Published by IOP Publishing for SISSA

Received: *September 16, 2008*
Accepted: *October 14, 2008*
Published: *October 28, 2008*

Nonsupersymmetric flux vacua and perturbed $\mathcal{N} = 2$ systems

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Abstract: We geometrically engineer $\mathcal{N} = 2$ theories perturbed by a superpotential by adding 3-form flux with support at infinity to local Calabi-Yau geometries in type IIB. This allows us to apply the formalism of Ooguri, Ookouchi, and Park [arXiv:0704.3613] to demonstrate that, by tuning the flux at infinity, we can stabilize the dynamical complex structure moduli in a metastable, supersymmetry-breaking configuration. Moreover, we argue that this setup can arise naturally as a limit of a larger Calabi-Yau which separates into two weakly interacting regions; the flux in one region leaks into the other, where it appears to be supported at infinity and induces the desired superpotential. In our endeavor to confirm this picture in cases with many 3-cycles, we also compute the CIV-DV prepotential for arbitrary number of cuts up to fifth order in the glueball fields.

Keywords: Flux compactifications, Supersymmetric gauge theory, Supersymmetry Breaking.

JHEP10(2008)102

Contents

1. Introduction

Over the last years much progress has been made in studying flux compactifications in string theory; see [1, 2] for recent reviews. By now there is strong evidence that there is a huge number of supersymmetric vacua with negative cosmological constant in which all scalar moduli are stabilized, the so called landscape of string theory. Typical constructions start with a warped Calabi-Yau compactification of type IIB string theory to four dimensions. Some of the scalar moduli are stabilized by the addition of fluxes through the compact cycles of the internal manifold and others by various quantum effects. Since supersymmetry is broken in the real world, to make contact with phenomenology it is necessary to extend the previous constructions to non-supersymmetric (meta)stable vacua with small positive cosmological constant. For this we need to understand the mechanism of supersymmetry breaking in string theory. By now several methods of supersymmetry breaking for string vacua have been proposed, such as the introduction of anti-branes [3, 4], or simply the existence of metastable points of the flux-induced potential [5, 6]. The main drawback of these constructions is that, in most cases, they are not under complete quantitative control.

While the question of supersymmetry breaking should be ultimately understood in an honest compactification, that is in a theory including gravity in four dimensions, it is technically easier to study simpler systems where the gravitational dynamics has been decoupled from the gauge theory degrees of freedom. This typically happens in the limit where a local singularity develops in the Calabi-Yau manifold. In such a situation all the interesting dynamics related to the degrees of freedom of the singularity takes place at energy scales much lower than the four dimensional Planck scale. Assuming that supersymmetry breaking is related to these light degrees of freedom, it is then possible to zoom in towards the singularity and forget about the rest of the Calabi-Yau. This leads us to the study of supersymmetry breaking and string phenomenology in the context of local Calabi-Yau geometries possibly with the addition of probe D3-branes [7 – 11].

Meanwhile a new important aspect of supersymmetry breaking in gauge theories was developed after the discovery of Intriligator, Seiberg and Shih [12] that even simple supersymmetric gauge theories can exhibit dynamical supersymmetry breaking in metastable

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vacua. From a phenomenological point of view this possibility is quite attractive, and a lot of activity has been concentrated around extensions of the ISS model and various related string theory constructions [13 – 22] (see also [23]). A certain class of gauge theories where supersymmetry breaking in metastable vacua can be studied with good control is that of $\mathcal{N} = 2$ gauge theories perturbed by a small superpotential, initiated by [24]. In such theories the exact Kähler metric on the moduli space is known, which allows one to compute the scalar potential produced by the perturbation of the theory by a small superpotential exactly to first order in the perturbation. It was shown that generically there are metastable supersymmetry breaking vacua generated by appropriate superpotentials. We will refer to this as the OOP mechanism for supersymmetry breaking in $\mathcal{N} = 2$ theories.

String theory in a local Calabi-Yau singularity realizes geometric aspects of supersymmetric gauge theories. In particular the question of supersymmetry breaking in these two systems should be related. The first goal of our paper is to make this connection more precise by giving a geometric realization of the OOP supersymmetry breaking mechanism in IIB on a local Calabi-Yau singularity. To realize OOP one first has to engineer the (IR of the) $\mathcal{N} = 2$ gauge theory and then to find a way of introducing the appropriate superpotential. The first step is achieved by the standard geometric engineering of $\mathcal{N} = 2$ gauge theories by IIB on noncompact Calabi-Yau manifolds [25, 26]. It is well known that the moduli space of the Calabi-Yau compactification encodes the geometry of the Coulomb branch of the gauge theory and that the Seiberg-Witten solution can be rederived by the complex geometry of the Calabi-Yau.

The introduction of superpotential to this system is less straightforward and, to our knowledge, has not been studied in the literature before, in this context. Our main proposal is that the superpotential can be introduced by turning on 3-form flux in the local Calabi-Yau, which is not piercing its compact 3-cycles, but which is growing in the noncompact direction of the Calabi-Yau. In other words, it is flux which has support at infinity. While this flux is not directly piercing the compact cycles we show that, once appropriately regularized, it does introduce an effective superpotential for the complex structure moduli, which is generalization of the usual Gukov-Vafa-Witten superpotential [27 – 29] to 3-form flux with noncompact support. This is a way to introduce a general superpotential in a geometrically engineered $\mathcal{N} = 2$ gauge theory. In particular, we explain that in certain cases it is possible to engineer the OOP-type superpotential, which guarantees the existence of metastable, supersymmetry breaking vacua for the complex structure moduli.

The second goal of our paper is to find a "natural" way to generate the supersymmetry breaking flux configurations described above, starting from a more standard setup. In this process, we also clarify the meaning of flux which has noncompact support and the various subtleties related to it.[1] The natural interpretation of the flux described in the previous paragraph emerges once we embed the previous supersymmetry-breaking local singularity into a bigger IIB compactification with standard flux of compact support. As shown in figure 1, the physical idea is to start with a Calabi-Yau manifold with a set of three-cycles which are isolated from the other three-cycles by a large distance. We turn on 3-form flux

[1] For example the fact that naively the total energy density in four dimensions diverges.

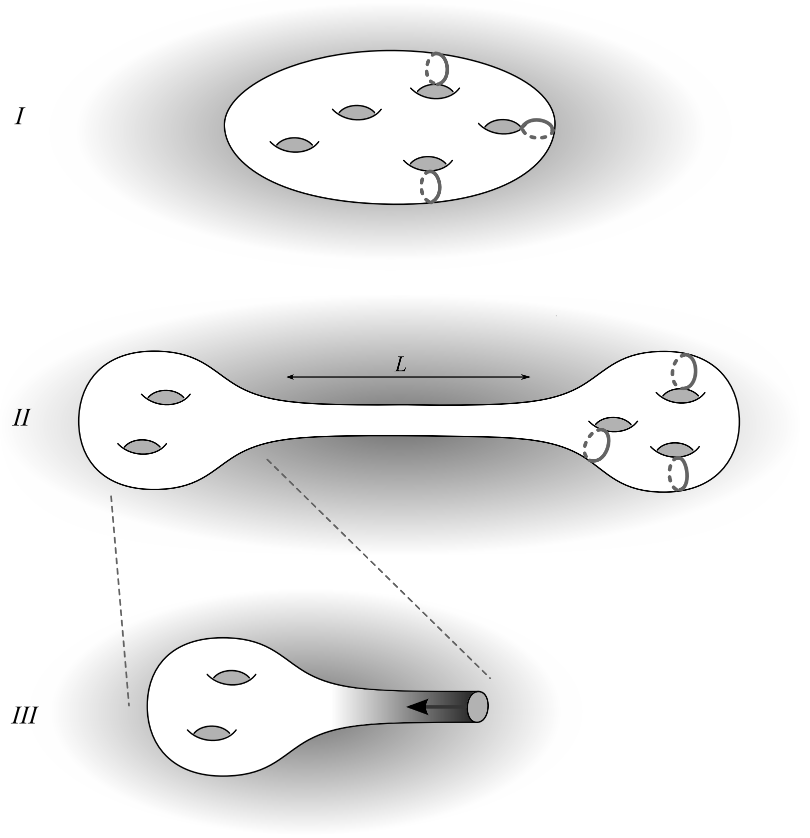

Figure 1: Idea of the paper: In I we start with a generic Calabi-Yau with flux piercing through some of its 3-cycles, while making the distance between the cycles with and without flux very large in II. This is seen as flux from infinity in the left sector without compact flux in III, and generates an OOP-like potential in that sector.

on all cycles except for the isolated set. While the flux that we have turned on is not piercing the isolated cycles, it does leak into their region[2] and it produces a potential for their complex structure moduli. In the limit where the distance between the two sets of cycles of the Calabi-Yau becomes very large, which we will refer to as the *factorization limit*, the flux leaking towards the isolated set will start to look like the flux coming from infinity, mentioned in the previous paragraph. In this way we manage to embed the scenario of the previous paragraph into a well defined system. While this factorization idea should work even in the case where the total Calabi-Yau is a compact manifold divided into two parts,[3] in this paper we will only analyze it in the local case, as it is technically easier.

As a check of this, we consider the example of a local Calabi-Yau based on a hyperelliptic Riemann surface. In this case the factorization can be studied more explicitly. Matrix model techniques can be used to compute the prepotential in the factorization limit. Our

[2]This means that the 3-form field strength is nonzero in the region around the isolated set of 3-cycles, but once integrated over one of these 3-cycles the integral is zero.

[3]Because of no-go theorems [30 – 32], in such compact setups one will need extra ingredients such as O-planes, which we do not consider in the present paper.

results verify the general intuition of the last paragraph.

The plan of this paper is as follows. In section 2, we review some general aspects of IIB flux compactifications and the potentials that can be generated by noncompact fluxes in the local limit. We also discuss the general mechanism by which such fluxes may be used to stabilize complex structure moduli in metastable supersymmetry-breaking configurations. After this, we turn in section 3 to the study of local Calabi-Yau geometries based on Riemann surfaces, providing a more detailed description of the generation of metastable vacua in this context and providing an explicit example. Section 4 then addresses the second point of this paper, namely the ability to obtain our local geometries with noncompact fluxes from larger Calabi-Yau with compact ones by taking a factorization limit. Section 5 supplements this general discussion by providing an explicit demonstration, using matrix model techniques, of the factorization limit in the class of local geometries studied in section 3. Finally, we finish in section 6 with some concluding remarks concerning the generalization of our story to other $\mathcal{N}=2$ contexts, such as M and F-theory compactifications. Some supplementary material and technical details are contained in four appendices. In appendix D, in particular, we study prepotential for the Cachazo-Intriligator-Vafa/Dijkgraaf-Vafa geometry using matrix model as a step towards confirming the above scenario of realizing metastable vacua by factorization. We compute the prepotential for an arbitrary number of cuts up to fifth order in glueball field.

As this paper was being prepared for publication, a paper appeared [33] where a similar system was studied as an example from a different perspective.

2. IIB compactifications with flux at infinity

2.1 Compact Calabi-Yau

Compactification of IIB on a Calabi-Yau threefold $\mathcal{M}$ leads to an $\mathcal{N}=2$ supergravity theory in 4d. The number of vector multiplets is $h^{2,1}$, their scalar components correspond to the complex structure moduli of $\mathcal{M}$. We also have $h^{1,1}+1$ hypermultiplets, whose scalars correspond to the Kähler moduli of $\mathcal{M}$ and the axion-dilaton. The two sets of multiplets are decoupled, and in the rest of the paper we will concentrate on the dynamics of the vector multiplets.

A Calabi-Yau threefold has a nowhere vanishing holomorphic $(3,0)$ form Ω which is unique up to scale. Consider a symplectic basis of 3-cycles $\{\mathcal{A}^I,\mathcal{B}_J\}$ with $I,J=0,1,\ldots,h^{2,1}$. We define the periods of Ω as

$$X^I=\int_{\mathcal{A}^I}\Omega,\qquad F_I=\int_{\mathcal{B}_I}\Omega. \tag{2.1}$$

The $\mathcal{A}$-periods X^I are projective coordinates on the complex structure moduli space of $\mathcal{M}$, and the F_I are functions of X^I. The metric on the complex structure moduli space is *special Kähler* and the Kähler potential is given by

$$K=-\log\left(i\int\Omega\wedge\overline{\Omega}\right). \tag{2.2}$$

This is an exact result which does not receive any α' or g_s corrections.

The easiest way to lift the phenomenologically unrealistic moduli space of such compactifications is to turn on fluxes through the compact cycles of the Calabi-Yau. In the case of IIB we can turn on RR and NS-NS 3-form flux F_3 and H_3 through the 3-cycles of the threefold. This generates a superpotential for the complex structure moduli [27 − 29] given by

$$W = \int G_3 \wedge \Omega, \tag{2.3}$$

where $G_3 = F_3 - \tau H_3$ and $\tau = C_0 + i/g_s$. The scalar potential is computed by the standard $\mathcal{N} = 1$ supergravity expression[4]

$$V = e^{\widetilde{K}} \left(G^{a\bar{b}} D_a W \overline{D_b W} - 3|W|^2 \right), \tag{2.4}$$

where $G_{a\bar{b}}$ is the metric on the moduli space derived from the Kähler potential $\widetilde{K}$, and where we have introduced the Kähler covariant derivative $D_a W = \partial_a W + (\partial_a \widetilde{K})W$.

The F_3 and H_3 fluxes generate charge for the F_5 form via a Chern-Simons coupling in the 10d IIB supergravity action. The F_5 flux has nowhere to end, so we are lead to the tadpole cancellation condition for IIB compactifications

$$\frac{1}{l_s^4} \int F_3 \wedge H_3 + Q_{D3} = 0, \tag{2.5}$$

where Q_{D3} receives positive contribution from probe D3 branes and negative contribution from induced charge on D7 and orientifold planes.

2.2 Local limit

It is well known that there are points on the complex structure moduli space of Calabi-Yau manifolds where the manifold develops a singularity [34]. The simplest example is the conifold singularity, where we have a 3-cycle whose size goes to zero. More generally, a more complicated set of cycles may become very small in some region of the moduli space. As we approach this region, the local dynamics of the singularity decouples from the rest of the fields. What this means is that in 4d the typical energy scale for the dynamics of the fields corresponding to the singularity becomes much smaller than any other scale, in particular much smaller than the Planck mass M_p in 4 dimensions. In this sense, the dynamics of the singularity is decoupled from gravity. Moreover to study the relevant dynamics, we can zoom in close to the singularity and forget about the rest of the Calabi-Yau. In this limit the Calabi-Yau looks noncompact, and it becomes technically easier to study the low energy dynamics.

A typical example of such a local Calabi-Yau is a complex manifold of the form of a hypersurface in $\mathbb{C}^4$

$$\mathcal{M}: \quad uv - F(x, y) = 0, \tag{2.6}$$

[4]In this expression the indices a, b run over complex structure moduli, Kähler moduli and the axion-dilaton. We denote by $\widetilde{K}$ the total Kähler potential for all moduli and by K, as in (2.2), the one for the complex structure moduli alone.

where $F(x,y)$ is a polynomial. In this case the holomorphic 3-form is

$$\Omega = \frac{du}{u} \wedge dx \wedge dy. \tag{2.7}$$

By taking the local limit to go from a compact Calabi-Yau to a noncompact one, the structure of special geometry described above reduces to what is called *rigid* special geometry [35, 36], which is relevant for the low energy dynamics of $\mathcal{N} = 2$ gauge theories. In this case the Kähler potential reduces to

$$K = i \int \Omega \wedge \overline{\Omega}, \tag{2.8}$$

and the Kähler covariant derivative D_i reduces to the ordinary derivative ∂_i.

An important point is the distinction between normalizable and non-normalizable complex structure moduli in the case of noncompact Calabi-Yau manifolds. To be more precise let us consider the example (2.6). The coefficients $\{t^i\}$ of the polynomial $F(x,y)$ characterize the complex structure of $\mathcal{M}$, so $\{t^i\}$ are the complex structure moduli of $\mathcal{M}$. However not all of them are dynamical. Some of them control the complex structure of 3-cycles which are localized in the "interior" of the singularity and are dynamical, while others describe how the singularity is embedded in the bigger Calabi-Yau and become frozen when we take the decoupling limit.

To determine if a specific complex structure modulus t is dynamical or not, one has to compute the corresponding Kähler metric

$$g_{t\bar{t}} = \partial_t \overline{\partial_t} K = i \partial_t \overline{\partial_t} \int \Omega \wedge \overline{\Omega}. \tag{2.9}$$

If this expression is finite, then the modulus t is dynamical, otherwise it is decoupled and should be treated as a parameter of the theory. We will refer to the first set of moduli as *normalizable* and to the second as *non-normalizable*.

2.3 Adding flux

As in the compact case, the addition of fluxes to the local Calabi-Yau introduces a superpotential for the moduli. The dynamics of the Kähler moduli and the dilaton decouple, and we can concentrate on the normalizable complex structure moduli. The superpotential is still given by (2.3), but now the scalar potential is computed by the rigid $\mathcal{N} = 2$ expression

$$V = G^{i\bar{j}} \partial_i W \overline{\partial_j W} = \int G_3 \wedge * \overline{G_3}. \tag{2.10}$$

Since we are in a noncompact Calabi-Yau it is not necessary to impose the tadpole cancellation condition. Instead, the quantity

$$\int F_3 \wedge H_3 \tag{2.11}$$

represents the F_5 flux going off to infinity and remains constant as we vary the moduli. We will use this to simplify the potential in the next section.

In most treatments of fluxes in noncompact Calabi-Yau manifolds the assumption is made that the flux is threading the compact cycles of the singularity and is going to zero at infinity. As we explained in the introduction the goal of our paper is to study the dynamics in the case where the flux is actually coming in from infinity and is not supported on the compact three-cycles. Of course, in a local singularity inside a bigger compact Calabi-Yau, what is meant by infinity is the rest of the Calabi-Yau and we should think of flux coming from infinity as flux leaking towards the singularity from the other compact cycles.

More precisely, in a noncompact Calabi-Yau we consider the vector space $H^3(\mathcal{M})$ of harmonic 3-forms which do not necessarily have compact support, so they can grow at infinity. The harmonic 3-forms of compact support form a linear subspace $H^3_{\mathrm{cpct}}(\mathcal{M}) \subset H^3(\mathcal{M})$. There is a natural way to define the complement subspace $H^3_\infty(\mathcal{M}) \subset H^3(\mathcal{M})$ as the harmonic forms with vanishing integrals on the compact 3-cycles.[5] Then we have the decomposition

$$H^3(\mathcal{M}) = H^3_\infty(\mathcal{M}) \oplus H^3_{\mathrm{cpct}}(\mathcal{M}). \tag{2.12}$$

We will also refer to the forms in $H^3_{\mathrm{cpct}}(\mathcal{M})$ as harmonic 3-forms with compact support and to those in $H^3_\infty(\mathcal{M})$ as 3-forms with support at infinity.

Now we want to consider the case where the 3-form field strength that we have turned on has support at infinity

$$G_3 \in H^3_\infty(\mathcal{M}), \tag{2.13}$$

which means that G_3 has zero flux through the compact cycles

$$\int_{\mathcal{A}^i} G_3 = \int_{\mathcal{B}_i} G_3 = 0. \tag{2.14}$$

The intuitive picture that one should keep in mind, is that this flux at infinity represents usual flux piercing other 3-cycles which are very far away from the singularity in the big Calabi-Yau. As we will see in more detail in the next section, in this case and if one zooms into the local singularity it is a good approximation to treat the flux from the distant 3-cycles as flux which "diverges" at infinity. In other words both $H^3_\infty(\mathcal{M})$ and $H^3_{\mathrm{cpct}}(\mathcal{M})$ correspond to the usual $H^3_{\mathrm{cpct}}(\widetilde{\mathcal{M}})$ of the bigger Calabi-Yau $\widetilde{\mathcal{M}}$ in which the singularity $\mathcal{M}$ develops.

What is maybe more surprising is that the 3-form flux G_3 with support at infinity generates a potential for the complex structure moduli of the singularity $\mathcal{M}$, even though it is not directly piercing the compact cycles of $\mathcal{M}$, as can be seen from (2.14). Our starting point for the computation of this potential is the energy stored in the 3-form field

$$\widetilde{V} = \int G_3 \wedge *\overline{G_3}. \tag{2.15}$$

[5]We should clarify that we are not interested in the most general harmonic 3-form with noncompact support, but only in a restricted subset characterized by 3-forms which grow in a "controlled" way at infinity. This means that we want to consider forms which have at most a "pole" of finite order at infinity, and not essential singularities. This statement has a nice interpretation in the example where we have a local Calabi-Yau based on a Riemann surface that we will study later. Another way to state this restriction is that we will consider harmonic 3-forms on a local Calabi-Yau which do have a lift to the original Calabi-Yau that we started with before we took the local limit near its singularity.

Since G_3 has noncompact support, this is a divergent integral meaning that the energy of the flux is infinite. This was to be expected and is not really a problem, since we are interested in the *changes* of this energy as we vary the sizes of the 3-cycles in the neighborhood of the singularity. We would like to throw away the divergent, moduli independent piece of this quantity and keep the finite, moduli dependent one. A nice way to achieve this is to use the fact that the net F_5 form flux leaking off at infinity, being a topological quantity, has to be kept constant as we vary the moduli. It is easy to show that we can write

$$\int G_3 \wedge \overline{G_3} = (\tau - \overline{\tau}) \int F_3 \wedge H_3, \tag{2.16}$$

and the left hand side must be constant for the reason we explained. Since it is a constant we can subtract it from the potential and define

$$V \equiv \int G_3 \wedge *\overline{G_3} - \int G_3 \wedge \overline{G_3}. \tag{2.17}$$

It is easy to show that this is equal to

$$V = \int G_3^- \wedge *\overline{G_3^-}, \tag{2.18}$$

where G_3^- is the imaginary anti-self dual part of the G_3 flux

$$*G_3^- = -iG_3^-. \tag{2.19}$$

The expression (2.18) is the finite and moduli dependent piece of the potential (2.15).

