod_i \int d^2 z_i V_{j_i}(z_i, \bar{z}_i; x_i, \bar{x}_i) \rangle_{\text{worldsheet}}. \tag{6.2}$$

There are two contributions: one comes from the variation of the action, yielding $T(\phi, \gamma, \bar{\gamma}; x)$, while the other comes from the variation of the vertex operators and has the form

$$\begin{aligned} \delta_\xi V_{j_i} &= -\left( \frac{-j_i}{(x - x_i)^2} + \frac{1}{(x - x_i)} \frac{\partial}{\partial x_i} \right) V_{j_i}(x_i) \\ &\quad - \frac{j_i(\gamma - x_i)^2}{(\gamma - x)^3 (x - x_i)^2} (e^\phi (\gamma - x_i)(\bar{\gamma} - \bar{x}_i) + e^{-\phi})^{2j_i - 1} R, \end{aligned} \tag{6.3}$$

where

$$R = e^{-\phi}(\gamma - 3x + 2x_i) + e^\phi (\gamma - x_i)(\bar{\gamma} - \bar{x}_i)(\gamma - x). \tag{6.4}$$

In the first line of (6.3) we recognize the operator product expansion of $T(x)$ with $V_{j_i}(x_i)$. Using the results (4.7) and (4.14) from the semi-classical analysis, we determine that the remainder, *i.e.*, the second line in (6.3), gives a vanishing contribution to the correlation function. Indeed, the leading term in the background field expansion vanishes, as do all terms containing quantum fields, after taking into account the renormalization factor (4.9). The main reason for this is the explicit factor of $(\gamma - x_i)^2$ in the second line of (6.3).

We have shown that

$$\langle T(\phi, \gamma, \bar{\gamma}; x) \prod_i \int d^2 z_i V_{j_i}(z_i, \bar{z}_i; x_i, \bar{x}_i) \rangle_{\text{worldsheet}} \tag{6.5}$$

is equal to

$$\sum_i \left( \frac{h_i}{(x - x_i)^2} + \frac{1}{(x - x_i)} \frac{\partial}{\partial x_i} \right) \langle \prod_i \int d^2 z_i V_{j_i}(z_i, \bar{z}_i; x_i, \bar{x}_i) \rangle_{\text{worldsheet}}, \tag{6.6}$$

where $h_i = -j_i$. Since both correlation functions have a corresponding meaning in the boundary *CFT*, this proves the Virasoro Ward identities of the boundary *CFT*, to all orders in the string worldsheet theory.

This analysis confirms that only the leading large $\phi$ behavior of the Brown-Henneaux diffeomorphisms is relevant. Had we chosen any other representative, we would still have obtained the correct Virasoro Ward identity. This is because the insertion of a graviton vertex operator corresponding to a diffeomorphism that decays faster, at large $\phi$, than the Brown-Henneaux diffeomorphism automatically yields zero. Again, this is as expected.

We can now also make contact with the contour representation of the Virasoro generators in [9]. To do this, we rewrite $T$ in (5.5) as the sum of total derivative and equation of motion terms. The equation of motion terms can be dropped if we view the UV regularization as cutting discs of radius $\epsilon$ out of the worldsheet around each of the vertex operators $V_{j_i}$. The equation of motion terms have only contact-term interactions with the $V_{j_i}$, and can therefore be neglected. What remains is the total derivative terms. In the presence of the $V_{j_i}$, the regularized worldsheet acquires a boundary, consisting of the boundaries of the small discs. The total derivative terms thus turn into a sum of contour integrals encircling each of the vertex operators. The relevant contour integrals for $T(x)$ are

$$\sum_i \oint_{z_i} dz \left( \frac{-1}{\gamma - x} e^{2\phi} \partial\bar{\gamma} + \frac{-1}{2(\gamma - x)^2} \partial\phi \right) + \oint_{z_i} d\bar{z} \left( \frac{-1}{(\gamma - x)^3} \bar{\partial}\gamma + \frac{1}{2(\gamma - x)^2} \bar{\partial}\phi \right) \qquad (6.7)$$

