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Subleading shape-function effects and the extraction of $|V_{ub}|$

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We derive a class of formulae relating moments of $B \to X_u \ell \bar{\nu}$ to $B \to X_s \gamma$ in the shape-function region, where $m_X^2 \sim m_b \Lambda_{\rm QCD}$. We also derive an analogous class of formulae involving the decay $B \to X_s \ell^+ \ell^-$. These results incorporate $\Lambda_{\rm QCD}/m_b$ power corrections, but are independent of leading and subleading hadronic shape functions. Consequently, they enable one to determine $|V_{ub}|/|V_{tb} V_{ts}^*|$ to subleading order in a model-independent way.

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I. INTRODUCTION

The study of decays of the B meson allows us to probe QCD and flavor physics. The program's goals include, on the one hand, precision measurements of standard model parameters and, on the other hand, searches for new physics. Short-distance physics is encoded in Wilson coefficients of local operators. By comparing measurements of these coefficients with theoretical predictions, signals of new physics may be found. High sensitivity to new physics is provided by the so-called rare decays, namely, those channels involving flavor-changing neutral currents, since they do not occur at tree level in the standard model. Measurements of the inclusive rare process $B \to X_s \gamma$ [1–5] have provided significant constraints on extensions to the standard model. The more complicated decay $B \to X_s \ell^+ \ell^-$ is complementary to $B \to X_s \gamma$, as its effective Hamiltonian includes two extra operators. Moreover, additional observables are available, such as the q^2 spectrum and the forward-backward asymmetry, which have been the focus of much work. Recently, it was noted that an angular decomposition provides a third observable sensitive to a different combination of Wilson coefficients [6]. Belle and *BABAR* have already made initial measurements of $B \to X_s \ell^+ \ell^-$ [7,8].

Precision measurements also provide determinations of elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix or, equivalently, the angles and sides of the unitarity triangle. By overconstraining these, the flavor structure of the standard model is subjected to rigorous examination. For the decay $B \to X_c \ell \bar{\nu}$, experimental and theoretical uncertainties are under control, and consequently $|V_{cb}|$ is one of the best-determined elements of the CKM matrix. From $B \to X_u \ell \bar{\nu}$, we can also determine $|V_{ub}|$ [9–12].

However, inclusive B decays often require a trade-off between theoretical and experimental difficulty: if phase-space cuts are necessary experimentally, then the spectra will be less inclusive and the corresponding theory more complicated. In this respect, $B \to X_c \ell \bar{\nu}$ and $B \to X_u \ell \bar{\nu}$ are markedly different. The former is sufficiently inclusive to enable the use of a local operator product expansion (OPE) [13], in which nonperturbative corrections appear as an expansion in inverse powers of m_b. This formalism has been calculated to order $1/m_b^3$ [14] (and recently to order $1/m_b^4$ [15]), with the relevant nonperturbative matrix elements defined via the heavy quark effective theory (HQET) [16–18]. In contrast, in $B \to X_u \ell \bar{\nu}$ experimental cuts (e.g. cuts on E_ℓ or m_X^2) are required in order to eliminate the dominant $b \to c$ background. In many cases, we are restricted to a region in which $m_X^2 \sim m_b \Lambda_{\rm QCD}$ and the local OPE breaks down. In this so-called endpoint or shape-function region [19], the set of outgoing hadronic states becomes jetlike and the relevant degrees of freedom are collinear and ultrasoft modes. The soft-collinear effective theory (SCET) [20–23] is then a powerful theoretical method.

Similarly, $B \to X_s \gamma$ measurements employ a cut on the photon energy. In Refs. [24,25] it was shown that the shape-function region is also relevant for $B \to X_s \ell^+ \ell^-$. Here, cuts are made in the dileptonic mass spectrum to remove the largest $c\bar{c}$ resonances, namely, the J/Ψ and Ψ'. These leave two perturbative windows, the low-q^2 and high-q^2 regions. At low q^2, where the rate is higher, an additional cut is needed: a hadronic invariant-mass cut is imposed in order to eliminate the background $b \to c(\to s\ell^+ \nu)\ell^- \bar{\nu}$.

At leading order (LO) in $\Lambda_{\rm QCD}/m_b$, decay rates now depend upon a nonperturbative, and hence analytically incalculable, shape function. However, this function is process independent and appears in both $B \to X_u \ell \bar{\nu}$ and $B \to X_s \gamma$, for example. One can thus measure the leading-order shape function from the photon energy spectrum of $B \to X_s \gamma$ and use the result in the $B \to X_u \ell \bar{\nu}$ spectrum, or, more directly, express the semileptonic rate in terms of the radiative rate instead of the shape function [26–29]. In this way, model dependence can be avoided in the determination of $|V_{ub}|$.

At subleading order, the situation is far more complicated, with several universal shape functions occurring in different combinations [30–35]. In this paper, we construct combinations of shape-function-dependent decay rates that

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are protected from nonperturbative effects to second order in the power expansion. Through this procedure, we obtain formulae for $|V_{ub}|/|V_{tb}V_{ts}^*|$ that are free from the hadronic uncertainties arising from the leading and subleading shape functions. This method uses moments of the fully differential decay spectra of $B \to X_u \ell \bar{\nu}$ and $B \to X_s \gamma$ (and, optionally, $B \to X_s \ell^+ \ell^-$).

The rest of this paper is organized as follows. In Sec. II, together with Appendices A and B, we present the basic formalism needed for our work. This includes power corrections for the triply differential decay spectra of the semileptonic processes and the photon energy spectrum of $B \to X_s \gamma$. In Sec. III, we derive and discuss our results, eliminating shape functions from expressions for $|V_{ub}|$ at next-to-leading order (NLO). We conclude in Sec. IV.

II. FORMALISM

In this section, we briefly review the formalism and results from Refs. [24,32,36] that we shall use in this paper (see these references for further details).

The inclusive decay rate for $\bar{B} \to X_u \ell \bar{\nu}$ ($\bar{B} \to X_s \gamma$) is proportional to $W_{\mu\nu} L^{\mu\nu}$, where $L^{\mu\nu}$ is the leptonic (photonic) tensor and $W_{\mu\nu}$ is the hadronic tensor, which can be written as

$$\begin{aligned} W_{\mu\nu} &= \frac{1}{2m_B} \sum_X (2\pi)^3 \delta^4(p_B - q - p_X) \langle \bar{B} | J_\mu^\dagger | X \rangle \langle X | J_\nu | \bar{B} \rangle \\ &= -g_{\mu\nu} W_1 + v_\mu v_\nu W_2 + i\epsilon_{\mu\nu\alpha\beta} v^\alpha q^\beta W_3 \\ &\quad + q_\mu q_\nu W_4 + (v_\mu q_\nu + v_\nu q_\mu) W_5. \end{aligned} \tag{1}$$

Here, v^μ is the velocity of the B meson and q^μ is the $\ell\bar{\nu}$ (γ) momentum. We use the hadronic current J (e.g. $J_\mu^u = \bar{u}\gamma_\mu P_L b$ for $B \to X_u \ell \bar{\nu}$) and relativistic normalization for the $|\bar{B}\rangle$ states. Similarly, the inclusive decay rate for $\bar{B} \to X_s \ell^+(p^+) \ell^-(p^-)$ is proportional to $(W^L_{\mu\nu} L_L^{\mu\nu} + W^R_{\mu\nu} L_R^{\mu\nu})$, where $L^{\mu\nu}_{L(R)} = 2[p_+^\mu p_-^\nu + p_-^\mu p_+^\nu - g^{\mu\nu} p_+ \cdot p_- \mp i\epsilon^{\mu\nu\alpha\beta} p_{+\alpha} p_{-\beta}]$ and $W^{L(R)}_{\mu\nu}$ can be defined analogously to Eq. (1), in terms of a current $J^{L(R)}$ [37].

Contracting $L^{\mu\nu}$ with $W^{\mu\nu}$ and neglecting the mass of the leptons give the differential decay rates

$$\begin{aligned} \frac{d\Gamma^s}{dE_\gamma} &= \Gamma_0^s \frac{8E_\gamma}{m_B^3}(4W_1^s - W_2^s - 2E_\gamma W_5^s), \\ \frac{d^3\Gamma^u}{dE_\ell dq^2 dE_\nu} &= \Gamma_0^u \frac{96}{m_B^5}[q^2 W_1^u + (2E_\ell E_\nu - q^2/2) W_2^u \\ &\quad + q^2(E_\ell - E_\nu) W_3^u] \theta(4E_\ell E_\nu - q^2), \\ \frac{d^3\Gamma^{\ell\ell}}{dq^2 dE_+ dE_-} &= \Gamma_0^{\ell\ell} \frac{96}{m_B^5}[q^2 W_1^{\ell\ell} + (2E_- E_+ - q^2/2) W_2^{\ell\ell} \\ &\quad + q^2(E_- - E_+) W_3^{\ell\ell}] \theta(4E_- E_+ - q^2), \end{aligned} \tag{2}$$

for $B \to X_s \gamma$, $B \to X_u \ell \bar{\nu}$, and $B \to X_s \ell^+ \ell^-$, respectively, where $W^{\ell\ell}_{1(2)} = W^L_{1(2)} + W^R_{1(2)}$, $W_3^{\ell\ell} = W_3^L - W_3^R$, $W_i = W_i(q^2, v \cdot q)$, and the normalization factors are

$$\begin{aligned} \Gamma_0^s &= \frac{G_F^2 m_B^3}{32\pi^4} |V_{tb} V_{ts}^*|^2 \alpha_{\rm em} [\bar{m}_b(m_b)]^2 |C_7^{\rm eff(0)}(m_b)|^2, \\ \Gamma_0^u &= \frac{G_F^2 m_B^5}{192\pi^3} |V_{ub}|^2, \qquad \Gamma_0^{\ell\ell} = \frac{G_F^2 m_B^5}{192\pi^3} \frac{\alpha_{\rm em}^2}{16\pi^2} |V_{tb} V_{ts}^*|^2. \end{aligned} \tag{3}$$

In SCET, it is natural to use light-cone coordinates, where we introduce vectors n and $\bar{n}$ such that $n^2 = \bar{n}^2 = 0$ and $n \cdot \bar{n} = 2$. A four-vector then has components $(p^+, p^-, p_\perp) = (n \cdot p, \bar{n} \cdot p, p_\perp^\mu)$. In the region of interest, the set of hadronic states X is jetlike, i.e. $p_X^+ \ll p_X^-$. For convenience we define the dimensionless variables

$$\begin{aligned} x_H^\gamma &= \frac{2E_\gamma}{m_B}, \qquad x_H = \frac{2E_\ell}{m_B}, \\ \bar{y}_H &= \frac{\bar{n} \cdot p_X}{m_B}, \qquad u_H = \frac{n \cdot p_X}{m_B}. \end{aligned} \tag{4}$$

In terms of these variables, the decay rates are

$$\begin{aligned} \frac{d\Gamma^s}{dx_H^\gamma} &= \Gamma_0^s \frac{2x_H^\gamma}{m_B}\{4W_1^s - W_2^s - m_B x_H^\gamma W_5^s\}, \\ \frac{1}{\Gamma_0^u} \frac{d^3\Gamma^u}{dx_H d\bar{y}_H du_H} &= 24 m_B (\bar{y}_H - u_H) \bigg\{ (1-u_H)(1-\bar{y}_H) W_1^u + \frac{1}{2}(1 - x_H - u_H)(x_H + \bar{y}_H - 1) W_2^u \\ &\quad + \frac{m_B}{2}(1-u_H)(1-\bar{y}_H)(2x_H + u_H + \bar{y}_H - 2) W_3^u \bigg\}, \\ \frac{1}{\Gamma_0^{\ell\ell}} \frac{d^3\Gamma^{\ell\ell}}{dx_H d\bar{y}_H du_H} &= 24 m_B (\bar{y}_H - u_H) \bigg\{ (1-u_H)(1-\bar{y}_H) W_1^{\ell\ell} + \frac{1}{2}(1 - x_H - u_H)(x_H + \bar{y}_H - 1) W_2^{\ell\ell} \\ &\quad + \frac{m_B}{2}(1-u_H)(1-\bar{y}_H)(2x_H + u_H + \bar{y}_H - 2) W_3^{\ell\ell} \bigg\}, \end{aligned} \tag{5}$$

where $W_i = W_i(u_H, \bar{y}_H)$. The full phase-space limits are given in Table II of Ref. [32].

