Orbit spaces of low-dimensional representations of simple compact connected Lie groups and extrema of a group-invariant scalar potential[a]

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Orbit spaces of low-dimensional representations of classical and exceptional Lie groups are constructed and tabulated. We observe that the orbit spaces of some single irreducible representations (adjoints, second-rank symmetric and antisymmetric tensors of classical Lie groups, and the defining representations of F_4 and E_6) are warped polyhedrons with (locally) more protrudent boundaries corresponding to higher level little groups. The orbit spaces of two irreducible representations have different shapes. We observe that dimension and concavity of different strata are not sharply distinguished. We explain that the observed orbit space structure implies that a physical system tends to retain as much symmetry as possible in a symmetry breaking process. In Appendix A, we interpret our method of minimization in the orbit space in terms of conventional language and show how to find all the extrema (in the representation space) of a general group-invariant scalar potential monotonic in the orbit space. We also present the criterion to tell whether an extremum is a local minimum or maximum or an inflection point. In Appendix B, we show that the minimization problem can always be reduced to a two-dimensional one in the case of the most general Higgs potential for a single irreducible representation and to a three-dimensional one in the case of an even degree Higgs potential for two irreducible representations. We explain that the absolute minimum condition prompts the boundary conditions enough to determine the representation vector.

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1. INTRODUCTION

Since the discovery of the Higgs mechanism,[1] it has been employed almost exclusively in the gauge symmetry breaking problem because it breaks a local gauge symmetry without damaging the renormalizability.[2] Though it is not the only known mechanism to do such a job,[3,4] certainly it is the only tractable one. Partly due to its tractability it drew considerable attention from theoretical physicists despite some ugly features. There is a consensus that, though it may not be a fundamental mechanism, it would describe the effective phenomena arising from some unknown fundamental interactions. It was applied to the unification of electromagnetic and weak interactions[5] with great success and subsequently to fancier grand unification theories.[6] Here some major difficulties arose, namely, the gauge hierarchy problem[7] and proliferation of Higgs parameters, etc. The spontaneous symmetry breaking problem in supersymmetric theories[8] is one of the most popular problems these days. Since the spontaneous symmetry breaking mechanism was devised by Landau[9] to explain continuous structural phase transitions in crystals, the mechanism has been widely employed in condensed matter physics.[10] Spontaneous symmetry breaking is one of the most fundamental phenomena observed in nature. It is no wonder that there exist several extensive review articles on this subject.[11,12]

The technical problem of minimizing the scalar potential or the thermodynamic potential and finding the symmetry of the vacuum or the equilibrium state has been considered to be a formidable task among theoretical physicists. Unification theorists could only check the list of possible symmetry breaking directions without knowing whether and when the symmetry is broken in certain directions. Condensed matter theorists had to use an abbreviated potential. Our geometrical method provides the most appropriate language for the problem. It gives accurate minimizing solutions for a general Higgs potential of single irreducible representations and for a general even-degree Higgs potential of two irreducible representations. We leave to an interested reader the analysis of a general even-degree Higgs potential of three irreducible representations, which will give needed solutions to some unification models, e.g., the E_6 model with a Higgs assignment in 27, 78, and 351 representations.[13]

The geometrical method of minimizing the Landau–Higgs potentials, devised by the author,[14–17] reduces the problem to one of finding "contours" of directional minima. It is based on the observation[18] that the orbits and the conjugacy classes of subgroups are the relevant quantities to describe the minimum of the Landau–Higgs potential which is invariant under a linear transformation of a compact Lie group on the scalar fields (or a finite group on the order parameters). Hilbert[19] proved that there is a basic set of invariants such that all other invariants are expressed as their polynomials and provided a systematic method to find all

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the basic invariants. It has been known[20] that invariants specify orbits, i.e., one can view an orbit as a point in an $(l+1)$-dimensional vector space, $(l+1)$ being the number of independent basic invariants. How can we describe a direction in such a space? Indeed there is a set of parameters[21] that can be used for such purpose. "Orbit parameters" are defined to be dimensionless ratios of invariant polynomials. These parameters can be considered as some set of generalized angles specifying a direction in the representation space. Their ranges being bounded, they occupy a localized region (called the "orbit space") in the orbit parameter space, which can be regarded as an l-dimensional vector space.

Since the scalar potential is a group invariant function, it can be expressed in terms of the basic invariants. But a classical Higgs potential is restricted to be a fourth-degree polynomial of the scalar fields due to renormalizability. Because of this restriction it is normal that only a subset of all the basic invariants appear in the Higgs potential, and that the orbit parameters formed from this subset appear linearly.[22] The potential can be written in terms of the norm of the field and a few orbit parameters. For a given set of orbit parameters we can survey the behavior, particularly extrema, of the potential along the corresponding direction in the field component space. By varying the orbit parameters, we can survey the whole space in search of the absolute minimum. Because of the linearity, the absolute minimum of the potential occurs on the most protrudent portions of the boundary of the orbit space formed from the fourth-degree Higgs potential, which is a projection of the complete orbit space.

The potential can be minimized abstractly for a general representation of a general compact group. The difficult part of extremizing the potential in the conventional methods[23,24] has as its counterpart in our method the problem of finding the orbit space boundary, which is unique for each different representation. In our original works our method for constructing the projected orbit space was empirical and we used the Michel–Radicati conjecture[25] for one irreducible representation (irrep)[26] and the Gell-Mann–Slansky conjecture[27] for two irreps as a guide for finding the orbit space boundary. Then our results were tested with the boundary conditions. It was realized that we need not solve high degree algebraic equations to find orbit space boundaries. The procedure is facilitated by some general mathematical results.[28,29,16] Using these results, we look for branching rules[27,30] and singlet forms of the given representation under various subgroups, starting from the highest level to successively lower levels. In any case we need to know at least this much information to specify the absolute minimum. "Usable boundaries" (where the potential may have a minimum) correspond to higher symmetry groups. In practice one finds the whole boundary before he reaches the lowest level.

Much work has been done by mathematicians[31,32,29] on the structure of the complete orbit space. Their results were originally derived from the properties of linear actions of compact transformation groups. However, Ref. 32 deals with the relationship between orbits and invariants. Recently a comprehensive review article[29] has been published for physicists. Our formalism is entirely based on the invariant polynomials and requires less mathematical background, presenting a concrete and intuitive picture without losing generality. The main result is that the orbit space consists of some l-dimensional volume occupied by the generic stratum of the lowest level symmetry group with all the other strata of higher symmetries forming the singular boundaries. Equivalently, the generic stratum occupies an open, dense, topologically connected region and thus the boundaries must belong to the lower dimensional strata. It was explicitly shown[12,29] that a lower dimensional stratum of a higher symmetry is a subspace which is spanned by the gradients of basic invariant polynomials. (This is equivalent to our boundary conditions.[16])

In this paper we present concrete examples, showing that lower dimensional strata of higher symmetry groups always form the boundaries of higher dimensional strata of lower symmetry groups. We also observe that high symmetry strata are normally[33] more protrudent than lower symmetry ones, which was conjectured in our earlier works.[14–16] This protrusiveness of orbit spaces (defined in terms of ratios of invariant polynomials) is important physically because it indicates that a physical system tries to retain the highest symmetries possible when the spontaneous symmetry breaking takes place, which is the spirit of the Michel–Radicati and the Gell-Mann–Slansky conjectures. It also makes our method powerful. The hierarchy of protrusiveness on the boundaries is essential to predict how small the little groups can be in the presence of nonmonotonic orbit parameters.[17]

In Sec. 2, we briefly review the minimization problem in orbit space. In Sec. 3, we construct orbit spaces of adjoint representations. We observe that they form polyhedrons as conjectured in Ref. 16. In Sec. 4, second-rank symmetric and antisymmetric tensors of all the classical Lie groups and other low-dimensional single irreps are analyzed. We observe that the tensors have the same orbit spaces as adjoint representations of other groups and that there is an interesting relationship between the number of maximal little groups and the degrees of basic invariants. We discuss the implications of the observed properties in the minimization problem. In Sec. 5, orbit spaces of two irreps are shown. We find that the generic stratum is semiclosed. It is shown that dimension and concavity of different strata are not sharply distinguished. In Appendix A, we compare our method to the conventional one to help the reader to understand the workings of our method. We also show how to find all the extrema (in the representation space) of a general Higgs potential. If a Higgs potential contains more than four independent invariant polynomials it seems difficult to locate the absolute minimum visually. In Appendix B, we show that the minimization problem can always be reduced to a two-dimensional one in the case of the most general Higgs potential for a single irrep and to a three-dimensional one in the case of an even-degree Higgs potential for two irreps. Thus we can visually minimize the Higgs potentials of these two types using the contours of directional minima we derived previously.

Once the orbit space is constructed, the absolute minimum of the Landau–Higgs potential for a given representation can be read off the list right away using the results derived in Refs. 14–16.

2. HIGGS PROBLEM AND ORBIT PARAMETERS

In a nonabelian gauge theory, where the scalar potential has a symmetry $G \times$reflection and the scalars transform as an n-dimensional irreducible representation $\bar{R}$ of a semi-simple compact Lie group G, the Higgs potential can be written as

$$V(\varphi) = -\frac{1}{2} m^2 \sum_{i=1}^{n} \varphi_i^* \varphi_i + \frac{1}{4} A \left(\sum_{i=1}^{n} \varphi_i^* \varphi_i \right)^2 + \frac{1}{4} A_1 f_{ijkl} \varphi_i^* \varphi_j \varphi_k^* \varphi_l + \frac{1}{4} A_2 g_{ijkl} \varphi_i^* \varphi_j \varphi_k^* \varphi_l + \cdots . \quad (1)$$

$V(\varphi)$ is invariant under a group transformation, $\varphi_j' = \Sigma_{i=1}^n T(\vartheta)_{ji} \varphi_i$. $T(\vartheta)$ is an $n \times n$ matrix representing a group element.[34] It can be written in general, $T(\vartheta) = \exp(-i \Sigma_{i=1}^N \vartheta_L X_L)$. X_L are $n \times n$ matrices representing the generators of the group, and ϑ_L are real or complex parameters specifying an element of the group. Our objective is to find the field configuration and the corresponding symmetry that yield the minimum energy. We set the scalar fields constant in space-time and minimize the resulting potential.