These contour integrals are just the canonical worldsheet generators of the Brown-Henneaux diffeomorphisms. Therefore, the contour integral can be worked out semi-classically, resulting in in (6.3). All corrections to this semi-classical result vanish as we take the regulator to zero. The contour integral representation of the Virasoro generators in [9] is a slight modification of (6.7), namely,

$$\sum_i \oint_{z_i} dz \left( \frac{-1}{\gamma - x} e^{2\phi} \partial\bar{\gamma} + \frac{-1}{(\gamma - x)^2} \partial\phi + \frac{1}{(\gamma - x)^3} \partial\gamma \right). \qquad (6.8)$$

The difference between (6.7) and (6.8) is annihilated when acting on $V_{j_i}$'s. In the free-field approximation, the integrand of (6.8) contains purely holomorphic operators, and it is valid to use free-field OPE's in computing contour integrals around the $V_{j_i}$. Again we recover (6.3) up to terms that vanish as the regulator is taken to zero. This shows precisely how and when the free-field representation is exact.

## 7 $T(x)T(y)$ OPE and Central Charge

To evaluate the insertion of two or more boundary stress tensors in a correlation function, one might consider, along the lines of the above procedure, performing consecutive Brown-Henneaux diffeomorphisms and studying the resulting Ward identities. The only novel feature would be the variation of the stress tensor under a Brown-Henneaux diffeomorphism. As it will turn out, this is not the whole story and has to be supplemented by an additional ingredient. The variation of the stress tensor can be computed using the contour integral representation (6.7). It is easiest to vary a mode of (6.7),

$$L_n \equiv \sum_i \oint_{x_i} dz \left( -\gamma^{n+1} e^{2\phi} \partial\bar{\gamma} + \frac{1}{2}(n+1)\gamma^n \partial\phi \right) +$$

$$+\sum_i \oint_{x_i} d\bar{z}\left(-\frac{1}{2}n(n+1)\gamma^{n-1}\bar{\partial}\gamma - \frac{1}{2}(n+1)\gamma^n\bar{\partial}\phi\right), \qquad (7.1)$$

under the Brown-Henneaux diffeomorphism (5.3) corresponding to $L_m$. This yields

$$\begin{aligned}\delta_m L_n &= (m-n)L_{m+n} - (m^3-m)\sum_i \oint_{z_i} \gamma^{m+n-1}\partial\gamma + \\ &+\frac{1}{2}m(m+1)\sum_i\left(\oint_{z_i} dz\gamma^{n+m}\partial\phi + \oint_{z_i} d\bar{z}\gamma^{n+m}\bar{\partial}\phi\right) + \\ &+\frac{1}{4}m(m+1)(n+2m-1)\sum_i\left(\oint_{z_i} dz\gamma^{n+m-1}\partial\gamma + \oint_{z_i} d\bar{z}\gamma^{n+m-1}\bar{\partial}\gamma\right). \qquad (7.2)\end{aligned}$$

The two last lines in this expression vanish as we send the regulator to zero. The last term in the first line is similar to the expression for the central charge proposed in [9]. However, since we insert the boundary Virasoro generators at points different from the insertion points of the primary fields, this term does not contribute. The $L_n$ correspond to insertions of $T$ at 0 or $\infty$, and

$$\oint_{z_i} \gamma^{m+n-1}\partial\gamma = 0, \qquad (7.3)$$

if $x_i \neq 0, \infty$.

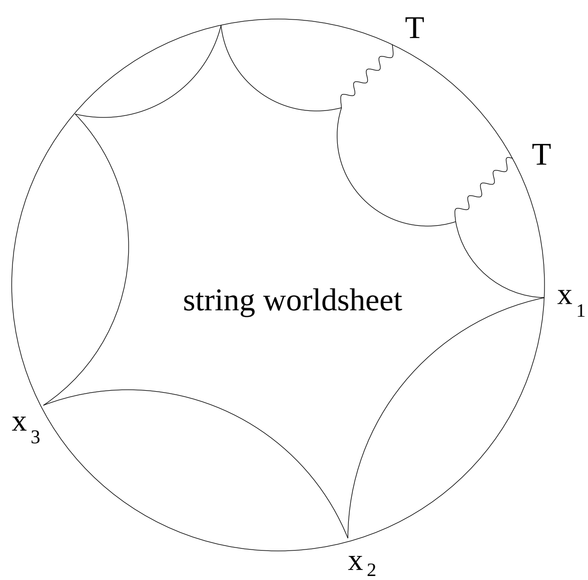


Figure 3: A single string worldsheet contributing to the $\langle TTV_1 \ldots V_n\rangle$ correlator. This diagram does not contribute the central charge of the Virasoro algebra.