2.4 Simplifying the potential

In this section we simplify the expression (2.18) for the potential. In general we have the following relation between the Hodge decomposition and the $*$ operator on a threefold

$$\begin{aligned} * H^{3,0} &= -iH^{3,0}, \qquad & *H^{1,2} &= -iH^{1,2}, \\ * H^{2,1} &= iH^{2,1}, \qquad & *H^{0,3} &= iH^{0,3}. \end{aligned} \tag{2.20}$$

Before we proceed we would like to analyze the relation between the decomposition (2.12) and the Hodge decomposition. In general we have the following decomposition[6]

$$H^3(\mathcal{M}) = H^{3,0}_\infty \oplus H^{3,0}_{\rm cpct} \oplus H^{2,1}_\infty \oplus H^{2,1}_{\rm cpct} \oplus \{c.c.\}. \tag{2.21}$$

Harmonic forms in $H^{p,q}_{\rm cpct}$ have compact support, while those in $H^{p,q}_\infty$ do not, and are chosen to have vanishing $\mathcal{A}$-periods on the compact cycles.[7] Since we do not want to break supersymmetry explicitly by the boundary conditions of the system, we want our configuration to be supersymmetric at infinity, which means that the flux at infinity has to be imaginary self dual so

$$G_3 \in H^{2,1}_\infty \oplus H^{2,1}_{\rm cpct} \oplus H^{1,2}_{\rm cpct}. \tag{2.22}$$

[6]Again, we are only considering a certain subset of all harmonic 3-forms with noncompact support, as explained in footnote 5.

[7]A harmonic (p,q)-form cannot have vanishing periods on all compact cycles unless it is identically zero.

where the subscript ∞ means that we have to consider the elements of the cohomology with noncompact support. We pick a basis

$$\Xi_m \in H^{2,1}_{\infty}, \qquad \Omega_i \in H^{2,1}_{\mathrm{cpct}} \tag{2.23}$$

with the following periods

$$\begin{aligned} \int_{\mathcal{A}^i} \Xi_m &= 0, & \int_{\mathcal{A}^i} \Omega_j &= \delta^i_j, \\ \int_{\mathcal{B}_i} \Xi_m &= K_{im}, & \int_{\mathcal{B}_i} \Omega_j &= \tau_{ij}, \end{aligned} \tag{2.24}$$

where τ_{ij} is the period matrix of the Calabi-Yau, and K_{im} are holomorphic functions of the normalizable-complex structure moduli.

The flux has an expansion of the form

$$G_3 = T^m \Xi_m + h^i \Omega_i + \overline{l^i}\,\overline{\Omega_i}. \tag{2.25}$$

The parameters T^m are fixed by the boundary conditions and have to be kept constant as we vary the normalizable moduli. We have also assumed that

$$\int_{\mathcal{A}^i} G_3 = \int_{\mathcal{B}_i} G_3 = 0. \tag{2.26}$$

which means

$$\begin{aligned} T^m \int_{\mathcal{A}^i} \Xi_m + h^j \int_{\mathcal{A}^i} \Omega_j + \overline{l^j} \int_{\mathcal{A}^i} \overline{\Omega_j} &= 0 \\ T^m \int_{\mathcal{B}_i} \Xi_m + h^j \int_{\mathcal{B}_i} \Omega_j + \overline{l^j} \int_{\mathcal{B}_i} \overline{\Omega_j} &= 0. \end{aligned} \tag{2.27}$$

The first equation of (2.27) implies that

$$\overline{l^j} = -h^j. \tag{2.28}$$

and the second

$$h^i = -\frac{1}{2i}\left(\frac{1}{\mathrm{Im}\,\tau}\right)^{ij}\left(K_{jm}T^m\right). \tag{2.29}$$

As we explained before, only the imaginary anti-self dual part of the flux $G_3^- = \overline{l^i}\,\overline{\Omega_i}$ contributes to the regularized potential and we have

$$\begin{aligned} V &= \int G_3^- \wedge \overline{G_3^-} \\ &= \frac{1}{4}\,\overline{(K_{im}T^m)}\left(\frac{1}{\mathrm{Im}\,\tau}\right)^{ij}\left(K_{jn}T^n\right). \end{aligned} \tag{2.30}$$

In this final expression the period matrix τ^{ij} and K_{im} are functions of the normalizable complex structure moduli, while T^m's have to be considered as constants which play the role of external parameters. This potential is in general very complicated and can have local nonsupersymmetric minima for appropriate choices of the parameters T^m as we will explain later.[8]

2.5 Properties of the potential

The potential (2.30) should look somewhat familiar as it shares the same basic structure as the scalar potential that arises when one adds a small superpotential to Seiberg-Witten theory. This connection can be made even more transparent by noting that K_{im} can in general be written as a total derivative with respect to the special coordinates X^i[9]

$$K_{im} = \frac{\partial}{\partial X^i}\kappa_m(X^j), \qquad\qquad X^i = \oint_{\mathcal{A}^i} \Omega. \tag{2.31}$$

With this notation, (2.30) takes the standard form

$$V = \frac{1}{4}\overline{\left(\frac{\partial W_{\text{eff}}(X^k)}{\partial X^i}\right)} \left(\frac{1}{\operatorname{Im}\tau}\right)^{ij} \left(\frac{\partial W_{\text{eff}}(X^k)}{\partial X^j}\right), \tag{2.32}$$

where

$$W_{\text{eff}}(X^k) = T^m \kappa_m(X^k) \tag{2.33}$$

is in fact proportional to the Gukov-Vafa-Witten superpotential induced by the flux G_3.

2.5.1 OOP mechanism

Equation (2.32) makes manifest the relation between our flux-induced potential (2.30) and that which arises in deformed Seiberg-Witten theory and allows us to utilize the technology developed by Ooguri, Ookouchi, and Park [24] in that context[10] for engineering supersymmetry-breaking vacua. In particular, if we want to realize a nonsupersymmetric minimum at some point $X^{i\,(0)}$ in the moduli space, the OOP procedure tells us to first construct Kähler normal coordinates [38–40], around $X^{i\,(0)}$

$$z^i = \Delta X^i + \tilde{g}^{i\bar{j}} \sum_{n=2}^{\infty} \frac{1}{n!} \partial_{i_3} \ldots \partial_{i_n} \tilde{\Gamma}_{j i_1 i_2} \Delta X^{i_1} \Delta X^{i_2} \ldots \Delta X^{i_n}, \tag{2.34}$$

where $\Delta X^i = X^i - X^{i\,(0)}$ and $\tilde{}$ means evaluation at $X = X^{(0)}$. We then build the potential V in (2.32) from a superpotential W_{eff} consisting of a linear combination of the z^i

$$W_{\text{eff}} = k_i z^i, \qquad k_i\text{: constant}. \tag{2.35}$$

[8] Although we do not discuss this in the present paper, from the viewpoint of flux compactification it is a natural generalization to consider fluxes through the compact 3-cycles, relaxing the condition (2.26). Such flux will make additional contribution to the superpotential of the form $N^i F_i - \alpha_i X^i$, $\alpha_i = \int_{\mathcal{B}_i} \Omega$, which cannot be controlled by external parameters and makes realization of OOP-like vacua more difficult.

[9] One quick way to see this is to use the identity $\int_{\mathcal{M}} \Xi_m \wedge \partial_i \overline{\Omega} = 0$ to derive $K_{im} \sim \int_{\partial\mathcal{M}} \Lambda_m \wedge \partial_i \overline{\Omega}$ for a 2-form Λ_m satisfying $d\Lambda_m = \Xi_m$ on the boundary (at infinity) of $\mathcal{M}$. Because the divergent contributions to Λ_m at infinity can be chosen independent of the dynamical moduli, we can pull the derivative outside of everything.

[10] See also related work by Pastras [37].

Stability can then be demonstrated by expanding V near p

$$V = k_i \bar{k}_{\bar{j}} \tilde{g}^{i\bar{j}} + k_i \bar{k}_{\bar{j}} \tilde{R}^{i\bar{j}}{}_{k\bar{l}} z^k \bar{z}^{\bar{l}} + \mathcal{O}(z^3). \tag{2.36}$$

The curvature of special Kähler manifolds, of which the complex structure moduli space is an example, is positive definite at generic points. As a result, any potential of the form (2.32) that agrees with (2.35) near $X^{i\,(0)}$ to cubic order will engineer a nontrivial vacuum at $X^{i\,(0)}$.[11]

One can obtain a nice physical picture for this mechanism by noting, as in [41], that the series (2.34) can be summed exactly and inserted into (2.35) to yield

$$W_{\text{eff}} \sim e_i X^i + m^i F_i, \tag{2.37}$$

where e_i and m^j satisfy

$$e_i + m^j \overline{\tilde{\tau}}_{ji} = 0. \tag{2.38}$$

From this, we see that the superpotential (2.37) built from Kähler normal coordinates is of precisely the form that we would have obtained had we instead simply turned on compactly-supported fluxes m^i and e_i threading the cycles $\mathcal{A}^i$ and $\mathcal{B}_i$, respectively. The condition (2.38), however, combined with the requirement that $\operatorname{Im}\tau$ be positive definite, implies that e_i and m^i can never satisfy the condition $e_i + m^j \tilde{\tau}_{ji} = 0$ that is required for preservation of the manifest $\mathcal{N} = 1$ supersymmetry.

It is well-known that the flux-induced superpotential (2.37) only breaks the full $\mathcal{N} = 2$ supersymmetry spontaneously, though, so there is a second $\mathcal{N} = 1$ in the game that is not manifest in this formalism. The relation (2.38) is, in fact, nothing other than the condition that the vacuum at $X^{i\,(0)}$ preserves precisely these non-manifest supersymmetries [42, 41]. As such, the vacuum at $X^{i\,(0)}$ in the presence of the superpotential (2.35) is stable for a good reason — it is secretly supersymmetric!

In general, our noncompactly supported fluxes will not generate potentials with W_{eff} exactly equivalent to (2.35). Rather, the W_{eff}'s that arise are globally well-defined functions on the moduli space[12] which we can then tune to agree with (2.35) to cubic order within a neighborhood of the point $X^{i\,(0)}$. The delicate manner by which the superpotential (2.35) managed to realize a non-manifest $\mathcal{N} = 2$ supersymmetry is crucially dependent on the full infinite series expansion about $X^{i\,(0)}$ so, by failing to exactly reproduce (2.35), we are able to explicitly break, at the level of the Lagrangian, the half of supersymmetry which would otherwise have been preserved by the vacuum at $X^{i\,(0)}$. Stability of the $X^{i\,(0)}$ vacuum, on the other hand, depends only on the local behavior of W_{eff} so our procedure will retain this property, leaving us with a locally stable supersymmetry-breaking vacuum.

In the end, what we are doing to engineer a supersymmetry-breaking vacuum at $X^{i\,(0)}$ is actually a quite intuitive procedure. We first turn on a collection of noncompactly supported fluxes which explicitly break half of the $\mathcal{N} = 2$ supersymmetry. We then tune

[11] For non-generic $X^{(0)}$, the curvature may have a zero eigenvalue in which case higher order agreement with (2.35) is required (that $X^{i\,(0)}$ is a stable for superpotential exactly equivalent to (2.35) will follow from the discussion below).

[12] Contrast this with (2.37), which manifestly suffer from monodromies for constant (non-transforming) k_i.

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these fluxes so that, near $X^{i\,(0)}$, their interactions with the dynamical complex structure moduli mimic those of the compactly supported fluxes that would generate a vacuum at $X^{i\,(0)}$ which preserves the opposite half of supersymmetries.

2.5.2 Supersymmetric vacua

In addition to possessing supersymmetry-breaking vacua when the T^m are suitably tuned, the potential (2.32) also typically contains a wide variety of supersymmetric vacua. As discussed in [24], these vacua fall into two different classes. First, because there is no flux directly threading the compact cycles, the energy cost associated with shrinking them is necessarily finite. Because the period matrix τ_{ij} diverges, the potential vanishes at these singular points and we obtain stable vacua which are in fact supersymmetric.

This argument is of course rather crude because we are neglecting the new light degrees of freedom that enter as 3-cycles degenerate but, as is well-known, this is easily fixed. In particular, the light D3 branes which wrap the degenerating cycles give rise to hypermultiplets [43, 44] comprised of pairs of $\mathcal{N}=1$ chiral superfields Q_i and $\tilde{Q}_i$ with bilinear superpotential couplings to special coordinates. For the simple case of degenerating $\mathcal{A}^i$ cycles, the superpotential takes the schematic form

$$W = W_{\text{eff}}(X^i) + (Q\tilde{Q})_i X^i, \tag{2.39}$$

and allows a supersymmetric vacuum at $X^i = 0$ through condensation of $Q\tilde{Q}$

$$(Q\tilde{Q})_i = -\frac{\partial W_{\text{eff}}}{\partial X^i}\left(X^j = 0\right). \tag{2.40}$$

The second class of supersymmetric vacua correspond to solutions of the F-term equations

$$\partial_i W_{\text{eff}}(X^j) = 0. \tag{2.41}$$

In general, there may be many solutions to these equations, as we will explore later in the example of section 3.2.

2.5.3 Lifetime of supersymmetry-breaking vacua

Because we have managed to achieve supersymmetry-breaking vacua while freezing all non-normalizable moduli, the energies V_0 will in general be finite and independent of the cutoff scale Λ_0 that we use to regulate the local geometry. This means that our vacua are truly metastable, even within this local model, and can decay to any of the supersymmetric vacua that exist in these models. Because the number the supersymmetric vacua is potentially large and their properties quite model-dependent, it is difficult to make general statements about the lifetime of our OOP vacua. Nevertheless, we recall here one observation from [24], namely that the decay rates will in general scale like

$$e^{-S}, \qquad S \sim \frac{(\Delta z)^4}{V_+}, \tag{2.42}$$

where Δz is the distance in field space between the initial and final vacuum state and V_+ is the difference in their energies. By simultaneously scaling all T^m by a common factor,

$T^m \to \epsilon T^m$, we can retain our supersymmetry-breaking vacua while decreasing V_+ by the same factor, $V_+ \to \epsilon V_+$. In this manner, we see that, just as with OOP vacua in deformed Seiberg-Witten theory, these OOP flux vacua can be made arbitrarily long-lived.[13]

3. Metastable flux vacua in local Calabi-Yau

In the previous section, we saw that, starting from a compact Calabi-Yau and taking a decoupling limit, one ends up with a local Calabi-Yau with noncompact flux with support at infinity, which is nothing but the flux leaking from the rest of the full Calabi-Yau that have been decoupled, towards "our" local Calabi-Yau. Furthermore, this noncompact flux induces potential (2.30) for the complex structure moduli in the local Calabi-Yau. Depending on the noncompact flux, this potential can be very complicated and create nonsupersymmetric metastable vacua in the local Calabi-Yau; the OOP mechanism [24] reviewed in 2.5.1 tells us exactly how this can be done. In this section, we will take specific examples of local Calabi-Yau and demonstrate that one can generate such OOP vacua as IIB flux geometries.

In subsection 3.1, we review constructions of typical local Calabi-Yau geometries, taking Seiberg-Witten and Dijkgraaf-Vafa geometries as examples. The focus will be on the form of the potential for the moduli which is induced by flux at infinity. We also make remarks on the gauge theory interpretation of the physics of these geometries. In subsection 3.2, we proceed to an explicit construction of metastable flux vacua in a Dijkgraaf-Vafa geometry, by tuning superpotential appropriately. In subsection 3.3, we estimate how much control of flux at infinity is required to create OOP vacua.

3.1 Local Calabi-Yau based on Riemann surface

A large group of examples of noncompact Calabi-Yau manifold in IIB is defined by an equation of the form

$$uv - F(x, y) = 0, \tag{3.1}$$

where x, y can both be variables in $\mathbb{C}$ or $\mathbb{C}^*$. Compactifying on such a Calabi-Yau leaves $\mathcal{N} = 2$ supersymmetry unbroken in four dimensions. An important role in these Calabi-Yau's is played by the underlying one-dimensional complex curve in the x, y-plane defined by $F(x, y) = 0$ [25, 45]. In most of our examples this curve is smooth, and we will refer to it as the Riemann surface Σ. The total Calabi-Yau space will be named $\mathcal{M}_\Sigma$. The holomorphic 3-form of $\mathcal{M}_\Sigma$ is given, e.g. for $x, y \in \mathbb{C}$, by

$$\Omega = \frac{du \wedge dx \wedge dy}{\partial F/\partial v} = \frac{du}{u} \wedge dx \wedge dy. \tag{3.2}$$

[13] Because we should really think of the local Calabi-Yau as sitting inside some larger compact geometry, one important caveat to this statement of longevity is that the noncompact fluxes T^m in reality derive from a suitable set of compact fluxes in the full Calabi-Yau. This means that there will be a series of quantization conditions that must be imposed that may affect the degree to which they may be tuned.

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Notice that the total threefold can be described as a local (or decompactified) elliptic fibration over the x, y-plane. Over generic points in the base x, y-space, its fiber is described by a hyperboloid satisfying the equation $uv = \mu$ where μ is nonzero, which may be viewed as a decompactified compact torus, making its B-cycle very large. On the other hand, when $(x, y) \in \Sigma$, the noncompact fiber degenerates into a cone $uv = 0$, which one obtains by decompactifying a pinched torus, corresponding to an A_0 geometry.

Many important properties of the noncompact Calabi-Yau threefold $\mathcal{M}_\Sigma$ have an interpretation in terms of the underlying Riemann surface Σ. For example, the compact 3-cycles $\{\mathcal{A}^i, \mathcal{B}_j\}$ in $\mathcal{M}_\Sigma$ are lifts of compact 1-cycles on Σ, which we denote by $\{A^i, B_j\}$. If $(x, y) \in \mathbb{C}^2$, these 3-cycles may be constructed by filling in a disk D in $\mathbb{C}^2$ whose boundary ∂D is the 1-cycle on Σ. Now consider an S^1-fibration over D where S^1 is the compact circle in the uv-fiber. Since this circle shrinks over Σ, the total 3-cycle has the topology of an S^3. If one of the variables x or y is $\mathbb{C}^*$-valued, the disk D will be punctured. In such a situation differences of 1-cycles have to be considered. We will see an example of this shortly. Notice that the one-to-one correspondence between 3- and 1-cycles shows an equivalence between the complex structure moduli on $\mathcal{M}_\Sigma$ and Σ.

A basis of (2,1)-forms with compact support on $\mathcal{M}_\Sigma$ is given by derivatives of Ω with respect to the normalizable complex structure moduli: $\{\Omega_i = \partial_i \Omega\}$. If $\mathcal{M}_\Sigma$ were compact, these derivatives ∂_i would be Kähler covariant derivatives D_i on the moduli space. Being noncompact instead, the moduli space is described by rigid special geometry and, as we saw before, the covariant derivatives simplify into partial derivatives. Another reduction over the compact 3-cycles in the Calabi-Yau shows that all these compactly supported $(2,1)$-forms Ω_i reduce to a basis of holomorphic 1-forms ω_i on Σ. Similarly, $(1,2)$-forms $\overline{\partial_i \Omega}$ in $\mathcal{M}_\Sigma$ reduce to antiholomorphic forms $\overline{\omega}_i$ on Σ. ω_i satisfy the following relations, which are reductions of (2.24):

$$\frac{1}{2\pi i} \int_{A^i} \omega_j = \delta^i_j, \qquad \frac{1}{2\pi i} \int_{B_i} \omega_j = \tau_{ij}. \tag{3.3}$$

The relation between the 3-cycles/3-forms on $\mathcal{M}_\Sigma$ and the 1-cycles/1-forms on Σ through the trivial uv-fibration being understood, we can rewrite the various relations in section 2 in terms of the Riemann surface Σ. First of all, the holomorphic 3-form Ω of $\mathcal{M}_\Sigma$, which is given e.g. for $x, y \in \mathbb{C}$ by (3.2), is easily seen to reduce to a meromorphic 1-form $\lambda = y\, dx$ on the Riemann surface in this case [25, 45]. The special coordinates (2.1) parametrizing complex structure moduli are

$$X^i = \frac{1}{2\pi i} \int_{A^i} \lambda, \qquad F_i = \frac{1}{2\pi i} \int_{B_i} \lambda, \tag{3.4}$$

and the Kähler potential (2.8) is given by

$$K = i \int_\Sigma \lambda \wedge \overline{\lambda}. \tag{3.5}$$

Recall that, in the special coordinates $\{X^i\}$, the moduli space metric takes a particularly simple form:

$$ds^2 = \left(\frac{\partial^2 K}{\partial X^i \partial \overline{X^j}} \right) dX^i d\overline{X^j} = (\operatorname{Im} \tau)_{ij}\, dX^i d\overline{X^j}, \tag{3.6}$$

as can be shown using $\partial_i \lambda = \omega_i$ and the Riemann bilinear relation.

Now we want to consider a very small deformation of the system breaking supersymmetry to $\mathcal{N} = 1$, thus generating a potential V for the moduli. As we saw before, this can be accomplished by turning on 3-form flux G_3 with support at infinity in the local Calabi-Yau. This flux can be thought of as leaking from the other part of the full compact Calabi-Yau, which has been frozen in the decoupling limit. We assume that the decoupling limit was taken consistently with the elliptic fibration structure; namely, we assume that the noncompact flux is supported at the asymptotic infinities of Σ, while being compact in the direction of the uv-fibers.

