SIAM J. APPL. MATH.
Vol. 35, No. 3, November 1978

0036-1399/78/3503-0011 $01.00/0

ROTATING SPIRAL WAVE SOLUTIONS OF REACTION-DIFFUSION EQUATIONS*

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Abstract. We resolve the question of existence of regular rotating spiral waves as a consequence of only the processes of chemical reaction and molecular diffusion. We prove rigorously the existence of these waves as solutions of reaction-diffusion equations, and we exhibit them by means of numerical computations in several concrete cases. Existence is proved via the Schauder fixed point theorem applied to a class of functions with sufficient structure that, in fact, important constructive properties such as asymptotic representations and frequency of rotation are obtained.

1. Introduction.

Rotating spiral waves arise naturally and as models of spatially organized activity in various chemical and biochemical processes. The Belousov–Zhabotinsky reaction [1], [2] provides a classic example. Experiments with this reaction in a two-dimensional medium (i.e., a thin layer in a Petri dish) produce spiral concentration patterns which rotate with constant frequency about a fixed center [3], [4]. A. T. Winfree [2], [5] proposes that these waves result from an interplay between the chemical process of reaction and the physical process of molecular diffusion, but rotating spiral patterns have not yet been obtained from reaction-diffusion equations. We prove the existence of these waves and exhibit them here.

Our demonstration of the existence of such rotating spiral waves which are smooth from the origin (the fixed center of the spiral) to infinity resolves the following important issue: Previous authors [6], [7] have found asymptotic solutions which represent spiral waves far from a fixed origin, but no analysis is given to show that these asymptotic spirals correspond to solutions that are smooth at the origin. In view of this failure, arguments have been advanced that a mechanism in addition to reaction and diffusion must be present to produce and possibly maintain spiral waves in the core of the spiral. Our results show that a rotating spiral wave can be maintained by reaction and diffusion alone. However, whether an additional mechanism (e.g., local precipitation) occurs in the actual chemistry is of course still an open question.

N. Kopell and L. N. Howard [8] have introduced a simple mathematical model of a reaction-diffusion process called a λ-ω system. The equations are

$$\begin{aligned} U_t &= \nabla^2 U + \lambda(R)U - \omega(R)V, \\ V_t &= \nabla^2 V + \omega(R)U + \lambda(R)V, \end{aligned} \tag{1.1}$$

where λ and ω are given functions of $R = \sqrt{U^2 + V^2}$. $\lambda(R)$ is assumed to be a decreasing function that passes through zero when $R = 1$, so that the spatially independent solutions of (1.1) asymptotically approach a limit cycle with amplitude $R = 1$ and frequency $\omega = \omega(1)$. We rigorously prove the existence of smooth spiral wave solutions for a certain class of λ-ω systems. Although it is commonly claimed that the λ-ω systems do not actually correspond to any particular physical situation, we show that in fact, a λ-ω system arises naturally as the dominant system in the

* Received by the editors January 10, 1978.

† Department of Applied Mathematics, California Institute of Technology, Pasadena, California 91125. This work was supported in part by the U.S. Army Research Office under Contract DAHC-04-68-0006 and in part by the National Science Foundation under Grant GP32157X2.

asymptotic analysis of more general reaction diffusion systems actually describing specific physical processes.

In § 4, we prove our main result that for a certain class of functions $\lambda(\rho)$ and $\omega(\rho)$, the system (1.1) has a solution

$$\begin{aligned} U &= \rho(r)\cos(\Omega t+\theta+\psi(r)), \\ V &= \rho(r)\sin(\Omega t+\theta+\psi(r)). \end{aligned} \tag{1.2}$$

Here, r and θ are polar coordinates of the plane. The functions $\rho(r)$, $\psi(r)$ satisfy

$$\begin{aligned} &\rho''+\frac{\rho'}{r}-\left(\psi'^2+\frac{1}{r^2}\right)\rho+\rho\lambda(\rho)=0, \\ &\psi''+\left(\frac{1}{r}+\frac{2\rho'}{\rho}\right)\psi'=\Omega-\omega(\rho), \end{aligned} \tag{1.3}$$

with boundary conditions $\rho(0)=0$, $\psi'(0)=0$, and $\rho\to\rho(\infty)$ as $r\to\infty$, where $\rho(\infty)$ is related to the rotation rate Ω via $\omega(\rho(\infty))=\Omega$. A consequence of the proof is that as $r\to\infty$

$$\begin{aligned} U &\sim \rho(\infty)\cos(\Omega t+\theta+c\log r), \\ V &\sim \rho(\infty)\sin(\Omega t+\theta+c\log r). \end{aligned} \tag{1.4}$$

Equations (1.4) represent a rotating spiral wave with U and V constant along the logarithmic spirals $\Omega t+\theta+c\log r=\text{constant}$.

In § 2, we derive the equations and boundary conditions for $\rho(r)$ and $\psi(r)$, and in § 3, we present numerical computations and graphs of spiral waves. For various choices of $\omega(\rho)$, we obtain both logarithmic and Archimedian spiral waves. The logarithmic spiral waves have $\psi(r)\sim c\log r$ as $r\to\infty$, while Archimedian spiral waves have $\psi(r)\sim cr$ as $r\to\infty$. Finally, in § 5 we show how the model λ-ω system (1.1) arises naturally as the dominant system in the asymptotic analysis of realistic reaction-diffusion systems.