The optical theorem relates the W_i to forward-scattering amplitudes, which can be calculated by taking time-ordered products of currents. An important part of the analysis is the separation of short- and long-distance contributions. The results, known as factorization theorems, may be written schematically in the form

$$d\Gamma = H \times \mathcal{J} \otimes f,$$

where $\otimes$ denotes a convolution. The hard (H) and jet ($\mathcal{J}$) functions encode perturbative corrections that appear at two different scales, $\mu_b \sim m_b$ and $\mu_i \sim \sqrt{m_b \Lambda_{\rm QCD}}$, respectively, whereas the shape function (f) represents nonperturbative physics.

SCET involves a power expansion in the small parameter $\lambda = \sqrt{\Lambda_{\rm QCD}/m_b}$. At leading order in λ, rates depend on one shape function, which we denote by $f^{(0)}$:

$$W_i^{(0)} = h_i(p_X^-, m_b, \mu) \int_0^{p_X^+} dk^+ \mathcal{J}^{(0)}(p_X^- k^+, \mu) \times f^{(0)}(k^+ + \bar{\Lambda} - p_X^+, \mu), \tag{6}$$

where $\bar{\Lambda} = m_B - m_b + (\lambda_1 + 3\lambda_2)/(2m_b) + \dots$. The first subleading shape functions occur at order λ^2 and we denote these by $f^{(2)}_{0-2}$, $f^{(4)}_{3,4}$, and $f^{(6)}_{5,6}$. These are common to the three decays, but appear in different combinations, and are convoluted with jet functions $\mathcal{J}^{(0)}$, $\mathcal{J}^{(-2)}$, and $\mathcal{J}^{(-4)}$, respectively, as shown in Eq. (B8). Note that we also have $u_H/\bar{y}_H \sim \lambda^2$ in the shape-function region.

The shape functions are given by B-meson matrix elements of nonlocal ultrasoft operators. The definitions used here follow Ref. [32] and are included in Appendix A. At tree level, the jet functions are

$$\mathcal{J}^{(0)}(k^+) = \delta(k^+), \qquad \mathcal{J}^{(-2)}(k_j^+) = \frac{\delta(k_1^+) - \delta(k_2^+)}{k_2^+ - k_1^+},$$
$$\mathcal{J}^{(-4)}(k_j^+) = 4\pi\alpha_s(\mu_i)\Big[\frac{\delta(k_1^+)}{(k_2^+)(k_3^+)} + \frac{\delta(k_2^+)}{(k_1^+)(k_3^+)} + \frac{\delta(k_3^+)}{(k_1^+)(k_2^+)} - \pi^2 \delta(k_1^+)\delta(k_2^+)\delta(k_3^+)\Big]. \tag{7}$$

At one-loop order, we have

$$\mathcal{J}^{(0)}(\omega, k^+, \mu) = \Big\{\delta(k^+)\Big[1 + \frac{\alpha_s(\mu)C_F}{4\pi}\Big(2\ln^2\frac{\omega p_X^+}{\mu^2} - 3\ln\frac{\omega p_X^+}{\mu^2} + 7 - \pi^2\Big)\Big] + \frac{\alpha_s(\mu)C_F}{4\pi}\Big[\Big(\frac{4\ln(k^+/p_X^+)}{k^+}\Big)_+ + \Big(4\ln\frac{\omega p_X^+}{\mu^2} - 3\Big)\frac{1}{(k^+)_+}\Big]\Big\} \times \theta(p_X^+ - k^+)\theta(k^+), \tag{8}$$

where $\omega = \bar{n}\cdot p$ is the large partonic momentum.

For convenience we define

$$F(p^+, p^-) = \int_0^{p_X^+} dk^+ \mathcal{J}^{(0)}(p^-, k^+, \mu) f^{(0)}(k^+ + \bar{\Lambda} - p^+, \mu) + \frac{1}{2m_B} f_0^{(2)}(\bar{\Lambda} - p^+) - \frac{\lambda_1 + 3\lambda_2}{2m_B} f^{(0)\prime}(\bar{\Lambda} - p^+),$$
$$F_{1,2}(p^+) = f_{1,2}^{(2)}(\bar{\Lambda} - p^+), \tag{9}$$

where a prime denotes a derivative, as well as

$$F_{3,4}(p^+) = \int dk_1^+ dk_2^+ \Big[\frac{\delta(k_1^+) - \delta(k_2^+)}{k_2^+ - k_1^+}\Big] f_{3,4}^{(4)}(k_j^+ + \bar{\Lambda} - p^+),$$
$$F_{5,6}(p^+) = \int dk_1^+ dk_2^+ dk_3^+ \Big[\frac{\delta(k_1^+)}{(k_2^+)(k_3^+)} + \frac{\delta(k_2^+)}{(k_1^+)(k_3^+)} + \frac{\delta(k_3^+)}{(k_1^+)(k_2^+)} - \pi^2\delta(k_1^+)\delta(k_2^+)\delta(k_3^+)\Big] \times f_{5,6}^{(6)}(k_j^+ + \bar{\Lambda} - p^+). \tag{10}$$

If we use the tree-level expression for $\mathcal{J}^{(0)}$, then $F(p^+, p^-) = F(p^+)$ is a function of p^+ only. Then, for $B \to X_s\gamma$, the rate $d\Gamma^s/dx_H^\gamma$ in the endpoint region is [32][1]

$$\frac{1}{\Gamma_0^s}\frac{d\Gamma^s}{dx_H^\gamma}\Big|_{x_H^\gamma > x_H^c} = m_B(C^{(t)})^2[1 - 3(1 - x_H^\gamma)]F(m_B(1 - x_H^\gamma), m_B) + [m_B(1 - x_H^\gamma) - \bar{\Lambda}]F(m_B(1 - x_H^\gamma)) + F_2(m_B(1 - x_H^\gamma)) - F_3(m_B(1 - x_H^\gamma)) + F_4(m_B(1 - x_H^\gamma)) - 8\pi\alpha_s(\mu_i)F_5^s(m_B(1 - x_H^\gamma)), \tag{11}$$

where $1 - x_H^c \sim \lambda^2$ and

$$C^{(t)} = 1 + \Delta_\gamma(m_b, \varrho) - \frac{\alpha_s(m_b)C_F}{4\pi}\Big\{\frac{\pi^2}{12} + 6\Big\},$$
$$\Delta_\gamma(m_b, \varrho) = \frac{1}{C_7^{\rm eff(0)}(m_b)}\Big\{\frac{\alpha_s}{4\pi}C_7^{\rm eff(1)}(m_b) + \sum_k C_k^{\rm eff(0)}(m_b) r_k(\varrho)\Big\}. \tag{12}$$

The triply differential decay rate for $B \to X_u\ell\bar{\nu}$ at NLO [32] is obtained by substituting the W_i^u listed in Appendix B into Eq. (5). At tree level, this becomes

[1] This includes $\mathcal{O}_7 - \mathcal{O}_7$ and $\mathcal{O}_7 - \mathcal{O}_2$ contributions only. In Ref. [38] subleading corrections from $\mathcal{O}_7 - \mathcal{O}_8$ are studied and estimated to contribute between -0.3% and -3% to the total flavor-averaged decay rate. We do not consider such corrections in this work.

$$\frac{1}{\Gamma_0^u}\frac{d^3\Gamma^u}{dx_H d\bar{y}_H du_H} = 6(1-u_H)(x_H+\bar{y}_H-1)\Big\{2m_B(2-x_H-\bar{y}_H-u_H)F(m_Bu_H) - \frac{1}{\bar{y}_H-u_H}(\bar{y}_H^2-(2-x_H)\bar{y}_H+2(1-x_H)-u_H(2-x_H-u_H))F_1(m_Bu_H) + \frac{2}{\bar{y}_H(\bar{y}_H-u_H)}(\bar{y}_H^3-(2-x_H)\bar{y}_H^2-(4-u_H)(x_H+u_H)\bar{y}_H+2(x_H+2\bar{y}_H+u_H-1))F_2(m_Bu_H) + \frac{2}{\bar{y}_H}(x_H+\bar{y}_H+u_H-2)F_3(m_Bu_H) - \frac{2}{\bar{y}_H(\bar{y}_H-u_H)}(\bar{y}_H^2-(2-x_H)\bar{y}_H+2(1-x_H)-u_H(2-x_H-u_H))F_4(m_Bu_H) - \frac{4}{\bar{y}_H(\bar{y}_H-u_H)}(1-\bar{y}_H)(x_H+\bar{y}_H-1)4\pi\alpha_s(\mu_i)F_5^u(m_Bu_H) + \frac{4}{\bar{y}_H(\bar{y}_H-u_H)}(1-u_H)(1-x_H-u_H)4\pi\alpha_s(\mu_i)F_6^u(m_Bu_H)\Big\}. \tag{13}$$

Note that we can use the relation [30]

$$F_1(m_Bu_H) = 2(\bar{\Lambda}-m_Bu_H)F(m_Bu_H) + \mathcal{O}(\lambda^4) \tag{14}$$

to eliminate $F_1(m_Bu_H)$, as was done in Eq. (11).

The triply differential decay rate for $B\to X_s\ell^+\ell^-$ was calculated in Refs. [24,36]. The $W_i^{\ell\ell}$ appearing in Eq. (5) are also listed in Appendix B.