The basis of (2,1)-forms with noncompact support, $\{\Xi_m\}$, in the Calabi-Yau $\mathcal{M}_\Sigma$ descend to meromorphic 1-forms $\{\xi_m\}$ on the Riemann surface Σ, satisfying the relations

$$\int_{A^i} \xi_m = 0, \qquad \int_{B_i} \xi_m = K_{im}, \tag{3.7}$$

which are reductions of (2.24). Therefore, the 3-form flux with noncompact support, G_3, on $\mathcal{M}_\Sigma$ as given in (2.25) descends to a harmonic 1-form flux

$$\begin{aligned} g &= g_H + \overline{g_A}, \\ g_H &= T^m \xi_m + h^i \omega_i, \qquad \overline{g_A} = \overline{l^i}\,\overline{\omega_i}, \end{aligned} \tag{3.8}$$

which will have poles at the punctures (or asymptotic legs) of Σ. A 3-form flux G_3 in $\mathcal{M}_\Sigma$ induces superpotential (2.3), which reduces to an integral on Σ:

$$W = \int_\Sigma g \wedge \lambda, \tag{3.9}$$

while the associated scalar potential (2.18) reduces to an integral on Σ:

$$V = \int_\Sigma g_A \wedge \overline{g_A}. \tag{3.10}$$

If we require the condition (2.26) that the flux (3.8) is zero through compact 3-cycles of $\mathcal{M}_{\mathrm{SW}}$, which translates into

$$\int_{A^i} g = \int_{B_i} g = 0, \tag{3.11}$$

then by the exactly same argument we did for general Calabi-Yau's in the previous section now reduced to the Riemann surface Σ (or simply by borrowing the result (2.30)), we can rewrite (3.10) in terms of periods on Σ:

$$V = \frac{1}{4}\overline{(K_{im}T^m)} \left(\frac{1}{\mathrm{Im}\,\tau}\right)^{ij} K_{jn}T^n. \tag{3.12}$$

For convenience, the relation between 3- and 1-forms in $\mathcal{M}_\Sigma$ and Σ is summarized in table 1.

Now we will turn to more specific examples of local Calabi-Yau geometries based on Riemann surfaces which have been studied in the context of string theory.

[15] As explained in footnote 5, we do not mean here that Ξ_m's span a complete basis of 3-forms in $\mathcal{M}_\Sigma$ with support at infinity; we are only considering a certain subset of all 3-forms diverging at infinity, which are the lifts of meromorphic 1-forms ξ_m on $\mathcal{M}$.

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	special forms	noncompact flux inducing superpotential W	compact flux entering potential V
$\mathcal{M}_\Sigma$	$\Omega \in H^{3,0}(\mathcal{M}_\Sigma)$	$T^m \Xi_m \in H^{2,1}_\infty(\mathcal{M}_\Sigma)$	$G_3^- = \overline{l^i}\,\overline{\Omega_i} \in H^{1,2}_{\mathrm{cpct}}(\mathcal{M}_\Sigma)$
Σ	$\lambda \in H^{1,0}(\Sigma)$	$T^m \xi_m \in H^{1,0}_\infty(\Sigma)$	$\overline{g_A} = \overline{l^i}\,\overline{\omega_i} \in H^{0,1}_{\mathrm{cpct}}(\Sigma)$

Table 1: The summary of the relation between 3-forms[15] in Calabi-Yau $\mathcal{M}_\Sigma$ and 1-forms on Riemann surface Σ

3.1.1 Seiberg-Witten geometries

An illustrative example of the general Calabi-Yau's (3.1) is given by SU(N) Seiberg-Witten (SW) geometries. In type IIB, these correspond to compactifications on noncompact Calabi-Yau threefold $\mathcal{M}_{\mathrm{SW}}$ defined by

$$\mathcal{M}_{\mathrm{SW}}: \qquad uv - F_{\mathrm{SW}}(x,y) = 0, \qquad x \in \mathbb{C}, \quad y \in \mathbb{C}^*, \tag{3.13}$$

where the underlying Riemann surface Σ_{SW} is a hyperelliptic curve

$$\Sigma_{\mathrm{SW}}: \qquad F_{\mathrm{SW}}(x,y) \equiv \Lambda^N \left(y + \frac{1}{y}\right) - P_N(x) = 0 \tag{3.14}$$

and $P_N(x)$ is a polynomial of degree N with the coefficient of x^{N-1} being zero:

$$P_N(x) = \prod_{i=1}^{N}(x - \alpha_i), \qquad \sum_{i=1}^{N} \alpha_i = 0. \tag{3.15}$$

The coefficients of $P_N(x)$, or equivalently α_i, are normalizable moduli, while Λ is a fixed parameter. The holomorphic 3-form on $\mathcal{M}_{\mathrm{SW}}$ is $\Omega_{\mathrm{SW}} = \frac{du}{u} \wedge dx \wedge \frac{dy}{y}$ and reduces to [25]

$$\lambda_{\mathrm{SW}} = x\frac{dy}{y} \tag{3.16}$$

on the Riemann surface Σ_{SW}.

Type IIB string theory compactified on the Calabi-Yau (3.13) without flux geometrically engineers [46, 25] an $\mathcal{N} = 2$ Seiberg-Witten theory [35, 47]. In particular, the SU(N) Seiberg-Witten curve of gauge theory [48, 49] is geometrically identified with the curve (3.14) underlying the Calabi-Yau. A T-duality along the compact circle in the uv-fiber, followed by a lift to M-theory, translates [50] this geometry into a system of an M5-brane which wraps the Riemann surface Σ_{SW} and fills $\mathbb{R}^{3,1}$. In the IIA limit, this system is related to a Hanany-Witten type brane configuration in type IIA, where one has two NS5-branes with N D4-branes stretching between them [51, 52]. From this last IIA/M-theory point of view, it is easy to see the relation of the system to $\mathcal{N} = 2$ SU(N) super Yang-Mills as the worldvolume theory on the D4-branes. In particular, α_i's correspond to the eigenvalues of the adjoint scalar Φ on the Coulomb branch. In passing we also note that the geometries (3.13) are related to toric geometries in IIA by mirror symmetry [26, 53].

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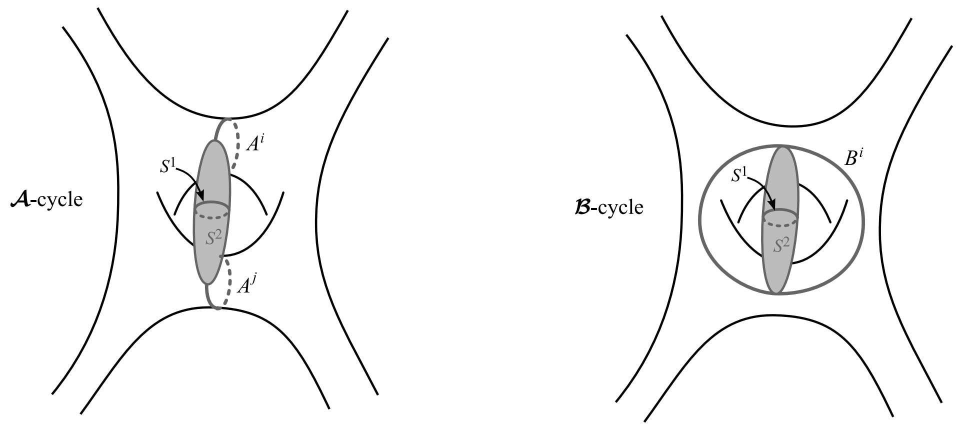

Figure 2: The relation between 3-cycles in the Calabi-Yau $\mathcal{M}_{\mathrm{SW}}$ and 1-cycles on the Riemann surface Σ_{SW} for the Seiberg-Witten geometry. For the $\mathcal{A}$-cycle, by fibering S^1 over the line segment whose endpoints are at a point on A^i and a point on $-A^j$, one obtains S^2. By moving the endpoints over A^i and $-A^j$, one obtains $S^2 \times S^1$. For the $\mathcal{B}$-cycle, similarly moving the S^2 ending on B_i, one obtains S^3.

Now let us look at the homological structure of the Seiberg-Witten geometry (3.13), focusing on the relation between 1-cycles on the hyperelliptic curve Σ_{SW} (3.14) and 3-cycles in the Calabi-Yau $\mathcal{M}_{\mathrm{SW}}$ (3.13). The Riemann surface Σ_{SW} may be compactified by adding two points at infinity. If we represent the curve (3.14) as a two-sheeted x-plane branched over $2N$ points, those infinities correspond to $x = \infty$ on the two sheets. It is thus a hyperelliptic curve of genus $N-1$ with two punctures. Therefore, its first homology $H_1(\Sigma_{\mathrm{SW}})$ is formed by $N-1$ pairs of compact A and B-cycles, (A^i, B_j), $i, j = 1, \ldots, N-1$, with in addition a closed 1-cycle A^∞ around one of the punctures which is dual to an open 1-cycle B_∞ connecting the two points. How can these 1-cycles be lifted to 3-cycles in $\mathcal{M}_{\mathrm{SW}}$? The fact that $y \in \mathbb{C}^*$ means that A-cycles on Σ_{SW} are not contractible on the x, y-plane (recall that we are regarding Σ_{SW} as embedded in the x, y-plane). Instead, compact A-cycles in the noncompact Calabi-Yau threefold will reduce to differences of A-cycles on Σ_{SW}. Indeed, notice that a point on the 1-cycle A^i and one on another 1-cycle $-A^j$, with opposite orientation, are connected by a $\mathbb{P}^1$ in the Calabi-Yau. The resulting 3-cycle therefore has the topology of $S^2 \times S^1$. For the B-cycles this subtlety does not arise, and compact B-cycles in the Calabi-Yau have S^3 topology and reduce to compact 1-cycles connecting the two hyperelliptic planes. See figure 2. This discussion is equivalent to page 10 of [25], and in particular shows the equivalence between 3-cycles on the Calabi-Yau's and 1-cycles on the Seiberg-Witten curve.

As seen in section 3.1, the complex structure moduli space is conveniently parametrized by the special coordinates (3.4), which in the Seiberg-Witten case is conventionally denoted by a^i, $i = 1, \ldots, N-1$:[16]

$$a^i = \frac{1}{2\pi i} \int_{A^i} \lambda_{\mathrm{SW}} = \frac{1}{2\pi i} \int_{A^i} x \frac{dy}{y}. \tag{3.17}$$

[16]Because of the subtlety mentioned above about how to take 1-cycles that lifts to compact 3-cycles in the Calabi-Yau, we should think of the A^i appearing in (3.17) e.g. as $\widetilde{A}^i \equiv A^i - A^N$, where $i = 1, \ldots, N-1$. For simplicity of presentation, we write $\widetilde{A}^i$ as A^i.

As in (3.6), the moduli space metric takes the special form for these:

$$ds^2 = (\mathrm{Im}\,\tau_{ij})\, da^i d\overline{a^j}. \tag{3.18}$$

Using a^i, the normalized basis of holomorphic 1-forms ω_i can be obtained as follows. Differentiating (3.17) with respect to a^j,

$$\delta^i_j = \frac{1}{2\pi i}\frac{\partial}{\partial a^j}\int_{A^i} x\frac{dy}{y}. \tag{3.19}$$

Comparing with the first equation in (3.3), this means that

$$\omega_i = \frac{\partial}{\partial a^i}\left(x\frac{dy}{y} + d\eta\right), \tag{3.20}$$

where the total derivative term $d\eta$ is fixed by requiring that $\omega_i = \mathcal{O}(x^{-2})dx$ as $x \to \infty$. Specifically, this leads to $d\eta = d(-x\log y)$ and ω_i is given by

$$\omega_i = \frac{\partial}{\partial a^i}(-\log y\, dx) = -\frac{\partial P_N(x)/\partial a^i}{\sqrt{P_N(x)^2 - 4\Lambda^{2N}}}dx. \tag{3.21}$$

Although $\log y$ may appear problematic because it is not single-valued on the Riemann surface, its a^i derivative is single-valued and does not cause any problem.

As we discussed in general in section 3.1, turning on noncompact flux breaks $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$ by inducing a superpotential. In the present case where the Riemann surface is hyperelliptic, we can take $\{\xi_m\}$ and $\{\omega_i\}$ to be the specific ones given in appendix A.1. As in (3.8), the 3-form flux in $\mathcal{M}_{\mathrm{SW}}$ reduces to a harmonic 1-form on Σ_{SW}:

$$g = \sum_{m\geq 1} T^m \xi_m + \sum_{i=1}^{N-1} h^i\omega_i + \sum_{i=1}^{N-1}\overline{l^i}\overline{\omega_i}. \tag{3.22}$$

Under the condition that the compact flux vanishes, (3.11), this leads to the scalar potential (3.12).

We can write the superpotential we are adding to the system in a form that will be useful later. By manipulating the quantity $K_{jn}T^n$ appearing in (3.12),

$$\begin{aligned} K_{jn}T^n &= T^n\oint_{B_j}\xi_n = -2T^n\oint_\infty x^n\omega_j = 2T^n\frac{\partial}{\partial a_j}\left(\oint_\infty x^n\log y\,dx\right) \\ &= -\frac{2T^n}{N+1}\frac{\partial}{\partial a_j}\left(\oint_\infty x^{n+1}\frac{dy}{y}\right). \end{aligned} \tag{3.23}$$

Here we used (3.7), (A.14), and (3.21). By examining (3.12) and (3.18), one sees that the superpotential is given by:

$$W_{\mathrm{SW}} = \sum_m T^m u_{m+1}, \tag{3.24}$$

where we defined

$$u_m \equiv \frac{1}{2\pi i m}\oint_\infty x^{m-1}\lambda_{\mathrm{SW}} = \frac{1}{2\pi i m}\oint_\infty x^m\frac{dy}{y}. \tag{3.25}$$

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So far everything was about geometry. Now let us turn to the gauge theory interpretation of these. As we mentioned above, the local Calabi-Yau geometry (3.13) without flux realizes $\mathcal{N} = 2$ Seiberg-Witten theory, with the hyperelliptic curve (3.14) identified with the $\mathcal{N} = 2$ curve of gauge theory. The special coordinates a^i defined in (3.17) correspond to the U(1) adjoint scalars in the IR and parametrize the Coulomb moduli space. The superpotential (3.24) also has a simple gauge theory interpretation. To see it, we need the relation between the vev of the adjoint scalar Φ and the curve $\Sigma_{\rm SW}$, given by [54, 55]:

$$\left\langle \mathrm{tr}\, \frac{dx}{x-\Phi} \right\rangle = \frac{dy}{y} = \frac{P_N'(x)}{\sqrt{P_N(x)^2 - 4\Lambda^{2N}}} dv. \tag{3.26}$$

In other words, u_m defined geometrically in (3.25) has an interpretation in gauge theory as follows:

$$u_m = \frac{1}{m} \langle \mathrm{tr}\, \Phi^m \rangle. \tag{3.27}$$

From this, one immediately sees that the superpotential (3.24) can be written as

$$W_{\rm SW} = \sum_m \frac{T^m}{m+1} \mathrm{tr}\, \Phi^{m+1} = \mathrm{tr}[W(\Phi)], \tag{3.28}$$

where we defined

$$W(x) = \sum_m \frac{T^m}{m+1} x^{m+1}. \tag{3.29}$$

In (3.28), Φ is understood as the chiral superfield whose lowest component is the adjoint scalar.

Therefore, the $\mathcal{N} = 2$ gauge theory perturbed by the single-trace superpotential (3.28) corresponds to the geometry (3.13) with the flux g obeying the following asymptotic boundary condition:

$$g \sim \sum_m m T^m x^{m-1}\, dx = W''(x) dx, \tag{3.30}$$

where we used (A.11). Note that this equivalence holds for any configurations, supersymmetric or nonsupersymmetric, because we have shown the equality of the full off-shell scalar potential. The perturbed $\mathcal{N} = 2$ theory is precisely the system which was shown in [24, 37] to have nonsupersymmetric metastable vacua if the superpotential is chosen appropriately.[17] Therefore, it tautologically follows that the IIB Seiberg-Witten geometry with flux at infinity also has metastable vacua, if we tune the parameters T^m appropriately.

As we mentioned above, this IIB Seiberg-Witten geometry is dual to a IIA brane configuration of NS5-branes and D4-branes which can be lifted to an M5-brane configuration. In [56], it was shown that superpotential perturbation corresponds in the M-theory setup

[17]It was shown in [24] to be possible to create metastable vacua by a single-trace superpotential of the form (3.28) at any point in the Coulomb moduli space for SU(2) and at least at the origin of the moduli space for SU(N).

to "curving" the $\mathcal{N} = 2$ configuration of the M5-brane at infinity in a way specified by the superpotential. The metastable gauge theory configuration of [24, 37] was realized as a metastable M5-brane configuration and its local stability was given a geometrical interpretation. The above proof of (3.28) is exactly in parallel to the one given in [56] for the M-theory system. In passing, it is also worth mentioning that the M-theory analysis of [56] revealed that at strong coupling the nonsupersymmetric configuration "backreacts" on the boundary condition and it is no longer consistent to impose a holomorphic boundary condition specified by a holomorphic superpotential, which is in accord with [17]. Therefore, also in the IIB flux setting, it is expected that if we go beyond the approximation that the flux does not backreact on the background metric, nonsupersymmetric flux configurations will backreact and it will be impossible to impose a holomorphic boundary condition of the type (3.22).

Although we do not do it in the present paper, from the viewpoint of flux compactification, it is a natural generalization to consider fluxes through the compact 3-cycles. Such flux will make additional contribution to the superpotential of the form $e_i a^i + m^i F_i$ (see 2.5.1). On the gauge theory side, in the Seiberg-Witten theory, this can be interpreted as perturbation one adds at the far IR, but its UV interpretation is not clear [41].

3.1.2 Dijkgraaf-Vafa (CIV-DV) geometries

Another example of geometries of the type (3.1) is type IIB on

$$\mathcal{M}_{\rm DV}: \qquad uv - F_{\rm DV}(x,y) = 0, \qquad x, y \in \mathbb{C}, \tag{3.31}$$

where the underlying Riemann surface $\Sigma_{\rm DV}$ is a hyperelliptic curve

$$\Sigma_{\rm DV}: \qquad F_{\rm DV}(x,y) \equiv y^2 - \left[P_n(x)^2 - f_{n-1}(x)\right] = 0 \tag{3.32}$$

and $P_n(x)$ and $f_{n-1}(x)$ are polynomials of degree n and $n-1$, respectively. If we write

$$f_{n-1}(x) = \sum_{i=1}^{n-1} b_i x^i, \tag{3.33}$$

then the coefficients of $P_n(x)$ as well as b_{n-1} are nonnormalizable and fixed,[18] while b_i, $i = 0, \ldots, n-2$ are normalizable complex structure moduli. The holomorphic 3-form is $\Omega_{\rm DV} = \frac{du}{u} \wedge dx \wedge dy$ which reduces to

$$\lambda_{\rm DV} = x\, dy \tag{3.34}$$

on the Riemann surface $\Sigma_{\rm DV}$. The geometry (3.31) was studied by Cachazo, Intriligator and Vafa (CIV) [57] (see also [58]) in the context of large N transition [59, 60] and further generalized in [61, 62]. The Dijkgraaf-Vafa (DV) conjecture [63–65] was also based on the same geometry. We will refer to this geometry as the CIV-DV geometry (3.31) or as the Dijkgraaf-Vafa geometry henceforth.

[18] More precisely, b_{n-1} is log-normalizable and can be treated as a variable modulus if one wishes, but in the present paper we treat it as a non-dynamical parameter.

The structure of the underlying hyperelliptic Riemann surface $\Sigma_{\rm DV}$ (3.32) is similar to the Seiberg-Witten case (3.14); $\Sigma_{\rm DV}$ is a genus $n-1$ surface with two punctures at infinity. If we represent $\Sigma_{\rm DV}$ as a two-sheeted x-plane branched over $2n$ points, those infinities correspond to $x=\infty$ on the two sheets. The coefficients of $P_n(x)$, which are nonnormalizable, determine the position of the n cuts on the x-plane, while the coefficients of $f_{n-1}(x)$, which are normalizable, are related to the sizes of the cuts. The first homology $H_1(\Sigma_{\rm DV})$ is spanned by $n-1$ pairs of compact A- and B-cycles (A^i, B_j), $i,j=1,\dots,n-1$ with in addition a closed cycle A^∞ around one of the infinities which is dual to the noncompact B-cycle B_∞ connecting two infinities. Because $x,y\in\mathbb{C}$, compact A- and B-cycles on $\Sigma_{\rm DV}$ are all contractible in the x,y-plane and hence all compact 1-cycles on $\Sigma_{\rm DV}$ lifts to 3-cycles in $\mathcal{M}_{\rm DV}$ with S^3 topology.

The special coordinates (3.4) in this case is conventionally denoted by S^i, Π_i:

$$S^i=\frac{1}{2\pi i}\int_{A^i}\lambda_{\rm DV}, \qquad \Pi_i=\frac{1}{2\pi i}\int_{B_i}\lambda_{\rm DV}, \tag{3.35}$$

for which, as in (3.6), the moduli space metric takes the special form:

$$ds^2=(\operatorname{Im}\tau_{ij})\,dS^i d\overline{S^j}. \tag{3.36}$$

Just as in (3.21), we can write the basis of holomorphic 1-forms ω_i using S^i as:

$$\omega_i=\frac{\partial}{\partial S^i}(-y\,dx)=\frac{\partial f_{n-1}(x)/\partial S^i}{2\sqrt{P_n(x)^2-f_{n-1}(x)}}\,dx. \tag{3.37}$$

Adding flux at infinity and breaking $\mathcal{N}=2$ supersymmetry to $\mathcal{N}=1$ go just as in the Seiberg-Witten case. The Riemann surface $\Sigma_{\rm DV}$ is hyperelliptic and we take $\{\xi_m\}$ and $\{\omega_i\}$ to be the ones given in appendix A.1. Just like (3.8) and (3.22), the 3-form flux in $\mathcal{M}_{\rm DV}$ reduces to a harmonic 1-form on $\Sigma_{\rm DV}$:

$$g=\sum_{m\geq 1}T^m\xi_m+\sum_{i=1}^{N-1}h^i\omega_i+\sum_{i=1}^{N-1}\overline{l^i}\overline{\omega_i}. \tag{3.38}$$

Under the condition that the compact flux vanishes (eq. (3.11)), the 1-form (3.38) leads to the scalar potential (3.12) which, just as we derived (3.24), can be shown to correspond to the following superpotential:

$$W_{\rm DV}=\sum_m T^m\Sigma_{m+1}, \tag{3.39}$$

where we defined

$$\Sigma_m\equiv\frac{1}{2\pi i m}\oint_\infty x^{m-1}\lambda_{\rm DV}=\frac{1}{2\pi i m}\oint_\infty x^m\,dy. \tag{3.40}$$

The 1-form $\lambda_{\rm DV}$ depends on the complex structure moduli S^i of the Riemann surface (3.32). Therefore, by changing the parameters T^m, we can generate a superpotential which is a quite general function of S^i's. The OOP mechanism [24] states that, if one tunes superpotential appropriately, one can create a metastable vacuum at any point of the $\mathcal{N}=2$

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moduli space. Therefore, also for this Dijkgraaf-Vafa geometry, we expect to be able to create metastable vacua by appropriately tuning T^m, i.e., flux at infinity. Indeed, in the next subsection we will demonstrate the existence of metastable vacua in a simple example.