2. Equations governing spiral waves. It is convenient to introduce polar variables (R,Θ) via the change of variables $U=R\cos\Theta$, $V=R\sin\Theta$. Then, system (1.1) becomes

$$\begin{aligned} R_t &= \nabla^2 R - R|\nabla\Theta|^2+R\lambda(R), \\ \Theta_t &= \nabla^2\Theta+\frac{2\nabla R\cdot\nabla\Theta}{R}+\omega(R). \end{aligned} \tag{2.1}$$

We seek solutions of the form

$$R=\rho(r),\qquad \Theta=\Omega t+\theta+\psi(r), \tag{2.2}$$

where (r,θ) are polar coordinates of the plane. Such solutions correspond to rotating waves in the concentrations. The corresponding values of U and V given by

$$\begin{aligned} U(r,\theta,t) &= \rho(r)\cos(\Omega t+\theta+\psi(r)), \\ V(r,\theta,t) &= \rho(r)\sin(\Omega t+\theta+\psi(r)), \end{aligned} \tag{2.3}$$

represent a spiral wave that rotates with frequency Ω about $r=0$. Upon substituting

(2.2) into (2.1), we find that $\rho(r)$ and $\psi(r)$ must satisfy

$$\rho''+\frac{\rho'}{r}-\rho\left(\psi'^2+\frac{1}{r^2}\right)+\rho\lambda(\rho)=0, \tag{2.4}$$
$$\psi''+\left(\frac{1}{r}+\frac{2\rho'}{\rho}\right)\psi'=\Omega-\omega(\rho).$$

Physical considerations dictate the proper boundary conditions. We seek solutions with ρ and ψ' bounded, so that the concentrations U, V given in (2.3) will have bounded values and gradients. Solutions of (2.4) that are regular at $r=0$ have $\rho(0)$, $\psi'(0)=0$. These are the boundary conditions at $r=0$. As $r\to\infty$, we demand that $\rho(r)$ asymptotes to a nonzero constant value $\rho(\infty)$. The value of $\rho(\infty)$ must be related to the rotation rate Ω by $\omega(\rho(\infty))=\Omega$. We can see this in the following way: Given $\rho(r)$, we can integrate the second of equations (2.4) to find

$$\psi'(r)=\frac{1}{r\rho^2(r)}\int_0^r s\rho^2(s)(\Omega-\omega(\rho(s)))\,ds. \tag{2.5}$$

If $\Omega-\omega(\rho(\infty))\neq 0$, then (2.5) implies

$$\psi'(r)\sim\frac{1}{r}\int_0^r s(\Omega-\omega(\rho(\infty)))\,ds=\frac{1}{2}(\Omega-\omega(\rho(\infty)))r \tag{2.6}$$

as $r\to\infty$. This violates the boundedness of ψ'. Hence $\Omega-\omega(\rho(\infty))=0$.

3. Numerical solutions. Various bounds and estimates in our proof of § 4 have been useful in helping to devise a numerical scheme to actually compute the spiral waves. We have computed $\rho(r)$ and $\psi'(r)$ for $\lambda(\rho)=1-\rho$ and various choices of $\omega(\rho)$. If $\Omega-\omega(\rho)=\varepsilon(1-\rho)^2$, where ε is a constant, we obtain the logarithmic spiral wave solutions whose existence we prove in § 4. Figures 1 and 2 are graphs of ρ and ψ' for $\varepsilon=1$. Figure 3 and 4 are contour plots of the concentration $U(r,\theta,0)$. Each contour represents a constant value of U. The values of U range from -0.8 to 0.8 in increments of 0.2. Some features of these graphs require comment. Notice that there are at least two solutions for $\varepsilon=1$, one with monotone increasing $\rho(r)$, and another with one oscillation in $\rho(r)$. This situation plainly reveals the need for a stability

