A New Elastic Potential Function for Rubbery Materials

(strain energy density/strain measure/rubbers/elastomers)

P. J. BLATZ*, S. C. SHARDA†, AND N. W. TSCHOEGL†‡

* Shock Hydrodynamics, Sherman Oaks, California 91403; and † California Institute of Technology, Pasadena, California 91109

ABSTRACT   A new four-parameter elastic potential function is proposed which represents data on the elastic deformation of rubbery materials with the same parameters in various deformation fields up to break.

We wish to report on a new elastic potential function for rubbery materials which permits one to predict deformations up to break in various deformation fields if the four material parameters of the potential function are known. Our elastic potential (or strain energy density) function is based on the generalized measure of strain introduced by Seth (1). In Lagrangean coordinates, this measure of strain is expressed as

$$E_\alpha = (\lambda_\alpha{}^n - 1)/n \qquad \alpha = 1, 2, 3 \qquad [1]$$

where E is the strain, λ is the stretch ratio, the αs denote the three principal directions, and n is a material parameter. Seth's measure of strain is based on the realization that there is no unique definition of strain; rather, the most convenient strain measure is a property of the material and of the geometry of the deformation.

The elastic potential of an isotropic material is customarily formulated in terms of three invariants of the stretch ratios. These are generally taken to be the principal invariants, I_1, I_2, and I_3, of the right Cauchy–Green deformation tensor. However, this choice is not unique. One may choose any three symmetric functions of the stretch ratios. In a general sense one can define invariants on perfectly arbitrary functions of the stretch ratios, namely

$$I = \sum_\alpha w(\lambda_\alpha) \qquad [2]$$

$$II = \frac{1}{2} \sum_{\alpha,\beta} w(\lambda_\alpha)w(\lambda_\beta) \qquad \alpha \neq \beta \qquad [3]$$

and

$$III = \prod_\alpha w(\lambda_\alpha) \qquad [4]$$

The most general elastic potential, therefore, is

$$W = W(I, II, III) \qquad [5]$$

in which I, II, and III are given by Eqs. 2 through 4. A special subset of the possible forms of the elastic potential represented by Eq. 5 is

$$W = W(I) \qquad [6]$$

‡ To whom reprint requests should be addressed.

We have replaced I in Eq. 6 by

$$I_E = \sum_\alpha E_\alpha \qquad [7]$$

the first principal invariant of the Lagrangean strain tensor based on Seth's definition of strain. By requiring that W conform to Hooke's law in infinitesimal deformation, we obtain

$$W = (2G/n)I_E \qquad [8]$$

where G is the shear modulus of the material in infinitesimal deformation. This relatively simple elastic potential function accurately represents data on elastomeric materials in different deformation fields (simple tension, biaxial tension, simple compression, pure shear, torsion, etc.) up to stretch ratios of about 3 in simple tension.

A particularly simple form of Eq. 6, namely,

$$W = I \qquad [9]$$

was chosen by Valanis and Landel (2) to represent data on natural rubber. The same approach has since been used by Dickie and Smith (3) and by Kawabata (4).

Eq. 9, when written out explicitly, becomes

$$W = w(\lambda_1) + w(\lambda_2) + w(\lambda_3) \qquad [10]$$

This equation was found (2–4) to represent data in different deformation fields up to about the same stretch ratios for which Eq. 8 also gives an adequate description of the data. Eq. 10 implies separability of the elastic potential into three principal contributions based on the single function, $w(\lambda_\alpha)$. Eq. 8 contains the same feature and it is, in fact, easily shown that

$$w(\lambda_\alpha) = 2G(\lambda_\alpha{}^n - 1)/n \qquad [11]$$

One can evaluate $w(\lambda_\alpha)$ as a function of λ_α experimentally (2–5). In moderate deformations, therefore, the two approaches represented by Eqs. (8) and (10) are equivalent, and a choice can be made only on the grounds of simplicity. There is no known basis in thermodynamics or statistical mechanics for concluding that the elastic potential must be separable. Indeed, we know of no demonstration that a separable potential function can represent data in different deformation fields up to the point of rupture.