We have been focusing on the case where there is flux at infinity but there is *no* flux through *compact* cycles. However, let us digress a little while and think about the case where there *is* flux through compact cycles but there is *no* flux at infinity. In this case, the IIB system has a standard interpretation [57, 58, 63–65] as describing the IR dynamics of $\mathcal{N}=2$ SU(N) theory[19] broken to $\mathcal{N}=1$ by a superpotential $W=\mathrm{tr}[W_n(\Phi)]$, $W_n'(x)=P_n(x)$, with the moduli S^i identified with glueball fields. More precisely, if there are N^i units of flux through the cycle A^i, where $N=\sum_i N^i$, then the system corresponds to the supersymmetric ground state of SU(N) gauge theory broken to $\left[\prod_i \mathrm{SU}(N^i)\right]\times \mathrm{U}(1)^{n-1}$. It is important to note that this equivalence between the Dijkgraaf-Vafa flux geometry and gauge theory is guaranteed to work only for holomorphic dynamics, or for the F-term. On the geometry side, one is considering the underlying geometry (3.31) determined by $P_n(x)$ and small flux perturbation on it. On the gauge theory side, this corresponds to the limit of large superpotential, where one has no control of the D-term. Therefore, there is no *a priori* reason to expect that the D-term of the Dijkgraaf-Vafa geometry, which governs e.g. existence of nonsupersymmetric vacua, and that of gauge theory are the same, even qualitatively. After all, two systems are different theories and it is only the holomorphic dynamics that is shared by the two.[20]

Despite such subtlety, it is interesting to ask what is the gauge theory interpretation of adding flux at infinity, in addition to flux through compact cycles. It is known that the curve (3.32) is related to the vev in gauge theory as [54, 63–66]:

$$-\frac{1}{32\pi^2}\left\langle \mathrm{tr}\,\frac{\mathcal{W}^2}{x-\Phi}\right\rangle dx = y\,dx = \sqrt{P_n(x)^2 - f_{n-1}(x)}\,dx. \tag{3.41}$$

where $\mathcal{W}^2=\mathcal{W}_\alpha\mathcal{W}^\alpha$ and $\mathcal{W}_\alpha$ is the gaugino field. Comparing this with (3.40), one finds that the quantity Σ_m defined geometrically in (3.40) has the following interpretation:

$$\Sigma_m = \frac{1}{32\pi^2}\langle \mathrm{tr}(\mathcal{W}^2\Phi^{m-1})\rangle. \tag{3.42}$$

Therefore the superpotential (3.39) can be written as

$$W_{\mathrm{DV}} = \frac{1}{32\pi^2}\sum_m T^m\,\mathrm{tr}[\mathcal{W}^2\Phi^m] = \frac{1}{32\pi^2}\,\mathrm{tr}[\mathcal{W}^2 M(\Phi)], \tag{3.43}$$

where we defined

$$M(x) = \sum_m T^m x^m. \tag{3.44}$$

[19]This is the case when we treat b_{n-1} as non-dynamical. If we regard this as dynamical, this system realizes U(N) theory.

[20]Of course, it is a logical possibility that even the D-terms of the two systems are identical, or closely related to each other.

Therefore, flux at infinity of the following asymptotic form:

$$g \sim \sum_m m T^m x^{m-1}\, dx = M'(x)dx, \tag{3.45}$$

corresponds in gauge theory to adding a novel superpotential of the form (3.43). This type of superpotential was recently studied in the gauge theory context in [67, 68]. Again, this correspondence must be taken with a grain of salt, since it holds only for holomorphic physics.

Note also that flux through compact cycles will induce glueball superpotential [57] of the form $\alpha_i S^i + N^i \Pi_i(S)$ added to (3.39). Because this part does not contain tunable parameters such as T^m that can be made very small, it is difficult, if not possible, to use the OOP mechanism to produce metastable vacua in that case.

Now let us come back to the main focus of the present paper, the case where there is no flux through compact cycles. In this case, we do not have an interpretation of the system as such an SU(N) theory described above, simply because $N = \sum_i N^i = 0$. Below, we take the Dijkgraaf-Vafa geometry with flux at infinity and no flux through compact cycles as an example, and see that we can generate metastable vacua by adjusting the parameters T^m using the OOP mechanism outlined in the previous section.

3.2 Metastable flux vacua in CIV-DV geometries — an example

To demonstrate that one can truly realize supersymmetry-breaking via the OOP mechanism in type IIB Dijkgraaf-Vafa flux geometries, we turn our attention now to a simple example, namely the geometry relevant for SU(2)

$$uv - F_{\rm DV}(x,y) = 0, \qquad \text{with} \quad F_{\rm DV}(x,y) = y^2 - \left[P_2(x)^2 - b_1 x - b_0\right], \tag{3.46}$$

where we choose

$$P_2(x) = x^2 - \frac{\Delta^2}{4}. \tag{3.47}$$

For simplicity, we will impose a $\mathbb{Z}_2$ symmetry on the Calabi-Yau under which $x \leftrightarrow -x$, the effect of which is to set the log-normalizable modulus b_1 to zero

$$b_1 = 0. \tag{3.48}$$

As usual, we can focus our attention on the associated Riemann surface, $\Sigma_{\rm DV}$, which in this example has genus 1 and is determined by the equation

$$F_{\rm DV}(x,y) = y^2 - \left[P_2(x)^2 - b_0\right] = 0. \tag{3.49}$$

This geometry admits a single dynamical modulus, b_0. This, in turn, can locally be traded for the special coordinate S^1 which, for notational simplicity, we refer to as S in the remainder of this section

$$S \equiv S^1 = \frac{1}{2\pi i}\oint_{A^1} x\, dy. \tag{3.50}$$

Alternatively, we can parametrize the moduli space by the globally well-defined coordinate Σ_2 (3.40), which we choose to denote simply by Σ

$$\Sigma \equiv \Sigma_2 = \frac{1}{4\pi i} \oint_{x=\infty} x^2 \, dy. \tag{3.51}$$

To this geometry, we now consider turning on flux given by

$$g = \sum_{m \geq 1} T^m \xi_m + h\omega + \bar{l}\overline{\omega}, \tag{3.52}$$

where ω is the unique holomorphic 1-form on $\Sigma_{\rm DV}$. As we have seen, this induces a nontrivial potential for Σ of the form

$$V \sim \left(\frac{\partial W_{\rm DV}(\Sigma)}{\partial \Sigma}\right) \left(\frac{\partial \Sigma}{\partial S}\right) \left(\frac{1}{\operatorname{Im}\tau}\right) \overline{\left(\frac{\partial \Sigma}{\partial S}\right)}\, \overline{\left(\frac{\partial W_{\rm DV}(\Sigma)}{\partial \Sigma}\right)}, \tag{3.53}$$

where

$$W_{\rm DV}(\Sigma_2) = \sum_m T^m \Sigma_{m+1}(\Sigma_2). \tag{3.54}$$

To engineer a metastable vacuum at a fixed point, $\Sigma^{(0)}$, we need only choose the T^m so that $W_{\rm DV}(\Sigma)$ is a cubic polynomial in Σ obtained by truncating the expansion of a Kähler normal coordinate associated to $\Sigma^{(0)}$ at cubic order. To determine the requisite T^m, we proceed in two steps. First, we must determine the relation between Σ and the higher Σ_m (3.40). This is rather trivial. Second, however, we must obtain an expression for the Kähler normal coordinate associated to a generic point $\Sigma^{(0)}$. This will be slightly messier.

3.2.1 Relating Σ_m and Σ_2

Evaluating generic Σ_m for $\Sigma_{\rm DV}$ is relatively easy to do given the defining equation (3.49) and leads to the result

$$\begin{aligned}
\Sigma_{2q-1} &= 0, \\
\Sigma_{4q} &= \sum_{n=0}^{q} \frac{\Gamma\left(2q-n+\frac{1}{2}\right)}{2\sqrt{\pi}\,(2q-2n+1)!\,n!} \left(\frac{\Delta^2}{2}\right)^{2(q-n)+1} \left(b_0 - \frac{\Delta^4}{16}\right)^n, \\
\Sigma_{4q-2} &= \frac{1}{2q-1}\left(b_0 - \frac{\Delta^2}{16}\right)^q \left[\frac{\Gamma\left(q+\frac{1}{2}\right)}{q!\sqrt{\pi}}\right] \\
&\quad + \sum_{n=0}^{q-1} \frac{\Gamma\left(2q-n-\frac{1}{2}\right)}{2\sqrt{\pi}\,(2q-2n)!n!} \left(\frac{\Delta^2}{2}\right)^{2(q-n)} \left(b_0 - \frac{\Delta^4}{16}\right)^n.
\end{aligned} \tag{3.55}$$

From this, we first see that Σ is proportional to b_0

$$\Sigma = \frac{b_0}{2}. \tag{3.56}$$

More importantly, however, we are also able to immediately read off the degree of each nonzero Σ_m when viewed as a polynomial in Σ

$$\Sigma_{4q} \sim \Sigma^q + \mathcal{O}(\Sigma^{q-1}), \qquad \Sigma_{4q-2} \sim \Sigma^q + \mathcal{O}(\Sigma^{q-1}). \tag{3.57}$$

Consequently, the lowest m for which Σ_m contains a term proportional to Σ^q is $m = 4q-2$. This means that to introduce terms of order Σ^3 into $W_{\rm DV}(\Sigma)$, it will be necessary to include Σ_m up to $m = 10$, leading to much more singular flux than one might have otherwise thought. This is the first example of a general lesson we will have more to say about later, namely that when engineering OOP vacua, the requisite noncompactly supported flux can have a large degree of divergence which, to the best of our knowledge, is not easy to determine by any simple arguments.

Because one can introduce quadratic (cubic) terms using any Σ_m with $6 \leq m \leq 9$ $(10 \leq m \leq 13)$ there is some choice as to which T^m we can turn on to achieve a particular desired $W_{\rm DV}(\Sigma)$. For the purposes of our example, we will only turn on T^1, T^5, and T^9, thereby adding terms proportional to

$$\begin{aligned}
\Sigma_2 &\equiv \Sigma, \\
\Sigma_6 &= \frac{1}{16}\left(\Sigma\Delta^4 + 8\Sigma^2\right), \\
\Sigma_{10} &= \frac{1}{256}\left(\Delta^8\Sigma + 48\Delta^4\Sigma^2 + 128\Sigma^3\right).
\end{aligned} \tag{3.58}$$

3.2.2 Kähler normal coordinate for Σ_0

We now proceed to the second step, namely computing the first few terms of the Kähler normal coordinate expansion about a generic point $\Sigma^{(0)}$

$$z = \left(\Sigma - \Sigma^{(0)}\right) + a_2\left(\Sigma - \Sigma^{(0)}\right)^2 + a_3\left(\Sigma - \Sigma^{(0)}\right)^3 + \cdots, \tag{3.59}$$

where we have implicitly defined the coefficients

$$\begin{aligned}
a_2 &= \frac{1}{2}\Gamma^{\Sigma}_{\Sigma\Sigma}, \\
a_3 &= \frac{1}{6}g^{\Sigma\overline{\Sigma}}\partial_\Sigma\left(g_{\Sigma\overline{\Sigma}}\Gamma^{\Sigma}_{\Sigma\Sigma}\right),
\end{aligned} \tag{3.60}$$

with $g_{\Sigma\overline{\Sigma}}$ the metric associated to the Σ coordinate

$$g_{\Sigma\overline{\Sigma}} = \left|\frac{\partial S}{\partial\Sigma}\right|^2 \operatorname{Im}\tau, \tag{3.61}$$

and $\Gamma^{\Sigma}_{\Sigma\Sigma}$ the associated nonvanishing Christoffel symbol. Computations of quantities such as $\partial S/\partial\Sigma$ in Dijkgraaf-Vafa geometries are often performed using a perturbative expansion about the singular point $S = 0$. For engineering OOP type vacua, though, we need to consider instead the neighborhood of a generic, nonsingular point Σ_0 away from $S = 0$. Fortunately, in the simple case of a genus 1 curve, we can actually obtain exact results without too much work by taking advantage of the parametric description reviewed in appendix B. As described there, one finds that both S and Σ can be expressed directly as functions of τ

$$S = \frac{\Delta^3}{2\pi i\left[12\wp(\tau/2)\right]^{3/2}}\left(\frac{2g_2}{3} - 4\wp(\tau/2)\left[\wp(\tau/2) - 2\eta_1\right]\right), \tag{3.62}$$

$$\Sigma = \frac{\Delta^4\left(12\wp(\tau/2)^2 - g_2\right)}{288\wp(\tau/2)^2}, \tag{3.63}$$

where $\wp(z)$ is the Weierstrass $\wp$-function, g_2 is the Weierstrass elliptic invariant appearing in the relation

$$\left(\frac{\partial \wp(z)}{\partial z}\right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \tag{3.64}$$

and η_1 is one of the half-periods of the Weierstrass ζ-function

$$\eta_1 = \zeta\left(\frac{1}{2}\right). \tag{3.65}$$

From (3.62) and (3.63), we can apply the differentiation formulae of appendix B to write both $\partial\Sigma/\partial S$ and $g_{\Sigma\overline{\Sigma}}$ as functions of τ

$$\frac{\partial \Sigma}{\partial S} = -\frac{i\pi\Delta}{\sqrt{3\wp(\tau/2)}} \qquad \Longrightarrow \qquad g_{\Sigma\overline{\Sigma}} = \frac{3}{\pi^2}\left|\Delta^2\wp(\tau/2)\right| \operatorname{Im}\tau. \tag{3.66}$$

It is now straightforward to determine the coefficients of the Kähler normal coordinate expansion (3.59) in terms of the value of τ at $\Sigma^{(0)}$

$$\begin{aligned} a_2 &= \frac{36\wp(\tau/2)^2\left(g_2 + 12\eta_1\wp(\tau/2) - 6\wp(\tau/2)^2\right) - \frac{216\pi\wp(\tau/2)^3}{\operatorname{Im}\tau}}{\Delta^4\left(g_2 - 3\wp(\tau/2)^2\right)\left(g_2 - 12\wp(\tau/2)^2\right)}, \\ a_3 &= \frac{864\wp(\tau/2)^4}{\Delta^8\left(g_2 - 3\wp(\tau/2)^2\right)^2\left(g_2 - 15\wp(\tau/2)^2\right)^2} \\ &\quad\times\left[\frac{360\pi\wp(\tau/2)^3 - 48\pi g_2\wp(\tau/2)}{\operatorname{Im}\tau}\right. \\ &\qquad \left. + 5g_2^2 + 96\eta_1 g_2\wp(\tau/2) - 63g_2\wp(\tau/2)^2 - 720\eta_1\wp(\tau/2)^3 + 252\wp(\tau/2)^4\right]. \end{aligned} \tag{3.67}$$

3.2.3 Noncompact flux for engineering OOP vacuum

We are finally ready to explicitly write the noncompact flux needed to engineer an OOP vacuum at a generic point $\Sigma^{(0)}$. In particular, we seek to specify values for the coefficients T^m which render

$$W_{\rm DV}(\Sigma) = \sum_m T^m \Sigma_m(\Sigma) \tag{3.68}$$

equivalent, up to a constant shift, to a truncation of the Kähler normal coordinate expansion (3.59) about $\Sigma^{(0)}$ at order Σ^3. Using (3.58), it is easy to see that the following choice of nonzero T^m does the job

$$\begin{aligned} T^1 &= 2 - \frac{a_2}{4}\left(\Delta^4 + 16\Sigma_0\right) + 2a_3\left(\frac{5\Delta^8}{128} + \frac{3\Delta^4\Sigma_0}{8} + 3\Sigma_0^2\right), \\ T^5 &= \frac{12a_2}{5} - \frac{9a_3}{10}\left(\Delta^4 + 8\Sigma_0\right), \\ T^9 &= \frac{20a_3}{9}, \end{aligned} \tag{3.69}$$

where a_2 and a_3 given by the expressions in (3.67) evaluated at the value of τ corresponding to $\Sigma^{(0)}$.

These expressions, while nice and exact, are a little cumbersome so let us also consider a special case where things simplify. To that end, we try to engineer an OOP vacuum at the special point $\tau = i$ corresponding to a square torus. In this case, several elliptic quantities simplify

$$\eta_1\big|_{\tau=i} = \frac{\pi}{2} \qquad g_3\big|_{\tau=i} = 0 \implies g_2\big|_{\tau=i} = 4\wp(\tau/2)^2\big|_{\tau=i} . \tag{3.70}$$

The value of Σ at $\tau = i$ can be obtained by applying (3.70) to (3.63)

$$\Sigma_0 = \frac{\Delta^4}{36}. \tag{3.71}$$

This means that the curve (3.49) is given at this point by

$$y^2 = x^4 - \frac{\Delta^2}{2}x^2 + \frac{\Delta^4}{144} = \left[x^2 - \left(\frac{1}{4} + \frac{1}{3\sqrt{2}}\right)\Delta^2\right]\left[x^2 - \left(\frac{1}{4} - \frac{1}{3\sqrt{2}}\right)\Delta^2\right]. \tag{3.72}$$

The coefficients a_2 and a_3 (3.67) appearing in the Kähler Normal Coordinate expansion (3.59) simplify to

$$a_2\big|_{\tau=i} = \frac{9}{\Delta^4} \qquad \text{and} \qquad a_3\big|_{\tau=i} = \frac{1080}{\Delta^8} \tag{3.73}$$

Inserting these into (3.59), we find that our desired effective superpotential $W_{\mathrm{DV}}(\Sigma)$ is given by

$$W_{\mathrm{DV}}(\Sigma) = \text{constant} + 3\Sigma - \frac{81\Sigma^2}{\Delta^4} + \frac{1080\Sigma^3}{\Delta^8}, \tag{3.74}$$

while plugging into (3.69) yields the T^m that do the job

$$T^2 = \frac{885}{8}, \qquad T^6 = -\frac{5832}{5\Delta^4}, \qquad T^{10} = \frac{2400}{\Delta^8}. \tag{3.75}$$

While metastability of the vacuum at $\tau = i$ is guaranteed by the OOP procedure, it is also gratifying to see it graphically by explicitly plotting the potential near $\tau = i$ as in figure 3.

3.3 Degree of superpotential required for metastable vacua

As we have seen in the above example, there is an issue about the degree of superpotential we have to consider in order to create OOP metastable vacua. In this subsection, we analyze this issue.

As one can see from (2.34), (2.35), in order to create an OOP vacuum at a specific point $X^{(0)}$ in the moduli space, one must be able to adjust the coefficients in the superpotential up to cubic terms in $\Delta X = X - X^{(0)}$. If the dimension of the moduli space is d, this means that we generically need to tune

$$d + \frac{d(d+1)}{2} + \frac{d(d+1)(d+2)}{6} - d = \frac{d(d+1)(d+5)}{6} \equiv C_d \tag{3.76}$$

parameters in the superpotential.[21] The last term is subtracting the degrees of freedom to choose the vector k_i.

[21] For having a metastable vacuum, the superpotential does not have to be *exactly* the same as the ones given in (2.35); if the coefficients are very close to the ones given in (2.34), (2.35), one still expect to have metastable vacua. However, this does not generically affect the number of parameters we need to tune.

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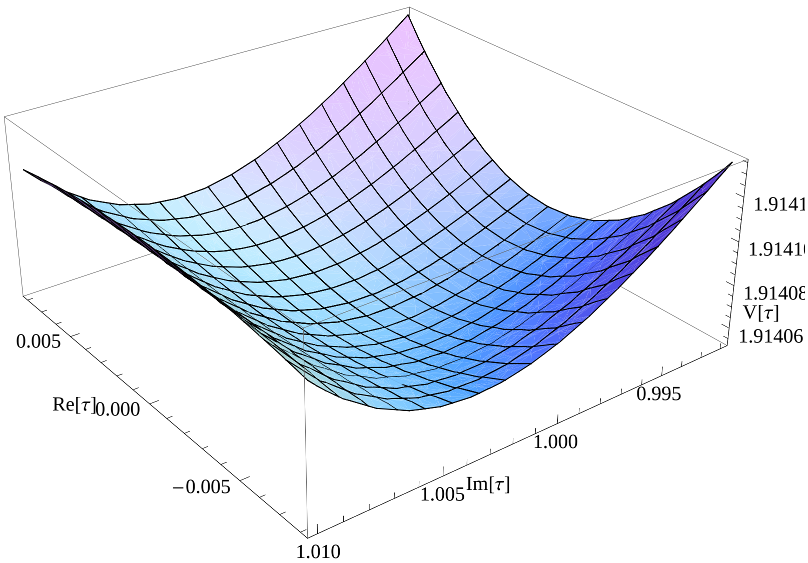

Figure 3: Plot of $V(\tau)$ in the neighborhood of our engineered OOP minimum at $\tau = i$

In the local Calabi-Yau geometries we have been considering, the superpotential is parametrized by the coefficients T^m. For example, in the Dijkgraaf-Vafa geometry, the superpotential was given by (3.39):

$$W_{\rm DV}(S) = \sum_{m\geq 1} T^m \Sigma_{m+1}(S), \qquad \Sigma_m(S) = \frac{1}{2\pi i m} \oint_\infty x^m dy(S), \tag{3.77}$$

where we wrote the dependence of Σ_m's on the moduli $S = \{S^i\}$ explicitly. Therefore, if $\Sigma_m(S)$'s are generic functions of S then, by tuning C_{n-1} parameters[22] $T^2, T^3, \ldots T^{C_{n-1}+1}$, one can create a metastable vacuum at a generic point $S = S^{(0)}$. More precisely, the OOP mechanism requires that, when we expand $\Sigma_m(S)$'s around $S^{(0)}$ in $\Delta S = S - S^{(0)}$, the coefficients of $\Delta S, (\Delta S)^2, (\Delta S)^3$ terms are all independent and by taking linear combinations of $\Sigma_m(S)$'s we can obtain the superpotential (2.35).