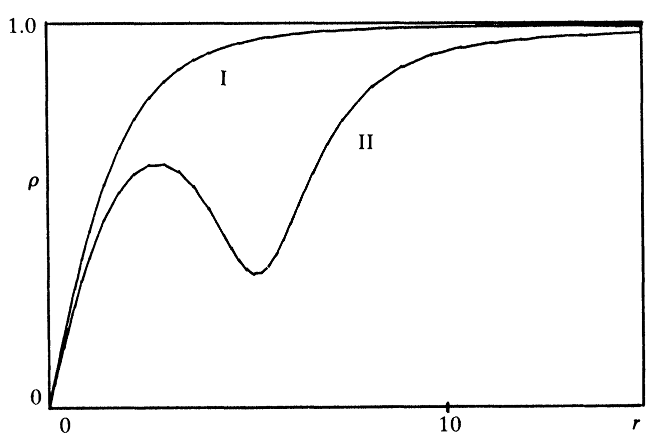

FIG. 1

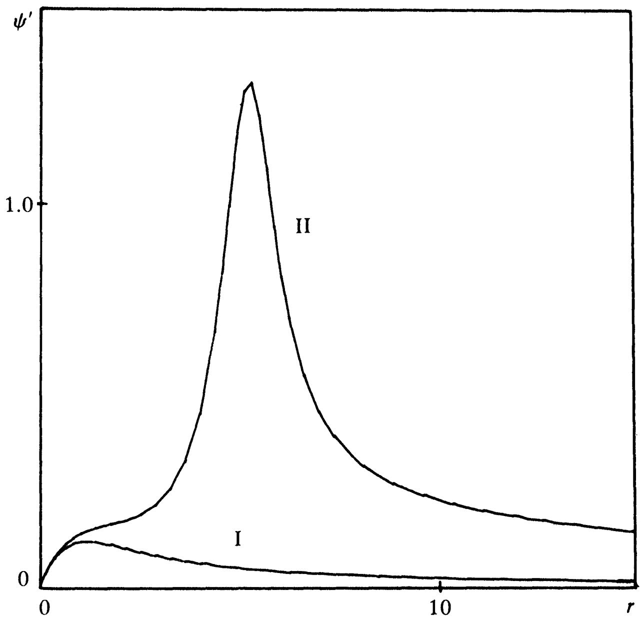

FIG. 2

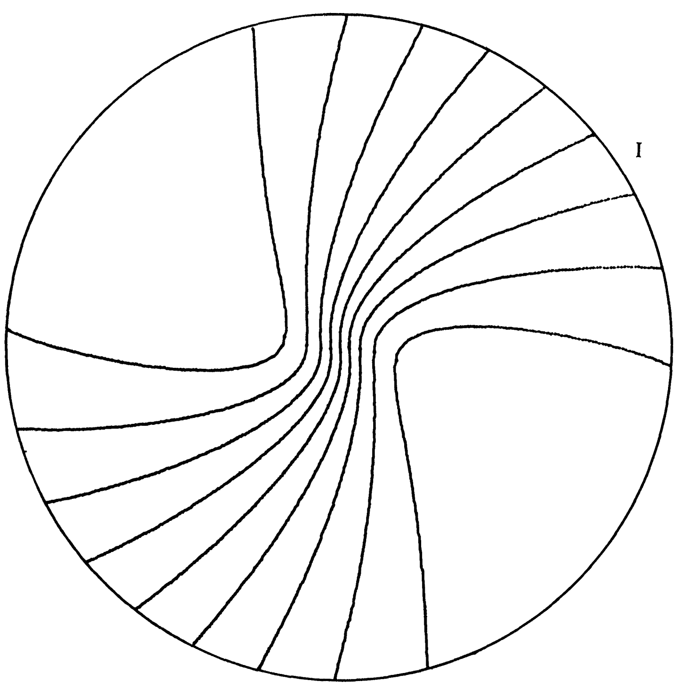

FIG. 3

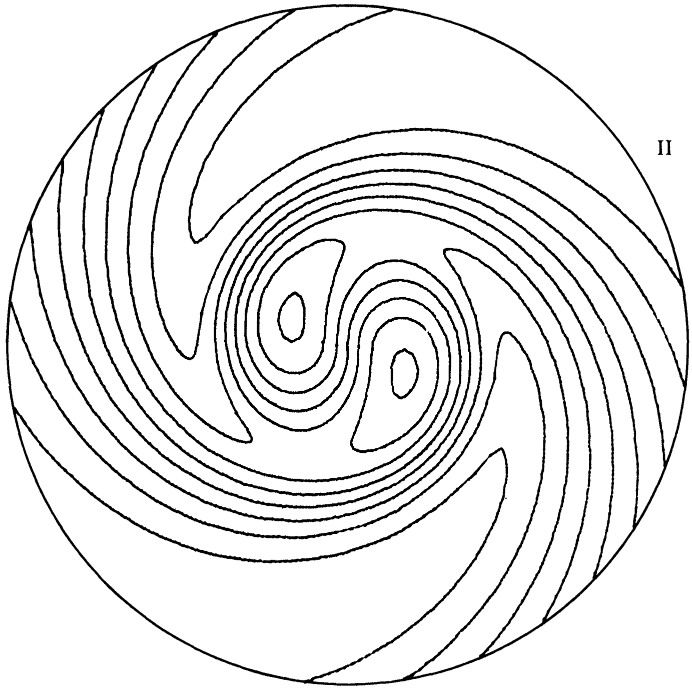

FIG. 4

analysis that determines which of the possible solutions for $\rho(r)$ and $\psi'(r)$ correspond to stable solutions of the λ-ω system.

If $\Omega-\omega(\rho)=\varepsilon(\rho(\infty)-\rho)$, one obtains Archimedian spiral wave solutions, in which $\psi'(r)$ asymptotes to a nonzero, constant value $\psi'(\infty)$ as $r\to\infty$. Figures 5 and 6 shows ρ and ψ' for $\varepsilon=0.5$, $\rho(\infty)=0.5$. From the first of equations (2.4) we see that $\rho(\infty)$ and $\psi'(\infty)$ must be related by $\psi'^2(\infty)=1-\rho(\infty)$. For $\rho(\infty)=0.5$, $\psi'(\infty)=\pm1/\sqrt{2}$. In Fig. 6, $\psi'(r)$ appears to asymptote to $-1/\sqrt{2}\simeq-0.707$. In Fig. 5, notice that $\rho(r)$ oscillates before flattening out to its value at $r=\infty$. This accounts for the "islands" that appear in the corresponding contour plot of $U(r,\theta,0)$ shown in Fig. 7.