III. $|V_{ub}|$ AT NLO

A. Relations between $B\to X_u\ell\bar{\nu}$ and $B\to X_s\gamma$

Consider first the process $B\to X_u\ell\bar{\nu}$. We wish to isolate or eliminate the subleading shape functions that appear in the rates. In the following, we shall work at tree level. Inspection of Eqs. (B2) and (B9) shows that the shape functions appear in the hadronic structure functions W_1 to W_3 in only two combinations, namely,

$$m_B\mathcal{F}_I = m_BF + \frac{1}{2}F_1 - F_2 - \frac{1}{\bar{y}_H}(F_3-F_4+8\pi\alpha_s(\mu_i)F_5^u),$$
$$m_B\mathcal{F}_{II} = F_1 - \frac{2(\bar{y}_H(2-u_H)-1)}{\bar{y}_H(1-u_H)}F_2 + \frac{2}{\bar{y}_H}(F_4-4\pi\alpha_s(\mu_i)F_5^u-4\pi\alpha_s(\mu_i)F_6^u), \tag{15}$$

where we have suppressed the argument m_Bu_H. Specifically,

$$W_1 = \frac{1}{4}\mathcal{F}_I, \qquad W_2 = \frac{1-u_H}{\bar{y}_H-u_H}\mathcal{F}_I - \frac{(1-u_H)^2}{(\bar{y}_H-u_H)^2}\mathcal{F}_{II},$$
$$W_3 = \frac{1}{2m_B(\bar{y}_H-u_H)}\mathcal{F}_I. \tag{16}$$

Nevertheless, taking integrals of the form

$$\int_{u_H}^1 d\bar{y}_H\int_{1-\bar{y}_H}^{1-u_H} dx_H K^u(x_H,\bar{y}_H,u_H)\frac{d^3\Gamma^u}{dx_Hd\bar{y}_Hdu_H}, \tag{17}$$

with suitable choices of the weight function $K^u(x_H,\bar{y}_H,u_H)$, we can isolate the following four linearly independent combinations of the F_i:

$$(4-2u_H)m_BF + F_1, \tag{18a}$$
$$(1-u_H)m_BF + F_2, \tag{18b}$$
$$F_3 - F_4 + 8\pi\alpha_s(\mu_i)F_5^u, \tag{18c}$$
$$m_BF - \tfrac{1}{2}F_3 - \tfrac{1}{2}F_4 + 4\pi\alpha_s(\mu_i)F_6^u. \tag{18d}$$

[Recall that we can apply Eq. (14) so that the first combination involves only the leading-order shape function.] Here, the treatment of the u_H dependence in the rate requires care. Expanding Eq. (13) in $u_H\sim\lambda^2$ when obtaining the weight function will typically result in excessively large coefficients in the $u_HF_{1\text{–}6}(m_Bu_H)$ terms (which are formally of order λ^4). For example, choosing $K^u(x_H,\bar{y}_H) = -21x_H + 21\bar{y}_H + 45x_H\bar{y}_H - \frac{75}{2}\bar{y}_H^2$, we obtain

$$\frac{1}{\Gamma_0^u}\iint dx_Hd\bar{y}_HK^u(x_H,\bar{y}_H)\frac{d^3\Gamma^u}{dx_Hd\bar{y}_Hdu_H} = (1\text{–}7u_H)m_BF(m_Bu_H) + \frac{1}{4}F_1(m_Bu_H) + \mathcal{O}(\lambda^4), \tag{19}$$

so this eliminates all but the leading-order shape function up to $\mathcal{O}(\lambda^4)$ corrections. However, we then have the additional contributions

$$\frac{5}{4}u_HF_1(m_Bu_H) - \frac{49}{2}u_HF_2(m_Bu_H) - \frac{109}{4}u_HF_3(m_Bu_H) + \frac{57}{4}u_HF_4(m_Bu_H) - \frac{83}{2}u_H\times4\pi\alpha_s(\mu_i)F_5^u(m_Bu_H) + 13u_H\times4\pi\alpha_s(\mu_i)F_6^u(m_Bu_H). \tag{20}$$

TABLE I. Some choices of $K^u(x_H, \bar{y}_H, u_H)$ for which the weighted integral Eq. (17) equals $m_B F + F_1/(4 - 2u_H)$.

(1)	$K^u_{\rm I} = \frac{5}{9}\frac{1}{(2-u_H)}\frac{1}{(1-u_H)^8}[10(7-u_H)(1-u_H)(4+3u_H)\bar{y}_H - (454+247u_H-71u_H^2)x_H\bar{y}_H - 4(1-u_H)(109-4u_H)\bar{y}_H^2 + 105(7-u_H)x_H\bar{y}_H^2]$
(2)	$K^u_{\rm I} = \frac{5}{32}\frac{1}{(2-u_H)}\frac{1}{(1-u_H)^8}[-10(7-u_H)(1-u_H)(34-27u_H)\bar{y}_H + 2(1-u_H)(2759-449u_H)x_H\bar{y}_H - 525(7-u_H)x_H^2\bar{y}_H + 2(1-u_H)(341-131u_H)\bar{y}_H^2]$
(3)	$K^u_{\rm I} = \frac{15}{41}\frac{1}{(2-u_H)}\frac{1}{(1-u_H)^8}[-2(1-u_H)^2(288-29u_H)\bar{y}_H + (1426-1793u_H+157u_H^2)x_H\bar{y}_H - 10(109-4u_H)x_H^2\bar{y}_H + (341-131u_H)x_H\bar{y}_H^2]$

TABLE II. Some choices of $K^u(x_H, \bar{y}_H, u_H)$ for which the weighted integral Eq. (17) equals $(1-u_H)m_B F + F_2$.

(A)	$K^u_{\rm II} = \frac{5}{9}\frac{1}{(1-u_H)^7}[-2(1-u_H)(7-15u_H)\bar{y}_H - (34+71u_H)x_H\bar{y}_H - 16(1-u_H)\bar{y}_H^2 + 105x_H\bar{y}_H^2]$
(B)	$K^u_{\rm II} = \frac{5}{32}\frac{1}{(1-u_H)^7}[-2(1-u_H)(266-135u_H)\bar{y}_H + 898(1-u_H)x_H\bar{y}_H - 525x_H^2\bar{y}_H + 262(1-u_H)\bar{y}_H^2]$
(C)	$K^u_{\rm II} = \frac{15}{41}\frac{1}{(1-u_H)^7}[-58(1-u_H)^2\bar{y}_H + (26-157u_H)x_H\bar{y}_H - 40x_H^2\bar{y}_H + 131x_H\bar{y}_H^2]$

TABLE III. Some choices of $K^u(x_H, \bar{y}_H, u_H)$ for which the weighted integral Eq. (17) equals $F_3 - F_4 + 2\tilde{F}_5^u$.

(a)	$K^u_{\rm III} = -\frac{10}{9}\frac{1}{(1-u_H)^8}[-2(1-u_H)(58+32u_H+15u_H^2)\bar{y}_H + (158+104u_H+53u_H^2)x_H\bar{y}_H + (1-u_H)(149+61u_H)\bar{y}_H^2 - 105(2+u_H)x_H\bar{y}_H^2]$
(b)	$K^u_{\rm III} = -\frac{15}{16}\frac{1}{(1-u_H)^8}[2(1-u_H)(92-12u_H-45u_H^2)\bar{y}_H - 2(1-u_H)(246+139u_H)x_H\bar{y}_H + 175(2+u_H)x_H^2\bar{y}_H - 2(1-u_H)(18+17u_H)\bar{y}_H^2]$
(c)	$K^u_{\rm III} = -\frac{15}{41}\frac{1}{(1-u_H)^8}[2(1-u_H)^2(166+93u_H)\bar{y}_H - 2(483-320u_H-268u_H^2)x_H\bar{y}_H + 5(149+61u_H)x_H^2\bar{y}_H - 6(18+17u_H)x_H\bar{y}_H^2]$

For this reason, when calculating K^u, we keep the full dependence on u_H in the rate, rather than dropping terms that are formally subleading in a strict SCET expansion in $u_H/\bar{y}_H \sim \lambda^2$. (The analysis of m_X-cut effects in $B \to X_s \ell^+\ell^-$ [24,25] also retained the full u_H dependence, since doing so facilitates making contact with the total rate in the local OPE [39–41].) Thus, subleading shape functions are eliminated to all orders in u_H, and the issue is resolved. One straightforward method for obtaining $K^u(x_H, \bar{y}_H, u_H)$ is then to take different moments of the rate with respect to x_H and $\bar{y}_H$, and solve the resulting set of linear equations in the F_i. In Eq. (17), we consider the case where a cut is imposed on p_X^+, i.e. $p_X^+ < m_D^2/m_B$. Different or additional cuts will change the limits of integration, calling for different weight functions. Table I lists several examples of K^u's that isolate the combination $m_B F + F_1/(4 - 2u_H)$, while Tables II and III give examples that result in (18b) and (18c), respectively.

Now, the subleading shape functions $F_{5,6}$ depend upon the light-quark flavor (see Appendix A). We indicate this difference between the $F_{5,6}$'s appearing in $B \to X_u \ell \bar{\nu}$ and $B \to X_s \gamma$ by using the superscripts "u" and "s." In order to cancel the F_5^s contribution to the latter decay,[2] we can use approximate $SU(3)$ flavor symmetry, namely, the fact that

$$\frac{F_5^u - F_5^s}{F_5^s} \sim \frac{m_s}{\Lambda_{\rm QCD}} \tag{21}$$

is suppressed. This enables us to relate the semileptonic process to the radiative process and thereby derive an expression for Γ_0^u, or equivalently $|V_{ub}|$, to subleading order. We can write

[2]The authors of Refs. [33,34] have used model-dependent arguments to estimate that the effects of $f_{5,6}$, when integrated over a sufficiently large region, are comparatively small ($\sim 5\%$), but that they may cause large corrections in the $d\Gamma/dp_X^+$ spectrum for $p_X^+ \leq 0.5$ GeV. We avoid any need to consider the reliability of these numerics by simply eliminating $f_{5,6}$, along with the other tree-level shape functions.