However, as we saw in the example above, the situation is not generic for small n and we need a more detailed analysis about how high degrees one should go, which is done in appendix C. The result (eq. (C.4)) is that, if we would like to make a critical point at a generic point in the moduli space, we have to tune on T^m at least up to $m = m_{\rm min}$, where[23]

$$\begin{aligned} n = 2: &\quad m_{\rm min} = 10, \\ n = 3: &\quad m_{\rm min} = 15, \\ n \geq 4: &\quad m_{\rm min} = \frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3} + 3. \end{aligned} \tag{3.78}$$

[22] Note that the number of moduli is $n-1$ because we are treating b_{n-1} dynamical.

[23] This result is for the case where b_{n-1} is treated nondynamical. For the result in the case where b_{n-1} is regarded as a modulus, see appendix C.

There is certain genericily assumption on the dependence of Σ_m on the moduli (see appendix C), and hence the actual degree m one must consider can be larger than the one given above.

Therefore, in order to stabilize metastable vacua made of n cuts by the OOP superpotential, we have to consider Σ_m's up to rather high degree $m_{\min}$ given by (3.78) at least. Because the degree m corresponds to the order of divergence of the flux at infinity (ξ_m), the noncompactly supported flux must diverge at infinity at the corresponding speed.

4. Factorization

4.1 The basic idea

In the previous sections we described how we can generate a supersymmetry breaking potential for the complex structure moduli of a local Calabi-Yau singularity by the introduction of 3-form flux which has support at infinity. Allowing flux with noncompact support may lead to various conceptual difficulties, such as the divergence of the total energy density. To clarify these difficulties we would like to sketch how such a system can be interpreted as an approximation of a larger Calabi-Yau threefold with flux of compact support in a certain factorization limit.

More precisely, we start with a Calabi-Yau with a subset of cycles pierced by usual 3-form flux of compact support. In another region of the manifold we have a second subset of cycles. The flux from the first cycles will generate a potential for the complex structure moduli of the second set. In the limit where the cycles are separated by a large distance, and where we zoom in towards the second set, the flux from the first subset will look as if it is coming from "infinity".[24] In this sense, the noncompact setup considered in the previous section can be considered as a small part of a larger Calabi-Yau with compactly supported flux.

In this section we would like to understand this embedding into a bigger Calabi-Yau in more detail. Our goal is to see how the potential (2.30) arises starting from the standard Gukov-Vafa-Witten superpotential for 3-form flux in the larger Calabi-Yau.

For simplicity we will work with a noncompact Calabi-Yau $\mathcal{M}$,

$$\mathcal{M}: \quad uv - F(x,y) = 0, \tag{4.1}$$

which is based on a Riemann surface Σ given by $F(x,y)=0$. As we explained before the complex parameters entering the defining equation of the Riemann surface correspond to complex structure moduli of the Calabi-Yau. Some of them are non-normalizable and can be considered as external parameters. We want to tune these parameters to approach the limit where the surface Σ factorizes into two surfaces Σ_L and Σ_R connected by long tubes. This factorization lifts to the entire Calabi-Yau $\mathcal{M}$ and divides it into two regions $\mathcal{M}_L$ and $\mathcal{M}_R$ that are widely separated. We introduce 3-form flux G_3 of compact support on the 3-cycles of $\mathcal{M}_R$. The superpotential and scalar potential are given by

$$W = \int G_3 \wedge \Omega \qquad \text{and} \qquad V = G^{I\bar{J}} \partial_I W \overline{\partial_J W}, \tag{4.2}$$

[24] As we will see in more detail later, we also have to scale the flux in an appropriate way.

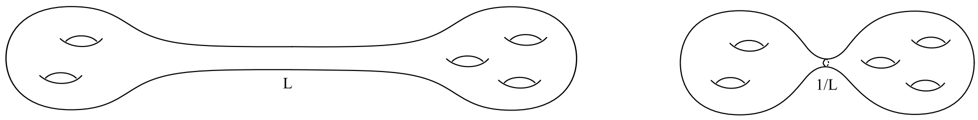

Figure 4: Two conformally equivalent ways of viewing the factorization of a Riemann surface into two parts. Physically though, we should distinguish both points of view, since particle masses depend on the size of the cycles. Because in our situation no new massless appears in the factorization limit, the left diagram represents our point of view best.

where the indices I, J run over all complex structure moduli of the total threefold $\mathcal{M}$. Using the properties of the Kähler metric $G_{I\overline{J}}$ in the factorization limit we show that the part of the potential (4.2) which depends on the complex structure moduli of $\mathcal{M}_L$ is of the form (2.30). Furthermore, we find an understanding of the effective value of the parameters T^m.

4.2 Geometry of factorization

In this subsection we study the degeneration of a Riemann surface Σ into two components Σ_L and Σ_R, depicted in figure 4.[25] In this factorization data of the full Riemann surface is expressed in terms of the complex structure of the individual surfaces. It is well known that in the limit where the length of the tubes $L = 1/\epsilon$ goes to infinity the period matrix of the full surface becomes block diagonal

$$\tau = \begin{pmatrix} \tau^{LL} & 0 \\ 0 & \tau^{RR} \end{pmatrix} + \mathcal{O}\left(\epsilon\right). \tag{4.3}$$

While the off-diagonal components τ^{LR} go to zero in the factorization limit, their subleading behavior is quite important in our analysis since it expresses the weak interaction between the two sectors. The period matrix τ^{LR} can be computed systematically in an expansion in ϵ from data on each of the two surfaces as we explain below.

Technically, we describe the factorization of the Riemann surface with the *plumbing fixture* method [69]. So consider two Riemann surfaces Σ_L and Σ_R of genus g_L and g_R respectively. On the left surface Σ_L we have g_L holomorphic differentials ω_i, while on the right surface Σ_R similarly g_R holomorphic differentials $\omega_{i'}$. The complex structure of the left surface is determined by the periods of the holomorphic differentials

$$\frac{1}{2\pi i}\int_{A^i}\omega_j = \delta^i_j, \qquad \frac{1}{2\pi i}\int_{B_i}\omega_j = \tau^{LL}_{ij}, \tag{4.4}$$

where τ^{LL}_{ij} is the period matrix of Σ_L, and we choose our definitions similarly for the right surface.

The plumbing fixture method works after choosing a puncture P on Σ_L and P' on Σ_R. It connects the two surfaces by a long tube of length L which is glued onto neighborhoods

[25] In general these components could be connected in a non-trivial way. We restrict our computations in this section to the case in which they are linked by just one long tube. These should be easily extendible to more general cases.

of the punctures P and P'. More precisely, we pick a local holomorphic coordinate z around the puncture P such that $z(P) = 0$ and a holomorphic coordinate z' near P' with $z'(P') = 0$. Then we identify points in these neighborhoods as

$$zz' = \epsilon. \tag{4.5}$$

Now we want to compute the period matrix of the full Riemann surface in terms of complex structure data of the two surfaces. For this we need to understand how the differentials ω_i and $\omega_{i'}$ extend to well-defined holomorphic differentials on the full surface $\Sigma = \Sigma_L \cup \Sigma_R/\sim$, where $\sim$ is the above identification. Let us first consider how to lift the differential ω_i. Around the puncture P it may be expanded as

$$\omega_i = \sum_{m=1}^{\infty} K^P_{im} z^{m-1} dz, \tag{4.6}$$

where the functions K^P_{im} are given by (A.7). Once we write this in terms of z' we observe that, as seen from the right surface, the differential has a Laurent expansion. So ω_i will be written as a linear combination of the meromorphic differentials $\xi^{P'}_m$ of the right surface. A meromorphic differential has the following expansion around the puncture

$$\xi^P_m = \left(\frac{m}{z^{m+1}} + \sum_{n=1}^{\infty} h^P_{mn} z^{n-1} \right) dz. \tag{4.7}$$

Here we have introduced the functions h^P_{mn}, which depend on the complex structure moduli of the surface and the position of P. So in general the differential ω_i will lift to a differential $\widetilde{\omega_i}$ on the full surface which can be written as

$$\widetilde{\omega_i} = \begin{cases} \omega_i + \sum_{m=1}^{\infty} x_{im} \xi^P_m & \text{on} \quad \Sigma_L, \\ \sum_{m=1}^{\infty} y_{im} \xi^{P'}_m & \text{on} \quad \Sigma_R. \end{cases} \tag{4.8}$$

for some coefficients x_{im} and y_{im}. Matching the differential on the two sides we find the following conditions

$$x_{im} = -\frac{\epsilon^m}{m} \sum_{n=1}^{\infty} y_{in} h^{P'}_{nm}, \qquad y_{im} = -\frac{\epsilon^m}{m} \left(K^P_{im} + \sum_{n=1}^{\infty} x_{in} h^P_{nm} \right). \tag{4.9}$$

This allows us to compute the cross-period matrix as

$$\begin{aligned} \tau^{LR}_{ij'} = \int_{b_{j'}} \omega_i = \sum_{m=1}^{\infty} K^{P'}_{j'm} y_{im} = -\sum_{m,n=1}^{\infty} \frac{\epsilon^n}{n} K^P_{im} G^{-1}_{mn} K^{P'}_{j'n}, \\ G_{mn} \equiv \delta_{mn} - \sum_{l=1}^{\infty} \frac{\epsilon^{n+l}}{nl} h'_{ml} h_{ln}. \end{aligned} \tag{4.10}$$

From this equation we can read off all order ϵ-corrections to the off-diagonal piece of the period matrix when a surface Σ degenerates.

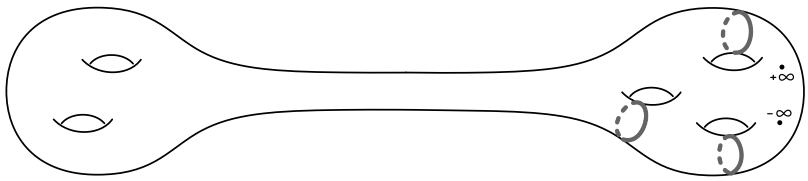

Figure 5: Turning on flux on the right part of the factorized Calabi-Yau.

Also, this procedure gives a clear understanding of the term "flux at infinity". We see that the flux at infinity is generated by regular forms on the degenerated surface, and therefore will at most have finite order poles at the punctures.

Notice that for a Calabi-Yau threefold that is based on a Riemann surface, the factorization region is described by the deformed conifold geometry

$$uv + x^2 + y^2 = \epsilon, \quad \text{or equivalently} \quad uv + zz' = \epsilon. \tag{4.11}$$

Usually, this is described as a 3-sphere shrinking to zero-size when $\epsilon \to 0$. However, as for the complex 1-dimensional plumbing fixture case we want the two sectors to be far apart from each other. Therefore we consider the conformally equivalent setup where the 3-sphere is scaled to be of finite size, while the transverse directions are made very large. The finite size three-sphere reduces to the cross-section of the tube on the left in figure 4, whereas the transverse directions reduce to the tube-length.

To describe the left and right neighborhoods of the degeneration, we can fix $x = \sqrt{\epsilon - y^2 - uv}$ on the left and $x = -\sqrt{\epsilon - y^2 - uv}$ on the right. In the limit that $\epsilon \to 0$ these neighborhoods will not just intersect in a point, but in the divisor $uv + y^2 = 0$. This is the region where regular forms on the total threefold will develop poles when the degeneration starts.

4.3 Dynamics

Now we consider turning on flux on the threefold. For simplicity we again take a Calabi-Yau (4.1) that is based on a factorized Riemann surface. We turn on 3-form flux $G_3 = F_3 - \tau H_3$ which is only piercing the set of A-cycles corresponding to Σ_R, as can be seen in figure 5, and write down the corresponding (super) potential. For regularization issues later, we take two more punctures on the right surface labeled by $\pm\infty$ and turn on some flux α through the noncompact $\mathcal{B}_\infty$ cycle running from $+\infty$ to $-\infty$.

A basis of $\mathcal{A}$ and $\mathcal{B}$ cycles is given by the compact 3-cycles on the left and the right, together with the lift $\mathcal{A}^\infty$ of the A-cycle enclosing $+\infty$ and $\mathcal{B}_\infty$. So the flux is determined by

$$\begin{aligned} &\int_{\mathcal{A}^i} G_3 = 0, \quad \int_{\mathcal{A}^{i'}} G_3 = N^{i'}, \quad \int_{\mathcal{A}^\infty} G_3 = 0, \\ &\int_{\mathcal{B}_i} G_3 = 0, \quad \int_{\mathcal{B}'_i} G_3 = 0, \quad\ \int_{\mathcal{B}_\infty} G_3 = \alpha. \end{aligned} \tag{4.12}$$

Let us denote the complex structure moduli and their duals by X^I and F_I, which are the $\mathcal{A}^I$ resp. $\mathcal{B}_I$ periods of the holomorphic 3-form Ω. Here we use the capital indices

$I = \{i, i', \infty\}$ to run over both the left and the right sides. Then the GVW superpotential for the complex structure moduli is given by

$$W = \int G_3 \wedge \Omega = \alpha X^\infty + \sum_{i'} N^{i'} F^R_{i'}, \tag{4.13}$$

and the corresponding scalar potential by

$$V = \sum_{I,J} G^{I\bar{J}} \partial_I W \overline{\partial_J W}. \tag{4.14}$$

Since X^∞ corresponds to a log-normalizable period and the derivatives in the above potential just correspond to normalizable modes, the α-factor decouples. This shows that

$$\begin{aligned} V = &\sum_{i,j,k',l'} \left(N^{k'}\tau^{LR}_{k'i}\right)\left(\frac{1}{\operatorname{Im}\tau}\right)^{ij}_{LL} \overline{\left(N^{l'}\tau^{LR}_{l'j}\right)} + \sum_{i,j',k',l'} \operatorname{Re}\left[\left(N^{k'}\tau^{LR}_{k'i}\right)\left(\frac{1}{\operatorname{Im}\tau}\right)^{ij'}_{LR} \overline{\left(N^{l'}\tau^{RR}_{l'j'}\right)}\right] \\ &+ \sum_{i',j',k',l'} \left(N^{k'}\tau^{RR}_{k'i'}\right)\left(\frac{1}{\operatorname{Im}\tau}\right)^{i'j'}_{RR} \overline{\left(N^{l'}\tau^{RR}_{l'j'}\right)}. \end{aligned} \tag{4.15}$$

Thus the total potential is the sum of three terms, which we denote in the obvious way by $V = V_1 + V_2 + V_3$.

Next we consider what happens in the limit where the distance L between the two sets of 3-cycles gets very large. As explained before the period matrices τ^{LL} and τ^{RR} remain of order one in this limit and become almost independent of the moduli X_R and X_L, respectively.

On the other hand, τ^{LR} goes to zero which would make the first term V_1 in the potential vanish in the limit that $\epsilon \to 0$, at least if we don't scale the fluxes $N^{i'}$ appropriately. Since V_1 describes the interaction between the two sides of the Calabi-Yau, we really want to scale the fluxes $N^{i'}$ to go to infinity in such a way that the term V_1 remains finite.

Then it becomes clear that the term V_3 of the potential dominates over the other two contribution to V. This implies that in the limit $\epsilon \to 0$ the term V_3 should be minimized first, i.e.,

$$\sum_{k'} N^{k'} \tau^{RR}_{k'i'} = 0, \quad \forall i', \tag{4.16}$$

which is a set of n_R equations for the n_R moduli $x^{j'}$. The solutions of this system correspond to supersymmetric vacua for the 3-cycles on the right side. Once we have fixed all $X^{j'}$ to their supersymmetric values $\widehat{X}^{j'}$, we can consider the effect of the backreaction of the right side to the left. This is purely expressed through the potential V_1, since the term V_2 vanishes as well at the supersymmetric point.

So effectively the potential for the complex structure moduli X^i_L of the left surface is

$$V_1 = \sum_{i,j,k',l'} \left(N^{k'}\tau^{LR}_{k'i}\right)\left(\frac{1}{\operatorname{Im}\tau}\right)^{ij}_{LL} \overline{\left(N^{l'}\tau^{LR}_{l'j}\right)}. \tag{4.17}$$

This may be written as $V_1 = \sum_{i,j} \partial_i W_{\text{eff}} (1/\operatorname{Im}\tau)^{ij}_{LL} \overline{\partial_j W_{\text{eff}}}$, where we define the effective "superpotential" for the left complex structure moduli as

$$\partial_i W_{\text{eff}} \equiv \sum_{k'} N^{k'} \tau^{LR}_{k'i}. \tag{4.18}$$

Comparing with expression (4.10) it is clear that the fluxes on the right should be scaled in such a way that the coefficients

$$T^m = \epsilon^m \sum_{k'} N^{k'} K'_{k'm} \tag{4.19}$$

remain constant. In that situation the effective superpotential is

$$\partial_i W_{\text{eff}} = \sum_m T^m K_{im} \tag{4.20}$$

to leading order in ϵ, which is precisely of the form (2.30).

4.4 Genericity of potential and metastable vacua

Let us summarize what we have demonstrated so far. We started with a large Calabi-Yau that consists of two parts $\mathcal{M}_L$ and $\mathcal{M}_R$ separated by a large distance, and turned on a large 3-form flux on one of the sides, say $\mathcal{M}_R$. This flux generates a large potential for the complex structure moduli of $\mathcal{M}_R$, which are therefore set to their supersymmetric minima. The flux on $\mathcal{M}_R$ is also weakly backreacting to the other side $\mathcal{M}_L$, inducing a small superpotential for the complex structure moduli of $\mathcal{M}_L$. We computed this superpotential in equations (4.18) and (4.20) and found that it is of the form (2.30). The main point is that the side $\mathcal{M}_L$ only knows about $\mathcal{M}_R$ via the parameters T^m given by (4.19).

In this section we discuss two questions. The first to which degree we can tune the parameters T^m independently. And the second is whether these T^m's can be chosen to realize an OOP supersymmetry breaking superpotential.

As we can see from (4.19), the values of the parameters T^m depend on the fluxes $N^{l'}$ on the cycles of $\mathcal{M}_R$ and also on the value of the (generalized) period matrix $K'_{l'm}$. The last one depends on the choice of the supersymmetric vacuum $\widehat{X}^{j'}$ on the right side. For given large fluxes $N^{l'}$ there is a huge number of supersymmetric vacua, or solutions of (4.16), with different values of $\widehat{X}^{j'}$ and consequently of $K'_{l'm}$. The density of such supersymmetric vacua over the complex structure moduli space of $\mathcal{M}_R$ has been studied before [6, 70 – 73], and it is believed that the vacua become dense in the moduli space in the limit where the fluxes are very large.

The coefficients $K'_{l'm}$ are holomorphic functions over the complex structure moduli space of $\mathcal{M}_R$. So naively one would conclude that when the dimension of this moduli space is large enough, meaning that the number of 3-cycles in $\mathcal{M}_R$ is large, we can always find supersymmetric points where the $K'_{l'm}$'s have the desired values. However the functions $K'_{l'm}$ are not "generic" and there may be relations between them which affect the naive counting. We have not analyzed this problem in detail but we think the following statement is true. Any number of the T^m's in the superpotential (4.20) can be tuned by considering a

Calabi-Yau whose right side $\mathcal{M}_R$ has a sufficiently large number of 3-cycles, and there will be some supersymmetric vacua with right values of $K'_{l'm}$ to reproduce the desired T^m's to good accuracy.

This claim is made more intuitive by the following physical interpretation of equation (4.19). Start by turning on fluxes $N^{l'}$ on the cycles of $\mathcal{M}_R$, which is based on the Riemann surface Σ_R. When reduced on the Riemann surface the flux looks like the electric field produced by a charge in two dimensions. The set of fluxes $N^{l'}$ resembles a charge distribution on the cycles of the Riemann surface. To compute the field produced by these charges in the distant region of the other set of cycles Σ_L, one has to consider a multipole expansion. Since the matrix $K'_{l'm}$ computes the mth multipole expansion of a charge distributed along the l'th cycle, the coefficients T^m are exactly the multipole moments of the charge distribution. In this formulation our first question reads whether we can arrange a charged distribution to have the desired multipole moments given by the coefficients T^m. We expect that the answer is positive.

The second question is more subtle. To realize a metastable nonsupersymmetric vacuum via the OOP mechanism, one has to tune the superpotential in a way which is determined by properties of the Kähler metric at that point. As we saw in section 3.2 one has to tune the coefficients of the effective superpotential only up to cubic order in an expansion around the candidate metastable point. Since we have a very large number of parameters T^m at our disposal it seems that generically we should be able to tune them to generate metastable vacua at most points on the moduli space. However we do not have a proof of this statement and it is possible that various relations between the period matrices and the Kähler metric invalidate the naive counting.[26]

5. Factorization II: an example

In the previous section, we argued, based on the factorization of the Riemann surface and Calabi-Yau, that it is possible to embed the nonsupersymmetric metastable vacua we found in 3 in a "larger" Calabi-Yau, the idea being that the flux threading compact cycles on one side of the Calabi-Yau looks like flux coming from infinity from the viewpoint of the other side of the Calabi-Yau. In this section, we will discuss the Dijkgraaf-Vafa geometries of subsection 3.1.2:

$$\Sigma_{\rm DV}: \quad y^2 = P_n(x)^2 - f_{n-1}(x), \qquad P_n(x) = \prod_{I=1}^{n}(x-\alpha_I), \tag{5.1}$$

as an example where our proposal can in principle be implemented, and make some steps towards actually confirming our proposal.

[26]This question is similar to whether one can realize the OOP mechanism with a single trace superpotential for the adjoint scalar in an SU(N) gauge theory. In [24] it was demonstrated that for SU(2) a metastable vacuum can be generated anywhere on the moduli space by a single trace superpotential, and for SU(N) at the center of the moduli space. It was not fully analyzed whether this is possible in generality.