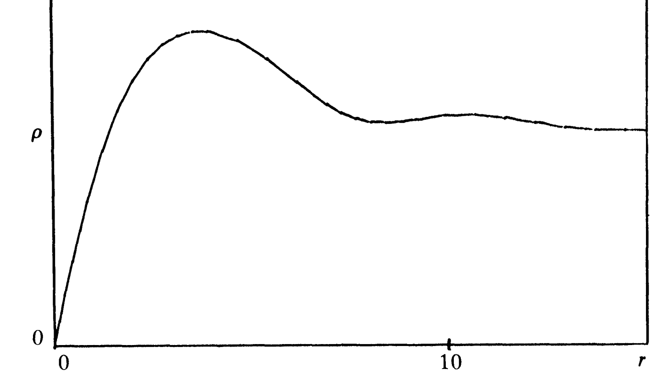

FIG. 5

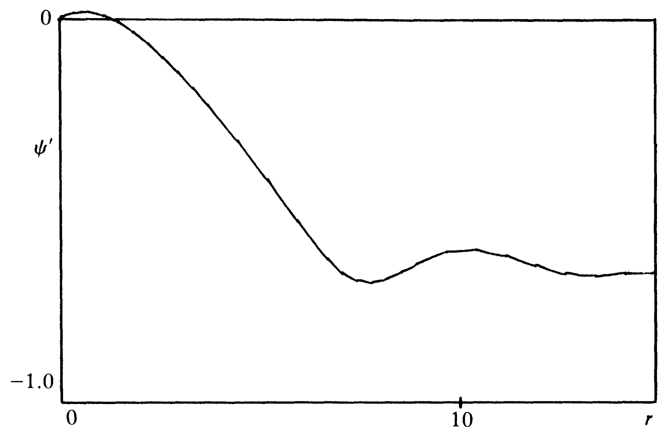

FIG. 6

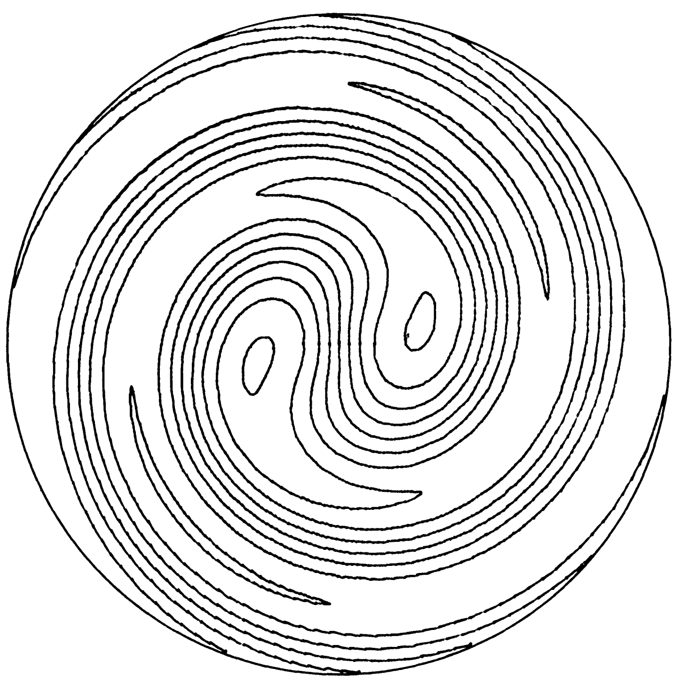

FIG. 7

The above solutions are computed by the shooting method, where the shooting parameter is the derivative of ρ at $r=0$. The iterations are continued until the exponential instabilities of the numerical solution are delayed until $r=20$. We then use the computed solution for $0<r<15$. By making small changes in the shooting parameter, we are able to make the exponential instability positive or negative. This gives confidence that the procedure actually approximates a true solution.

4. Existence of spiral waves. We now prove that the reaction-diffusion system (1.1) possesses rotating logarithmic spiral waves. Consider system (2.4), and assume

H.1. $\lambda=\lambda(\rho)$ is defined and continuously differentiable on $0\leqq\rho\leqq a$ for some $a>0$, $\lambda(\rho)>0$ for $0\leqq\rho<a$, $\lambda(a)=0$, and $\lambda'(a)<0$.

H.2. $\omega=\omega(\rho)$ is defined and continuous for $0\leqq\rho\leqq a$, and furthermore, there exist $\varepsilon\geqq 0$ and $\mu>0$ such that

$$|\omega(a)-\omega(\rho)|\leqq\varepsilon(a-\rho)^{1+\mu},\qquad 0\leqq\rho\leqq a. \tag{4.1}$$

Under these conditions we have the following result:

THEOREM 4.1. *For ε sufficiently small there exist a number $\Omega=\omega(\rho(\infty))=\omega(a)$ and functions $\rho=\rho(r)$ and $\psi=\psi(r)$, twice continuously differentiable on $0\leqq r<\infty$, satisfying (2.4), and*

$$0<\rho(r)<a\quad \text{for } 0<r<\infty, \tag{4.2}$$

$$\rho(r)=\begin{cases} O(r) & \text{as } r\to 0,\\ a+O(r^{-2}) & \text{as } r\to\infty,\end{cases} \tag{4.3}$$

$$\psi''(r)=\begin{cases} O(r) & \text{as } r\to 0,\\ cr^{-1}+O(r^{-1-2\mu}) & \text{as } r\to\infty,\end{cases} \tag{4.4}$$

where

$$c=\frac{1}{a^2}\int_0^\infty s\rho^2(s)[\omega(a)-\omega(\rho(s))]\,ds.$$

Remark I. We will also show that from (4.2)–(4.4) it follows that

$$\begin{gathered}\rho'(0)>0,\quad \rho''(0)=0,\quad \rho(r)=1+\frac{1+c^2}{\lambda'(a)r^2}+o(r^{-2}),\\ \rho'(r)=o(r^{-2})\quad\text{and}\quad \rho''(r)=o(r^{-2})\quad\text{as } r\to\infty,\end{gathered} \tag{4.5}$$

$$\psi(r)=c\ln r+\text{constant}+O(r^{-2\mu})\quad\text{as } r\to\infty. \tag{4.6}$$

Upon substituting (4.3) and (4.6) into (1.2), we obtain (1.4).

Remark II. Numerical computations in addition to those presented in § 3 suggest that the limitation on the size of ε in our proof is necessary, i.e. for ε large there are no solutions. Moreover the solution, when it exists, may not be unique.

Remark III. We have expressed $\Omega=\omega(\rho(\infty))$ as a consequence of our analysis. However, as we saw in § 2, we could equally well have taken this as one of the boundary conditions.

We proceed now to prove Theorem 4.1. We shall reduce the differential equations to integral equations and then apply Schauder's fixed point theorem. The unboundedness of the interval $0\leqq r<\infty$ introduces some complications, and rather precise a priori bounds for the solution for large r are needed to satisfy the compactness requirements necessary for Schauder's theorem to apply.

Proof. Without loss of generality we may assume $a=1$. Let now $m>0$ be such that $m^2\rho+\rho\lambda(\rho)$ is increasing in $0\leqq\rho\leqq 1$; that is, take

$$m^2\geqq\max_{0\leqq\rho\leqq 1}\{-\lambda(\rho)-\rho\lambda'(\rho)\}, \tag{4.7}$$

and let $\varphi=\varphi(z)$ be any thrice continuously differentiable function on $0\leqq z<\infty$ such that

$$\varphi''(z)<0\quad\text{for } 0<z<\infty,\quad \varphi(0)=0,\quad \varphi''(0)=0\quad\text{and}\quad \varphi'''(0)<0, \tag{4.8}$$

$$\varphi(z) \sim 1 - Mz^{-2}, \quad \varphi'(z) \sim 2Mz^{-3}, \tag{4.9}$$
$$\varphi''(z) = -6Mz^{-4} \quad \text{as } z \to \infty, \quad \text{for some constant } M > 0.$$

For example, we may take $\varphi(z) = 1 - (1 + z + z^2)^{-1}$, with $M = 1$ and $\varphi'''(0) = -6$. It immediately follows that

$$\varphi' > 0 \text{ is strictly decreasing and } z\varphi' < \varphi \text{ for } 0 < z < \infty, \tag{4.10}$$

$$\varphi \text{ is strictly increasing and } 0 < \varphi < 1 \text{ for } 0 < z < \infty. \tag{4.11}$$

Now, take $0 < \alpha < \beta < \infty$ and define $F = F(r)$, $\delta_F = \delta_F(r)$, $f = f(r)$ and $\delta_f = \delta_f(r)$ by

$$F(r) = \varphi(\alpha r), \qquad 0 \leqq r < \infty, \tag{4.12}$$

$$\begin{aligned} \delta_F(r) &= F'' + \frac{1}{r}F' - \frac{1}{r^2}F + F\lambda(F) \\ &= -\alpha^2 g(\alpha r) + \varphi(\alpha r)\lambda(\varphi(\alpha r)), \qquad 0 \leqq r < \infty, \end{aligned} \tag{4.13}$$

where $g(z) = -\varphi''(z) - (1/z)\varphi'(z) + (1/z^2)\varphi(z)$, and similarly for f and δf with β replacing α. We note that F and f have the same properties as φ, and that, from (4.11), $F < f$. The function g is continuous and, from (4.8) and (4.10), positive for $0 < z < \infty$. Furthermore, from (4.8) and (4.9),

$$g(z) = -\tfrac{4}{3}\varphi'''(0)z + o(z) \quad \text{as } z \to 0, \tag{4.14}$$

$$g(z) = z^{-2} + O(z^{-4}) \quad \text{as } z \to \infty. \tag{4.15}$$

Since $\varphi\lambda(\varphi)$ is also positive for $0 < z < \infty$ and

$$\varphi\lambda(\varphi) = \varphi'(0)\lambda(0)z + O(z^2) \quad \text{as } z \to 0, \tag{4.16}$$

$$\varphi\lambda(\varphi) = -\lambda'(1)Mz^{-2} + o(z^{-2}) \quad \text{as } z \to \infty, \tag{4.17}$$

we see that for α small enough and for β large enough we will have

$$\delta_F \geqq \tfrac{3}{4}F\lambda(F), \qquad \delta_f \leqq 0, \quad 0 \leqq r < \infty. \tag{4.18}$$

Now, let S be the set of all continuous functions $\rho = \rho(r)$ on $0 \leqq r \leqq \infty$ such that $F \leqq \rho \leqq f$. Then, S is a nonempty closed convex set in the Banach space B of all the continuous functions on $0 \leqq r < \infty$, with the maximum norm $\|\rho\| = \max_{0 \leqq r < \infty} |\rho(r)|$. Any function in S satisfies (4.2) and (4.3).