$$\frac{1}{\Gamma_0^u}\iint[K_{\rm II}^u - K_{\rm III}^u + \rho K_{\rm I}^u]\frac{d^3\Gamma^u}{dx_H d\bar{y}_H du_H}dx_H d\bar{y}_H = m_B F(m_B u_H) - \frac{1}{2}[F_1(m_B u_H) - 2F_2(m_B u_H)] - [F_3(m_B u_H) - F_4(m_B u_H) + 2\tilde{F}_5^u(m_B u_H)], \quad (22)$$

where

$$\rho(u_H) = \frac{(2-u_H)(u_H - \frac{\bar{\Lambda}}{2m_B})}{(1-u_H+\frac{\bar{\Lambda}}{2m_B})} \quad (23)$$

and $\tilde{F}_5^u = 4\pi\alpha_s(\mu_i)F_5^u$. $K_{\rm I}^u$, $K_{\rm II}^u$, and $K_{\rm III}^u$ are any weight functions that give the combinations $m_B F + F_1/(4-2u_H)$, $(1-u_H)m_B F + F_2$, and $F_3 - F_4 + 2\tilde{F}_5^u$, respectively (examples of which are presented in Tables I, II, and III). The shape functions in Eq. (22) appear in the same linear combination as in the rate $d\Gamma^s/du_H$. Hence, at NLO we obtain

$$\frac{1}{\Gamma_0^u}\iint[K_{\rm II}^u - K_{\rm III}^u + \rho K_{\rm I}^u]\frac{d^3\Gamma^u}{dx_H d\bar{y}_H du_H}dx_H d\bar{y}_H = -\frac{1}{(1-u_H)^3}\frac{1}{\Gamma_0^s}\frac{d\Gamma^s}{du_H}. \quad (24)$$

More generally, we can construct K^u such that

$$\hat{M}^u \equiv \frac{1}{\Gamma_0^u}M^u \equiv \frac{1}{\Gamma_0^u}\iint K^u(x_H,\bar{y}_H,u_H)\frac{d^3\Gamma^u}{dx_H d\bar{y}_H du_H}dx_H d\bar{y}_H = m_B F(m_B u_H) + \kappa_1^u(u_H)F_1(m_B u_H) + \kappa_2^u(u_H)[F_2(m_B u_H) - F_3(m_B u_H) + F_4(m_B u_H) - 2\tilde{F}_5^u(m_B u_H)]. \quad (25)$$

For example, we can use

$$K^u = K_{\rm IV}^u - \kappa_2^u K_{\rm III}^u, \quad (26)$$

where $K_{\rm IV}^u$ is a weight function that gives the linear combination $m_B F + \kappa_1^u F_1 + \kappa_2^u F_2$, examples of which are given in Table IV in Appendix C (with the corresponding values of κ_1^u and κ_2^u shown there). We can also use

$$K^u = \beta K_{\rm I}^u + \frac{1-\beta}{1-u_H}(K_{\rm II}^u - K_{\rm III}^u), \quad (27)$$

with β an arbitrary real number [in which case $\kappa_1^u = \beta/(4-2u_H)$ and $\kappa_2^u = (1-\beta)/(1-u_H)$]. For any such K^u, we have

$$\hat{M}^u + \kappa_2^u(1-u_H)^{-3}\hat{M}^s = \left\{(1-\kappa_2^u) + \left(\frac{\bar{\Lambda}}{m_B} - u_H\right)(2\kappa_1^u + \kappa_2^u)\right\}m_B F(m_B u_H) + \mathcal{O}(\alpha_s, \lambda^4), \quad (28)$$

where $\hat{M}^s = (1/\Gamma_0^s)M^s = (1/\Gamma_0^s)(d\Gamma^s/du_H)$, i.e. combining $\hat{M}^u$ and $\hat{M}^s$ in this way gives an expression dependent only on the leading-order shape function. Taking the ratio of two such expressions (two choices of K^u) at $u_H \neq 0$ then provides us with a relation independent of both leading and subleading shape functions. We shall use the superscripts (i) and (ii) when we need to distinguish between quantities in the two expressions. We then obtain

$$\frac{\Gamma_0^u}{\Gamma_0^s} = -\frac{[b_0^{(ii)}M^{u(i)} - b_0^{(i)}M^{u(ii)}]}{[b_0^{(ii)}\kappa_2^{u(i)} - b_0^{(i)}\kappa_2^{u(ii)}](1-u_H)^{-3}M^s}, \quad (29)$$

where

$$b_0 = (1-\kappa_2^u) + \left(\frac{\bar{\Lambda}}{m_B} - u_H\right)(2\kappa_1^u + \kappa_2^u). \quad (30)$$

Since the right-hand side of Eq. (29) is measurable, it enables an experimental determination of the CKM ratio on the left-hand side. Additionally, the factor $|V_{tb}V_{ts}^*|$ in this ratio can be eliminated by normalizing the photon spectrum by the total $B \to X_s\gamma$ rate, which is given in a local OPE.

There will be loop and power (λ^4-suppressed) corrections to the rates and hence also to Eq. (29). While these are not fully known, one can show that the corrections to Eq. (29) are proportional to

$$-\frac{b_0^{(i)}}{b_0^{(ii)} - b_0^{(i)}} + \frac{b_0^{(i)}\kappa_2^{u(ii)}}{b_0^{(ii)}\kappa_2^{u(i)} - b_0^{(i)}\kappa_2^{u(ii)}} + \cdots \quad (31)$$

(multiplied by α_s or λ^4). This needs to be taken into account when selecting $\{K^{u(i)}, K^{u(ii)}\}$: one should avoid pairs of weight functions that result in Eq. (31) being excessively large, lest parametrically suppressed terms acquire excessively large coefficients. For example, one appropriate choice is to use Eq. (27) for both K^u's, with $\beta^{(i)} = 1$ and $\beta^{(ii)} = 0$, after which the magnitude of Eq. (31) is less than 1/6 for $0 < u_H < m_D^2/m_B^2$.

B. Relations involving $B \to X_s\ell^+\ell^-$

We can also try to isolate shape functions in the process $B \to X_s\ell^+\ell^-$ by taking integrals of the form

$$\int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H \int_{1-\bar{y}_H}^{1-u_H} dx_H K^{\ell\ell}(x_H, \bar{y}_H, u_H) \frac{d^3\Gamma^{\ell\ell}}{dx_H d\bar{y}_H du_H}, \tag{32}$$

where

$$\bar{y}_{\min(\max)} = 1 - \frac{y_H^{\max(\min)}}{(1-u_H)}. \tag{33}$$

Here, $y_H = q^2/m_B^2$ and the low-q^2 region corresponds to $1\ \mathrm{GeV}^2 \le q^2 \le 6\ \mathrm{GeV}^2$. However, determining $K^{\ell\ell}(x_H, \bar{y}_H, u_H)$ in the straightforward manner described above proves to be problematic in practice. Therefore, we resort to another method, which is based on the following observation. Under the transformation $x_H \to x'_H = 2 - u_H - \bar{y}_H - x_H$, we find that $\int_{1-\bar{y}_H}^{1-u_H} dx_H = \int_{1-\bar{y}_H}^{1-u_H} dx'_H$ and

$$(1 - x_H - u_H) \leftrightarrow (x_H + \bar{y}_H - 1),$$
$$(2x_H + u_H + \bar{y}_H - 2) \leftrightarrow -(2x_H + u_H + \bar{y}_H - 2).$$

This symmetry or antisymmetry can be exploited to obtain $K^{\ell\ell}$. For example, if $K^{\ell\ell}$ changes sign under the transformation, then we can see from the triply differential rate, Eq. (5), that integration over x_H eliminates the W_1 and W_2 terms, whereas the W_3 term remains. Now, Eq. (B10) shows that F_3, F_4, and F_5^s occur in W_3 in the same linear combination as in the $B \to X_s\gamma$ rate.

This still leaves the integration over $\bar{y}_H$, and if we choose $K^{\ell\ell}(x_H, \bar{y}_H, u_H) = (2x_H + u_H + \bar{y}_H - 2)\tilde{K}^{\ell\ell}(\bar{y}_H, u_H)$, where $\tilde{K}^{\ell\ell}(\bar{y}_H, u_H)$ satisfies

$$\int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H (\bar{y}_H - u_H)^3 \frac{1}{\bar{y}_H} (2\,\mathrm{Re}[C_{10a}C_{7a}^*] + \mathrm{Re}[C_{10a}C_{9a}^*](1-\bar{y}_H^2))\tilde{K}^{\ell\ell}(\bar{y}_H, u_H) = 0, \tag{34}$$

then all of the subleading shape functions in Eq. (32) appear in the same combination as in the $B \to X_s\gamma$ rate, which can thus be used to eliminate these functions. Table V in Appendix C shows several examples of $K^{\ell\ell}$ of this form. We observe that $z = \cos\theta = (2x_H + u_H + \bar{y}_H - 2)/(\bar{y}_H - u_H)$, where θ is the angle between the B and ℓ^+ in the center-of-mass frame of the $\ell^+\ell^-$. This means that a choice of $K^{\ell\ell} \propto (2x_H + u_H + \bar{y}_H - 2)$ is equivalent to taking moments of the forward-backward asymmetry,

$$\frac{d^2A_{FB}}{d\bar{y}_H du_H} = \int_{-1}^{1} dz \frac{\mathrm{sign}(z)}{\Gamma_0} \frac{d^3\Gamma}{d\bar{y}_H du_H dz} = \frac{3}{2\Gamma_0}\int_{-1}^{1} dz\, z \frac{d^3\Gamma}{d\bar{y}_H du_H dz}. \tag{35}$$

Note also that C_{9a} is a function of q^2, and hence of $\bar{y}_H$ (see Appendix B), but in the low-q^2 region $|C_{9a}|$ varies by less than $\pm 1\%$ and we take it to be constant. There is no problem taking into account the exact dependence, but integrals over regions of $\bar{y}_H$ must then be performed numerically.

Let $\Gamma_0^s \hat{M}^s = d\Gamma^s/du_H$, and let $M^u = \Gamma_0^u \hat{M}^u$ and $\Gamma_0^{\ell\ell}\hat{M}^{\ell\ell}$ denote the integrals (17) and (32) respectively, with weight functions from Tables I and V. Then we obtain

$$\Gamma_0^u = \frac{1+\kappa_3^{\ell\ell}}{1+2(\frac{\bar{\Lambda}}{m_B} - u)\kappa_1^u} \frac{M^u}{\hat{M}^{\ell\ell} - \kappa_3^{\ell\ell}(1-u_H)^{-3}\hat{M}^s}, \tag{36}$$

where κ_1^u ($\kappa_3^{\ell\ell}$) is the coefficient of F_1 (F_3) in $\hat{M}^u$ ($\hat{M}^{\ell\ell}$).