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5.1 Factorization limit in practice

As explained in 3.1.2, α_I are non-normalizable parameters which represent the positions of the cuts on the x-plane, while the coefficients in $f_{n-1}(x)$, or equivalently variables S^I defined in (3.35), are normalizable (or at least log-normalizable) and hence are dynamical variables describing the size of those cuts. Therefore, in this Dijkgraaf-Vafa case (5.1), α_I are the parameters we want to adjust in order to approach the factorization limit where $\Sigma_{\rm DV}$ degenerates into two subsectors.

So, what we should do is clear: we divide the n cuts into two parts as $n = n_L + n_R$, the ones on the left indexed by i and on the right by i', and send these two groups apart from each other by a large factor $L = 1/\epsilon$ so that

$$\alpha_i - \alpha_{i'} = \mathcal{O}(L) \qquad (\text{when } L \to \infty). \tag{5.2}$$

In the $L \to \infty$ limit, the left and right sides will be very far apart and the factorization we discussed in the previous section must be achieved. For example, the period matrix of the total Riemann surface must diagonalize as in (4.3) up to $1/L$ correction.

There is one thing we should be careful about when taking the $L \to \infty$ limit. If we try to separate the two sets of cuts by naively taking the typical difference between α_i and $\alpha_{i'}$ to be of order L while keeping the size of the cuts fixed, then a simple estimate of the scaling of $S_i^L, S_{i'}^R$ using (3.35) shows that the physical size of the 3-cycles in the Calabi-Yau blows up. What we want instead is to end up with two sets of 3-cycles of finite size, separated by a large distance, so that we are left with nontrivial dynamics of $S_i^L, S_{i'}^R$. To achieve this we must also scale the size of the cuts, as we send $L \to \infty$. Let x_L and x_R be local coordinates in the left and right sectors, respectively, and set

$$\widetilde{x}_L = L^r x_L, \qquad \widetilde{x}_R = L^{r'} x_R, \tag{5.3}$$

where

$$r = \frac{n_R}{n_L+1}, \qquad r' = \frac{n_L}{n_R+1}. \tag{5.4}$$

Then, from (3.35), it is not difficult to see that we can keep $S_i^L, S_{i'}^R$ finite if we keep $\widetilde{x}_L$, $\widetilde{x}_R$ finite while taking the $L \to \infty$ limit. A similar rescaling of local coordinates must be also necessary when taking a factorization limit in any other examples than (5.1).

5.2 Computation of period matrix

In the Dijkgraaf-Vafa geometry (5.1), the period matrix is given by

$$\tau_{IJ} = \frac{\partial^2 \mathcal{F}_0}{\partial S^I \partial S^J}, \tag{5.5}$$

Here, $\mathcal{F}_0$ is the B-model prepotential, which by the Dijkgraaf-Vafa relation [63, 74] is related to matrix models. The precise way to scale various quantities to take the factorization limit being understood from subsection 5.1, it is in principle possible to confirm our proposal for the Dijkgraaf-Vafa geometry using (5.5). For doing that, it is important to be able to compute the prepotential $\mathcal{F}_0$ for a large number of cuts n. The results from section 3 show that generating a metastable vacuum requires quite a lot of coefficients T^m. Since we roughly need the same number of cuts on the right as the number of tuned Σ_m's on the left, the total Riemann surface must have quite a large number of cuts. So, in this subsection we will explain the way to compute $\mathcal{F}_0$ and thus τ_{IJ} for an arbitrary n.

For Dijkgraaf-Vafa geometries (5.1) the prepotential $\mathcal{F}_0$ may in fact be computed for any number of cuts n in a number of ways. The most direct way is evaluating the period integrals on the hyperelliptic curve. This has been done up to cubic order in S^I in [75]. Duality with a U(N) matrix model [63, 74]

$$Z = \exp\left[\sum_{g=0}^{\infty} g_s^{2g-2}\mathcal{F}_g(S)\right] = \int d^{N^2}\Phi \, \exp\left[\frac{1}{g_s}\,\mathrm{tr}\, W(\Phi)\right], \tag{5.6}$$

where the matrix model action is given by

$$W'(x) = P_n(x) = \prod_{I=1}^{n}(x-\alpha_I) \tag{5.7}$$

makes this computation quite a bit simpler. Let us quickly show this argument [74].

The field Φ is an $N \times N$ matrix. Say N^I eigenvalues of Φ are placed at the critical point $x = \alpha_I$ and divide the matrix Φ into $N^I \times N^J$ blocks Φ_{IJ}, where $\sum_{I=1}^{n} N^I = N$. One can go to the gauge $\Phi_{IJ} = 0$ for $I \neq J$ by introducing fermionic ghosts in the matrix model action. This produces the following extra term in the action, where $\Phi_I \equiv \Phi_{II}$:

$$W_{\text{ghost}} = \sum_{I\neq J} \mathrm{tr}(B_{JI}\Phi_I C_{IJ} + C_{JI}\Phi_I B_{IJ}). \tag{5.8}$$

To write down Feynman diagrams, we expand Φ_I around $x = \alpha_I$ as $\Phi_I = \alpha_I + \phi_I$. A Taylor series of $W(\Phi_I) = W(\alpha_I + \phi_I)$ around α_I yields the propagator and p-vertices for ϕ_I. In particular, this shows that the propagator for ϕ_I is given by

$$\langle \phi_I \phi_I \rangle = \frac{1}{W''(\alpha_I)} = \frac{1}{\Delta_I}, \tag{5.9}$$

where $\Delta_I = W''(\alpha_I) = \prod_{J\neq I}^{n} \alpha_{IJ}$. Moreover, expanding the ghost action determines the ghost propagator to be

$$\langle B_{JI} C_{IJ} \rangle = \frac{1}{\alpha_{IJ}}, \tag{5.10}$$

and gives the Yukawa interactions between ϕ_I, B_{JI} and C_{IJ}.

The contribution to the prepotential $\mathcal{F}_0$ of order three in the S^I's is given by planar diagrams with three holes. Writing down the expressions $g_{I,3}$ and $g_{I,4}$ in terms of α's and Δ's shows that

$$\mathcal{F}_{0,3} = \sum_{I=1}^{n} u_I S_I^3 + \sum_{I\neq J}^{n} u_{I;J} S_I^2 S_J + \sum_{I<J<K}^{n} u_{IJK} S_I S_J S_K, \tag{5.11}$$

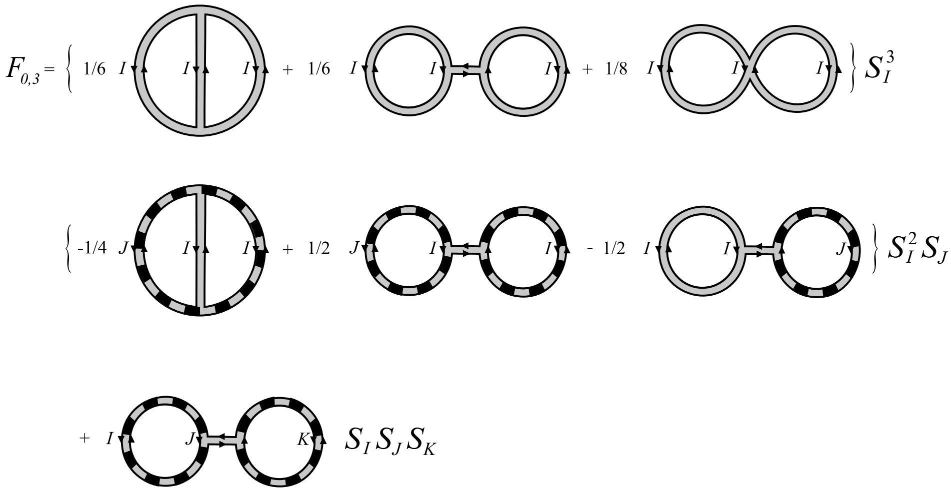

Figure 6: The contribution to $\mathcal{F}_{0,3}$ given in terms of matrix diagrams. Gray double lines represent ϕ_I fields, while black-and-gray double lines represent BC ghosts.

where

$$
\begin{aligned}
u_I &= \frac{2}{3}\left(-\sum_{J\neq I}\frac{1}{\alpha_{IJ}^2\Delta_J}+\frac{1}{4\Delta_I}\sum_{\substack{J<K\\ J,K\neq i}}\frac{1}{\alpha_{IJ}\alpha_{IK}}\right),\\
u_{I;J} &= -\frac{3}{\alpha_{IJ}^2\Delta_I}+\frac{2}{\alpha_{IJ}^2\Delta_J}-\frac{2}{\alpha_{IJ}\Delta_I}\sum_{K\neq I,J}\frac{1}{\alpha_{IK}} \quad \text{and}\\
u_{IJK} &= 4\left(\frac{1}{\alpha_{IJ}\alpha_{IK}\Delta_I}+\frac{1}{\alpha_{JI}\alpha_{JK}\Delta_J}+\frac{1}{\alpha_{KI}\alpha_{KJ}\Delta_K}\right).
\end{aligned}
$$

In appendix D, we discuss the generalization of this result to higher order in S^I. In particular, we compute $\mathcal{F}_0$ up to S^5 terms.

5.3 Scaling of period matrix

The method explained in subsection 5.2 allows one in principle to compute the period matrix to any order in S^I for general Dijkgraaf-Vafa curves (5.1). Then the factorization limit can be achieved simply by taking the $L \to \infty$ limit of the result and one can start looking for metastable vacua. In this subsection, as a step towards it, let us pursue a more modest goal of seeing the factorized behavior of the period matrix, (4.3).

The form of the scaling can be elegantly derived for any possible contributing matrix model diagram to $\mathcal{F}_0$. First note that Δ_i scales as L^{2r} as $L \to \infty$, and $\Delta_{i'}$ as $L^{2r'}$. All propagators with indices from either side of the surface have an expansion in terms of α_{IJ}'s and Δ_I's, and thus a scaling in L which is easy to determine. The total scaling of a planar diagram with an arbitrary number of these elements turns out to depend just on the number of ghost vertices that connect the left side to the right side. It is given by

$$
\frac{1}{L^{(1+r)N_{ii'}+(1+r')N_{i'i}}}, \tag{5.12}
$$

where $N_{ii'}$ is the number of ghost vertices with external ghost lines indexed by (i, i') and the external ϕ-line by (i, i). Note that in deriving this we assumed the scaling (5.3) and thus $S_L^i, S_R^{i'}$ are of order one.

This shows that a diagram with only indices on the left (or on the right) will be of order 1 in L. Since such diagrams contribute to the period matrix τ_{ij} (or $\tau_{i'j'}$), so this shows that the period matrix is of order 1 in L, with corrections in $1/L$ from diagrams that contain at least two loops indexed by i and j. On the other hand, the off-diagonal pieces of the period matrix $\tau_{ii'}$ and $\tau_{i'i}$ contain at least one ghost cross-vertex with indices i and i'. These parts will therefore scale at least as $1/L$. In particular, for large L the properties of the full Riemann surface Σ are determined by those of the two factors Σ_L, Σ_R, and the period matrix τ_{IJ} indeed diagonalizes as in (4.3).

Having checked the diagonalization (4.3), the problem of actually finding an example of a metastable vacuum then just amounts to solving equation (4.19) together with (4.16) using the data from matrix model, for T^m giving a metastable vacuum. Solving these equations is nontrivial, since the relation between the flux parameters $N^{i'}$ on the right and the coefficients in the superpotential T^m we want on the left are non-linear, although we expect that the solutions do exist by the multipole argument we gave in section 4. We leave matrix model computations up to requisite orders as well as finding the actual metastable vacua by solving those equations for the future work.

6. Conclusion and generalizations

Summarizing, we found that turning on flux with support at infinity in local Calabi-Yau in type IIB induces superpotential for the moduli in the local Calabi-Yau, thus breaking $\mathcal{N} = 2$ of the Calabi-Yau compactification down to $\mathcal{N} = 2$. Then we demonstrated that one can create metastable vacua by tuning the flux at infinity using the OOP mechanism, using a Dijkgraaf-Vafa (CIV-DV) geometry as a primary example. The metastable vacua known to exist [24, 37] in perturbed Seiberg-Witten theory can also be understood in terms of metastable flux configuration.

Flux diverging at infinity may appear problematic, but in reality a local Calabi-Yau must be regarded as a local approximation of a larger compact Calabi-Yau and the flux at infinity has a natural interpretation there; there is flux floating around in the rest of the Calabi-Yau, which "leaks" into our local Calabi-Yau and just appear to be coming in from infinity. This, furthermore, motivates a more natural setting to realize metastable flux vacua: in a part, say on the right side, of the full Calabi-Yau $\mathcal{M}$, there are some 3-cycles threaded by flux (and possibly O-planes to cancel net charge if $\mathcal{M}$ is compact) and on the left side there are some 3-cycles without flux through them. If the distance between the left and right sectors is large, the full Calabi-Yau $\mathcal{M}$ factorizes into an almost decoupled system of $\mathcal{M}_L$ and $\mathcal{M}_R$, and the flux in $\mathcal{M}_R$ appears to be flux at infinity from the viewpoint of $\mathcal{M}_L$ and induces superpotential in $\mathcal{M}_L$. By adjusting the number of fluxes in $\mathcal{M}_R$, we can tune the superpotential and generate metastable vacua in $\mathcal{M}_L$. This is a very well controlled setting to analyze flux vacua, which may shed light on the structure of the nonsupersymmetric landscape of string vacua. We also made some steps toward actually embedding metastable vacua in a larger Calabi-Yau as sketched above in the case of Dijkgraaf-Vafa geometry by computing certain matrix model amplitudes. Actually finding explicit vacua along that line is an interesting open problem.

Note that we needed just two main ingredients to achieve this result. The OOP mechanism requires that the complex structure moduli space is special Kähler, and it is important that a superpotential for flux is very much controllable by tuning the flux, such as the Gukov-Vafa-Witten superpotential. This means that we can generalize the above story to any setting which fulfills these two requirements. Other possibilities therefore include M-theory and F-theory on Calabi-Yau fourfolds [76, 28]. Let us finish by saying a few words on these two setups.

Compactifying M-theory on a Calabi-Yau fourfold $\mathcal{M}_4$ with fluxes yields a three-dimensional low energy theory with 4 supercharges. The complex structure moduli of the Calabi-Yau are part of the chiral supermultiplets and are described by variations of the holomorphic $(4,0)$-form Ω. In the local limit where the fourfold becomes noncompact, the Kähler potential on the moduli space is given by

$$K = \int_{\mathcal{M}_4} \Omega \wedge \overline{\Omega}, \tag{6.1}$$

so that the metric on the moduli space is indeed special Kähler. Moreover, it is well-known that the complex moduli may be stabilized by turning on 4-form flux F_4, which introduces the superpotential

$$W = \int_{\mathcal{M}_4} F_4 \wedge \Omega. \tag{6.2}$$

The condition for unbroken supersymmetry is $W = dW = 0$, so that F_4 has to be a $(2,2)$-form. Stabilizing the Kähler moduli as well requires that the flux is primitive under the Lefschetz decomposition (and in particular self-dual). Turning on primitive $(2,2)$ flux on some compact 4-cycles, we can now follow an equivalent procedure as in IIB.

M-theory compactified on $\mathcal{M}_4$ is equivalent to compactifying F-theory on $\mathcal{M}_4 \times S^1$, at least if $\mathcal{M}_4$ is an elliptically fibered Calabi-Yau. This leads to a four-dimensional space-time with 4 supercharges. So again, the Kähler potential is given by (6.1), and the flux F_4 is a primitive $(2,2)$-form. The relation with IIB consistently reduces F_4 to a harmonic $(2,1)$-flux G_3. The extra seven-branes that must be inserted in IIB when reducing over a singular T^2 do not contribute to the superpotential and thus don't play an important role here.

In particular, consider as an example the local Calabi-Yau fourfold

$$u^2 + v^2 + w^2 + F(x,y) = 0, \tag{6.3}$$

where all variables are $\mathbb{C}$ (or $\mathbb{C}^*$) valued, and $F(x,y)$ defines a smooth curve in the x,y-plane. Its holomorphic four-form is given by

$$\Omega = \frac{du \wedge dv}{w} \wedge dx \wedge dy. \tag{6.4}$$

The u, v, w-fiber defines a two-sphere over each point in the x, y-plane, which shrinks to zero-size over the curve $F(x, y) = 0$.[27] Four-cycles can be constructed as an S^2 fibration over some disk D ending on the curve and have the topology of a four-sphere (when x and $y \in \mathbb{C}$). Notice that the intersection lattice is symmetric now and not simply symplectic anymore, so that the bilinear identity takes a more complicated form. However, like in the threefold case all relevant quantities reduce to the Riemann surface, and the analysis is similar as before.

Acknowledgments

We would like to thank J. de Boer, R. Dijkgraaf, Y. Ookouchi, and K. Saraikin for valuable discussions. J.M. would also like to thank the SITP at Stanford University for their kind hospitality during the completion of this work. The work of L.H. and M.S. was supported by an NWO Spinoza grant. The work of J.M. was supported in part by Department of Energy grant DE-FG03-92ER40701 and by a John A McCone postdoctoral fellowship. The work of K.P. was supported by Foundation of Fundamental Research on Matter (FOM).

A. Some basic results on Riemann surfaces

In this appendix we summarize some basic properties of Riemann surfaces [77].

A compact Riemann surface Σ_g is a one-dimensional compact complex manifold and its topology is completely characterized by its genus g. The middle cohomology group has $\dim H^1(\Sigma_g) = 2g$. The intersection form on $H_1(\Sigma_g, \mathbb{Z})$ is antisymmetric and by Poincaré duality unimodular, meaning that we can pick a basis of 1-cycles A^i, B_j with intersection:

$$A^i \cap A^j = 0, \quad B_i \cap B_j = 0, \quad A^i \cap B_j = \delta^i_j, \qquad i, j = 1, \dots, g. \tag{A.1}$$

Such a basis is unique up to a symplectic transformation in $\mathrm{Sp}(2g, \mathbb{Z})$.

Σ_g has a complex structure moduli space $\mathcal{M}_g$ with $\dim \mathcal{M}_g = 3g - 3$, $g \geq 2$.

A 1-form ω on a Riemann surface is called a *holomorphic differential* if in a local coordinate patch it has the form:

$$\omega = f(z) dz, \qquad f(z) : \text{ holomorphic.} \tag{A.2}$$

We will also consider *meromorphic differentials*, for which we allow the function $f(z)$ to have poles at certain points on the surface. Now we present a standard basis for holomorphic and meromorphic differentials on a general Riemann surface:

Holomorphic differentials[28] $\boldsymbol{\omega_i}$: Once we pick a symplectic basis of one-cycles, there is a canonical basis of holomorphic differentials ω_i, $i = 1, \dots, g$, with the following periods:

$$\frac{1}{2\pi i} \oint_{A^i} \omega_j = \delta^i_j, \qquad \frac{1}{2\pi i} \oint_{B_i} \omega_j = \tau_{ij}. \tag{A.3}$$

[27] Like in the Calabi-Yau threefold case, the real part of $F(x, y)$ changes sign when crossing the Riemann surface. This flop changes the parametrization of the compact S^2 in the T^*S^2-fiber from a "real" S^2 into an "imaginary" S^2.

[28] These are also called meromorphic differentials of the first kind.

The (symmetric) matrix τ_{ij} is the period matrix of the surface, which depends on the complex structure of Σ_g.

Meromorphic differentials of the second kind,[29] $\boldsymbol{\xi^P_{m\geq 1}}$**:** These are characterized by a point P on the surface where the differential has a pole of order $m+1$ with $m \geq 1$. They are normalized so that in local complex coordinates z where $z(P) = 0$ they have the Laurent expansion:

$$\xi^P_m \sim m\frac{dz}{z^{m+1}} + \text{regular}. \tag{A.4}$$

Meromorphic differentials of the third kind, $\boldsymbol{\xi_0^{P,P'}}$**:** characterized by two points P, P', where the differential has first order poles with opposite residues. Around P we have:

$$\xi_0^{P,P'} \sim \frac{dz}{z} + \text{regular} \tag{A.5}$$

and similarly around P' with the opposite sign.

Notice that we can always shift a meromorphic differential by a holomorphic differential without changing the singular part of the Laurent expansions (A.4), (A.5). We can eliminate this ambiguity by demanding that the A periods of the meromorphic differentials vanish:

$$\oint_{A^i} \xi^P_m = 0. \tag{A.6}$$

In general, it is not possible to simultaneously set the B periods to zero. Instead we have:

$$\oint_{B_i} \xi^P_m = K^P_{im}, \tag{A.7}$$

where the matrix K^P_{im} depends on the complex structure moduli of the Riemann surface and the position of the puncture P.

A.1 Hyperelliptic case

Let us consider the case where Σ_g is hyperelliptic. For example, the curve appearing in the Dijkgraaf-Vafa case, (3.32), can be written as:

$$y^2 = P_n(x)^2 - f_{n-1}(x), \qquad P_n(x) = \prod_{i=1}^n (x - \alpha_i). \tag{A.8}$$

This curve can be regarded as a two-sheeted x-plane with n cuts and two punctures, the latter corresponding to infinities on the two x-planes. Let us denote these points by ∞ and $\widetilde{\infty}$.

A basis of holomorphic differentials ω_i, $i = 1, \ldots, n-1$ can be constructed by

$$\omega_i = \frac{Q_i(x)}{y}dx = \frac{Q_i(x)}{\sqrt{P_n(x)^2 - f_{n-1}(x)}}dx, \tag{A.9}$$

[29] A more common notation in the literature for meromorphic differentials of the second and third kinds is $\boldsymbol{d\Omega^P_m}$ and $\boldsymbol{d\Omega_0^{P,P'}}$.

where $Q_i(x)$ is a polynomial of degree up to $n-2$ chosen so that (A.3) holds. Note that this ω_i goes as $\sim \mathcal{O}(x^{-2})dx$ as $x \to \infty, \widetilde{\infty}$, which means that this is regular at $x = \infty, \widetilde{\infty}$.