We introduce some useful group theoretical concepts, nicely explained by O'Raifeartaigh.[11] The orbit of φ_a is defined to be the set of vectors $\varphi^{(a)}$ that can be expressed as $\varphi^{(a)} = T(\vartheta)\varphi_a$ with $T(\vartheta)$ an element of G. The little group of φ_a is defined to be the subgroup G_a' of G that leaves φ_a invariant: $T(\vartheta)\varphi_a = \varphi_a$ for $T(\vartheta) \in G_a' \subset G$. The vectors on an orbit are in one-to-one correspondence with the coset G/G_a'. It can easily be shown that the little group G_b' of any vector φ_b on the orbit of φ_a is conjugate to G_a'. If the $T(\vartheta)$ are unitary, then all the vectors $\varphi^{(a)}$ have the same norm $\varphi_a^* \varphi_a$. In general, there is a continuum of distinct orbits respecting the same little group up to conjugation. The set of all such orbits is called the stratum of the little group. Note that if the little groups of two orbits are distinct, then the orbits are distinct. However, the converse is not true, i.e., if two orbits are distinct, their little groups are not necessarily different.

By definition an invariant polynomial is constant on an orbit and thus is a function of orbits. A classical Higgs potential is a polynomial of some algebraically independent invariant polynomials. When we seek a solution to the Higgs problem, we are actually seeking the orbit that minimizes the potential, and its little group.

However we need to find a better way to specify an orbit, because to an orbit there corresponds a trajectory of vectors in the φ-space. Aronhold[20] realized that invariant polynomials specify orbits, which we adopt for our purposes. One is naturally led to the fact that there are only a finite number of independent invariants because there are only a finite number of real parameters specifying a vector in the representation space. Mathematicians have more to say. Hilbert[19] proved that there exists a set of invariant polynomials $I_a(\varphi)$, called the integrity basis, such that every invariant polynomial $P(\varphi)$ can be expressed as a polynomial of I_a: $P(\varphi) = \overline{P}\,[I_a(\varphi)]$. The invariants in the integrity basis are not necessarily independent, and indeed, for some representations, called noncoregular representations, there are polynomial identities among them, called syzygies. We will call the complete set of lowest degree independent invariants, "basic invariants." The number $(l+1)$ of basic invariants is different for each different representation. We can visualize an orbit as a point in the $(l+1)$-dimensional space of I_a.

The dimensionless ratios of invariants to the magnitude of the φ vector, for example,

$$\lambda = f_{ijkl} \varphi_i^* \varphi_j \varphi_k^* \varphi_l \Big/ \left(\sum_{i=1}^{n} \varphi_i^* \varphi_i \right)^2, \quad (2)$$

can be used to specify strata, and yield a powerful tool in the minimization problem. We will call the dimensionless ratios *orbit parameters*. They can be considered as a set of generalized angles containing all the directional information. From the definition we can readily see that their ranges are bounded, and thus they occupy a localized region (called the *orbit space*)[35] in the orbit parameter space, which can be regarded as an l-dimensional vector space.

Our method reduces the minimization problem to one of finding "contours" of directional minima (the minimum of the potential in the direction specified by a set of orbit parameters). The "contour" for the most general Higgs potential of one irrep has been analyzed in Ref. 16. It is a curve somewhat similar to a parabola or a surface made by translating the curve. The absolute minimum of the potential occurs at the most protrudent portions of the projected orbit space boundary, corresponding to higher level little groups. The "contour" for the most general even degree Higgs potential of two irreps has been analyzed in Ref. 15. It is a cone and again the absolute minimum occurs at the most protrudent portions of the projected orbit space boundary. This result yields a powerful method for locating the absolute minimum.

When the orbit space dimension is less than four, one can visually locate the absolute minimum. When the dimension is higher than three, one would compare the potential values at different extremum points and pick the lowest one to find the absolute minimum. In Appendix A, we show, by derivation, that there are only a finite number of extrema (in the representation space) of a general Higgs potential, including the ones corresponding to lower level little groups. The absolute minimum normally occurs at the stratum of one of the maximal or maximaximal little groups because they normally correspond to the most protrudent portions of the orbit space boundary. However, we can still visually locate the potential minimum by projecting the orbit space further. In Appendix B, we show that the minimization problem can be reduced to a two-dimensional one in the case of the most general Higgs potential for a single irrep and to a three-dimensional one in the case of an even-degree Higgs potential for two irreps.

When the potential is not monotonic in orbit parameters, the situation is more complicated. We have shown[17] that, in a nontrivial case, each time an orbit parameter appears in the potential nonmonotonically the problem reduces to the same form on the constraint "surface" introduced by the nonmonotonicity. Thus the absolute minimum is now most likely to occur on the less (by one level) protrudent portions of the orbit space boundary. However, we cannot totally exclude trivial cases where the maximal or maximaximal little groups are still favored.

Deeper knowledge of the orbit space structure is essential to understand the Michel–Radicati and the Gell-Mann–Slansky conjectures on the minimal symmetry breaking principle and to see how the principle (not the conjectures themselves) works in the presence of nonmonotonic orbit parameters. The above conjectures seem to hold most of the time but counterexamples[36,37,17] have already been found recently.

In the following we tabulate orbit spaces of all the coregular representations that admit less than (or equal to) five independent basic invariants. Although the orbit space in its original sense is a uniquely defined mathematical object, there is some arbitrariness in defining it using invariant functions. Mathematicians would say that any $(l+1)$ independent smooth invariants would do the job. Physicists would try to be more specific and to make a definition useful for their own needs, preferably a visual and compact one. Our definition was formed in ignorance: dimensionless ratios of lowest degree independent invariant polynomials in the integrity basis to the unique quadratic invariant. This set itself is not uniquely defined because any linear combinations of the same-degree invariant polynomials are equally qualified. Since the concavity of a geometrical object does not change upon linear transformations of the coordinates, our definition is safe.

3. COMPLETE ORBIT SPACES OF ADJOINT REPRESENTATIONS

As we shall see, the orbit spaces of adjoint representations are prototypes for many other representations. We describe them in detail. Let us briefly review some group theoretical results[38] to set up our notation. For the algebra of order N and rank $(l+1)$ we choose a Cartan–Weyl basis, so that the commutation relations assume the standard form:

$$[H_i,H_j]=0, \quad i,j=1,2,\dots,(l+1), \tag{3a}$$

$$[H_i,E_{\pm\alpha}]=\pm r_i(\alpha)E_{\pm\alpha'} \quad \alpha=1,2,\dots,(N-l-1)/2, \tag{3b}$$

$$[E_\alpha,E_{-\alpha}]=\sum_{i=1}^{l+1} r_i(\alpha)H_i, \tag{3c}$$

$$[E_\alpha,E_\beta]=N_{\alpha\beta}E_{\alpha+\beta}, \tag{3d}$$

where $N_{\alpha\beta}\neq 0$ only if $\mathbf{r}(\alpha)+\mathbf{r}(\beta)$ is also a root. The Killing scalar products are

$$(H_i,H_i)=1, \quad (E_\alpha,E_{-\alpha})=1, \tag{4}$$

with all other scalar products being zero. Furthermore, the roots $\mathbf{r}(\alpha)$ satisfy the condition

$$\sum_\alpha r_i(\alpha)r_j(\alpha)=\delta_{ij}. \tag{5}$$

Using the generalized Casimir operators derived by Racah,[39] Gruber and O'Raifeartaigh[40] have derived forms for the Casimir invariants that are more useful in practice. [The field components can be reduced by a group transformation to $(l+1)$ (number of rank) irreducible components which correspond to H_i's in the Cartan–Weyl basis. Utilizing these results, we can readily write down the tractable form of each invariant.] The complete set of invariant polynomials for adjoint representations can be obtained by using the matrix form for the representation vector,

$$\varphi=\sum_{i=1}^{N}\varphi_i X_i, \tag{6}$$

where φ_i is the ith component of φ in vector notation and X_i is the matrix corresponding to the ith generator. Note that X_i can be based on any representation. Using the notation

$$I_m=\mathrm{Tr}\,\varphi^m, \tag{7}$$

we list the complete set of invariant polynomials in Table I along with other useful properties for each classical and exceptional Lie group. The I'_n of SO_{2n} is of a form similar to A_5 in Eq. (39) of Ref. 17.

Using the convention

$$\varphi=\sum_{i=1}^{l+1}\varphi_i H_i\equiv[a_1,a_2,\cdots], \tag{8}$$

where we have defined the square bracket as the diagonal elements of the matrix, we can directly write down the orbit parameters in the following generic form:

$$\alpha_m\equiv\frac{\mathrm{Tr}\,\varphi^m}{(\mathrm{Tr}\,\varphi^2)^{m/2}}, \tag{9a}$$

$$\alpha'_m\equiv\frac{2^m a_1a_2\cdots a_m}{(\mathrm{Tr}\,\varphi^2)^{m/2}}. \tag{9b}$$

A. Groups of rank two

There is only one orbit parameter for the adjoint representation of a group of rank 2, and thus the orbit space is a line.

SU(3)

We choose the vector representation for the basis of the matrices. Then the generic stratum and the orbit parameter are represented as follows:

$$U_1\times U_1:$$

$$\varphi=[a,b,-a-b],$$

$$\alpha_3=(a^3+b^3-(a+b)^3)/(a^2+b^2+(a+b)^2)^{3/2}. \tag{10}$$

TABLE I. List of Casimir invariants, order and rank of classical and exceptional Lie groups.