Define the operator $\chi : S \to B$ by

$$\chi(\rho)(r) = \frac{1}{r\rho^2(r)} \int_0^r s\rho^2(s)\{\omega(1) - \omega(\rho(s))\}\, ds, \qquad 0 < r < \infty. \tag{4.19}$$

Note that because of (4.1) and the bounds $F \leqq \rho \leqq f$, with F and f satisfying (4.8)–(4.11), $\chi(\rho)(r)$ is well defined and continuous for $0 < r < \infty$. Furthermore $\chi(\rho)(r) = O(r)$ as $r \to 0$ and as $r \to \infty$

$$\chi(\rho)(r) = \frac{1}{a^2}\left\{\int_0^\infty s\rho^2[\omega(1) - \omega(\rho)]\, ds\right\} r^{-1} + O(r^{-1-2\mu}). \tag{4.20}$$

The transformation $\chi: S \to B$ is continuous since we have for any $\rho, \eta \in S$:

$$|\chi(\rho)(r)-\chi(\eta)(r)| \leqq \frac{|\eta^2-\rho^2|}{r\eta^2\rho^2}\int_0^r s\rho^2|\omega(1)-\omega(\rho)|\,ds$$

$$+\frac{1}{r\eta^2}\int_0^r s|\rho^2-\eta^2|\,|\omega(1)-\omega(\rho)|\,ds+\frac{1}{r\eta^2}\int_0^r s\eta^2|\omega(\eta)-\omega(\rho)|\,ds$$

$$\leqq 2\varepsilon\frac{f}{F^2}\left(\frac{f^2}{F^2}+1\right)\int_0^r (1-F)^{1+\mu}\,ds\|\rho-\eta\|+\frac{f^2}{F^2}\int_0^\infty |\omega(\eta)-\omega(\rho)|\,ds.$$

The function multiplying $\|\rho-\eta\|$ in the last inequality is bounded and $|\omega(\eta)-\omega(\rho)| \leqq 2\varepsilon(1-F)^{1+\mu}$. Thus, from the dominated convergence theorem, $\|\chi(\rho)-\chi(\eta)\| \to 0$ when $\|\rho-\eta\| \to 0$, and this proves the continuity of χ. We note also that $\chi(S)$ is a set of uniformly bounded functions, the bound being that given in (4.23).

Next, define the transformation $T: S \to B$ by $T\rho = v$ for any $\rho \in S$, where v is the unique solution of the problem

$$\text{(4.21)} \qquad Lv = -v''-\frac{1}{r}v'+\left(\frac{1}{r^2}+m^2\right)v = [m^2+\lambda(\rho)-\chi^2(\rho)]\rho, \qquad 0<r<\infty,$$

with the boundary conditions $v(0)=0$, $v(\infty)=1$; i.e. v is given by

$$\text{(4.22)} \qquad v(r)=\frac{1}{m}\int_0^\infty G(mr, ms)\{m^2+\lambda[\rho(s)]-\chi^2(\rho)(s)\}\rho(s)\,ds,$$

where $G(r,s)=K_1(r)I_1(s)s$ for $s \leqq r$, $G(r,s)=I_1(r)K_1(s)s$ for $s \geqq r$, is the Green's function of the modified Bessel operator on the right hand side of (4.21) for $m=1$ (Abramowitz and Stegun [9, pp. 374–378]). T is a continuous transformation and the functions in $T(S)$ are twice continuously differentiable, with both derivatives uniformly bounded. Thus $T(S)$ is an equicontinuous family of functions. If $T(S) \subseteq S$ then $\bar{T}(S)$ will be a compact set. This follows from an argument entirely similar to that used to prove Arzela's theorem, where the unboundedness of the interval $[0,\infty)$ is compensated by the fact that all the functions in S tend to 1 uniformly as $r \to \infty$. In this case, when $T(S) \subseteq S$, it follows from Schauder's second fixed point theorem (Smart [10, p. 25]) that T has a fixed point, say ρ. This fixed point provides a solution of (2.4) satisfying (4.2), (4.3) and (4.4), where ψ' is given by $\chi(\rho)$. In this case, ρ being smooth, (4.19) can be differentiated and it is immediately seen that $\psi'=\chi(\rho)$ satisfies the second of equations (2.4). Then (4.4) follows from (4.20).

Let us now prove that $T(S) \subseteq S$ for ε small enough. For any $\rho \in S$ we have

$$\text{(4.23)} \qquad |\chi(\rho)(r)| \leqq \frac{\varepsilon}{rF^2(r)}\int_0^r sf^2(s)[1-F(s)]^{1+\mu}\,ds = \varepsilon H(r),$$

with $H(r)=O(r)$ as $r \to 0$, $H(r)=O(r^{-1})$ as $r \to \infty$ and $H(r)$ continuous. For ε small enough it now follows from (4.16)–(4.18) that $\delta_F \geqq \chi^2(\rho)$ for all $\rho \in S$. Then, (4.7), (4.18), and (4.21) imply that

$$\text{(4.24)} \qquad L(f-v)=\chi^2(\rho)\rho-\delta_f+[m^2f+f\lambda(f)]-[m^2\rho+\rho\lambda(\rho)] \geqq 0,$$

since $f \geqq \rho$, and

$$\text{(4.25)} \qquad L(F-v)=-\delta_F+\chi^2(\rho)\rho+[m^2F+F\lambda(F)]-[m^2\rho+\rho\lambda(\rho)] \leqq 0,$$

since $F \leqq \rho$. Thus, from the maximum principle for the operator L (i.e. the Green's function of L is positive) it follows that $F \leqq v \leqq f$ so that $v \in S$. This completes the proof of the theorem.