In the hyperelliptic case, it is convenient to take the meromorphic differentials of the second kind, ξ_m, as

$$\xi_m = \frac{R_m(x)}{y}dx = \frac{R_m(x)}{\sqrt{P_n(x)^2 - f_{n-1}(x)}}dx, \qquad m \geq 1. \tag{A.10}$$

Here, $R_m(x) = mx^{m+n-1} + \ldots$ is a polynomial and the coefficients of $x^{m+n-2}, \ldots, x^{n-1}$ are chosen so that

$$\xi_m = \pm \left[mx^{m-1} + \mathcal{O}(x^{-2})\right] dx, \qquad x \sim \infty, \widetilde{\infty} \tag{A.11}$$

is satisfied. This condition is similar to (A.4), but this ξ_m has poles at two points, $x = \infty, \widetilde{\infty}$, instead of one. The coefficients of $x^{n-2}, \ldots, x^0$ are chosen so that (A.6) is satisfied.

The meromorphic differentials of the third kind, ξ_0, can be defined likewise using a polynomial $R_0(x) = x^{n-1} + \ldots$, where the coefficients are chosen so that

$$\xi_0 = \frac{R_0(x)}{y}dx = \pm \left[\frac{1}{x} + \mathcal{O}(x^{-2})\right] dx, \qquad x \sim \infty, \widetilde{\infty} \tag{A.12}$$

holds and (A.6) is satisfied.

Let us derive a formula that will be useful in the main text. By expanding the right hand side of the trivial identity $0 = \int_{\Sigma_g} \omega_i \wedge \xi_m$ by the Riemann bilinear identity, one finds

$$\begin{aligned} 0 &= \sum_j \left(\int_{A^j} \omega_i \int_{B_j} \xi_m - \int_{A^j} \xi_m \int_{B_j} \omega_i \right) + \sum_{p=\infty,\widetilde{\infty}} \oint_p \omega_i \, d^{-1}\xi_m \\ &= K_{im} + \sum_{p=\infty,\widetilde{\infty}} \oint_p \omega_i \, d^{-1}\xi_m. \end{aligned} \tag{A.13}$$

Because the behaviors of ω_i, ξ_m at $x = \infty$ is the same as those at $x = \widetilde{\infty}$ up to a sign,

$$K_{im} = -\sum_{p=\infty,\widetilde{\infty}} \oint_p \omega_i \, d^{-1}\xi_m = -2 \oint_\infty \omega_i \, d^{-1}\xi_m = -2 \oint_\infty x^m \omega_i. \tag{A.14}$$

B. Parametric representation of genus 1 curves and sample computations

In this appendix, we review the parametric representation of the genus 1 Riemann surface Σ_{DV} of section 3.2 defined by

$$0 = F_{\mathrm{DV}}(x, y) = y^2 - \left[\left(x^2 - \frac{\Delta^2}{4}\right)^2 - b_0\right] \tag{B.1}$$

and its application to obtaining some of the results used therein. In particular, we think of Σ_{DV} as a copy of the standard fundamental domain with two marked points, a_1 and a_2, corresponding to the points at infinity on the two sheets. In figures 7(a) and 7(b), we depict both the standard visualization of Σ_{DV} as a double-sheeted cover of the x-plane as well as the parametric one, identifying the standard A and B cycles in the former and their realization in the latter.

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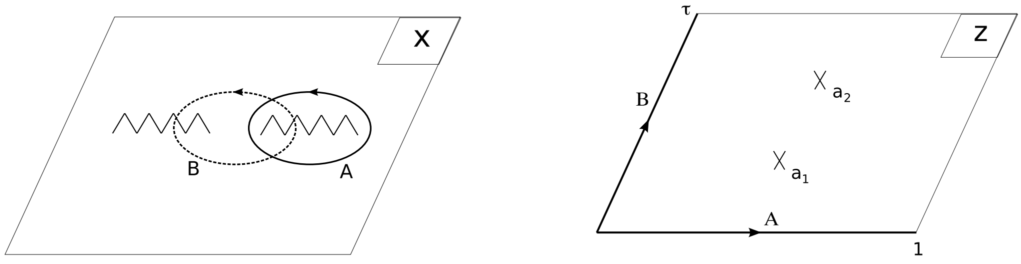


(a) (b)

Figure 7: (a) Depiction of $\Sigma_{\rm DV}$ as double cover of the x-plane with compact A and B cycles indicated. (b) Depiction of $\Sigma_{\rm DV}$ as fundamental domain in z-plane with the corresponding A and B cycles indicated.

B.1 Building blocks

The embedding of $\Sigma_{\rm DV}$ into xy space is obtained by specifying functions $x(z)$ and $y(z)$ which satisfy (B.1). The basic building blocks that we use to construct $x(z)$ and $y(z)$ are Janik's functions $F_i(z)$ [78]

$$F_i(z) \equiv \ln\theta(z - a_i - \tilde{\tau}) \qquad \tilde{\tau} = \frac{\tau+1}{2} \tag{B.2}$$

and their derivatives

$$F_i^{(n)}(z) \equiv \left(\frac{\partial}{\partial z}\right)^n F_i(z) \tag{B.3}$$

A detailed description of these functions, their properties, and several sample computations can be found in appendix C of [79]. For now, we simply note a few elementary facts. First, we point out that $F_i(z)$ introduces a branch point at a_i while $F_i^{(n)}$ introduces a pole of order n. For $n \geq 2$ these functions are elliptic while for $n = 0, 1$ they have the following monodromies

$$\begin{aligned} F_i(z+1) &= F_i(z) \\ F_i(z+\tau) &= F_i(z) + i\pi - 2\pi i(z - a_i) \\ F_i^{(1)}(z+1) &= F_i^{(1)}(z) \\ F_i^{(1)}(z+\tau) &= F_i^{(1)}(z) - 2\pi i \end{aligned} \tag{B.4}$$

It is also useful to record the relation between these functions and the Weierstrass σ, ζ, and $\wp$ functions

$$\begin{aligned} F(z) &= \ln\sigma(z) - \eta_1 z^2 + i\pi z + \ln\theta'(\tilde{\tau}) \\ F^{(1)}(z) &= \zeta(z) + i\pi - 2\eta_1 z \\ F^{(2)}(z) &= -\wp(z) - 2\eta_1 \\ F^{(n)}(z) &= -\left(\frac{\partial}{\partial z}\right)^{n-2}\wp(z) \qquad \text{n¿2} \end{aligned} \tag{B.5}$$

where

$$\eta_1 = \zeta\left(\frac{1}{2}\right) \tag{B.6}$$

Finally, we also recall the differential equation satisfied by $\wp(z)$

$$\left(\frac{\partial\wp(z)}{\partial z}\right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \tag{B.7}$$

which can also be taken as an implicit definition of the Weierstrass elliptic invariants g_2 and g_3.

B.2 The embedding functions $x(z)$ and $y(z)$

Using the building block functions $F_i^{(n)}(z)$, it is fairly easy to write down embedding functions $x(z)$ and $y(z)$ satisfying (B.1). Because $x(z)$ should be locally one-to-one near the marked points, we must construct it from functions with single poles, namely $F_1^{(1)}$ and $F_2^{(1)}$. On the other hand, $y(z) \sim x(z)^2$ near the marked points so it must contain functions with double poles, $F_1^{(2)}$ and $F_2^{(2)}$. This leads us to write[30]

$$\begin{aligned} x(z) &= X\left(F_1^{(1)} - F_2^{(1)} - \left[F^{(1)}(a) - i\pi\right]\right) \\ y(z) &= X^2\left(F_1^{(2)} - F_2^{(2)}\right) \end{aligned} \tag{B.8}$$

where

$$a \equiv a_2 - a_2 \tag{B.9}$$

Because elliptic functions such as $x(z)$ and $y(z)$ are completely determined by their pole structure, it is in fact quite easy to verify that

$$y(z)^2 = \left[\left(x(z)^2 - \frac{\Delta^2}{4}\right)^2 - b_1 x(z) - b_0\right] \tag{B.10}$$

where

$$\begin{aligned} \Delta^2 &= 12X^2\wp(a) \\ b_1 &= -4X^3\wp'(a) \\ b_0 &= X^4\left[12\wp(a)^2 - g_2\right] \end{aligned} \tag{B.11}$$

In order to obtain $b_1 = 0$ we set

$$a = \frac{\tau}{2} \tag{B.12}$$

We can solve for X in (B.11)

$$X^2 = \frac{12\wp(\tau/2)b_0}{\Delta^3\left[12\wp(\tau/2)^2 - g_2\right]} \tag{B.13}$$

and use this to eliminate X, thereby obtaining a direct relationship between Δ, b_0, and the complex modulus τ

$$b_0 = \frac{\Delta^4\left[12\wp(\tau/2)^2 - g_2\right]}{144\wp(\tau/2)^2} \tag{B.14}$$

[30] The constant term that we add to $x(z)$ is added for later convenience.

B.3 Two sample computations

We now describe two sample computations which illustrate the power of this approach. First, we will reproduce a result that is more easily obtained using the explicit representation (B.1). Next, we will consider a computation for which the parametric approach is simpler.

As our first example, let us consider the quantity

$$\Sigma = \frac{1}{4\pi i}\oint_{x=\infty} x^2\, dy \tag{B.15}$$

As we saw in section 3.2, this can be done quite easily using the explicit representation (B.1) with the result

$$\Sigma = \frac{b_0}{2} \tag{B.16}$$

We can also write this directly in terms of τ using (B.14)

$$\Sigma = \frac{\Delta^4\left[12\wp(\tau/2)^{-} g_2\right]}{288\wp(\tau/2)^2} \tag{B.17}$$

Let us now see how the result (B.16) can be obtained using the parametric representation. For this, we write

$$\Sigma = \frac{1}{4\pi i}\oint_{a_1} x(z)^2 \frac{\partial y(z)}{\partial z} dz = \frac{1}{4\pi i}\oint_{a_2} X^4 \left(F_1^{(1)} - F_2^{(1)} - \left[F^{(1)}(a) - i\pi\right]\right)^2 \left(F_1^{(3)} - F_2^{(3)}\right) dz \tag{B.18}$$

and expand the integrand near a_1. This is straightforward and leads to

$$x(z)^2\frac{\partial y(z)}{\partial z} \sim \frac{b_0}{12\wp(\tau/2)^2 - g_2}\left(\frac{2}{(z-a_1)^5} + \frac{4\wp(\tau/2)}{(z-a_1)^3} + \frac{12\wp(\tau/2)^2 - g_2}{z - a_1} + \mathcal{O}([z-a_1]^0)\right) \tag{B.19}$$

where we have used (B.11). The residue appearing in Σ is now easily read off with the desired result

$$\Sigma = \frac{b_0}{2} \tag{B.20}$$

Next, let us turn our attention to the computation of

$$S \equiv \frac{1}{2\pi i}\oint_{A^1} y\, dx \tag{B.21}$$

In the parametric formalism, we write this as

$$S \equiv \frac{1}{2\pi i}\int_{A^1} y(z)\,\frac{\partial x(z)}{\partial z}\, dz = \frac{1}{2\pi i}\int_{A^1} X^3\left(F_1^{(2)} - F_2^{(2)}\right)^2 \tag{B.22}$$

To evaluate this, we will write the integrand as a sum of quasi-elliptic functions and use their known monodromies (B.4). Given that the integrand has poles of degree at most 4 with even (odd) poles at a_1 and a_2 entering with identical (opposite) signs, the general form of this expansion is relatively simple

$$\left(F_1^{(2)} - F_2^{(2)}\right)^2 = a\left(F_1^{(4)} + F_2^{(4)}\right) + b\left(F_1^{(3)} - F_2^{(3)}\right) + c\left(F_1^{(2)} + F_2^{(2)}\right) + d\left(F_1^{(1)} - F_2^{(1)} + i\pi\right) + e \tag{B.23}$$

In terms of these expansion coefficients, the monodromies (B.4) lead to the simple result

$$S=\frac{X^3}{2\pi i}\left(i\pi d+e\right) \tag{B.24}$$

In practice, the coefficients $a,\ldots,e$ can be by comparing pole structures on the two sides of (B.23) with the following result when $a=\tau/2$

$$a=-\frac{1}{6},\quad b=0,\quad c=2\wp(\tau/2),\quad d=0,\quad e=\frac{2g_2}{3}+8\eta_1\wp(\tau/2)-4\wp(\tau/2)^2 \tag{B.25}$$

This means that S is actually given by the relatively simple expression

$$S=\frac{\Delta^3}{2\pi i\left[12\wp(\tau/2)\right]^{3/2}}\left(\frac{2g_2}{3}+8\eta_1\wp(\tau/2)-4\wp(\tau/2)^2\right) \tag{B.26}$$

B.4 Some useful identities

Finally, we close this appendix by listing a few derivative identities that were useful in section 3.2. First, some derivative identities

$$\begin{aligned}
\frac{\partial\zeta(z)}{\partial\tau}&=-\frac{1}{2\pi i}\left[\frac{1}{2}\wp'(z)+\zeta(z)\wp(z)-\frac{g_2z}{12}+2\eta_1\left(\zeta(z)-z\wp(z)\right)\right]\\
\frac{\partial\wp(z)}{\partial\tau}&=\frac{1}{2\pi i}\left[2\wp(z)^2+\zeta(z)\wp'(z)-\frac{g_2}{3}-2\eta_1\left(z\wp'(z)+2\wp(z)\right)\right]\\
\frac{\partial\eta_1}{\partial\tau}&=-\frac{1}{2\pi i}\left(2\eta_1^2-\frac{g_2}{24}\right)\\
\frac{\partial g_2}{\partial\tau}&=\frac{1}{2\pi i}\left(6g_3-8g_2\eta_1\right)
\end{aligned} \tag{B.27}$$

Several of these can be combined in order to derive the additional useful result

$$\frac{\partial}{\partial\tau}\left[\wp(\tau/2)\right]=\frac{1}{2\pi i}\left[2\wp(\tau/2)^2-\frac{g_2}{3}-4\eta_1\wp(\tau/2)\right] \tag{B.28}$$

We also remind the reader that the partial differential equation

$$\left(\frac{\partial\wp(z)}{\partial z}\right)^2=4\wp(z)^3-g_2\wp(z)-g_3 \tag{B.29}$$

combined with the fact that

$$\wp'(\tau/2)=0 \tag{B.30}$$

implies that the elliptic invariant g_3 can be written in terms of g_2 and $\wp(\tau/2)$ as

$$g_3=4\wp(\tau/2)^3-g_2\wp(\tau/2) \tag{B.31}$$

C. Independence of Σ_m's

In this appendix, we consider the Dijkgraaf-Vafa geometry and examine the dependence of Σ_m's on the moduli $S=\{S^i\}$, or equivalently, on the coefficients $b=\{b_i\}$ of the polynomial $f_{n-1}(x)$ as defined in (3.33). To apply the OOP mechanism and generate a metastable vacuum at a point $b_i = b_i^{(0)}$, it is needed that, when we expand $\Sigma_m(b)$'s around a point in $\Delta b_i \equiv b_i - b_i^{(0)}$, the coefficients of Δb_i, $\Delta b_i\,\Delta b_j$, $\Delta b_i\,\Delta b_j \Delta b_k$ terms are all independent and by taking linear combinations of $\Sigma_m(b)$'s we can obtain the OOP superpotential (2.35).

For simplicity, let us first discuss the case where we treat b_{n-1}, which is log-normalizable, as a dynamical modulus. In this case the number of moduli is n and the number of coefficients we would like to tune is, from (3.76),

$$C_n = \frac{n(n+1)(n+5)}{6}. \tag{C.1}$$

Explicitly, $\Sigma_m(b)$ is given by

$$\begin{aligned}\Sigma_m(b) &= \frac{1}{2\pi i m}\oint_\infty x^m dy = \frac{1}{m}\operatorname*{Res}_{x=\infty}\left[x^m \frac{2P_n(x)P_n'(x) - f_{n-1}'(x)}{2\sqrt{P_n(x)^2 - f_{n-1}(x)}}\right] \\ &= \frac{1}{M}\operatorname*{Res}_{x=\infty}\left[x^m\left(P_n'(x) - \frac{f_{n-1}'(x)}{2P_n(x)}\right)\right. \\ &\qquad\qquad \left.\times\left(1 + \frac{1}{2}\frac{f_{n-1}(x)}{P_n(x)^2} + \frac{3}{8}\left(\frac{f_{n-1}(x)}{P_n(x)^2}\right)^2 + \frac{5}{16}\left(\frac{f_{n-1}(x)}{P_n(x)^2}\right)^3 + \cdots\right)\right].\end{aligned} \tag{C.2}$$

So, $\Sigma_m(b)$ are polynomials in b_i's. If they are generic polynomials in b with high enough degree, then the expansion of $\Sigma_m(b)$ around a generic point $b^{(0)}$ in Δb will have different coefficients of Δb, $(\Delta b)^2$, $(\Delta b)^3$ terms, for different values of m. If this were the case, then the minimum number of Σ_m's we need to consider would be C_n in (C.1).

However, for small m, $\Sigma_m(p)$ is not a generic polynomial in b_i and we need to be careful. From (C.2), one can read off the following pattern of dependence of Σ_m's on b_i's:

- $\Sigma_{-n}, \ldots, \Sigma_0$ do not depend on b_k's, because the only contributions come from P_n'.
- A term with just one b_i ($i = 0, \ldots n-1$) appears in $P_n'(f_{n-1}'/P_n^2)$ and f_{n-1}'/P_n. Such a term has degree $i - n - 1$ in both cases and hence b_i first shows up in Σ_{n-i}. b_{n-1} appears in Σ_1 and b_0 appears in Σ_n.
- The combination $b_i b_j$ ($i, j = 0, \ldots n-1$) appears in $P_n'(f_{n-1}'/P_n^2)^2$ and $(f_{n-1}'/P_n)(f_{n-1}'/P_n^2)$. These terms have degree $i + j - 3n - 1$ and hence $b_i b_j$ first shows up in Σ_{3n-i-j}. b_{n-1}^2 appears in Σ_{n+2} and b_0^2 appears in Σ_{3n}.
- The combination $b_i b_j b_k$ ($i, j, k = 0, \ldots n-1$) appears in $P_n'(f_{n-1}'/P_n^2)^3$ and $(f_{n-1}'/P_n)(f_{n-1}'/P_n^2)^2$. These terms have degree $i + j + k - 5n - 1$ and hence $b_i b_j b_k$ first shows up in $\Sigma_{5n-i-j-k}$. b_0^3 appears in Σ_{2n+3} and b_0^3 appears in Σ_{5n}.

From these, we can see that we have to satisfy some requirements. Let us call m of Σ_m "order."

- We need all combinations of $\Delta b_i\,\Delta b_j\,\Delta b_k$, but b_0^3, which contains $(\Delta b_0)^3$, does not appear until order $A_n = 5n$.

- The number of possible cubic terms, $\Delta b_i\,\Delta b_j\,\Delta b_k$, is $n(n+1)(n+2)/6$. Cubic terms start to appear at order $2n+3$ and therefore, for all possible cubic terms to have chance of all showing up in a linear independent way, we need to wait until order $(2n+3)+n(n+1)(n+2)/6-1 \equiv B_n$.

- The number of possible quadratic terms and cubic terms is $n(n+1)/2+n(n+1)(n+2)/6$. Quadratic terms start to appear at order $n+2$ and cubic terms appear at higher order. Therefore, for all possible quadratic and cubic terms to have chance of all showing up in a linear independent way, we need to wait until order $(n+2)+n(n+1)/2+n(n+1)(n+2)/6-1 \equiv \widetilde{B}_n$.

- From (C.1), we need C_n independent coefficients. So, we need to wait until at least order C_n.

By looking at which of $A_n, B_n, \widetilde{B}_n, C_n$ is largest for given n, we find that we need Σ_m's at least up to $m_{\rm min}$, where

$$\begin{aligned} n&=1 &&\to && m_{\rm min}=5,\\ n&\geq 2 &&\to && m_{\rm min}=B_n=\frac{(n+1)(n+2)(n+3)}{6}. \end{aligned} \tag{C.3}$$

By a similar analysis, if b_{n-1} is regarded as a nondynamical parameter, we find the following:

$$\begin{aligned} n&=2 &&\to && m_{\rm min}=10,\\ n&=3 &&\to && m_{\rm min}=15,\\ n&\geq 4 &&\to && m_{\rm min}=\frac{n^3}{6}+\frac{n^2}{2}+\frac{n}{3}+3. \end{aligned} \tag{C.4}$$

D. Prepotential for Dijkgraaf-Vafa (CIV-DV) geometries

In this appendix, we first review different approaches to computing the prepotential $\mathcal{F}_0$ for the Dijkgraaf-Vafa (CIV-DV) geometries [57, 58, 63–65] given in eqs. (3.31), (3.32):

$$uv - F_{\rm DV}(x,y) = 0, \tag{D.1}$$

$$F_{\rm DV}(x,y) \equiv w^2 - \left[P_n(x)^2 - f_{n-1}(x)\right], \tag{D.2}$$

$$P_n(x) = W'(x) = g_{n+1}\prod_{i=1}^{n}(x-\alpha_i), \tag{D.3}$$

for arbitrary number of cuts n. Moreover, we will present $\mathcal{F}_0$ for general n up to S^5 terms. In the present paper, the prepotential is used in section 5 to evaluate the period matrix of the underlying hyperelliptic Riemann surface. However, the content of this appendix is almost independent of the main text and can be read separately.

The prepotential is physically important, because by putting fluxes in the Dijkgraaf-Vafa geometry one can realize supersymmetric $\mathcal{N}=1$ $\mathrm{U}(N)$ gauge theory, and its glueball superpotential which governs low energy dynamics can be computed from the prepotential [57, 58]. Furthermore, by the Dijkgraaf-Vafa relation [63–65], the prepotential is related to unitary matrix models. The relation to matrix models was studied also using supergraphs [80] and Konishi anomaly [54]. The same prepotential also underlies the physics of metastable brane-antibrane systems studied recently [9, 81, 79].

The first computation of the prepotential was performed in [57] for $n = 2$ (two cuts) up to S^5 terms by directly evaluating period integrals, where S is the glueball. For small values of n, the computation of $\mathcal{F}_0$ up to several orders in S is relatively easy, but computations for general number of cuts n require more systematic approaches. One such approach is to evaluate period integrals systematically; ref. [75] established a methodology, computing $\mathcal{F}_0$ for general n up to S^3 terms. Another approach is to use the relation to matrix models. One can evaluate the matrix integrals directly [82] or by a more sophisticated diagrammatic technique [74]. This matrix model approach turns out to be rather efficient in actual computations and indeed, in section D.1, we will compute $\mathcal{F}_0$ up to S^5 terms. Yet another approach is to use the relation to the Whitham hierarchy [83]. For other work on computations of $\mathcal{F}_0$ for general n, see [84–86].