More generally, by the same methods, we can find K^u and $K^{\ell\ell}$ such that

$$\begin{aligned}
\hat{M}^u &\equiv \frac{1}{\Gamma_0^u} M^u \equiv \frac{1}{\Gamma_0^u} \iint K^u(x_H, \bar{y}_H, u_H) \frac{d^3\Gamma^u}{dx_H d\bar{y}_H du_H} dx_H d\bar{y}_H = m_B F(m_B u_H) + \kappa_1^u(u_H) F_1(m_B u_H) + \kappa_2^u(u_H) F_2(m_B u_H),\\
\hat{M}^s &\equiv \frac{1}{\Gamma_0^s} M^s \equiv \frac{1}{\Gamma_0^s} \frac{d\Gamma^s}{du_H}\\
&= -(1-u_H)^3 \Big\{ m_B F(m_B u_H) - \frac{1}{2}[F_1(m_B u_H) - 2F_2(m_B u_H)] - [F_3(m_B u_H) - F_4(m_B u_H) + 2\tilde{F}_5^s(m_B u_H)]\Big\},\\
\hat{M}^{\ell\ell} &\equiv \frac{1}{\Gamma_0^{\ell\ell}} M^{\ell\ell} \equiv \frac{1}{\Gamma_0^{\ell\ell}} \iint K^{\ell\ell}(x_H, \bar{y}_H, u_H) \frac{d^3\Gamma^{\ell\ell}}{dx_H d\bar{y}_H du_H} dx_H d\bar{y}_H\\
&= m_B F(m_B u_H) + \frac{1}{2}\kappa_2^{\ell\ell}(u_H)[F_1(m_B u_H) - 2F_2(m_B u_H)] + \kappa_3^{\ell\ell}(u_H)[F_3(m_B u_H) - F_4(m_B u_H) + 2\tilde{F}_5^s(m_B u_H)],
\end{aligned} \tag{37}$$

where $\tilde{F}_5^s = 4\pi\alpha_s(\mu_i)F_5^s$. Tables IV and VI show (further) examples of such weight functions, along with the corresponding values of the coefficients $\kappa_{1,2}^u$ and $\kappa_{2,3}^{\ell\ell}$. Then

$$[\kappa_2^{\ell\ell} - \kappa_3^{\ell\ell}]\hat{M}^u + \kappa_2^u \hat{M}^{\ell\ell} - \kappa_2^u \kappa_3^{\ell\ell}(1-u_H)^{-3}\hat{M}^s = \Big\{(\kappa_2^{\ell\ell} - \kappa_3^{\ell\ell} + \kappa_2^u + \kappa_2^u\kappa_3^{\ell\ell}) + \Big(\frac{\bar{\Lambda}}{m_B} - u_H\Big)(\kappa_2^{\ell\ell} - \kappa_3^{\ell\ell})(2\kappa_1^u + \kappa_2^u)\Big\} m_B F(m_B u_H) + \mathcal{O}(\alpha_s, \lambda^4), \tag{38}$$

so in this case we have a combination of $\hat{M}^u$, $\hat{M}^s$, and $\hat{M}^{\ell\ell}$ that is dependent only on the leading-order shape function. Taking the ratio of two such expressions [two choices of $\{K^u, K^{\ell\ell}\}$, denoted by superscripts (i) and (ii) as previously] at $u_H \neq 0$ then provides us with another relation independent of both leading and subleading shape functions.

Specifically, let

$$\hat{\mathcal{M}}^u = \frac{1}{\Gamma_0^u}\mathcal{M}^u = \begin{cases} [\kappa_2^{\ell\ell} - \kappa_3^{\ell\ell}]\hat{M}^u, & \text{if } \kappa_2^u \neq 0 \\ \hat{M}^u, & \text{if } \kappa_2^u = 0 \end{cases},$$

$$\hat{\mathcal{M}}^{\ell\ell} = \frac{1}{\Gamma_0^{\ell\ell}}\mathcal{M}^{\ell\ell} = \begin{cases} \kappa_2^u\hat{M}^{\ell\ell}, & \text{if } \kappa_2^{\ell\ell} \neq \kappa_3^{\ell\ell} \\ \hat{M}^{\ell\ell}, & \text{if } \kappa_2^{\ell\ell} = \kappa_3^{\ell\ell} \end{cases},$$

$$\hat{\mathcal{M}}^s = \frac{1}{\Gamma_0^s}\mathcal{M}^s = \begin{cases} -\kappa_2^u\kappa_3^{\ell\ell}(1-u_H)^{-3}\hat{M}^s, & \text{if } \kappa_2^{\ell\ell} \neq \kappa_3^{\ell\ell} \\ -\kappa_3^{\ell\ell}(1-u_H)^{-3}\hat{M}^s, & \text{if } \kappa_2^{\ell\ell} = \kappa_3^{\ell\ell} \end{cases}, \tag{39}$$

and

$$c_0 = \begin{cases} (\kappa_2^{\ell\ell} - \kappa_3^{\ell\ell} + \kappa_2^u + \kappa_2^u\kappa_3^{\ell\ell}) + (\frac{\bar{\Lambda}}{m_B} - u_H)(\kappa_2^{\ell\ell} - \kappa_3^{\ell\ell})(2\kappa_1^u + \kappa_2^u), & \text{if } \kappa_2^u \neq 0 \quad \text{and} \quad \kappa_2^{\ell\ell} \neq \kappa_3^{\ell\ell} \\ 1 + 2(\frac{\bar{\Lambda}}{m_B} - u_H)\kappa_1^u, & \text{if } \kappa_2^u = 0 \\ (1 + \kappa_3^{\ell\ell}), & \text{if } \kappa_2^{\ell\ell} = \kappa_3^{\ell\ell}. \end{cases} \tag{40}$$

We find that

$$\frac{\Gamma_0^u}{\Gamma_0^s} = -\Bigg[\frac{c_0^{(ii)}\mathcal{M}^{u(i)} - c_0^{(i)}\mathcal{M}^{u(ii)}}{c_0^{(ii)}(\mathcal{M}^{s(i)} + r\mathcal{M}^{\ell\ell(i)}) - c_0^{(i)}(\mathcal{M}^{s(ii)} + r\mathcal{M}^{\ell\ell(ii)})}\Bigg], \tag{41}$$

where $r = \Gamma_0^s/\Gamma_0^{\ell\ell}$, or

$$\Gamma_0^u = -\Bigg[\frac{c_0^{(ii)}\mathcal{M}^{u(i)} - c_0^{(i)}\mathcal{M}^{u(ii)}}{c_0^{(ii)}(\hat{\mathcal{M}}^{s(i)} + \hat{\mathcal{M}}^{\ell\ell(i)}) - c_0^{(i)}(\hat{\mathcal{M}}^{s(ii)} + \hat{\mathcal{M}}^{\ell\ell(ii)})}\Bigg]. \tag{42}$$

In the special case where $\kappa_2^{u(i)} = 0$ and $\kappa_2^{\ell\ell(ii)} = \kappa_3^{\ell\ell(ii)}$, Eq. (42) reduces to Eq. (36).

The loop and power (λ^4-suppressed) corrections to Eq. (41) can be shown to be proportional to

$$-\frac{\tilde{c}_0^{(i)}[\kappa_2^{\ell\ell} - \kappa_3^{\ell\ell}]^{(ii)}}{\tilde{c}_0^{(ii)}[\kappa_2^{\ell\ell} - \kappa_3^{\ell\ell}]^{(i)} - \tilde{c}_0^{(i)}[\kappa_2^{\ell\ell} - \kappa_3^{\ell\ell}]^{(ii)}} + \frac{\tilde{c}_0^{(i)}[\kappa_2^u(1+\kappa_3^{\ell\ell})]^{(ii)}}{\tilde{c}_0^{(ii)}[\kappa_2^u(1+\kappa_3^{\ell\ell})]^{(i)} - \tilde{c}_0^{(i)}[\kappa_2^u(1+\kappa_3^{\ell\ell})]^{(ii)}} + \cdots, \tag{43}$$

where $\tilde{c}_0 = (\kappa_2^{\ell\ell} - \kappa_3^{\ell\ell} + \kappa_2^u + \kappa_2^u\kappa_3^{\ell\ell}) + (\bar{\Lambda}/m_B - u_H)(\kappa_2^{\ell\ell} - \kappa_3^{\ell\ell})(2\kappa_1^u + \kappa_2^u)$. When selecting $\{K^{u(i)}, K^{\ell\ell(i)}\}$, $\{K^{u(ii)}, K^{\ell\ell(ii)}\}$, one should avoid those sets of weight functions that result in Eq. (43) being excessively large. The following combinations of weight functions are suitable choices:

$$K^{u(i)} = (1),\ (2),\quad \text{or}\quad (3) \qquad [\text{Table I}]$$
$$\text{and}\quad K^{\ell\ell(ii)} = (7),\ (8),\quad \text{or}\quad (9) \qquad [\text{Table V}];$$

$$K^{u(i)} = (4),\ (5),\quad \text{or}\quad (6) \qquad [\text{Table IV}],$$
$$K^{\ell\ell(i)} = (10),\ (11),\quad \text{or}\quad (12) \qquad [\text{Table VI}],$$
$$\text{and}\quad K^{\ell\ell(ii)} = (7),\ (8),\quad \text{or}\quad (9) \qquad [\text{Table V}].$$

C. Perturbative corrections

Let us now consider the feasibility of incorporating perturbative corrections in our relations. In Ref. [32], the complete set of subleading corrections (to all orders in α_s) for the triply differential spectrum of $B \to X_u \ell \bar{\nu}$ was derived. It was shown that prohibitively many new shape functions appear at order $\alpha_s \Lambda_{QCD}/m_b$, and hence it is not phenomenologically viable to work to that order.[3] However, one may choose to work to order $(\alpha_s\lambda^0, \alpha_s^0\lambda^2)$, by including perturbative corrections to just the leading-power terms. Recall that there are two perturbative scales, $\mu_b \sim m_b$ (hard) and $\mu_i \sim \sqrt{m_b\Lambda_{QCD}}$ (jet). It is straightforward to take into account the relevant hard corrections. Including the effect of corrections to the jet function $\mathcal{J}^{(0)}$, which is convoluted with the shape function $f^{(0)}$, is more involved: one has to "invert" a distribution [see Eq. (8)]. An implementation akin to Refs. [26–29] is left for future

[3]Unless these shape functions appear in the rates in only a much smaller number of linear combinations.

work. Nevertheless, before this is done, we can still use the less direct approach mentioned in the introduction, using two instances of Eq. (28) or (38), with appropriately modified right-hand sides. For example, one can extract the leading-order shape function from the analogue of Eq. (38), with $K^{\ell\ell}$ from Table V, and substitute this function into a second choice, with K^u from Table I. Finally, we note that the extent to which Eq. (29) or (42) varies with respect to u_H or different combinations of the K^u's and $K^{\ell\ell}$'s will provide a measure of the effect of α_s and λ^4 corrections.

IV. CONCLUSION

In this paper, we have established a method for obtaining $|V_{ub}|/|V_{tb}V_{ts}^*|$ that includes $\mathcal{O}(\Lambda_{\rm QCD}/m_b)$ corrections in a model-independent way. Our approach relies upon a class of relations between the inclusive decays $B \to X_u \ell \bar{\nu}$ and $B \to X_s \gamma$ that are valid including the first-order power corrections [see Eqs. (24) and (29)]. Alternatively, one can use a separate class of relations involving $B \to X_s \ell^+ \ell^-$ [see Eqs. (36) and (42)]. Experimentally required cuts make shape-function effects important in these processes. Their differential decay spectra in the shape-function region have previously been derived to subleading order with the help of the soft-collinear effective theory. These rates involve a number of nonperturbative but universal shape functions in different linear combinations. We are able to eliminate these sources of hadronic uncertainty by taking suitable weighted integrals of the triply differential rates. Hence, our results incorporate NLO power corrections while avoiding model dependence. There are many possible weight functions [see e.g. Eqs. (26) and (27)]; different choices provide a consistency check on the determination of $|V_{ub}|$.