Group	Invariants	Order	Rank
SU_{n+1}	$I_2, I_3,\dots,I_{n+1}$	$n(n+2)$	n
SO_{2n+1}	$I_2, I_4,\dots,I_{2n}$	$n(2n+1)$	n
Sp_{2n}	$I_2, I_4, \dots,I_{2n}$	$n(2n+1)$	n
SO_{2n}	$I_2, I_4, \dots,I_{2n-2}, I'_n$	$n(2n-1)$	n
G_2	I_2, I_6	14	2
F_4	I_2, I_6, I_8, I_{12}	52	4
E_6	$I_2, I_5, I_6, I_8, I_9, I_{12}$	78	6
E_7	$I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}$	133	7
E_8	$I_2, I_8, I_{12}, I_{14}, I_{18}, I_{20}, I_{24}, I_{30}$	248	8

The stratum of each little group is represented as follows:

$SU_2 \times U_1$:

$$3 = 1[-2] + 2[1]\,, \qquad \varphi = [a,a,-2a]\,, \quad \alpha_3 = \pm 1/\sqrt{6}\,. \tag{11}$$

The orbit space consists of two end points corresponding to $[SU_2 \times U_1]$ and the interior corresponding to $[U_1 \times U_1]$.

SO(5) and Sp(4)

We choose the five-dimensional vector representation for the basis. The generic stratum and the orbit parameter are represented as follows:

$U_1 \times U_1$:

$$\varphi = [a,-a,b,-b,0]\,, \qquad \alpha_4 = (2a^4 + 2b^4)/(2a^2 + 2b^2)^2\,. \tag{12}$$

The stratum of each little group is represented as follows:

$SO_3 \times U_1$:

$$5 = 1[1] + 1[-1] + 3[0]\,, \qquad \varphi = [a,-a,0,0,0]\,, \quad \alpha_4 = \tfrac{1}{2}\,; \tag{13}$$

$SU_2 \times U_1$:

$$5 = 1[0] + 2[1] + 2[-1]\,, \qquad \varphi = [a,-a,a,-a,0]\,, \quad \alpha_4 = \tfrac{1}{4}\,. \tag{14}$$

The orbit space consists of two end points corresponding to $[SO_3 \times U_1]$, $[SU_2 \times U_1]$ and the interior corresponding to $[U_1 \times U_1]$.

SO(4)

Although SO_4 can be considered to be a direct product group $SU_2 \times SU_2$, we include it for completeness. We choose the vector representation for the basis. The generic stratum and the orbit parameter are represented as follows:

$U_1 \times U_1$:

$$\varphi = [a,-a,b,-b]\,, \qquad \alpha_2' = 2^2 ab/(2a^2 + 2b^2)\,. \tag{15}$$

The stratum of each little group is represented as follows:

$SU_2 \times U_1$:

$$(2,2) = 2[1] + 2[-1]\,, \qquad \varphi = [a,-a,a,-a]\,, \quad \alpha_2 = \pm 1\,. \tag{16}$$

The orbit space consists of two end points corresponding to $[SU_2 \times U_1]$ and the interior corresponding to $[U_1 \times U_1]$.

G(2)

We choose the seven-dimensional representation for the basis. The generic stratum and the orbit parameter are represented as follows:

$U_1 \times U_1$:

$$\varphi = [2a,0,-2a,a+b,a-b,-a+b,-a-b]\,, \qquad \alpha_6 = \frac{2(2a)^6 + 2(a+b)^6 + 2(a-b)^6}{[2(2a)^2 + 2(a+b)^2 + 2(a-b)^2]^3}\,. \tag{17}$$

The stratum of each little group is represented as follows:

$SO_3 \times U_1$:

$$7 = 3[0] + 2[1] + 2[-1]\,, \qquad \varphi = [a,-a,a,-a,0,0,0]\,, \qquad \alpha_6 = \tfrac{1}{16}\,; \tag{18}$$

$SU_2 \times U_1$:

$$7 = 1[0] + 1[2] + 1[-2] + 2[1] + 2[-1]\,, \qquad \varphi = [2a,0,-2a,a,-a,a,-a,]\,, \qquad \alpha_6 = \tfrac{33}{128}\,. \tag{19}$$

The orbit space consists of two end points corresponding to $[SO_3 \times U_1]$, $[SU_2 \times U_1]$ and the interior corresponding to $[U_1 \times U_1]$.

B. Groups of rank three

There are two orbit parameters for the adjoint representation of a Lie group of rank 3. The orbit space turns out to be a warped triangle.

SU(4)

We choose the four-dimensional representation for the basis of the matrices. The generic stratum and the orbit parameters are represented as follows:

$U_1 \times U_1 \times U_1$:

$$\varphi = [a,b,c,-a-b-c]\,, \qquad \alpha_3 = \frac{a^3 + b^3 + c^3 - (a+b+c)^3}{[a^2 + b^2 + c^2 + (a+b+c)^2]^{3/2}}\,, \qquad \alpha_4 = \frac{a^4 + b^4 + c^4 + (a+b+c)^4}{[a^2 + b^2 + c^2 + (a+b+c)^2]^2}\,. \tag{20}$$

The stratum of each little group is represented as follows:

$SU_3 \times U_1$:

$$4 = 1[-3] + 3[1]\,, \qquad \varphi = [a,a,a,-3a]\,, \qquad \alpha_3 = \pm 1/\sqrt{3},\ \alpha_4 = \tfrac{7}{12}\,; \tag{21}$$

$SU_2 \times SU_2 \times U_1$:

$$4 = (2,1)[1] + (1,2)[-1]\,, \qquad \varphi = [a,a,-a,-a]\,, \qquad \alpha_3 = 0,\ \alpha_4 = \tfrac{1}{4}; \tag{22}$$

$SU_2 \times U_1 \times U_1$:

$$4 = 1[1,1] + 1[1,-1] + 2[-1,0]\,, \qquad \varphi = [a,a,b,-2a-b]\,. \tag{23}$$

The orbit space is shown in Fig. 1. It is a warped triangle. Two cusps $\pm$ **P1** of $[SU_3 \times U_1]$ and cusp **P2** of $[SU_2 \times SU_2 \times U_1]$ are connected by the curve of $[SU_2 \times U_1 \times U_1]$. The cusps and the curve together form the boundary of the generic stratum of $[U_1 \times U_1 \times U_1]$ which occupies the interior.

SO(6)

Since SO_6 is isomorphic to SU_4, the orbit spaces of their adjoints are identical up to scale factors and locations. If we choose the vector representation for the basis, the generic stratum and the orbit parameters are represented as follows:

$U_1 \times U_1 \times U_1$:

$$\varphi = [a, -a, b, -b, c, -c],$$

$$\bar{\alpha}_4 = \frac{2a^4 + 2b^4 + 2c^4}{(2a^2 + 2b^2 + 2c^2)^2}, \tag{20'}$$

$$\bar{\alpha}_3' = \frac{2^3 abc}{(2a^2 + 2b^2 + 2c^2)^{3/2}}.$$

The stratum of each little group is represented as follows:

$SU_3 \times U_1$:

$$6 = 3[2] + \bar{3}[-2],$$

$$\varphi = [a, -a, a, -a, a, -a], \tag{21'}$$

$$\bar{\alpha}_4 = \tfrac{1}{6}, \quad \bar{\alpha}_3' = \pm 4/3\sqrt{6};$$

$SU_2 \times SU_2 \times U_1$:

$$6 = (1,1)[2] + (1,1)[-2] + (2,2)[0],$$

$$\varphi = [a, -a, 0,0,0,0], \tag{22'}$$

$$\bar{\alpha}_4 = \tfrac{1}{2}, \quad \bar{\alpha}_3' = 0;$$

$SU_2 \times U_1 \times U_1$:

$$6 = 1[2,0] + 1[-2,0] + 2[0,1] + 2[0,-1],$$

$$\varphi = [a, -a, a, -a, b, -b]. \tag{23'}$$

The orbit space of the SO_6 adjoint is obtained from that of SU_4 by the following substitutions: $\bar{\alpha}_4 = -\alpha_4 + \frac{3}{4}$ and $\bar{\alpha}_3' = -\alpha_3(4/3\sqrt{2})$.

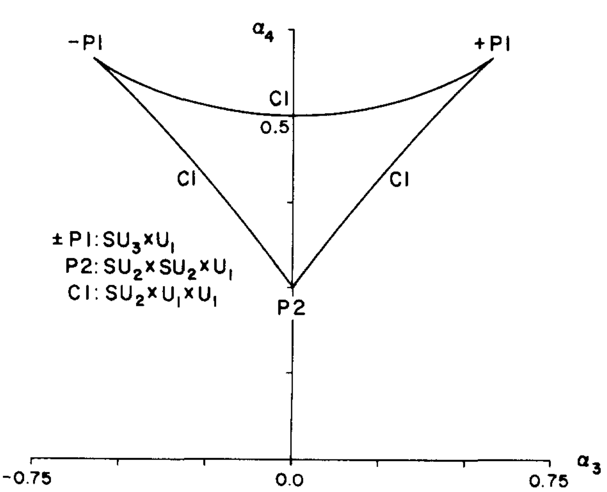

FIG. 1. The orbit space of the SU_4 (SO_6) adjoint representation.