All that remains from (7.2) is the Virasoro algebra with zero central charge. Therefore, performing two Brown-Henneaux variations gives us the correct Ward identity for the insertion of two stress tensors in a correlation function of primary fields, except for the central charge term.

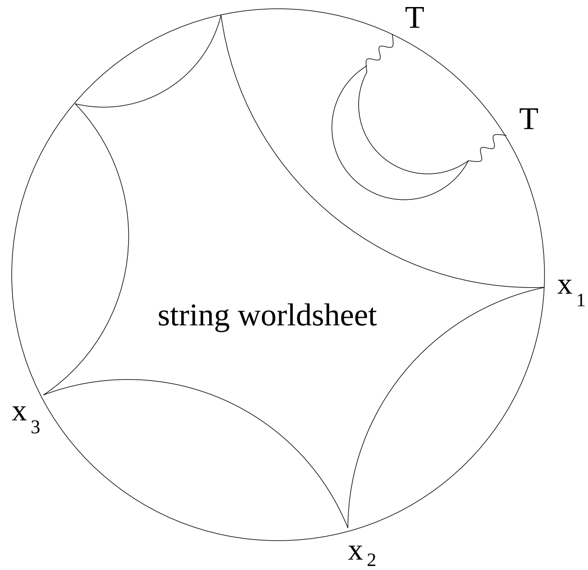


Figure 4: A multiple string worldsheet contributing to the $\langle\langle TTV_1 \ldots V_n\rangle\rangle$ correlator. The central charge $c = 6Q_1Q_5$ is obtained from this diagram.

The reason that the computation does not capture the central charge in this Ward identity is the following. In the $AdS/CFT$ duality, the string theory on $AdS$ is second-quantized. Therefore we need to sum over all possible string worldsheets, including disconnected ones. This corresponds in the supergravity limit [4] to the prescription to sum over all Feynman diagrams constructed out of bulk-bulk and bulk-boundary propagators, including disconnected Feynman diagrams. So far we have been focusing on a single string worldsheet, as illustrated in figure 3. Let us denote by $\langle\langle V_1 \ldots V_n\rangle\rangle$ the second-quantized string theory correlation function involving arbitrary multiple worldsheets, and by $\langle V_1 \ldots V_n\rangle$ the correlation function obtained from a single worldsheet. Then

$$\langle\langle V_1 \ldots V_n\rangle\rangle = \langle V_1 V_2 \cdots V_n\rangle + \langle V_1 V_2\rangle\langle V_3 \cdots V_n\rangle + \cdots. \tag{7.4}$$

It is $\langle\langle V_1 \ldots V_n\rangle\rangle$, rather than $\langle V_1 \ldots V_n\rangle$, that should be identified with a boundary $CFT$ correlation function. One can easily check that the Virasoro Ward identities still hold if we replace $\langle V_1 \ldots V_n\rangle$ by $\langle\langle V_1 \ldots V_n\rangle\rangle$. However, the correlation function $\langle\langle TTV_1 \ldots V_n\rangle\rangle$ containing two boundary stress-energy tensors includes a contribution from

$$\langle TT\rangle\langle\langle V_1 \ldots V_n\rangle\rangle, \tag{7.5}$$

as illustrated in figure 4. We have not yet computed the two-point function of stress tensors. The previous analysis of the Ward identities does not apply to $\langle TT\rangle$, because the contour integral representation of $T$ cannot be used in the absence of other vertex operators.