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APPENDIX A: SHAPE FUNCTIONS

The leading-order shape function is

$$f^{(0)}(\ell^+) = \tfrac{1}{2}\langle \bar{B}_v | \bar{h}_v \delta(\ell^+ - in\cdot D) h_v | \bar{B}_v \rangle, \tag{A1}$$

where h_v is the heavy quark field. The subleading shape functions are

$$\begin{aligned}
\langle \bar{B}_v | O_0(\ell^+) | \bar{B}_v \rangle &= f_0^{(2)}(\ell^+),\\
\langle \bar{B}_v | O_1^\beta(\ell^+) | \bar{B}_v \rangle &= \Big(v^\beta - \frac{n^\beta}{n\cdot v}\Big) f_1^{(2)}(\ell^+),\\
\langle \bar{B}_v | P_2^{\beta\lambda}(\ell^+) | \bar{B}_v \rangle &= \epsilon_\perp^{\beta\lambda} f_2^{(2)}(\ell^+),\\
\langle \bar{B}_v | O_3^{\alpha\beta}(\ell^+_{1,2}) | \bar{B}_v \rangle &= g_\perp^{\alpha\beta} f_3^{(4)}(\ell_1^+, \ell_2^+),\\
\langle \bar{B}_v | P_{4\lambda}^{\alpha\beta}(\ell^+_{1,2}) | \bar{B}_v \rangle &= -\epsilon_\perp^{\alpha\beta}\Big(v_\lambda - \frac{n_\lambda}{n\cdot v}\Big)\\
&\quad \times f_4^{(4)}(\ell_1^+, \ell_2^+),\\
n_\alpha n_\beta \langle \bar{B}_v | O_5^{\alpha\beta}(\ell^+_{1,2,3}) | \bar{B}_v \rangle &= f_5^{(6)}(\ell_1^+, \ell_2^+, \ell_3^+),\\
(g_{\alpha\beta}^\perp - i\epsilon_{\alpha\beta}^\perp)\langle \bar{B}_v | O_5^{\alpha\beta}(\ell^+_{1,2,3}) | \bar{B}_v \rangle &= f_6^{(6)}(\ell_1^+, \ell_2^+, \ell_3^+),
\end{aligned} \tag{A2}$$

where $g_\perp^{\mu\nu} = g^{\mu\nu} - (1/2)(n^\mu \bar{n}^\nu + n^\nu \bar{n}^\mu)$ and $\epsilon_\perp^{\mu\nu} = (1/2)\epsilon^{\mu\nu\alpha\beta}\bar{n}_\alpha n_\beta$. The ultrasoft operators are

$$\begin{aligned}
O_0^{(2)}(\ell^+) &= \int \frac{dx^-}{8\pi} e^{-(i/2)x^-\ell^+} \int d^4y\, T[\bar{h}_v(\tilde{x}) Y(\tilde{x},0) h_v(0) iO_h(y)],\\
O_1^\beta(\ell^+) &= \frac{1}{2}\bar{h}_v\{iD_{us}^\beta, \delta(\ell^+ - in\cdot D_{us})\} h_v,\\
P_{2\lambda}^\beta(\ell^+) &= \frac{i}{2}\bar{h}_v [iD_{us}^\beta, \delta(\ell^+ - in\cdot D_{us})]\gamma_\lambda^T \gamma_5 h_v,\\
O_3^{\alpha\beta}(\ell_1^+, \ell_2^+) &= \frac{1}{2}\bar{h}_v \delta(\ell_2^+ - in\cdot D_{us}) Y^\dagger \{iD_{us}^{\perp\alpha}, iD_{us}^{\perp\beta}\} Y \delta(\ell_1^+ - in\cdot D_{us}) h_v,\\
P_{4\lambda}^{\alpha\beta}(\ell_1^+, \ell_2^+) &= -\frac{1}{2}\bar{h}_v \delta(\ell_2^+ - in\cdot D_{us}) g G_{\perp us}^{\alpha\beta} \delta(\ell_1^+ - in\cdot D_{us}) \gamma_\lambda^T \gamma_5 h_v,\\
O_5^{\alpha\beta}(\ell^+_{1,2,3}) &= \frac{1}{2}\{\bar{h}_v \delta(\ell_3^+ - in\cdot D_{us})\gamma^\beta P_L T^A q^{\bar{n}}\}\delta(\ell_2^+ - in\cdot\partial)\{\bar{q}^{\bar{n}}\gamma^\alpha P_L \delta(\ell_1^+ - in\cdot D_{us}) T^A h_v\},
\end{aligned} \tag{A3}$$

where $\tilde{x}^\mu = \bar{n} \cdot x n^\mu/2$. Here, O_h is the NLO term in the HQET Lagrangian, Y is an ultrasoft Wilson line, $igG^{\mu\nu}_{us\perp} = [iD^{\perp\mu}_{us}, iD^{\perp\nu}_{us}]$ and $q^{\bar{n}}_{us} = (\bar{\not n}\not n)/4 q_{us}$. The operator $\bar{O}^{\alpha\beta}_5$, which appears in the definitions of $f_{5,6}$, depends upon the light-quark flavor, u or s.

APPENDIX B: HARD COEFFICIENTS

In this Appendix, we present expressions for the hard coefficients in $B \to X_u \ell \bar{\nu}$ and $B \to X_s \ell^+\ell^-$ [24,32,36]. At lowest order, we have

$$W_i^{(0)} = h_i(p_X^-, m_b, \mu) \int_0^{p_X^+} dk^+ \mathcal{J}^{(0)}(p_X^- k^+, \mu) \times f^{(0)}(k^+ + \bar{\Lambda} - p_X^+, \mu). \tag{B1}$$

For $B \to X_u \ell \bar{\nu}$, we have

$$\begin{aligned} h_1^u &= \frac{1}{4}[C_1^{(v)}]^2, \\ h_2^u &= \frac{(1-u_H)[(C_1^{(v)})^2 + C_1^{(v)}C_2^{(v)} + C_2^{(v)}C_3^{(v)}]}{(\bar{y}_H - u_H)} \\ &\quad + \frac{(C_2^{(v)})^2}{4} + \frac{(1-u_H)^2[(C_3^{(v)})^2 + 2C_1^{(v)}C_3^{(v)}]}{(\bar{y}_H - u_H)^2}, \\ h_3^u &= \frac{(C_1^{(v)})^2}{2m_B(\bar{y}_H - u_H)}, \end{aligned} \tag{B2}$$

where

$$\begin{aligned} C_1^{(v)}(\hat{\omega}, 1) &= 1 - \frac{\alpha_s(m_b)C_F}{4\pi}\Big\{2\ln^2(\hat{\omega}) + 2\mathrm{Li}_2(1-\hat{\omega}) \\ &\quad + \ln(\hat{\omega})\Big(\frac{3\hat{\omega}-2}{1-\hat{\omega}}\Big) + \frac{\pi^2}{12} + 6\Big\}, \\ C_2^{(v)}(\hat{\omega}, 1) &= \frac{\alpha_s(m_b)C_F}{4\pi}\Big\{\frac{2}{(1-\hat{\omega})} + \frac{2\hat{\omega}\ln(\hat{\omega})}{(1-\hat{\omega})^2}\Big\}, \\ C_3^{(v)}(\hat{\omega}, 1) &= \frac{\alpha_s(m_b)C_F}{4\pi}\Big\{\frac{(1-2\hat{\omega})\hat{\omega}\ln(\hat{\omega})}{(1-\hat{\omega})^2} - \frac{\hat{\omega}}{1-\hat{\omega}}\Big\}. \end{aligned} \tag{B3}$$

Here, $\hat{\omega} = \omega/m_b$.

For $B \to X_s \ell^+\ell^-$, we have

$$\begin{aligned} h_1^{\ell\ell} &= \frac{1}{2}(|\mathcal{C}_9|^2 + |\mathcal{C}_{10a}|^2) + \frac{2\,\mathrm{Re}[\mathcal{C}_7\mathcal{C}_9^*]}{(1-\bar{y}_H)} + \frac{2|\mathcal{C}_7|^2}{(1-\bar{y}_H)^2}, \\ h_2^{\ell\ell} &= \frac{2(1-u_H)}{(\bar{y}_H - u_H)}(|\mathcal{C}_9|^2 + |\mathcal{C}_{10a}|^2 + \mathrm{Re}[\mathcal{C}_{10a}\mathcal{C}_{10b}^*]) \\ &\quad + \frac{|\mathcal{C}_{10b}|^2}{2} - \frac{8|\mathcal{C}_7|^2}{(1-\bar{y}_H)(\bar{y}_H - u_H)}, \\ h_3^{\ell\ell} &= \frac{-4\,\mathrm{Re}[\mathcal{C}_{10a}\mathcal{C}_7^*]}{m_B(1-\bar{y}_H)(\bar{y}_H - u_H)} - \frac{2\mathrm{Re}[\mathcal{C}_{10a}\mathcal{C}_9^*]}{m_B(\bar{y}_H - u_H)}. \end{aligned} \tag{B4}$$

The full expressions for the coefficients $\mathcal{C}_{7,9,10a,10b}$ are given in Ref. [24]. When we ignore $\mathcal{O}(\alpha_s(m_b))$ corrections, they simplify to

$$\begin{aligned} \mathcal{C}_9 &= C_{9a} = C_9^{\mathrm{mix}}, & \mathcal{C}_7 &= C_{7a} = \frac{\bar{m}_b(\mu_0)}{m_B} C_7^{\mathrm{NDR}}(\mu_0), \\ \mathcal{C}_{10a} &= C_{10a} = C_{10}, & \mathcal{C}_{10b} &= 0, \end{aligned} \tag{B5}$$

where $\mu_0 \sim m_b$ and

$$\begin{aligned} C_9^{\mathrm{mix}}(\mu_0) &= C_9^{\mathrm{NDR}}(\mu_0) + \frac{2}{9}(3C_3 + C_4 + 3C_5 + C_6) \\ &\quad - \frac{1}{2}h(1,s)(4C_3 + 4C_4 + 3C_5 + C_6) \\ &\quad + h\Big(\frac{m_c}{m_b}, s\Big)(3C_1 + C_2 + 3C_3 + C_4 + 3C_5 \\ &\quad + C_6) - \frac{1}{2}h(0,s)(C_3 + 3C_4) + \mathcal{O}(\alpha_s(\mu_0)). \end{aligned} \tag{B6}$$

The function $h(z, s)$ is given by

$$\begin{aligned} h(z,s) &= \frac{8}{9}\ln\Big(\frac{\mu_0}{m_b}\Big) - \frac{8}{9}\ln z + \frac{8}{27} + \frac{4}{9}\zeta - \frac{2}{9}(2+\zeta) \\ &\quad \times \sqrt{|1-\zeta|}\Big[\theta(1-\zeta)\Big(-i\pi + \ln\frac{1+\sqrt{1-\zeta}}{1-\sqrt{1-\zeta}}\Big) \\ &\quad + \theta(\zeta - 1)2\arctan\frac{1}{\sqrt{\zeta-1}}\Big], \\ h(0,s) &= \frac{8}{27} + \frac{8}{9}\ln\Big(\frac{\mu_0}{m_b}\Big) - \frac{4}{9}\ln s + \frac{4}{9}i\pi, \end{aligned} \tag{B7}$$

with $\zeta = 4z^2/s$ and $s = q^2/m_b^2$.