SO(7)

We choose the vector representation for the basis. The generic stratum and the orbit parameters are represented as follows:

$U_1 \times U_1 \times U_1$:

$$\varphi = [a, -a, b, -b, c, -c, 0],$$

$$\alpha_4 = \frac{2a^4 + 2b^4 + 2c^4}{(2a^2 + 2b^2 + 2c^2)^2}, \tag{24}$$

$$\alpha_6 = \frac{2a^6 + 2b^6 + 2c^6}{(2a^2 + 2b^2 + 2c^2)^3}.$$

The stratum of each little group is represented as follows:

$SO_5 \times U_1$:

$$7 = 1[1] + 1[-1] + 5[0],$$

$$\varphi = [a, -a, 0,0,0,0,0], \tag{25}$$

$$\alpha_4 = \tfrac{1}{2}, \quad \alpha_6 = \tfrac{1}{4};$$

$SU_2 \times SO_3 \times U_1$:

$$7 = (1,3)[0] + (2,1)[1] + (2,1)[-1],$$

$$\varphi = [a, -a, a, -a, 0,0,0], \tag{26}$$

$$\alpha_4 = \tfrac{1}{4}, \quad \alpha_6 = \tfrac{1}{16};$$

$SU_3 \times U_1$:

$$7 = 1[0] + 3[1] + \bar{3}[-1],$$

$$\varphi = [a, -a, a, -a, a, -a, 0], \tag{27}$$

$$\alpha_4 = \tfrac{1}{6}, \quad \alpha_6 = \tfrac{1}{36};$$

$SU_2 \times U_1 \times U_1$:

$$7 = 1[0,0] + 1[0,1] + 1[0,-1] + 2[1,0] + 2[-1,0],$$

$$\varphi = [a, -a, a, -a, b, -b, 0]; \tag{28}$$

$SO_3 \times U_1 \times U_1$:

$$7 = 1[1,1] + 1[1,-1] + 1[-1,1] + 1[-1,-1] + 3[0,0], \tag{29}$$

$$\varphi = [a, -a, b, -b, 0,0,0].$$

The orbit space is shown in Fig. 2. It is again a warped triangle. Cusp **P1** of $[SO_5 \times U_1]$ and cusp **P2** of $[SU_2 \times SO_3 \times U_1]$ are connected by straight line **L1** of $[SO_3 \times U_1 \times U_1]$. All three cusps including cusp **P3** of

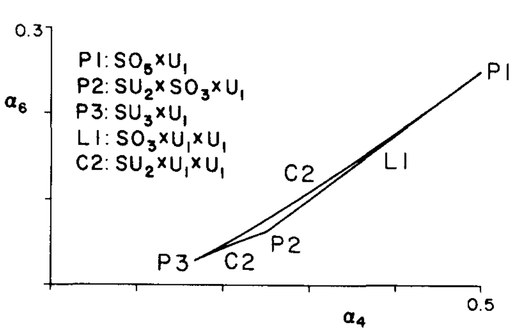

FIG. 2. The orbit space of the SO_7 adjoint representation.

$[SU_3 \times U_1]$ are connected by curve **C2** of $[SU_2 \times U_1 \times U_1]$. All the cusps and **L1** and **C2** together form the boundary of the generic stratum $[U_1 \times U_1 \times U_1]$ which occupies the interior.

Sp(6)

We choose the vector representation for the basis. The generic stratum and the orbit parameters are represented as follows:

$$U_1 \times U_1 \times U_1:$$
$$\varphi = [a, -a, b, -b, c, -c],$$
$$\alpha_4 = \frac{2a^4 + 2b^4 + 2c^4}{(2a^2 + 2b^2 + 2c^2)^2}, \tag{30}$$
$$\alpha_6 = \frac{2a^6 + 2b^6 + 2c^6}{(2a^2 + 2b^2 + 2c^2)^3}.$$

The stratum of each little group is represented as follows:

$$Sp_4 \times U_1:$$
$$6 = 1[1] + 1[-1] + 4[0],$$
$$\varphi = [a, -a, 0,0,0,0], \tag{31}$$
$$\alpha_4 = \tfrac{1}{2}, \quad \alpha_6 = \tfrac{1}{4};$$
$$SU_2 \times SU_2 \times U_1:$$
$$6 = (2,1)[0] + (1,2)[1] + (1,2)[-1],$$
$$\varphi = [a, -a, a, -a, 0,0], \tag{32}$$
$$\alpha_4 = \tfrac{1}{4}, \quad \alpha_6 = \tfrac{1}{16};$$
$$SU_3 \times U_1:$$
$$6 = 3[1] + \bar{3}[-1],$$
$$\varphi = [a, -a, a, -a, a, -a], \tag{33}$$
$$\alpha_4 = \tfrac{1}{6}, \quad \alpha_6 = \tfrac{1}{36};$$
$$SU_2 \times U_1 \times U_1(A):$$
$$6 = 1[0,1] + 1[0,-1] + 2[1,0] + 2[-1,0],$$
$$\varphi = [a, -a, a, -a, b, -b]. \tag{34}$$
$$SU_2 \times U_1 \times U_1(B):$$
$$6 = 1[1,1] + 1[1,-1] + 1[-1,1] + 1[-1,-1] + 2[0,0], \tag{35}$$
$$\varphi = [a, -a, b, -b, 0,0].$$

The orbit space is shown in Fig. 3. As we can see from Figs. 2 and 3 the orbit space of the Sp_6 adjoint is identical to that of the SO_7 adjoint. This identity persists between the Sp_{2n} adjoint and the SO_{2n+1} adjoint for any n because the orbit parameters are identically defined. Only the labeling of the little groups is different.

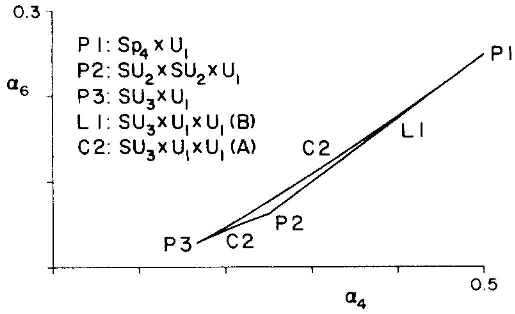

FIG. 3. The orbit space of the Sp_6 adjoint representation.

C. Groups of rank four

There are three orbit parameters for the adjoint representation of a Lie group of rank 4. The orbit space turns out to be a warped tetrahedron.

SU(5)

We choose the vector representation for the basis of the matrices. The generic stratum and the orbit parameters are represented as follows:

$$U_1 \times U_1 \times U_1 \times U_1:$$
$$\varphi = [a,b,c,d, -a-b-c-d],$$
$$\alpha_3 = \frac{a^3 + b^3 + c^3 + d^3 - (a+b+c+d)^3}{[a^2 + b^2 + c^2 + d^2 + (a+b+c+d)^2]^{3/2}},$$
$$\alpha_4 = \frac{a^4 + b^4 + c^4 + d^4 + (a+b+c+d)^4}{[a^2 + b^2 + c^2 + d^2 + (a+b+c+d)^2]^2}, \tag{36}$$
$$\alpha_5 = \frac{a^5 + b^5 + c^5 + d^5 - (a+b+c+d)^5}{[a^2 + b^2 + c^2 + d^2 + (a+b+c+d)^2]^{5/2}}.$$

The stratum of each little group is represented as follows:

$$SU_4 \times U_1:$$
$$5 = 1[-4] + 4[1],$$
$$\varphi = [a,a,a,a,-4a], \tag{37}$$
$$\alpha_3 = \pm \frac{3}{2\sqrt{5}}, \quad \alpha_4 = \frac{13}{20}, \quad \alpha_5 = \pm \frac{51}{40\sqrt{5}};$$
$$SU_3 \times SU_2 \times U_1:$$
$$5 = (3,1)[2] + (1,2)[-3],$$
$$\varphi = [2a,2a,2a,-3a,-3a], \tag{38}$$
$$\alpha_3 = \pm \frac{1}{\sqrt{30}}, \quad \alpha_4 = \frac{7}{30}, \quad \alpha_5 = \pm \frac{13}{30\sqrt{30}};$$
$$SU_3 \times U_1 \times U_1:$$
$$5 = 1[0,1] + 1[-3,-1] + 3[1,0], \tag{39}$$
$$\varphi = [a,a,a,b,-3a-b];$$
$$SU_2 \times SU_2 \times U_1 \times U_1:$$
$$5 = (1,1)[-2,-2] + (1,2)[1,0] + (2,1)[0,1], \tag{40}$$
$$\varphi = [a,a,b,b,-2a-2b];$$
$$SU_2 \times U_1 \times U_1 \times U_1:$$
$$5 = 1[0,1,0] + 1[0,0,1] + 1[-2,-1,-1] + 2[1,0,0], \tag{41}$$
$$\varphi = [a,a,b,c,-2a-b-c].$$

The orbit space is shown in Fig. 4. It is a thin warped tetrahedron. Cusps $\pm$ **P1** of $[SU_4 \times U_1]$ and cusps $\pm$ **P2** of $[SU_3 \times SU_2 \times U_1]$ are connected by both curves **C1** of $[SU_3 \times U_1 \times U_1]$ and curves **C2** of $[SU_2 \times SU_2 \times U_1 \times U_1]$.

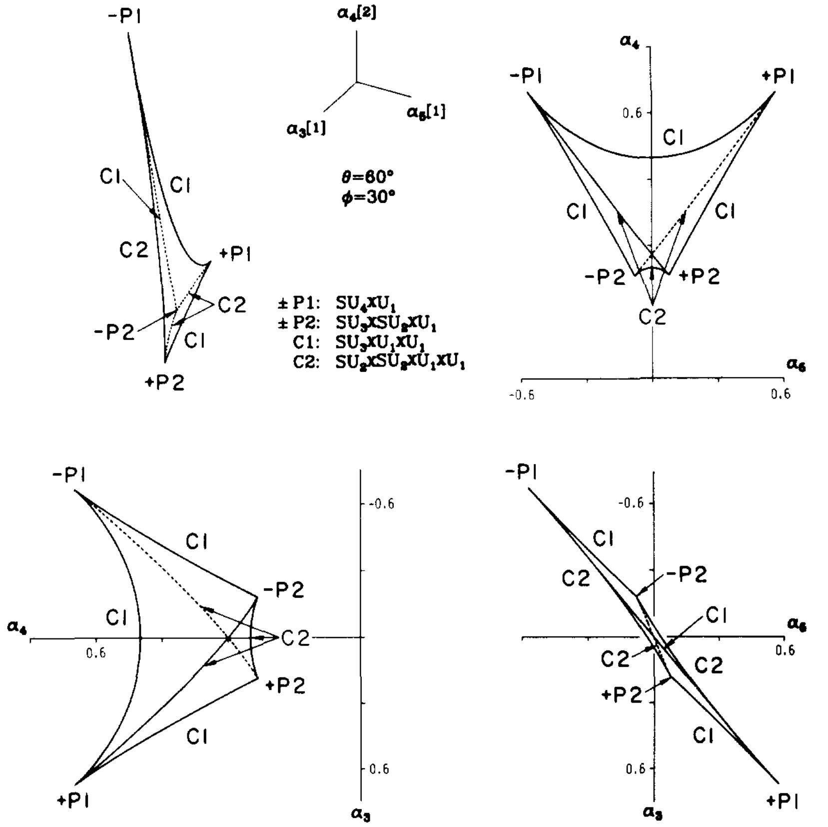

FIG. 4. The complete orbit space of the SU_5 adjoint representation. Shown at the upper left corner is a view from the direction oriented 30° from the α_3 axis and 60° from the α_4 axis. The numbers in the square brackets are the relative ratios of scale. Each projection is a view from the positive direction of the axis not shown in the picture. The dotted curves represent edges on the back (hidden) side of the orbit space.