We have found that such a representation is obtained from the potential function

$$W = (2G/n)I_E + BI_E{}^m \qquad [12]$$

where B and m are two additional material parameters. Eq. 12 is a special case of Eq. 6 in which the elastic potential
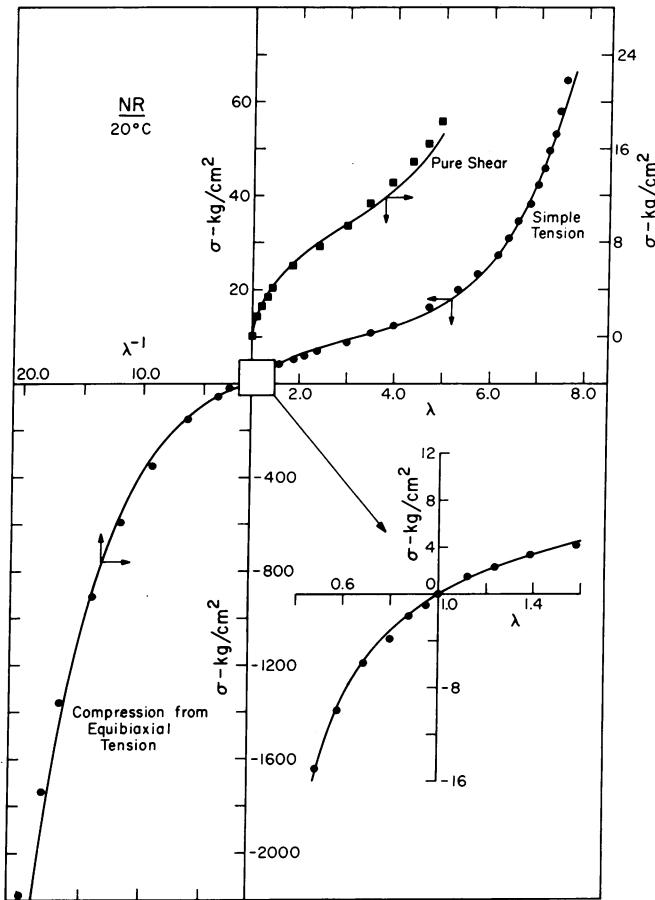
FIG. 1.   Data of Treloar on natural rubber fitted by Eq. **13**.

cannot be separated into three principal contributions. The potential is, however, distinguished by its simplicity. The

parameters G, n, B, and m may be obtained conveniently from measurements in simple tension. The same parameters will then describe the behavior in other deformation fields from the equation

$$(\bar{\sigma}_\alpha - \bar{\sigma}_\beta) = (\lambda_\alpha{}^n - \lambda_\beta{}^n)[(2G/n) + mBI_E{}^{m-1}]   \quad [13]$$

where $\bar{\sigma}$ is the true stress. Eq. **13** is thus a true constitutive equation.

A full discussion of Eqs. **12** and **13** and their application to experimental data will not be presented here. Here we only show the application of Eq. **13** to the data obtained by Treloar (6) on natural rubber at 20°C in simple tension, pure shear (maximum stress only), and two-dimensional tension. The latter data were recalculated for simple compression. The parameters, $G = 4.0\,\text{kg/cm}^2$, $n = 1.64$, $B = 8.3 \times 10^{-4}\,\text{kg/cm}^2$, and $m = 4.0$, were derived from the data in simple tension, and used to predict the observed behavior (*solid lines*) in simple tension and compression, and in pure shear. The fit is shown in Fig. 1. The behavior near $\lambda = 1$ is shown enlarged in the *insert*. A similarly excellent fit has been obtained on the data of Dickie and Smith (3) on styrene–butadiene rubber (Blatz, P. J., Sharda, S. C. & Tschoegl, N. W., unpublished).

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