In the expressions above, C_{1-6}, $C_{7,9}^{\mathrm{NDR}}$, C_{10} are the coefficients of the corresponding operators in the effective Hamiltonian for $b \to s\ell^+\ell^-$ (for which the next-to-leading-log calculations were done in Refs. [42,43]), while C_9^{mix} differs from $\tilde{C}_9^{\mathrm{eff}}$ of Ref. [42] by only an $\mathcal{O}(\alpha_s)$ piece.

Note that there is a complication in the perturbative power counting. Above the scale m_b, one usually expands in α_s, with $\alpha_s \log(m_W/m_b) = \mathcal{O}(1)$. Because of mixing with $\mathcal{O}_{1,2}$, $C_9 \sim \log(m_W/m_b) \sim 1/\alpha_s$, whereas $C_{7,10} \sim 1$. However, numerically $|C_9(m_b)| \sim C_{10}$. This problem is exacerbated by the fact that in the shape-function region only the rate is calculable, not the amplitude. The solution is to use a "split matching" procedure, which decouples the scale dependence above and below $\mu = m_b$ and thereby allows us to consider the coefficients as $\mathcal{O}(1)$ numbers in the latter region [24].

At next-to-leading order, we have

$$
\begin{aligned}
W_i^{(2)f} = {} & \frac{h_i^{0f}(\bar{n}\cdot p)}{2m_b}\int_0^{p_X^+} dk^+ \mathcal{J}^{(0)}(\bar{n}\cdot pk^+,\mu) f_0^{(2)}(k^+ + r^+,\mu) \\
& + \sum_{r=1}^{2}\frac{h_i^{rf}(\bar{n}\cdot p)}{m_b}\int_0^{p_X^+} dk^+ \mathcal{J}^{(0)}(\bar{n}\cdot pk^+,\mu) f_r^{(2)}(k^+ + r^+,\mu) \\
& + \sum_{r=3}^{4}\frac{h_i^{rf}(\bar{n}\cdot p)}{m_b}\int dk_1^+ dk_2^+ \mathcal{J}^{(-2)}(\bar{n}\cdot pk_j^+,\mu) f_r^{(4)}(k_j^+ + r^+,\mu) \\
& + \sum_{r=5}^{6}\frac{h_i^{rf}(\bar{n}\cdot p)}{\bar{n}\cdot p}\int dk_1^+ dk_2^+ dk_3^+ \mathcal{J}^{(-4)}(\bar{n}\cdot pk_{j'}^+,\mu) f_r^{(6)}(k_{j'}^+ + r^+,\mu) + \dots,
\end{aligned}
\tag{B8}
$$

where $j = 1, 2$ and $j' = 1, 2, 3$. The ellipses denote terms that have jet functions $\mathcal{J}$ that start at one-loop order or higher. (These terms are given in Ref. [32].) When we keep the full dependence on u_H, the h_{1-3}^{ru} are

$$
\begin{gathered}
h_1^{1u} = \frac{1}{8}, \qquad h_2^{1u} = -\frac{(1-u_H)(2-\bar{y}_H-u_H)}{2(\bar{y}_H-u_H)^2}, \qquad h_3^{1u} = \frac{1}{4m_B(\bar{y}_H-u_H)}, \qquad h_1^{2u} = -\frac{1}{4}, \\
h_2^{2u} = \frac{(1-u_H)((4-u_H)\bar{y}_H-\bar{y}_H^2-2)}{\bar{y}_H(\bar{y}_H-u_H)^2}, \qquad h_3^{2u} = -\frac{1}{2m_B(\bar{y}_H-u_H)}, \qquad h_1^{3u} = -\frac{1}{4\bar{y}_H}, \\
h_2^{3u} = -\frac{(1-u_H)}{\bar{y}_H(\bar{y}_H-u_H)}, \qquad h_3^{3u} = -\frac{1}{2m_B\bar{y}_H(\bar{y}_H-u_H)}, \qquad h_1^{4u} = \frac{1}{4\bar{y}_H}, \\
h_2^{4u} = -\frac{(1-u_H)(2-\bar{y}_H-u_H)}{\bar{y}_H(\bar{y}_H-u_H)^2}, \qquad h_3^{4u} = \frac{1}{2m_B\bar{y}_H(\bar{y}_H-u_H)}, \qquad h_1^{5u} = -\frac{1}{2}, \\
h_2^{5u} = \frac{2(1-u_H)(1-\bar{y}_H)}{(\bar{y}_H-u_H)^2}, \qquad h_3^{5u} = -\frac{1}{m_B(\bar{y}_H-u_H)}, \qquad h_1^{6u} = 0, \qquad h_2^{6u} = \frac{2(1-u_H)^2}{(\bar{y}_H-u_H)^2}, \qquad h_3^{6u} = 0,
\end{gathered}
\tag{B9}
$$

and the $h_{1-3}^{r\ell\ell}$ are

$$
\begin{aligned}
h_1^{1\ell\ell} &= -\frac{4|C_{7a}|^2 - (|C_{10a}|^2 + |C_{9a}|^2)(1-\bar{y}_H)^2}{4(1-\bar{y}_H)^2}, \\
h_2^{1\ell\ell} &= \frac{(2-\bar{y}_H-u_H)(4|C_{7a}|^2 - (|C_{10a}|^2 + |C_{9a}|^2)(1-\bar{y}_H)(1-u_H))}{(1-\bar{y}_H)(\bar{y}_H-u_H)^2}, \\
h_3^{1\ell\ell} &= -\frac{\mathrm{Re}[C_{10a}C_{9a}^*]}{m_B(\bar{y}_H-u_H)}, \\
h_1^{2\ell\ell} &= \frac{4|C_{7a}|^2 - (|C_{10a}|^2 + |C_{9a}|^2)(1-\bar{y}_H)^2}{2(1-\bar{y}_H)^2}, \\
h_2^{2\ell\ell} &= -\frac{2}{\bar{y}_H(\bar{y}_H-u_H)^2}\bigg[4|C_{7a}|^2\frac{2-\bar{y}_H^2-\bar{y}_Hu_H}{1-\bar{y}_H} + 4\,\mathrm{Re}[C_{7a}C_{9a}^*](2-\bar{y}_H-u_H) \\
&\quad + (|C_{10a}|^2 + |C_{9a}|^2)(2-4\bar{y}_H+\bar{y}_H^2+\bar{y}_Hu_H)(1-u_H)\bigg], \\
h_3^{2\ell\ell} &= \frac{2\,\mathrm{Re}[C_{10a}C_{9a}^*]}{m_B(\bar{y}_H-u_H)}, \\
h_1^{3\ell\ell} &= -\frac{4|C_{7a}|^2 + 4\,\mathrm{Re}[C_{7a}C_{9a}^*](1-\bar{y}_H) + (|C_{10a}|^2 + |C_{9a}|^2)(1-\bar{y}_H)^2}{2\bar{y}_H(1-\bar{y}_H)^2}, \\
h_2^{3\ell\ell} &= 2\frac{4|C_{7a}|^2 - (|C_{10a}|^2 + |C_{9a}|^2)(1-\bar{y}_H)(1-u_H)}{\bar{y}_H(1-\bar{y}_H)(\bar{y}_H-u_H)}, \\
h_3^{3\ell\ell} &= 2\frac{2\,\mathrm{Re}[C_{10a}C_{7a}^*] + \mathrm{Re}[C_{10a}C_{9a}^*](1-\bar{y}_H)}{m_B\bar{y}_H(1-\bar{y}_H)(\bar{y}_H-u_H)}, \qquad\qquad \text{(B10)} \\
h_1^{4\ell\ell} &= \frac{4|C_{7a}|^2 + 4\,\mathrm{Re}[C_{7a}C_{9a}^*](1-\bar{y}_H) + (|C_{10a}|^2 + |C_{9a}|^2)(1-\bar{y}_H)^2}{2\bar{y}_H(1-\bar{y}_H)^2}, \\
h_2^{4\ell\ell} &= -\frac{2}{\bar{y}_H(\bar{y}_H-u_H)^2}\bigg[4|C_{7a}|^2\frac{2-\bar{y}_H-u_H}{1-\bar{y}_H} + 8\,\mathrm{Re}[C_{7a}C_{9a}^*](1-u_H) + (|C_{10a}|^2 + |C_{9a}|^2)(2-\bar{y}_H-u_H)(1-u_H)\bigg], \\
h_3^{4\ell\ell} &= -2\frac{2\,\mathrm{Re}[C_{10a}C_{7a}^*] + \mathrm{Re}[C_{10a}C_{9a}^*](1-\bar{y}_H)}{m_B\bar{y}_H(1-\bar{y}_H)(\bar{y}_H-u_H)}, \\
h_1^{5\ell\ell} &= -\frac{4|C_{7a}|^2 + 4\,\mathrm{Re}[C_{7a}C_{9a}^*](1-\bar{y}_H) + (|C_{10a}|^2 + |C_{9a}|^2)(1-\bar{y}_H)^2}{(1-\bar{y}_H)^2}, \\
h_2^{5\ell\ell} &= 4\frac{(4|C_{7a}|^2 + 4\,\mathrm{Re}[C_{7a}C_{9a}^*](1-\bar{y}_H) + (|C_{10a}|^2 + |C_{9a}|^2)(1-\bar{y}_H)^2)(1-u_H)}{(1-\bar{y}_H)(\bar{y}_H-u_H)^2}, \\
h_3^{5\ell\ell} &= 4\frac{2\,\mathrm{Re}[C_{10a}C_{7a}^*] + \mathrm{Re}[C_{10a}C_{9a}^*](1-\bar{y}_H)}{m_B(1-\bar{y}_H)(\bar{y}_H-u_H)}, \qquad h_1^{6\ell\ell} = 0, \\
h_2^{6\ell\ell} &= 4\frac{4|C_{7a}|^2 + 4\,\mathrm{Re}[C_{7a}C_{9a}^*](1-u_H) + (|C_{10a}|^2 + |C_{9a}|^2)(1-u_H)^2}{(\bar{y}_H-u_H)^2}, \qquad h_3^{6\ell\ell} = 0.
\end{aligned}
$$

APPENDIX C: WEIGHT FUNCTIONS

TABLE IV. Some choices of $K^u(x_H, \bar{y}_H, u_H)$ for which the weighted integral Eq. (17) depends only on the shape functions F, F_1, and F_2. The coefficients $\kappa_1^u(u_H)$ and $\kappa_2^u(u_H)$ are defined in Eq. (37).