The two curves lie on the warped surfaces of $[SU_2\times U_1\times U_1\times U_1]$. All these cusps, curves, and surfaces together form the boundary of the generic stratum $[U_1\times U_1\times U_1\times U_1]$ which occupies the interior.

The curves are all concave. One of the principal curvatures of each surface is zero (the surface is flat in this direction) and the other is negative (the surface is concave in this direction).

SO(9)

We choose the vector representation for the basis. The generic stratum and the orbit parameters are represented as follows:

$$\begin{aligned}
&U_1\times U_1\times U_1\times U_1:\\
&\varphi=[a,-a,b,-b,c,-c,d,-d,0],\\
&\alpha_4=\frac{2a^4+2b^4+2c^4+2d^4}{(2a^2+2b^2+2c^2+2d^2)^2},\\
&\alpha_6=\frac{2a^6+2b^6+2c^6+2d^6}{(2a^2+2b^2+2c^2+2d^2)^3},\\
&\alpha_8=\frac{2a^8+2b^8+2c^8+2d^8}{(2a^2+2b^2+2c^2+2d^2)^4}.
\end{aligned} \tag{42}$$

The stratum of each little group is represented as follows:

$$\begin{aligned}
&SO_7\times U_1:\\
&9=1[1]+1[-1]+7[0],\\
&\varphi=[a,-a,0,0,0,0,0,0,0],\\
&\alpha_4=\tfrac{1}{2},\quad \alpha_6=\tfrac{1}{4},\quad \alpha_8=\tfrac{1}{8};
\end{aligned} \tag{43}$$

$$\begin{aligned}
&SO_5\times SU_2\times U_1:\\
&9=(5,1)[0]+(1,2)[1]+(1,2)[-1],\\
&\varphi=[a,-a,a,-a,0,0,0,0,0],\\
&\alpha_4=\tfrac{1}{4},\quad \alpha_6=\tfrac{1}{16},\quad \alpha_8=\tfrac{1}{64};
\end{aligned} \tag{44}$$

$$\begin{aligned}
&SU_3\times SU_2\times U_1:\\
&9=(3,1)[1]+(\bar{3},1)[-1]+(1,3)[0],\\
&\varphi=[a,-a,a,-a,a,-a,0,0,0],\\
&\alpha_4=\tfrac{1}{6},\quad \alpha_6=\tfrac{1}{36},\quad \alpha_8=\tfrac{1}{216};
\end{aligned} \tag{45}$$

$$\begin{aligned}
&SU_4\times U_1:\\
&9=1[0]+4[1]+\bar{4}[-1],\\
&\varphi=[a,-a,a,-a,a,-a,a,-a,0],\\
&\alpha_4=\tfrac{1}{8},\quad \alpha_6=\tfrac{1}{64},\quad \alpha_8=\tfrac{1}{512};
\end{aligned} \tag{46}$$

$$\begin{aligned}
&SU_3\times U_1\times U_1:\\
&9=1[0,0]+1[0,1]+1[0,-1]\\
&\qquad+3[1,0]+\bar{3}[-1,0],\\
&\varphi=[a,-a,a,-a,a,-a,b,-b,0];
\end{aligned} \tag{47}$$

$$\begin{aligned}
&SU_2\times SU_2\times U_1\times U_1:\\
&9=(1,1)[0,0]+(2,1)[1,0]+(2,1)[-1,0]\\
&\qquad+(1,2)[0,1]+(1,2)[0,-1],\\
&\varphi=[a,-a,a,-a,b,-b,b,-b,0];
\end{aligned} \tag{48}$$

$$\begin{aligned}
&SU_2\times SO_3\times U_1\times U_1:\\
&9=(1,1)[0,1]+(1,1)[0,-1]+(2,1)[1,0]\\
&\qquad+(2,1)[-1,0]+(1,3)[0,0],\\
&\varphi=[a,-a,a,-a,b,-b,0,0,0];
\end{aligned} \tag{49}$$

$SO_5 \times U_1 \times U_1$:

$$9 = 1[1,0] + 1[-1,0] + 1[0,1] + 1[0,-1] + 5[0,0]\,, \tag{50}$$

$$\varphi = [a, -a, b, -b, 0,0,0,0,0]\,.$$

$SU_2 \times U_1 \times U_1 \times U_1(A)$:

$$9 = 1[0,0,0] + 1[0,1,0] + 1[0,-1,0] + 1[0,0,1] + 1[0,0,-1] + 2[1,0,0] + 2[-1,0,0]\,, \tag{51}$$

$$\varphi = [a, -a, a, -a, b, -b, c, -c, 0]\,;$$

$SU_2 \times U_1 \times U_1 \times U_1(B)$:

$$9 = 1[1,0,0] + 1[-1,0,0] + 1[0,1,0] + 1[0,-1,0] + 1[0,0,1] + 1[0,0,-1] + 3[0,0,0]\,, \tag{52}$$

$$\varphi = [a, -a, b, -b, c, -c, 0,0,0]\,.$$

The orbit space is shown in Fig. 5. It is a thin and sharp tetrahedron. Cusp **P1** of $[SO_7 \times U_1]$ and cusp **P2** of $[SO_5 \times SU_2 \times U_1]$ are connected by curve **C1** of $[SO_5 \times U_1 \times U_1]$. Cusp **P3** of $[SU_3 \times SU_2 \times U_1]$ and cusp **P4** of $[SU_4 \times U_1]$ are connected by curve **C2** of $[SU_3 \times U_1 \times U_1]$ which connects also **P1** and **P4**. **P2** and **P4** are connected by curve **C3** of $[SU_2 \times SU_2 \times U_1 \times U_1]$. **P1**, **P2**, and **P3** are connected by curve **C4** of $[SU_2 \times SO_3 \times U_1 \times U_1]$. The stratum of $[SU_2 \times U_1 \times U_1 \times U_1(B)]$ occupies the warped triangular surface **P1–P2–P3** bounded by **C1** and **C4**. The stratum of $[SU_2 \times U_1 \times U_1 \times U_1(A)]$ closes the rest of the boundary of the generic stratum $[U_1 \times U_1 \times U_1 \times U_1]$ which occupies the interior.

Curves **C1** and **C3** are convex plane-curves and curves **C2** and **C4** are concave space-curves. Surface **P1–P2–P3** is convex along its length but it meets with a $\alpha_4 = \text{const}$ plane along a straight line. All the other surfaces meet with a $\alpha_4 = \text{const}$ plane along concave curves. Surface **P1–P3–P4** is totally concave. Each of the surfaces **P2–P3–P4** and **P1–P2–P4** have two principal curvatures of opposite sign, i.e., the surfaces are saddle-shaped.

Sp(8)

We choose the vector representation for the basis. The generic stratum and the orbit parameters are represented as follows:

$U_1 \times U_1 \times U_1 \times U_1$:

$$\varphi = [a, -a, b, -b, c, -c, d, -d]\,,$$

$$\alpha_4 = \frac{2a^4 + 2b^4 + 2c^4 + 2d^4}{(2a^2 + 2b^2 + 2c^2 + 2d^2)^2}\,,$$

$$\alpha_6 = \frac{2a^6 + 2b^6 + 2c^6 + 2d^6}{(2a^2 + 2b^2 + 2c^2 + 2d^2)^3}\,, \tag{53}$$

$$\alpha_8 = \frac{2a^8 + 2b^8 + 2c^8 + 2d^8}{(2a^2 + 2b^2 + 2c^2 + 2d^2)^4}\,.$$

The stratum of each little group is represented as follows:

$Sp_6 \times U_1$:

$$8 = 1[1] + 1[-1] + 6[0]\,,$$

$$\varphi = [a, -a, 0,0,0,0,0,0]\,, \tag{54}$$

$$\alpha_4 = \tfrac{1}{2}, \quad \alpha_6 = \tfrac{1}{4}, \quad \alpha_8 = \tfrac{1}{8}\,;$$