Let now $\rho = \rho(r)$ and $\psi = \psi(r)$ be a solution of (2.4) such as that provided by the statement of Theorem 4.1. Then, we can write

$$\psi'(r) = \sigma(r) = \frac{1}{r\rho^2(r)} \int_0^r s\rho^2(s)\{\omega(a) - \omega(\rho(s))\}\, ds, \tag{4.26}$$

$$\rho(r) = \frac{1}{m} \int_0^\infty G(mr, ms)\{m^2 + \lambda(\rho(s)) - \sigma^2(s)\}\rho(s)\, ds, \tag{4.27}$$

where $m > 0$ is large enough so that $m^2 + \lambda(\rho) - \sigma^2 > 0$ for all s. From (4.2)–(4.4) and the properties of $G(r, s)$ we can deduce from (4.27) that

$$\rho'(0) > 0, \quad \rho''(0) = 0, \quad \rho'(r) = O(r^{-2}) \tag{4.28a}$$

and

$$\rho''(r) = O(r^{-2}) \quad \text{as } r \to \infty. \tag{4.28b}$$

Introduce $\kappa(r) = r^2[a - \rho(r)]$, $0 \leqq r < \infty$. Then κ is twice continuously differentiable with κ, κ' and κ'' bounded and $\kappa(0) = \kappa'(0) = 0$. Substituting in the first of equations (2.4) we obtain for κ the equation

$$\kappa'' + a\lambda'(a)\kappa + a + c^2 a = h, \qquad \kappa(0) = 0, \quad \kappa \text{ bounded}, \tag{4.29}$$

where h is the continuous function $h = a[r^2\lambda(\rho) + \lambda'(a)\kappa] - \lambda(\rho)\kappa - (r^2\sigma^2 - c^2)\rho + (3/r)\kappa' + ((c-3)/r^2)\kappa$. Thus, if $\nu = \sqrt{-a\lambda'(a)}$ we have

$$\kappa(r) = -\frac{1+c^2}{\lambda'(a)} - e^{-\nu r}\int_0^r e^{\nu r} h(s)\frac{ds}{2\nu} - e^{-\nu r}\int_r^\infty e^{-\nu s} h(s)\frac{ds}{2\nu}$$
$$+ \left\{\frac{1+c^2}{\lambda'(a)} + \int_0^\infty e^{-\nu s} h(s)\frac{ds}{2\nu}\right\} e^{-\nu r}.$$

Now $h(r) = o(1)$ as $r \to \infty$. Thus, $\kappa(r) = -(1+c^2)/(\lambda'(a)) + o(1)$ as $r \to \infty$. It follows that

$$\begin{aligned} &\rho(r) = 1 + \frac{1+c^2}{\lambda'(a)r^2} + o(r^{-2}), \qquad \rho'(r) = o(r^{-2}), \\ &\rho''(r) = o(r^{-2}) \quad \text{as } r \to \infty, \end{aligned} \tag{4.30}$$

where the statements for ρ' and ρ'' follow from (4.27) upon use of the expansion for ρ just proved. This proves Remark I.

The process just used to prove (4.30) can be continued to higher orders of approximation for ρ and ψ and we have the following

THEOREM 4.2. *Assume that in addition to* H.1 *and* H.2, $\lambda = \lambda(\rho)$ *and* $\omega = \omega(\rho)$ *are* n *times differentiable at* $\rho = a$, *for some* $n \geqq 1$. *Then* (2.4) *has solutions satisfying* (4.2), (4.3) *and* (4.4) *such that*

$$\rho = a + \sum_{j=1}^{n} a_j r^{-2j} + o(r^{-2n}) \qquad \text{as } r \to \infty,$$

$$\rho' = -\sum_{j=1}^{n-1} 2ja_j r^{-2j-1} + o(r^{-2n}) \qquad \text{as } r \to \infty,$$

$$\rho'' = \sum_{j=1}^{n-1} 2j(2j+1) r^{-2j-2} + o(r^{-2n}) \text{ as } r \to \infty,$$

$$\psi' = \sum_{j=0}^{n-1} c_j r^{-2j-1} + o(r^{1-2n}) \qquad \text{as } r \to \infty,$$

where $c_0 = c$ *and the constants* a_j *and* c_j *can be obtained by formal substitution into the equations.*

We note that for $n > 1$ we must have $\mu \geqq 1$ and (4.1) is equivalent then to $\omega'(a) = 0$.