(4)	$K^u_{\mathrm{IV}} = \frac{1}{N(u_H)} \frac{(2x_H+u_H+\bar{y}_H-2)}{(\bar{y}_H-u_H)} \frac{(1-\bar{y}_H)(2\bar{y}_H-u_H-1)}{(1+u_H-\bar{y}_H)}$ $N(u_H) = \frac{1}{30}(1-u_H)^2(1-14u_H-94u_H^2-14u_H^3+u_H^4) - 2u_H^2(1-u_H^2)\log u_H$ $\kappa_1^u = \frac{1}{2}, \qquad \kappa_2^u = -1$
(5)	$K^u_{\mathrm{IV}} = \frac{6[-(7-u_H)\bar{y}_H+4x_H\bar{y}_H+6\bar{y}_H^2]}{(1-u_H)^7} \qquad \kappa_1^u = -\frac{1}{10}, \qquad \kappa_2^u = \frac{7-u_H}{5(1-u_H)}$
(6)	$K^u_{\mathrm{IV}} = -\frac{105}{101} \frac{[14(1-u_H)^2\bar{y}_H+5(2+7u_H)x_H\bar{y}_H-45x_H\bar{y}_H^2]}{(1-u_H)^8} \qquad \kappa_1^u = -\frac{2}{101}, \qquad \kappa_2^u = \frac{109-4u_H}{101(1-u_H)}$

TABLE V. Some choices of $K^{\ell\ell}(x_H, \bar{y}_H, u_H)$ for which $\kappa_2^{\ell\ell}(u_H) = \kappa_3^{\ell\ell}(u_H)$ in Eq. (37).[a] Here, $\mathcal{A} = -2\,\mathrm{Re}[C_{10a}C_{7a}^*]$, $\mathcal{B} = \mathrm{Re}[C_{10a}C_{9a}^*]$, and $\bar{y}_{H*} = \bar{y}_{\min} + \bar{y}_{\max}$.

(7)	$K^{\ell\ell} = \frac{1}{N(u_H)} \frac{(2x_H+u_H+\bar{y}_H-2)}{(\bar{y}_H-u_H)} \frac{(\bar{y}_{H*}-\bar{y}_H-u_H)^2}{(\bar{y}_{H*}-\bar{y}_H)} \{\mathcal{A} - \mathcal{B}[1-(\bar{y}_{H*}-\bar{y}_H)^2]\}(\bar{y}_{H*}-2\bar{y}_H)$ $N(u_H) = 8(1-u_H)\int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H (\bar{y}_H-u_H)^2 \frac{(\bar{y}_{H*}-\bar{y}_H-u_H)^2}{(\bar{y}_{H*}-\bar{y}_H)} (\bar{y}_{H*}-2\bar{y}_H)\{\mathcal{A}-\mathcal{B}(1-\bar{y}_H)\}\{\mathcal{A}-\mathcal{B}[1-(\bar{y}_{H*}-\bar{y}_H)^2]\}$ $\kappa_2^{\ell\ell} = -\frac{8(1-u_H)}{N(u_H)} \int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H (\bar{y}_H-u_H)^2(1-\bar{y}_H) \frac{(\bar{y}_{H*}-\bar{y}_H-u_H)^2}{(\bar{y}_{H*}-\bar{y}_H)} (\bar{y}_{H*}-2\bar{y}_H)\mathcal{B}\{\mathcal{A}-\mathcal{B}[1-(\bar{y}_{H*}-\bar{y}_H)^2]\}$
(8)	$K^{\ell\ell} = \frac{1}{N(u_H)}(2x_H+u_H+\bar{y}_H-2)(\bar{y}_{H*}-\bar{y}_H-u_H)^3\bar{y}_H(\bar{y}_{H*}-2\bar{y}_H)\{\mathcal{A}-\mathcal{B}[1-(\bar{y}_{H*}-\bar{y}_H)^2]\}$ $N(u_H) = 8(1-u_H)\int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H (\bar{y}_H-u_H)^3(\bar{y}_{H*}-\bar{y}_H-u_H)^3\bar{y}_H(\bar{y}_{H*}-2\bar{y}_H)\{\mathcal{A}-\mathcal{B}[1-(\bar{y}_{H*}-\bar{y}_H)^2]\}\{\mathcal{A}-\mathcal{B}(1-\bar{y}_H)\}$ $\kappa_2^{\ell\ell} = -\frac{8(1-u_H)}{N(u_H)} \int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H (\bar{y}_H-u_H)^3(1-\bar{y}_H)(\bar{y}_{H*}-\bar{y}_H-u_H)^3\bar{y}_H(\bar{y}_{H*}-2\bar{y}_H)\mathcal{B}\{\mathcal{A}-\mathcal{B}[1-(\bar{y}_{H*}-\bar{y}_H)^2]\}$
(9)	$K^{\ell\ell} = \frac{1}{N(u_H)} \frac{(2x_H+u_H+\bar{y}_H-2)}{(\bar{y}_H-u_H)} \frac{(\bar{y}_{H*}-\bar{y}_H-u_H)^2}{(\bar{y}_{H*}-\bar{y}_H)} \frac{(\bar{y}_{H*}-2\bar{y}_H)}{\{\mathcal{A}-\mathcal{B}(1-\bar{y}_H^2)\}}$ $N(u_H) = 8(1-u_H)\int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H (\bar{y}_H-u_H)^2 \frac{(\bar{y}_{H*}-\bar{y}_H-u_H)^2}{(\bar{y}_{H*}-\bar{y}_H)} (\bar{y}_{H*}-2\bar{y}_H) \frac{\{\mathcal{A}-\mathcal{B}(1-\bar{y}_H)\}}{\{\mathcal{A}-\mathcal{B}(1-\bar{y}_H^2)\}}$ $\kappa_2^{\ell\ell} = -\frac{8(1-u_H)}{N(u_H)} \int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H (\bar{y}_H-u_H)^2(1-\bar{y}_H) \frac{(\bar{y}_{H*}-\bar{y}_H-u_H)^2}{(\bar{y}_{H*}-\bar{y}_H)} (\bar{y}_{H*}-2\bar{y}_H) \frac{\mathcal{B}}{\{\mathcal{A}-\mathcal{B}(1-\bar{y}_H^2)\}}$

[a]Note that Example (9) requires a harsher cut, e. g. $2\ \mathrm{GeV}^2 \le q^2 \le 6\ \mathrm{GeV}^2$ (rather than $1\ \mathrm{GeV}^2 \le q^2 \le 6\ \mathrm{GeV}^2$), so that it is not singular.

TABLE VI. Some choices of $K^{\ell\ell}$ and the corresponding coefficients $\kappa_2^{\ell\ell}(u_H)$ and $\kappa_3^{\ell\ell}(u_H)$, which are defined in Eq. (37). Here, $\mathcal{A} = -2\,\mathrm{Re}[C_{10a}C_{7a}^*]$, $\mathcal{B} = \mathrm{Re}[C_{10a}C_{9a}^*]$, and $\bar{y}_{H*} = \bar{y}_{\min} + \bar{y}_{\max}$. C_{9a} may be taken to be constant, in which case the integrals can be evaluated analytically.

(10)	$K^{\ell\ell} = \frac{1}{N(u_H)} \frac{(2x_H+u_H+\bar{y}_H-2)}{4(1-u_H)} \qquad N(u_H) = 2\int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H(\bar{y}_H-u_H)^3\{\mathcal{A}-\mathcal{B}(1-\bar{y}_H)\}$ $\kappa_2^{\ell\ell} = -\frac{2}{N(u_H)} \int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H \mathcal{B}(1-\bar{y}_H)(\bar{y}_H-u_H)^3 \qquad \kappa_3^{\ell\ell} = -\frac{2}{N(u_H)} \int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H \frac{(\bar{y}_H-u_H)^3}{\bar{y}_H}\{\mathcal{A}-\mathcal{B}(1-\bar{y}_H)\}$
(11)	$K^{\ell\ell} = \frac{1}{N(u_H)} \frac{(2x_H+u_H+\bar{y}_H-2)\bar{y}_H}{4(1-u_H)} \qquad N(u_H) = 2\int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H\bar{y}_H(\bar{y}_H-u_H)^3\{\mathcal{A}-\mathcal{B}(1-\bar{y}_H)\}$ $\kappa_2^{\ell\ell} = -\frac{2}{N(u_H)} \int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H \mathcal{B}(1-\bar{y}_H)(\bar{y}_H-u_H)^3\bar{y}_H \qquad \kappa_3^{\ell\ell} = -\frac{2}{N(u_H)} \int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H (\bar{y}_H-u_H)^3\{\mathcal{A}-\mathcal{B}(1-\bar{y}_H)\}$
(12)	$K^{\ell\ell} = \frac{1}{N(u_H)} \frac{(2x_H+u_H+\bar{y}_H-2)}{4(1-u_H)(\bar{y}_H-u_H)} \frac{(\bar{y}_{H*}-\bar{y}_H-u_H)^2}{(\bar{y}_{H*}-\bar{y}_H)} (\bar{y}_{H*}-2\bar{y}_H)\{\mathcal{A}-\mathcal{B}(1-\bar{y}_{H*}+\bar{y}_H)\}$ $N(u_H) = 2\int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H(\bar{y}_H-u_H)^2 \frac{(\bar{y}_{H*}-\bar{y}_H-u_H)^2}{(\bar{y}_{H*}-\bar{y}_H)} (\bar{y}_{H*}-2\bar{y}_H)\{\mathcal{A}-\mathcal{B}(1-\bar{y}_H)\}\{\mathcal{A}-\mathcal{B}(1-\bar{y}_{H*}+\bar{y}_H)\}$ $\kappa_2^{\ell\ell} = -\frac{2}{N(u_H)} \int_{\bar{y}_{\min}}^{\bar{y}_{\max}} d\bar{y}_H (1-\bar{y}_H)(\bar{y}_H-u_H)^2 \frac{(\bar{y}_{H*}-\bar{y}_H-u_H)^2}{(\bar{y}_{H*}-\bar{y}_H)} (\bar{y}_{H*}-2\bar{y}_H)\mathcal{B}\{\mathcal{A}-\mathcal{B}(1-\bar{y}_{H*}+\bar{y}_H)\} \qquad \kappa_3^{\ell\ell} = 0$

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