We believe that the result of this appendix has various practical applications, including search for nonsupersymmetric vacua in $\mathcal{N} = 1$ gauge theories.

D.1 Matrix model

By the Dijkgraaf-Vafa relation [63–65], the prepotential $\mathcal{F}_0(S)$, $S = (S^1, \ldots, S^n)$, of the geometry (D.1) is related to the free energy of the associated $\mathrm{U}(N)$ matrix model,

$$Z = e^{-F_{mm}(g_s,N)} = \int d^{N^2}\Phi \, \exp\left[-\frac{1}{g_s} \operatorname{tr} W(\Phi)\right], \tag{D.4}$$

where Φ is an $N \times N$ matrix.[31] This matrix integral is performed around the vacuum where N^i eigenvalues of Φ sit at α_i. If we replace N^i in $F_{mm}(g_s, N)$ by S^i by the relation

$$g_s N^i = S^i, \tag{D.5}$$

then the free energy organize itself into a genus ('t Hooft) expansion. Namely,

$$F_{mm}\left(g_s, \frac{S}{g_s}\right) = \sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{F}_g(S). \tag{D.6}$$

As reviewed in section 5, one can evaluate the matrix integral (D.4) by perturbation theory using diagrams [74], as far as the perturbative part of F_{mm} is concerned. However, this quickly gets out of hand, particularly because for general n one can have p-point interaction vertices with arbitrarily large p, which makes the number of diagrams explode. Namely, if we expand Φ around the critical point $x = \alpha_i$, each coefficient $g_{i,p}$ in the expansion

$$W(\alpha_i + x) = W(\alpha_i) + \frac{m_i^2}{2} x^2 + \sum_{p=3}^{n+1} \frac{g_{i,p}}{p} x^p \tag{D.7}$$

[31] Here, the argument "N" in $F_{mm}(g_s, N)$ denotes $(N^1, \ldots, N^n)$ collectively, not to be confused with the rank $N = \sum_i N^i$ of the matrix Φ.

gives a p-vertex interaction, and p can be arbitrarily large for general n. Here,

$$m_i^2 = W''(\alpha_i), \qquad g_{i,p} = \frac{1}{(p-1)!} W^{(p)}(\alpha_i) \tag{D.8}$$

and $W^{(p)}$ is the pth derivative.

A more efficient method amenable to computer was proposed in [87–89], and here we generalize it to the case with an arbitrary number of cuts n. First note that the perturbative part of the matrix model free energy F_{mm} can be written as an expansion in the coupling constant g_s as:

$$F_{mm,pert}(g_s, N) = \sum_{k=1}^{\infty} g_s^k f_k(N). \tag{D.9}$$

Here, the order k amplitude $f_k(N)$ is a polynomial of degree $k+2$ in N^i's, which in turn has a genus expansion as follows:

$$f_k(N) = \sum_{g=0}^{[\frac{k+1}{2}]} A^{(k,g)}_{i_1 \dots i_{k-2g+2}} N^{i_1} \cdots N^{i_{k-2g+2}}, \tag{D.10}$$

where the coefficients $A^{(k,g)}_{i_1 i_2 \dots}$ are totally symmetric in $i_1, i_2, \dots,$ and $[x]$ is the integer part of x. For a given finite k, the number of coefficients $A^{(k,g)}_{i_1 i_2 \dots}$ in $f_k(N)$ is of course finite. Therefore, if we compute $f_k(N)$ for some small values of $\{N^i\}$ by computer, we can determine the coefficients $A^{(k,g)}_{i_1 i_2 \dots}$. Furthermore, there is symmetry under exchange of eigenvalues; for example, if we know $A^{(k,g)}_{1123}$, we can obtain $A^{(k,g)}_{2214}$ by the manipulation:

$$(\alpha_1, m_1, g_{1,p}) \leftrightarrow (\alpha_2, m_2, g_{2,p}), \qquad (\alpha_3, m_3, g_{3,p}) \to (\alpha_4, m_4, g_{4,p}).$$

This symmetry significantly reduces the number of "data points" $\{N^i\}$, for which we should evaluate the matrix integral in order to determine $f_k(N)$. In particular, this means that, if one knows $f_k(N)$ for $n = k+2$ cuts, then one can determine $f_k(N)$ for arbitrary number of cuts n by symmetry.

For actually evaluating matrix integrals, it is convenient to go to the eigenvalue basis [90]:

$$e^{-F_{mm}(g,N)} = \int d^N\lambda \left[\prod_{a<b}^{N} (\lambda_a - \lambda_b)^2 \right] \exp\left[-\frac{1}{g_s} \sum_{a=1}^{N} W(\lambda_a) \right], \tag{D.11}$$

where the Van der Monde determinant is from the change of variables [90]. We would like to compute this perturbatively around the vacuum where N^i of the eigenvalues λ_a's are equal to α_i, where $i = 1, \dots, n$. So, let us divide λ_a's into n groups and expand around α_i as:

$$\lambda_{ia} = \alpha_i + \mu_{ia}. \qquad i = 1, \dots, n, \qquad a = 1, \dots, N^i. \tag{D.12}$$

Then the matrix integral (D.11) is, up to a multiplicative constant,

$$\int d^N\mu \left[\prod_{i=1}^{n}\prod_{a<b}^{N^i}(\mu_{ia}-\mu_{ib})^2\right]\left[\prod_{i<j}^{n}\prod_{a=1}^{N^i}\prod_{b=1}^{N^j}(\mu_{ia}-\mu_{jb}+\alpha_{ij})^2\right] \\ \times \exp\left[-\frac{1}{g_s}\sum_{i=1}^{n}\sum_{a=1}^{N^i}\left(\frac{m_i^2}{2}\mu_{ia}^2+\sum_{p=3}^{n}\frac{g_{i,p}}{p}\mu_{ia}^p\right)\right], \tag{D.13}$$

where we used the expansion (D.7) and $\alpha_{ij}\equiv\alpha_i-\alpha_j$. Given $\{N^i\}$, we can evaluate this using computer by power expansion in g_s which, following the procedure sketched above, allows us to determine $f_k(N)$ order by order.

D.2 Result

By setting $\Delta_i\equiv m_i^2$, the first order result ($\mathcal{O}(N^3)$) is

$$f_1(N)=\sum_i\left(\frac{g_{i,4}}{2\Delta_i^2}-\frac{2g_{i,3}^2}{3\Delta_i^3}\right)(N^i)^3+\sum_{i\neq j}\left(\frac{2g_{i,3}}{\Delta_i^2\alpha_{ij}}+\frac{1}{\Delta_i\alpha_{ij}^2}-\frac{2}{\Delta_j\alpha_{ij}^2}\right)(N^i)^2N^j \\ +4\sum_{i<j<k}\left(\frac{1}{\Delta_i\alpha_{ij}\alpha_{ki}}+\frac{1}{\Delta_j\alpha_{jk}\alpha_{ij}}+\frac{1}{\Delta_k\alpha_{ki}\alpha_{jk}}\right)N^iN^jN^k \\ +\sum_i\left(\frac{g_{i,4}}{4\Delta_i^2}-\frac{g_{i,3}^2}{6\Delta_i^3}\right)N^i, \tag{D.14}$$

The coupling constants $g_{i,p}$ can be expressed in terms of α^i using (D.8). The terms cubic in N^i are planar amplitude, while the ones linear in N^i are genus 1 (torus) amplitude. This agrees with the known result [75], if we use identities

$$\frac{g_{i,4}}{\Delta_i^2}=g_{n+1}\frac{1}{\Delta_i}\sum_{\substack{j<k\\ j,k\neq i}}\frac{1}{\alpha_{ij}\alpha_{ik}},\qquad \frac{g_{i,3}}{\alpha_{ij}\Delta_i^2}=g_{n+1}\left(\frac{1}{\alpha_{ij}^2\Delta_i}+\frac{1}{\alpha_{ij}\Delta_i}\sum_{k\neq i,j}\frac{1}{\alpha_{ik}}\right), \\ \frac{g_{i,3}^2}{\Delta_i^3}=g_{n+1}^2\left(-\sum_{j\neq i}\frac{1}{\alpha_{ij}^2\Delta_j}+\frac{1}{\Delta_i}\sum_{\substack{j<k\\ j,k\neq i}}\frac{1}{\alpha_{ij}\alpha_{ik}}\right), \tag{D.15}$$

upon using which (D.14) becomes, after setting $g_{n+1}=1$,

$$f_1(N)=\frac{2}{3}\sum_i\left(\sum_{j\neq i}\frac{1}{\alpha_{ij}^2\Delta_j}-\frac{1}{4\Delta_i}\sum_{\substack{j<k\\ j,k\neq i}}\frac{1}{\alpha_{ij}\alpha_{ik}}\right)(N^i)^3 \\ +\sum_{i\neq j}\left(\frac{3}{\Delta_i\alpha_{ij}^2}-\frac{2}{\Delta_j\alpha_{ij}^2}+\frac{2}{\alpha_{ij}\Delta_i}\sum_{k\neq i,j}\frac{1}{\alpha_{ik}}\right)(N^i)^2N^j \\ +4\sum_{i<j<k}\left(\frac{1}{\Delta_i\alpha_{ij}\alpha_{ki}}+\frac{1}{\Delta_j\alpha_{jk}\alpha_{ij}}+\frac{1}{\Delta_k\alpha_{ki}\alpha_{jk}}\right)N^iN^jN^k \\ +\sum_i\left(\frac{1}{6}\sum_{j\neq i}\frac{1}{\alpha_{ij}\Delta_j^2}+\frac{1}{12\Delta_i}\sum_{\substack{j<k\\ j,k\neq i}}\frac{1}{\alpha_{ij}\alpha_{ik}}\right)N^i. \tag{D.16}$$

The second order result ($\mathcal{O}(N^4)$) is much more lengthy. Let us write it as:

$$\begin{aligned} f_2(N) = &\sum_i a_{iiii} N_i^4 + \sum_{i \neq j} a_{iiij} N_i^3 N_j + \sum_{i<j} a_{iijj} N_i^2 N_j^2 \\ &\qquad + \sum_{\substack{i,j,k \\ j<k}} a_{iijk} N_i^2 N_i N_j + \sum_{i<j<k<l} a_{ijkl} N_i N_j N_k N_l \\ &+ \sum_i b_{ii} N_i^2 + \sum_{i<j} b_{ij} N_i N_j . \end{aligned} \tag{D.17}$$

Then the coefficients a_{iiii}, etc. are:

$$a_{iiii} = + \frac{5 g_{i,6}}{6 \Delta_i^3} - \frac{3 g_{i,5} g_{i,3}}{\Delta_i^4} - \frac{9 g_{i,4}^2}{8 \Delta_i^4} + \frac{6 g_{i,4} g_{i,3}^2}{\Delta_i^5} - \frac{8 g_{i,3}^4}{3 \Delta_i^6},$$

$$\begin{aligned} a_{iiij} = &+ \frac{4 g_{i,5}}{\alpha_{ij} \Delta_i^3} - \frac{12 g_{i,4} g_{i,3}}{\alpha_{ij} \Delta_i^4} - \frac{2 g_{i,4}}{\alpha_{ij}^2 \Delta_i^3} + \frac{8 g_{i,3}^3}{\alpha_{ij} \Delta_i^5} + \frac{4 g_{i,3}^2}{\alpha_{ij}^2 \Delta_i^4} + \frac{8 g_{i,3}}{3 \alpha_{ij}^3 \Delta_i^3} - \frac{8 g_{j,3}}{3 \alpha_{ij}^3 \Delta_j^3} - \frac{4 g_{i,3}}{\alpha_{ij}^3 \Delta_i^2 \Delta_j} \\ &+ \frac{1}{\alpha_{ij}^4 \Delta_i^2} + \frac{4}{\alpha_{ij}^4 \Delta_j^2} - \frac{4}{\alpha_{ij}^4 \Delta_i \Delta_j}, \end{aligned}$$

$$\begin{aligned} a_{iijj} = &+ \frac{6 g_{i,4}}{\alpha_{ij}^2 \Delta_i^3} + \frac{6 g_{j,4}}{\alpha_{ij}^2 \Delta_j^3} - \frac{8 g_{i,3}^2}{\alpha_{ij}^2 \Delta_i^4} - \frac{8 g_{j,3}^2}{\alpha_{ij}^2 \Delta_j^4} - \frac{2 g_{i,3} g_{j,3}}{\alpha_{ij}^2 \Delta_i^2 \Delta_j^2} \\ &- \frac{8 g_{i,3}}{\alpha_{ij}^3 \Delta_i^3} + \frac{8 g_{j,3}}{\alpha_{ij}^3 \Delta_j^3} + \frac{2 g_{i,3}}{\alpha_{ij}^3 \Delta_i^2 \Delta_j} - \frac{2 g_{j,3}}{\alpha_{ij}^3 \Delta_i \Delta_j^2} - \frac{5}{\alpha_{ij}^4 \Delta_i^2} - \frac{5}{\alpha_{ij}^4 \Delta_j^2} + \frac{11}{\alpha_{ij}^4 \Delta_i \Delta_j}, \end{aligned}$$

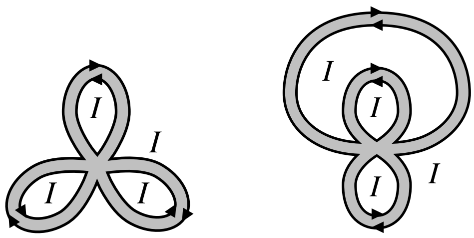
$$\begin{aligned} a_{iijk} = &+ \frac{12 g_{i,4}}{\alpha_{ij} \alpha_{ik} \Delta_i^3} - \frac{16 g_{i,3}^2}{\alpha_{ij} \alpha_{ik} \Delta_i^4} \\ &- \frac{8 g_{i,3}}{\alpha_{ij}^2 \alpha_{ik} \Delta_i^3} - \frac{8 g_{i,3}}{\alpha_{ik}^2 \alpha_{ij} \Delta_i^3} + \frac{8 g_{j,3}}{\alpha_{ij}^2 \alpha_{jk} \Delta_j^3} - \frac{8 g_{k,3}}{\alpha_{ik}^2 \alpha_{jk} \Delta_k^3} + \frac{4 g_{i,3}}{\alpha_{ij}^2 \alpha_{jk} \Delta_i^2 \Delta_j} - \frac{4 g_{i,3}}{\alpha_{ik}^2 \alpha_{jk} \Delta_i^2 \Delta_k} \\ &- \frac{8}{\alpha_{ij} \alpha_{ik} \alpha_{jk}^2 \Delta_j \Delta_k} - \frac{8}{\alpha_{ij}^3 \alpha_{jk} \Delta_j^2} + \frac{8}{\alpha_{ij}^3 \alpha_{ik} \Delta_i \Delta_j} + \frac{8}{\alpha_{ik}^3 \alpha_{jk} \Delta_k^2} + \frac{8}{\alpha_{ik}^3 \alpha_{ij} \Delta_i \Delta_k} \\ &- \frac{4}{\alpha_{ij}^3 \alpha_{ik} \Delta_i^2} + \frac{4}{\alpha_{ij}^3 \alpha_{jk} \Delta_i \Delta_j} - \frac{4}{\alpha_{ik}^3 \alpha_{ij} \Delta_i^2} - \frac{4}{\alpha_{ik}^3 \alpha_{jk} \Delta_i \Delta_k} \\ &+ \frac{4}{\alpha_{ij}^2 \alpha_{jk}^2 \Delta_j^2} - \frac{2}{\alpha_{ij}^2 \alpha_{ik}^2 \Delta_i^2} + \frac{4}{\alpha_{ik}^2 \alpha_{jk}^2 \Delta_k^2}, \end{aligned}$$

$$\begin{aligned} a_{ijkl} = &+ \frac{16 g_{i,3}}{\alpha_{ij} \alpha_{ik} \alpha_{il} \Delta_i^3} - \frac{16 g_{j,3}}{\alpha_{ij} \alpha_{jk} \alpha_{jl} \Delta_j^3} + \frac{16 g_{k,3}}{\alpha_{ik} \alpha_{jk} \alpha_{kl} \Delta_k^3} - \frac{16 g_{l,3}}{\alpha_{il} \alpha_{jl} \alpha_{kl} \Delta_l^3} \\ &- \frac{8}{\alpha_{ij}^2 \alpha_{il} \alpha_{jk} \Delta_i \Delta_j} - \frac{8}{\alpha_{ij}^2 \alpha_{ik} \alpha_{jl} \Delta_i \Delta_j} + \frac{8}{\alpha_{ik}^2 \alpha_{il} \alpha_{jk} \Delta_i \Delta_k} - \frac{8}{\alpha_{ik}^2 \alpha_{ij} \alpha_{kl} \Delta_i \Delta_k} \\ &+ \frac{8}{\alpha_{il}^2 \alpha_{ik} \alpha_{jl} \Delta_i \Delta_l} + \frac{8}{\alpha_{il}^2 \alpha_{ij} \alpha_{kl} \Delta_i \Delta_l} + \frac{8}{\alpha_{jk}^2 \alpha_{ik} \alpha_{jl} \Delta_j \Delta_k} + \frac{8}{\alpha_{jk}^2 \alpha_{ij} \alpha_{kl} \Delta_j \Delta_k} \\ &- \frac{8}{\alpha_{jl}^2 \alpha_{ij} \alpha_{kl} \Delta_j \Delta_l} + \frac{8}{\alpha_{jl}^2 \alpha_{il} \alpha_{jk} \Delta_j \Delta_l} - \frac{8}{\alpha_{kl}^2 \alpha_{il} \alpha_{jk} \Delta_k \Delta_l} - \frac{8}{\alpha_{kl}^2 \alpha_{ik} \alpha_{jl} \Delta_k \Delta_l} \\ &+ \frac{8}{\alpha_{ij}^2 \alpha_{ik} \alpha_{il} \Delta_i^2} + \frac{8}{\alpha_{ij}^2 \alpha_{jk} \alpha_{jl} \Delta_j^2} + \frac{8}{\alpha_{ik}^2 \alpha_{ij} \alpha_{il} \Delta_i^2} - \frac{8}{\alpha_{ik}^2 \alpha_{jk} \alpha_{kl} \Delta_k^2} \\ &+ \frac{8}{\alpha_{il}^2 \alpha_{ij} \alpha_{ik} \Delta_i^2} + \frac{8}{\alpha_{il}^2 \alpha_{jl} \alpha_{kl} \Delta_l^2} - \frac{8}{\alpha_{jk}^2 \alpha_{ij} \alpha_{jl} \Delta_j^2} - \frac{8}{\alpha_{jk}^2 \alpha_{ik} \alpha_{kl} \Delta_k^2} \end{aligned}$$

$$
\begin{aligned}
&- \frac{8}{\alpha_{jl}^2 \alpha_{ij} \alpha_{jk} \Delta_j^2} + \frac{8}{\alpha_{jl}^2 \alpha_{il} \alpha_{kl} \Delta_l^2} + \frac{8}{\alpha_{kl}^2 \alpha_{ik} \alpha_{jk} \Delta_k^2} + \frac{8}{\alpha_{kl}^2 \alpha_{il} \alpha_{jl} \Delta_l^2} \\
b_{ii} =& + \frac{5 g_{i,6}}{3 \Delta_i^3} - \frac{4 g_{i,5} g_{i,3}}{\Delta_i^4} + \frac{13 g_{i,4} g_{i,3}^2}{2 \Delta_i^5} - \frac{15 g_{i,4}^2}{8 \Delta_i^4} - \frac{7 g_{i,3}^4}{3 \Delta_i^6} \\
b_{ij} =& + \frac{2 g_{i,5}}{\alpha_{ij} \Delta_i^3} - \frac{2 g_{j,5}}{\alpha_{ij} \Delta_j^3} - \frac{4 g_{i,4} g_{i,3}}{\alpha_{ij} \Delta_i^4} + \frac{4 g_{j,3} g_{j,4}}{\alpha_{ij} \Delta_j^4} - \frac{g_{i,4}}{\alpha_{ij}^2 \Delta_i^3} - \frac{g_{j,4}}{\alpha_{ij}^2 \Delta_j^3} \\
&+ \frac{2 g_{i,3}^3}{\alpha_{ij} \Delta_i^5} - \frac{2 g_{j,3}^3}{\alpha_{ij} \Delta_j^5} + \frac{g_{i,3}^2}{\alpha_{ij}^2 \Delta_i^4} + \frac{g_{j,3}^2}{\alpha_{ij}^2 \Delta_j^4} + \frac{2 g_{i,3}}{3 \alpha_{ij}^3 \Delta_i^3} - \frac{2 g_{j,3}}{3 \alpha_{ij}^3 \Delta_j^3} \\
&- \frac{2}{\alpha_{ij}^4 \Delta_i \Delta_j} + \frac{1}{2 \alpha_{ij}^4 \Delta_i^2} + \frac{1}{2 \alpha_{ij}^4 \Delta_j^2}
\end{aligned}
\tag{D.18}
$$

The third order result ($\mathcal{O}(N^5)$) is too lengthy to be included here. The interested reader can find the result in the Mathematica file included in the source file for the current paper at arXiv.org.

It is easy to identify the matrix model diagrams corresponding to each term in the above result. For example, the first term in a_{iiii} in the second order result (D.18), $\frac{5g_{i,6}}{6\Delta_i^3}$ comes from the following two planar diagrams:



$-2 \cdot (-\frac{g_{i,6}}{6}) \cdot \frac{1}{\Delta_i^3}$ $\qquad$ $-3 \cdot (-\frac{g_{i,6}}{6}) \cdot \frac{1}{\Delta_i^3}$

Note that the definitions (D.4) means that the coupling constants $g_{i,p}$ enter the free energy F_{mm} with a sign as follows:

$$
- \prod_{i,p} (-g_{i,p}). \tag{D.19}
$$

This is in addition to the signs coming from the fermionic ghosts.

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