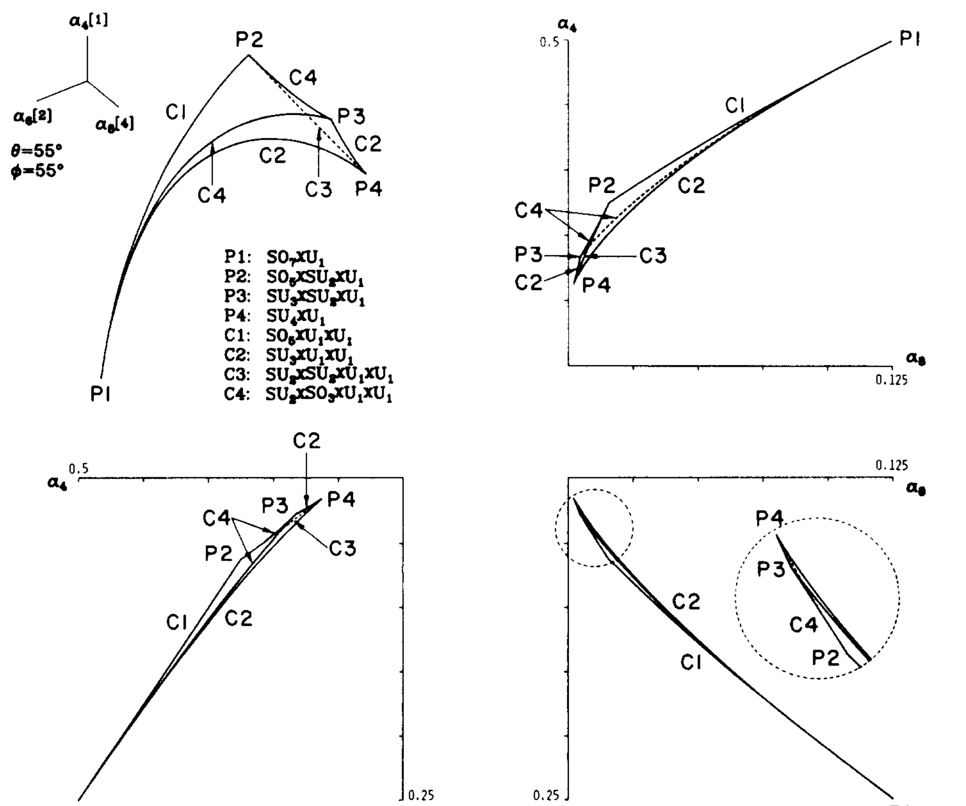

FIG. 5. The complete orbit space of the SO_9 adjoint representation. Shown at the upper left corner is a view from the direction oriented 55° from the α_6 axis and 55° from the α_4 axis. The numbers in the square brackets are the relative ratios of scale. Each projection is a view from the positive direction of the axis not shown in the picture. The dotted curves represent edges on the back (hidden) side of the orbit space.

$Sp_4 \times SU_2 \times U_1$:

$$8 = (4,1)[0] + (1,2)[1] + (1,2)[-1],$$
$$\varphi = [a,-a,a,-a,0,0,0,0], \tag{55}$$
$$\alpha_4 = \tfrac{1}{4}, \quad \alpha_6 = \tfrac{1}{16}, \quad \alpha_8 = \tfrac{1}{64};$$

$SU_3 \times SU_2 \times U_1$:

$$8 = (3,1)[1] + (\bar{3},1)[-1] + (1,2)[0],$$
$$\varphi = [a,-a,a,-a,a,-a,0,0], \tag{56}$$
$$\alpha_4 = \tfrac{1}{6}, \quad \alpha_6 = \tfrac{1}{36}, \quad \alpha_8 = \tfrac{1}{216};$$

$SU_4 \times U_1$:

$$8 = 4[1] + \bar{4}[-1],$$
$$\varphi = [a,-a,a,-a,a,-a,a,-a], \tag{57}$$
$$\alpha_4 = \tfrac{1}{8}, \quad \alpha_6 = \tfrac{1}{64}, \quad \alpha_8 = \tfrac{1}{512};$$

$SU_3 \times U_1 \times U_1$:

$$8 = 1[0,1] + 1[0,-1] + 3[1,0] + \bar{3}[-1,0], \tag{58}$$
$$\varphi = [a,-a,a,-a,a,-a,b,-b];$$

$SU_2 \times SU_2 \times U_1 \times U_1(A)$:

$$8 = (1,2)[1,0] + (1,2)[-1,0] + (2,1)[0,1] + (2,1)[0,-1], \tag{59}$$
$$\varphi = [a,-a,a,-a,b,-b,b,-b];$$

$SU_2 \times SU_2 \times U_1 \times U_1(B)$:

$$8 = (1,1)[0,1] + (1,1)[0,-1] + (1,2)[0,0] + (2,1)[1,0] + (2,1)[-1,0], \tag{60}$$
$$\varphi = [a,-a,a,-a,b,-b,0,0];$$

$Sp_4 \times U_1 \times U_1$:

$$8 = 1[1,0] + 1[-1,0] + 1[0,1] + 1[0,-1] + 4[0,0], \tag{61}$$
$$\varphi = [a,-a,b,-b,0,0,0,0];$$

$SU_2 \times U_1 \times U_1 \times U_1(A)$:

$$8 = [0,1,0] + 1[0,-1,0] + 1[0,0,1] + 1[0,0,-1] + 2[1,0,0] + 2[-1,0,0], \tag{62}$$
$$\varphi = [a,-a,a,-a,b,-b,c,-c];$$

$SU_2 \times U_1 \times U_1 \times U_1(B)$:

$$8 = 1[1,0,0] + 1[-1,0,0] + 1[0,1,0] + 1[0,-1,0] + 1[0,0,1] + 1[0,0,-1] + 2[0,0,0], \tag{63}$$
$$\varphi = [a,-a,b,-b,c,-c,0,0].$$

The orbit space of the Sp_8 adjoint is identical to the SO_9 case except for the labeling of the little groups.

SO(8)

We choose the vector representation for the basis. The generic stratum and the orbit parameters are represented as follows:

$U_1 \times U_1 \times U_1 \times U_1$:

$$\varphi = [a,-a,b,-b,c,-c,d,-d],$$
$$\alpha_4 = \frac{2a^4 + 2b^4 + 2c^4 + 2d^4}{(2a^2 + 2b^2 + 2c^2 + 2d^2)^2},$$
$$\alpha_6 = \frac{2a^6 + 2b^6 + 2c^6 + 2d^6}{(2a^2 + 2b^2 + 2c^2 + 2d^2)^3}, \tag{64}$$
$$\alpha_4' = \frac{2^4 abcd}{(2a^2 + 2b^2 + 2c^2 + 2d^2)^2}.$$

The stratum of each little group is represented as follows:

$SO_6 \times U_1$:

$$8 = 1[1] + 1[-1] + 6[0],$$
$$\varphi = [a,-a,0,0,0,0,0,0], \tag{65}$$
$$\alpha_4 = \tfrac{1}{2}, \quad \alpha_6 = \tfrac{1}{4}, \quad \alpha_4' = 0;$$

$SO_4 \times SU_2 \times U_1$:

$$8 = (2,1,1)[1] + (2,1,1)[-1] + (1,2,2)[0],$$
$$\varphi = [a,-a,a,-a,0,0,0,0], \tag{66}$$
$$\alpha_4 = \tfrac{1}{4}, \quad \alpha_6 = \tfrac{1}{16}, \quad \alpha_4' = 0;$$

$SU_4 \times U_1$:

$$8 = 4[1] + \bar{4}[-1],$$
$$\varphi = [a,-a,a,-a,a,-a,a,-a], \tag{67}$$
$$\alpha_4 = \tfrac{1}{8}, \quad \alpha_6 = \tfrac{1}{64}, \quad \alpha_4' = \pm\tfrac{1}{4};$$

$SU_3 \times U_1 \times U_1$:

$$8 = 1[0,1] + 1[0,-1] + 3[1,0] + \bar{3}[-1,0], \tag{68}$$
$$\varphi = [a,-a,a,-a,a,-a,b,-b];$$

$SU_2 \times SU_2 \times U_1 \times U_1$:

$$8 = (2,1)[1,0] + (2,1)[-1,0] + (1,2)[0,1] + (1,2)[0,-1], \tag{69}$$
$$\varphi = [a,-a,a,-a,b,-b,b,-b];$$

$SO_4 \times U_1 \times U_1$:

$$8 = (1,1)[1,0] + (1,1)[-1,0] + (1,1)[0,1] + (1,1)[0,-1] + (2,2)[0,0], \tag{70}$$
$$\varphi = [a,-a,b,-b,0,0,0,0];$$

$SU_2 \times U_1 \times U_1 \times U_1$:

$$8 = 1[0,1,0] + 1[0,-1,0] + 1[0,0,1] + 1[0,0,-1] + 2[1,0,0] + 2[-1,0,0], \tag{71}$$
$$\varphi = [a,-a,a,-a,b,-b,c,-c].$$

The orbit space is shown in Fig. 6. It is a warped tetrahedron. Cusp **P1** of $[SO_6 \times U_1]$ and cusp **P2** of $[SO_4 \times SU_2 \times U_1]$ are connected by line **L1** of $[SO_4 \times U_1 \times U_1]$. Cusps $\pm$ **P3** of $[SU_4 \times U_1]$ and **P2** are connected by line **L2** of $[SU_2 \times SU_2 \times U_1 \times U_1]$. **P1** and $\pm$ **P3** are connected by curve **C3** of $[SU_3 \times U_1 \times U_1]$. The stratum of $[SU_2 \times U_1 \times U_1 \times U_1]$ closes the boundary of the generic stratum $[U_1 \times U_1 \times U_1 \times U_1]$ which occupies the interior.