When $Q_5 \gg 1$, the two-point function of the energy-momentum tensor is computable in the semi-classical approximation giving

$$\langle T(x)T(y)\rangle_{\text{worldsheet}} = \frac{c/2}{(x-y)^4}, \tag{7.6}$$

with $c = 6Q_1Q_5$. Let us outline the derivation of this formula. As shown in section 5, the energy-momentum tensor $T(\phi,\gamma,\bar{\gamma};x)$ can be interpreted as the bulk-boundary Green's function for a graviton in $AdS_3$. Therefore, in the semi-classical approximation, $\langle T(x)T(y)\rangle_{\text{worldsheet}}$ can be identified with the two-point graviton amplitude in the $AdS_3$ supergravity. The relevant part of the supergravity action is (up to numerical coefficients)

$$S = \frac{1}{l_p}\int d\phi d\gamma d\bar{\gamma}\sqrt{g}(R + l_{AdS}^{-2}) + (\text{boundary term}). \tag{7.7}$$

If we perturb the metric by $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$, the action is expanded as

$$S = \frac{1}{l_p}\int d\phi d\gamma d\bar{\gamma}\sqrt{g}\partial h\partial h + \cdots. \tag{7.8}$$

Let us choose $h$ to be the bulk-boundary Green's function with sources at $x$ and $y$ on the boundary. Since $\sqrt{g} \sim l_{AdS}^3$ and $\partial^2 \sim l_{AdS}^{-2}$, the action scales as $S \propto l_{AdS}/l_p$. The $x, y$ dependence of the action is determined by the $SL(2,C)$ invariance, and we obtain*

$$S \sim \frac{l_{AdS}/l_p}{(x-y)^4} \sim \frac{Q_1Q_5}{(x-y)^4}. \tag{7.9}$$

Thus the Virasoro central charge indeed arises from the two-point graviton amplitude, which is a part of the disconnected diagram in figure 4.

It should also be possible to obtain (7.6) directly from a string worldsheet computation. In string theory, every genus-zero worldsheet carries an extra factor of $g_s^{-2}$. Therefore the disconnected diagram of figure 4 has an extra factor of $g_s^{-2} = Q_1\sqrt{Q_5}$ compared to the connected diagrams of figure 3. The worldsheet amplitude itself is a function of $l_{AdS}/l_s = \sqrt{Q_5}$ only. For the two-point function of the energy-momentum tensors, our preliminary computation (analogous to the spacetime computation in [22]) indicates that the only $l_{AdS}$ dependence comes from the measure of the $\phi$ zero mode integral. Thus we

*An explicit computation of this can be found in [21].

expect that this computation also reproduces (7.6) with $c \sim Q_1 Q_5$. It would be desirable to make this computation more precise in order to estimate finite $Q_5$ corrections to the central charge formula.

## 8 Discussion

In this paper we have studied string theory on $AdS_3$ and found that many properties of the $AdS/CFT$ duality can be understood from a semi-classical analysis. In particular, we found vertex operators in the worldsheet theory that correspond to the insertion of operators in the boundary $CFT$. The structure of these vertex operators is somewhat reminiscent of the master field for large $N$ field theory. We showed that the string worldsheet stretches to the boundary of $AdS_3$ in the presence of such vertex operators, and that the Virasoro generators of Brown and Henneaux directly give rise to the contour integral representation of the Virasoro algebra in [9]. We have explained why, in this representation, the contour is localized near the boundary of $AdS_3$, and deduced from this the Virasoro Ward identities of the boundary theory. This clarifies several aspects of [9]. However, in our formulation the central charge arises by a different mechanism than one put forth in [9]. We found no need to introduce fundamental strings at infinity and to consider worldsheets wrapping a certain number of times around the boundary of $AdS$. Instead the central charge arose from the disconnected diagram of the second-quantized string theory. It is conceivable that the two different pictures of the central charge are roughly analogous to the short and long string pictures that one encounters, for instance, in matrix string theory [23–25]. The precise meaning and definition of such a long string picture would require further clarification.

Several other issues deserve further investigation. We have not yet given a detailed derivation of the central charge from the worldsheet theory. It would be interesting to do this and to see whether the central charge satisfies a non-renormalization theorem in the case of superstrings on $AdS_3 \times S^3 \times M^4$. In addition, we would like to extend this analysis to Lorentzian signature $AdS_3$, and to have a more detailed understanding of the spectrum and the vertex operators in that case. Finally, we would like to see whether this formulation of string theory on $AdS_3$ can be used in a practical way to compute higher order $\alpha'$ corrections to supergravity results.