5. The general reaction-diffusion system. We now show that the model λ-ω system (1.1) arises naturally from more physically realistic reaction-diffusion equations as the leading term in an appropriate asymptotic analysis. Consider the general nonlinear system

$$\begin{aligned} u_t &= D_1 \nabla^2 u + F(u, v, \lambda), \\ v_t &= D_2 \nabla^2 v + G(u, v, \lambda), \end{aligned} \tag{5.1}$$

where λ is a nondimensional parameter. We assume that $u \equiv v \equiv 0$ corresponds to a static state and that $\lambda = \lambda_0$ is a bifurcation point at which the diffusionless system exhibits a (Hopf) bifurcation from the steady state to a limit cycle. Then, u and v should be interpreted as deviations of concentrations from a static state of the system (5.1), and near $\lambda = \lambda_0$ we can re-write the system (5.1) in the form

$$\begin{aligned} u_t &= D_1 \nabla^2 u + \alpha(\lambda) u - \beta(\lambda) v + f(u, v, \lambda), \\ v_t &= D_2 \nabla^2 v + \beta(\lambda) u + \gamma(\lambda) v + g(u, v, \lambda), \end{aligned} \tag{5.2}$$

where f and g are second order in u and v. The coefficients α, β, and γ, and the functions f and g are assumed to satisfy the following conditions:

P.1. $\alpha(\lambda)$, $\beta(\lambda)$, and $\gamma(\lambda)$ are smooth functions of λ satisfying $\alpha(\lambda_0) = \gamma(\lambda_0) = 0$, $\alpha(\lambda) > 0$ and $\gamma(\lambda) \geqq 0$ for $\lambda > \lambda_0$, and $\beta(\lambda) > 0$ for all λ.

P.2. $f(u, v, \lambda)$ and $g(u, v, \lambda)$ are smooth functions of u, v, and λ satisfying $f(u, v, \lambda)$, $g(u, v, \lambda) = o(|u| + |v|)$ as $|u| + |v| \to 0$.

The one-dimensional version of this problem (i.e., $\nabla^2 \equiv \partial^2/\partial x^2$) was studied by D. S. Cohen, F. C. Hoppensteadt, and R. M. Muira [11]. We shall generalize and extend their analysis to account for n-dimensional waves. Our approach is motivated by the observation that in various chemical and biochemical reactions, an oscillatory or periodic variation of reactants at each point in space will undergo a slow change or drift [11]. This takes place in the form of a slowly evolving envelope modulating the amplitude of rapid oscillations.

These observations motivate the study of (5.2) for solutions of special form. Thus, for $\lambda = \lambda_0 + \varepsilon$ where $0 < \varepsilon \ll 1$, we look for solutions of the reaction-diffusion equations (5.2) of the form

$$R(\xi, \tau) P(t^*), \tag{5.3}$$

where $P(t^*)$ represents a periodic oscillation on a fast time scale $t^* = (1 + O(\varepsilon^2))t$, and the amplitude modulation $R(\xi, \tau)$ evolves on slow space, $\xi = (\varepsilon x_1, \cdots, \varepsilon x_n)$, and slow time, $\tau = \varepsilon^2 t$, scales. Hence, we assume that

$$u = u(\xi, \tau, t^*) = \varepsilon F_1(\xi, \tau, t^*) + \varepsilon^2 F_2(\xi, \tau, t^*) + \cdots,$$
$$v = V(\xi, \tau, t^*) = \varepsilon G_1(\xi, \tau, t^*) + \varepsilon^2 G_2(\xi, \tau, t^*) + \cdots, \tag{5.4}$$

where $\varepsilon = \lambda - \lambda_0$, $0 < \varepsilon \ll 1$, and

$$\xi = (\varepsilon x_1, \cdots, \varepsilon x_n), \quad \tau = \varepsilon^2 t, \quad t^* = (1 - \varepsilon\omega(\varepsilon))^{-1} t. \tag{5.5}$$

With the definitions (5.5) we find that

$$\frac{\partial}{\partial t} = (1 - \varepsilon\omega(\varepsilon))^{-1} \frac{\partial}{\partial t^*} + \varepsilon^2 \frac{\partial}{\partial \tau}, \quad \nabla_x^2 = \varepsilon^2 \nabla_\xi^2. \tag{5.6}$$

Just as in [11] we carry out the multi-scale perturbation procedure. The actual manipulation is lengthy but relatively straightforward if we use the solvability and orthogonality conditions derived in [11], and thus, we simply state the results here. We find that

$$F_1 = R(\xi, \tau) \cos [\beta(\lambda_0) t^* + \Theta(\xi, \tau)],$$
$$G_1 = R(\xi, \tau) \sin [\beta(\lambda_0) t^* + \Theta(\xi, \tau)], \tag{5.7}$$

where R and Θ satisfy equations of the form

$$R_\tau = D[\nabla^2 R - R|\nabla\Theta|^2] + p(R),$$
$$\Theta_\tau = D\left[\nabla^2\Theta + \frac{2\nabla R \cdot \nabla\Theta}{R}\right] + q(R). \tag{5.8}$$

Here $D = \frac{1}{2}(D_1 + D_2)$, and $p(R)$ and $q(R)$ are nonlinear functions of R depending on the nonlinearities f and g in the original system (5.2). The actual form of $p(R)$ and $q(R)$ is not important here. All we need note is that the equations (5.8) governing the slowly varying amplitude R and phase Θ are of precisely the form (2.1) which itself is the polar form of the general λ-ω system (1.1).

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