The projected orbit space $\alpha_4 - \alpha_6$ is not closed by the

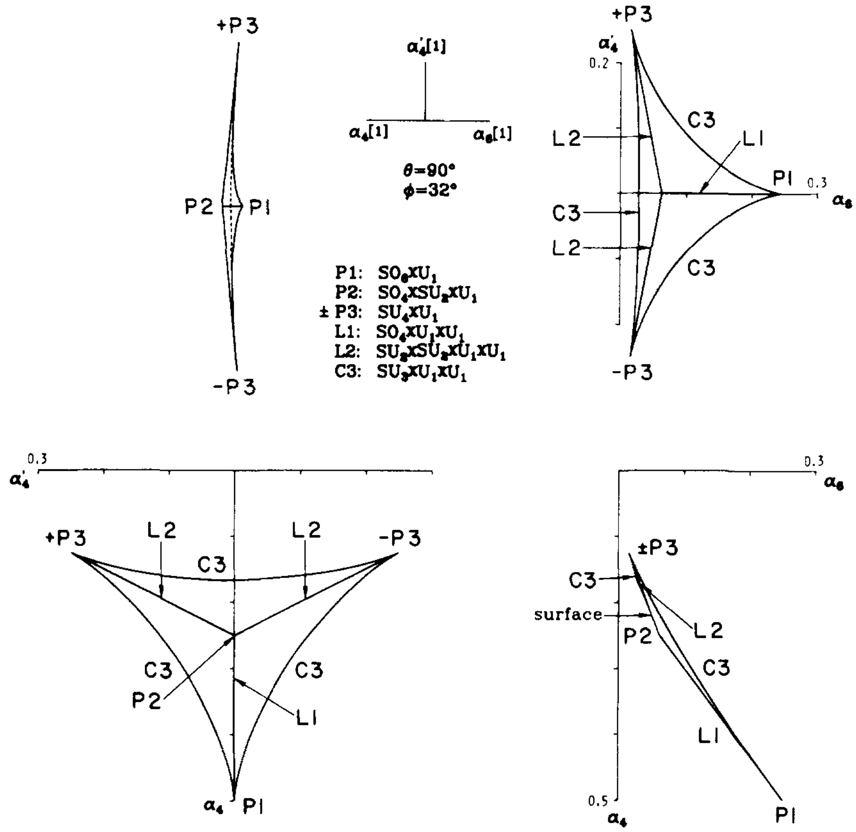

FIG. 6. The complete orbit space of the SO_8 adjoint representation. Shown at the upper left corner is a view from the direction oriented 32° from the α_4 axis and 90° from the α_4' axis. The numbers in the square brackets are the relative ratios of scale. Each projection is a view from the positive direction of the axis not shown in the picture except for the one viewed from the $-\alpha_6$ axis. The dotted curve represents the edge on the back (hidden) side of the orbit space.

one-dimensional strata **L1**, **L2**, and **C3**. The concave punctured portion belongs to the two-dimensional stratum. This is related to the fact that the triangular surface **P2**– + **P3**– – **P3** is convex in the direction + **P3**→ – **P3** but concave in the direction normal to it. All the surfaces that contain cusp **P2** are saddle-shaped. Surface **P1**– + **P3**– – **P3** is totally concave.

F(4)

We choose the 26-dimensional representation for the basis. The generic stratum and the orbit parameters are represented as follows:

$U_1 \times U_1 \times U_1 \times U_1$:

$$\begin{aligned}\varphi = [&2a,0,-2a,2c,0,-2c,b+d,b-d,-b+d,\\ &-b-d,a+b+c,a+b-c,a-b+c,-a+b+c,\\ &-a-b-c,-a-b+c,-a+b-c,a-b-c,\\ &a+c+d,a+c-d,a-c+d,-a+c+d,\\ &-a-c-d,-a-c+d,-a+c-d,a-c-d\,],\end{aligned} \tag{72}$$

$$\alpha_6 = \frac{\sum_{i=1}^{26} \varphi_i^6}{\left[\sum_{i=1}^{26} \varphi_i^2\right]^3}, \quad \alpha_8 = \frac{\sum_{i=1}^{26} \varphi_i^8}{\left[\sum_{i=1}^{26} \varphi_i^2\right]^4}, \quad \alpha_{12} = \frac{\sum_{i=1}^{26} \varphi_i^{12}}{\left[\sum_{i=1}^{26} \varphi_i^2\right]^6}.$$

The stratum of each little group is represented as follows:

$SO_7 \times U_1$:

$$26 = 1[0] + 1[2] + 1[-2] + 7[0] + 8[1] + 8[-1],$$

$$\varphi = [2a, -2a, 8\ a\text{'s}, 8\ (-a)\text{'s}, 8\ 0\text{'s}], \tag{73}$$

$$\alpha_6 = \tfrac{1}{96}, \quad \alpha_8 = \tfrac{11}{6912}, \quad \alpha_{12} = \tfrac{19}{442\,368};$$

$Sp_6 \times U_1$:

$$26 = 6[1] + 6[-1] + 14[0],$$

$$\varphi = [6\ a\text{'s}, 6\ (-a)\text{'s}, 14\ 0\text{'s}], \tag{74}$$

$$\alpha_6 = \tfrac{1}{144}, \quad \alpha_8 = \tfrac{1}{864}, \quad \alpha_{12} = \tfrac{1}{124\,416};$$

$SU_3 \times SU_2 \times U_1(A)$:

$$\begin{aligned}26 = {}&(8,1)[0] + (3,2)[1] + (3,1)[-2]\\ &+ (\bar{3},2)[-1] + (\bar{3},1)[2],\end{aligned}$$

$$\varphi = [6\ a\text{'s}, 6\ (-a)\text{'s}, 3\ (2a)\text{'s}, 3\ (-2a)\text{'s}, 8\ 0\text{'s}], \tag{75}$$

$$\alpha_6 = 11/1296, \quad \alpha_8 = 43/46\,656,$$

$$\alpha_{12} = 683/60\,466\,176;$$

$SU_3 \times SU_2 \times U_1(B)$:

$$\begin{aligned}26 = {}&(1,1)[0] + (1,2)[3] + (1,2)[-3]\\ &+ (1,3)[0] + (3,1)[-2] + (3,2)[1]\\ &+ (\bar{3},1)[2] + (\bar{3},2)[-1],\end{aligned}$$

$$\begin{aligned}\varphi = [&2\ (3a)\text{'s}, 2\ (-3a)\text{'s}, 6\ a\text{'s}, 6\ (-a)\text{'s},\\ &3\ (2a)\text{'s}, 3\ (-2a)\text{'s}, 4\ 0\text{'s}],\end{aligned} \tag{76}$$

$$\alpha_6 = 23/2592, \quad \alpha_8 = 193/186\,624,$$

$$\alpha_{12} = 14\,933/967\,458\,816;$$

$SO_5\times U_1\times U_1$:

$$26=1[0,0]+1[2,0]+1[-2,0]+1[0,2]+1[0,-2]+5[0,0]+4[1,1]+4[1,-1]+4[-1,1]+4[-1,-1]\,, \tag{77}$$

$$\varphi=[2a,-2a,2b,-2b,6\ 0\text{'s},\ 4\ (a+b)\text{'s},\ 4\ (a-b)\text{'s},\ 4\ (-a+b)\text{'s},\ 4\ (-a-b)\text{'s}]\,;$$

$SU_2\times SU_2\times U_1\times U_1$:

$$26=(1,1)[2,0]+(1,1)[0,0]+(1,1)[-2,0]+(1,3)[0,0]+(2,1)[0,1]+(2,1)[0,-1]+(1,2)[1,1]+(1,2)[1,-1]+(2,2)[1,0]+(1,2)[-1,1]+(1,2)[-1,-1]+(2,2)[-1,0]\,, \tag{78}$$

$$\varphi=[2a,-2a,b,b,-b,-b,(a+b),(a+b),(a-b),(a-b),(-a+b),(-a+b),(-a-b),(-a-b),4\ a\text{'s},\ 4\ (-a)\text{'s},\ 4\ 0\text{'s}]\,;$$

$SU_3\times U_1\times U_1(A)$:

$$26=3[1,1]+\bar{3}[1,-1]+3[-1,1]+\bar{3}[-1,-1]+3[0,-2]+\bar{3}[0,2]+8[0,0]\,,$$
$$\varphi=[3\ (a+b)\text{'s},\ 3\ (a-b)\text{'s},\ 3\ (-a+b)\text{'s},\ 3\ (-a-b)\text{'s},\ 3\ (2a)\text{'s},\ 3\ (-2a)\text{'s},\ 8\ 0\text{'s}]\,; \tag{79}$$

$SU_3\times U_1\times U_1(B)$:

$$26=1[2,0]+1[0,0]+1[-2,0]+1[0,0]+3[0,2]+\bar{3}[0,-2]+1[1,3]+3[1,-1]+1[1,-3]+\bar{3}[1,1]+1[-1,3]+3[-1,-1]+1[-1,-3]+\bar{3}[-1,1]\,, \tag{80}$$

$$\varphi=[2a,0,-2a,0,3\ (2b)\text{'s},\ 3\ (-2b)\text{'s},\ (a+3b),\ 3\ (a-b)\text{'s},(a-3b),\ 3\ (a+b)\text{'s},\ (-a+3b),\ 3\ (-a-b)\text{'s},(-a-3b),\ 3\ (-a+b)\text{'s}]\,,$$

$SU_2\times U_1\times U_1\times U_1(A)$:

$$26=1[2,0,0]+1[0,0,0]+1[-2,0,0]+3[0,0,0]+1[0,1,1]+1[0,1,-1]+1[0,-1,1]+1[0,-1,-1]+2[1,1,0]+2[1,-1,0]+2[1,0,1]+2[1,0,-1]+2[-1,1,0]+2[-1,-1,0]+2[-1,0,1]+2[-1,0,-1]\,; \tag{81}$$

$SU_2\times U_1\times U_1\times U_1(B)$:

$$26=1[2,0,0]+1[0,0,0]+1[-2,0,0]+1[0,0,2]+1[0,0,0]+1[0,0,-2]+2[0,1,0]+2[0,-1,0]+1[1,1,1]+1[1,1,-1]+1[1,-1,1]+1[1,-1,-1]+2[1,0,1]+2[1,0,-1]+1[-1,1,1]+1[-1,1,-1]+1[-1,-1,1]+1[-1,-1,-1]+2[-1,0,1]+2[-1,0,-1]\,. \tag{82}$$

The unspecified components of φ in Eqs. (81) and (82) can be obtained as follows: in order to get φ_i, multiply the first number in the ith square bracket by a, the second by b, the third by c, and sum all three.