**Note Added:**

Toward the completion of this paper, we received [26]. In that paper, string theory on $AdS_3 \times S^3 \times T^4$ is studied using the approach of [9], and a disagreement is found in the spectra of the $U(1)^4$ charges between the string theory on $AdS_3$ and the $CFT_2$ with target space $(T^4)^N/\mathcal{S}_N$. Since that computation depends crucially on the evaluation of the $U(1)$ central charge, it would be interesting to calculate the central charge from our point of view and see if the disagreement persists.

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## References


[1] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity," Commun. Math. Phys. 104 (1986) 207.

[2] J. M. Maldacena, "The Large $N$ Limit of Superconformal Field Theories and Supergravity," hep-th/9711200.

[3] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, "Gauge Theory Correlators from Non-Critical String Theory," hep-th/9802109.

[4] E. Witten, "Anti-de Sitter Space and Holography," hep-th/9802150.

[5] A. Strominger, "Black Hole Entropy from Near-Horizon Microstates," J. High Energy Physics 2 (1998) 9; hep-th/9712251.

[6] J. Maldacena and A. Strominger, "$AdS_3$ Black Holes and a Stringy Exclusion Principle" hep-th/9804085.

[7] E. Martinec, "Matrix Models of $AdS$ Gravity," hep-th/9804111.

[8] J. de Boer, "Six-Dimensional Supergravity on $S^3 \times AdS_3$ and $2d$ Conformal Field Theory," hep-th/9806104.

[9] A. Giveon, D. Kutasov and N. Seiberg, "Comments on String Theory on $AdS_3$," hep-th/9806194.

[10] N. Seiberg, "Notes on Quantum Liouville Theory and Quantum Gravity," Lectures given at the 1990 Yukawa International Seminar, "Common Trends in Mathematics and Quantum Field Theories," Prog. Theor. Phys. Suppl. 102 (1990) 319.

[11] L. Susskind and E. Witten, "The Holographic Bound in Anti-de Sitter Space," hep-th/9805114.

[12] A. Strominger and C. Vafa, Phys. Lett. B379 (1996) 99; hep-th/9601029.

[13] K. Gawędzki, "Non-Compact WZW Conformal Field Theories," in Proceedings of NATO Advanced Study Institute, "New Symmetry Principles in Quantum Field Theory," Cargese 1991, eds. J. Fröhlich, G. 't Hooft, A. Jaffe, G. Mack, P. K. Mitter and R. Stora, (Plenum Press, 1992); hep-th/9110076.

[14] K. Gawędzki and A. Kupiainen, "Coset Construction from Functional Integral," Nucl. Phys. B320 (1989) 351.

[15] J. Teschner, "The Mini-Superspace Limit of the $SL(2,C)/SU(2)$-WZNW Model," hep-th/9712258.

[16] K. Gawędzki, "Quadrature of Conformal Field Theory," Nucl. Phys. B328 (1989) 733.

[17] M. Roček and E. Verlinde, "Duality, Quotients, and Currents", Nucl. Phys. B373 (1992) 630, hep-th/9110053.

[18] J. Teschner, "On Structure Constants and Fusion Rules in the $SL(2,C)/SU(2)$ WZNW Model," hep-th/9712256; and private communication.

[19] P. Ginsparg and G. Moore, "Lectures on $2-D$ Gravity and $2-D$ String Theory," Lectures given at TASI summer school, Boulder CO, June 11-19, 1992; hep-th/9304011.

[20] J. Polchinski, String Theory Vol. 1, p. 196 (Cambridge Univ. Press, 1998).

[21] S. Hyun, W. T. Kim and J. Lee, "Statistical Entropy and AdS/CFT Correspondence in BTZ Black Holes," hep-th/9811005.

[22] M. Henningson and K. Skenderis, "The Holographic Weyl Anomaly," J. High Energy Phys. 9807 (1998) 023; hep-th/9806087.

[23] L. Motl, "Proposals for Nonperturbative Superstring Interactions," hep-th/9701025.

[24] T. Banks and N. Seiberg, "Strings from Matrices," Nucl. Phys. B497 (1997) 41; hep-th/9702187.

[25] R. Dijkgraaf, E. Verlinde and H. Verlinde, "Matrix String Theory," Nucl. Phys. B500 (1997) 43; hep-th/9703030.

[26] D. Kutasov, F. Larsen and R. G. Leigh, "String Theory in Magnetic Monopole Background," hep-th/9812027.