The orbit space is shown in Fig. 7. It is a very thin warped tetrahedron. Cusp **P1** of $[SO_7\times U_1]$ and cusp **P2** of $[Sp_6\times U_1]$ are connected by curve **C1** of $[SO_5\times U_1\times U_1]$. Cusp **P2** and cusp **P3** of $[SU_3\times SU_2\times U_1(A)]$ are connected by curve **C2** of $[SU_3\times U_1\times U_1(A)]$. Cusp **P1** and cusp **P4** of $[SU_3\times SU_2\times U_1(B)]$ are connected by curve **C3** of $[SU_3\times U_1\times U_1(B)]$. All the cusps are connected by curve **C4** of $[SU_2\times SU_2\times U_1\times U_1]$. Surface **S1** (**P1–P2–P3**) and surface **S2** (**P2–P3–P4**) belong to the stratum of $[SU_2\times U_1\times U_1\times U_1(A)]$. Surface **S3** (**P1–P3–P4**) and surface **S4** (**P1–P2–P4**) belong to the stratum of $[SU_2\times U_1\times U_1\times U_1(B)]$. The interior is occupied by the generic stratum of $[U_1\times U_1\times U_1\times U_1]$.

C2 and **C3** are convex plane-curves. **C1** and the portion of **C4** between **P3** and **P4** are concave space-curves. The other portions of **C4** are convex space-curves. Surface **P1–P3–P4** is totally concave. All the other surfaces are saddle-shaped.

4. SINGLE IRREDUCIBLE REPRESENTATIONS WITH LOW-DIMENSIONAL ORBIT SPACES

In this section we tabulate orbit spaces of all the single coregular irreducible representations which allow less than (or equal to)[41] five independent basic invariant polynomials. Since all the generators that do not leave a generic orbit invariant are consumed in simplifying the scalar fields through a global gauge transformation, the number I of independent basic invariants is given by

$$I=D-(\dim G-\dim G_g)\,, \tag{83}$$

where D is the dimension of the representation, $\dim G$ the number of generators of the symmetry group G, and $\dim G_g$ the number of generators of the little group of the generic orbit. The little group of the generic orbit is trivial for most irreps. Hsiang and Hsiang[31] listed the nontrivial little groups of generic strata of all the single irreps of compact connected Lie groups.

It is a nontrivial job to construct the invariant polynomials in the integrity basis for a given representation. Although there exists a systematic method[42] for constructing them, it is excessively laborious to build high degree invariant polynomials. However, we do have practical methods

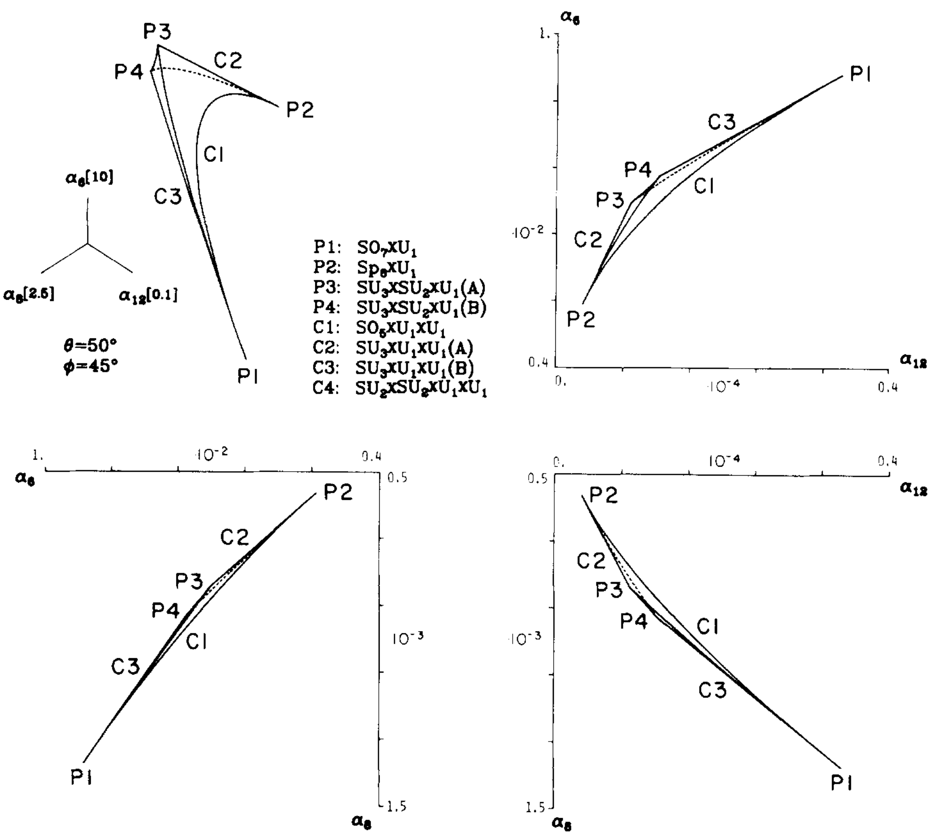

FIG. 7. The complete orbit space of the F_4 adjoint representation. Shown at the upper left corner is a view from the direction oriented 45° from the α_8 axis and 50° from the α_6 axis. The numbers in the square brackets are the relative ratios of scale. Each projection is a view from the positive direction of the axis not shown in the picture. The dotted curves represent edges on the back (hidden) side of the orbit space. The unlabeled curves are all portions of $C4$.

for some representations such as the examples considered in this paper.

Some valuable hints are available in the mathematical literature. Reference 32 lists the degrees and symmetry properties of the polynomials for coregular representations. Noncoregular representations admit polynomial identities (syzygies) among the members of the integrity basis. Patera and Sharp[43] developed a powerful method for finding character generating functions of finite group representations, which can be used for finding the degrees of polynomials in an integrity basis and the degrees of the syzygies among them.

It is convenient to have tables of maximal little groups[44] in carrying out classification of little groups.

A. Symmetric tensor representations

1. Symmetric tensors of SU(N)

Symmetric tensors ψ_{ij} of SU_N can be diagonalized through a group transformation, $\psi'_{ij} = U_{ik}(g)U_{jl}(g)\psi_{kl}$, where $U_{ij}(g)$ is a unitary matrix representing a group element. We abbreviate the diagonal elements as $\psi_{ij} = \mathrm{diag}(\psi_1,\psi_2,\ldots,\psi_N)\exp(i\delta)$ with ψ_i real. Thus there are $N+1$ independent basic invariants. They are given by

$$\begin{aligned} I_2 &= \psi_{ij}\psi^{ij}, \quad I_4 = \psi_{ij}\psi^{jk}\psi_{kl}\psi^{li},\ldots, \\ I'_N &= \epsilon^{ij\cdots kl}\psi_{1i}\psi_{2j}\cdots\psi_{(N-1)k}\psi_{Nl}, \\ I'^{*}_N &= \epsilon_{ij\cdots kl}\psi^{1i}\psi^{2j}\cdots\psi^{(N-1)k}\psi^{Nl}. \end{aligned} \tag{84}$$

We shall see that the cross section of the orbit space of an SU_N symmetric tensor at any phase angle is identical, except for different scale factors, to the orbit space of the SO_{2N} adjoint representation.

SU(3) symmetric tensor $6+\bar{6}$: The generic stratum is invariant under a finite group, $[Z_2\times Z_2]$, (Z_2: a finite group of order 2) and is represented by $\psi_{ij} = \mathrm{diag}(a,b,c)\exp(i\delta)$. Orbit parameters are defined as follows:

$$\begin{aligned} \alpha_4 &= (a^4+b^4+c^4)/(a^2+b^2+c^2)^2, \\ \alpha'_3 &= \exp(3i\delta)abc/(a^2+b^2+c^2)^{3/2}, \\ \alpha'^{*}_3 &= \exp(-3i\delta)abc/(a^2+b^2+c^2)^{3/2}. \end{aligned} \tag{85}$$

Each stratum and its little group are represented as follows:

SO_3: $3=3$:

$$\psi = \mathrm{diag}(a,a,a)\exp(i\delta), \tag{86}$$

$$\alpha_4 = \tfrac{1}{3},\ \alpha'_3 = \exp(3i\delta)/\sqrt{3};$$

$SU_2\times Z_2$: $3 = 1+2$:

$$\psi = \mathrm{diag}(a,0,0)\exp(i\delta), \tag{87}$$

$$\alpha_4 = 1,\quad \alpha'_3 = 0;$$

$U_1\times Z_2$:

$$\psi = \mathrm{diag}(a,b,b)\exp(i\delta). \tag{88}$$

The cross section of the orbit space at any angle δ is identical, except for different scale factors and locations, to the orbit space of the SO_6 adjoint representation (Fig. 1) with $[SO_3]$ at $\pm$ **P1**, $[SU_2\times Z_2]$ at **P2**, and $[U_1\times Z_2]$ at **C1**.

SU(4) symmetric tensor $10+\overline{10}$: The generic stratum is invariant under a finite group, $[Z_2\times Z_2\times Z_2]$, and is represented by $\psi_{ij} = \mathrm{diag}(a,b,c,d)\exp(i\delta)$. Orbit parameters are defined as follows:

$$\alpha_4 = (a^4 + b^4 + c^4 + d^4)/(a^2 + b^2 + c^2 + d^2)^2,$$
$$\alpha_6 = (a^6 + b^6 + c^6 + d^6)/(a^2 + b^2 + c^2 + d^2)^3,$$
$$\alpha_4' = \exp(4i\delta)abcd/(a^2 + b^2 + c^2 + d^2)^2, \tag{89}$$
$$\alpha_4'^* = \exp(-4i\delta)abcd/(a^2 + b^2 + c^2 + d^2)^2.$$

Each stratum and its little group are represented as follows:

$SU_3 \times Z_2$:
$$4 = 1 + 3,$$
$$\psi = \mathrm{diag}(a,0,0,0)\exp(i\delta), \tag{90}$$
$$\alpha_4 = 1, \alpha_6 = 1, \alpha_4' = 0;$$

$SU_2 \times U_1 \times Z_2$:
$$4 = 1[1] + 1[-1] + 2[0],$$
$$\psi = \mathrm{diag}(a,a,0,0)\exp(i\delta), \tag{91}$$
$$\alpha_4 = \tfrac{1}{2}, \quad \alpha_6 = \tfrac{1}{4}, \quad \alpha_4' = 0;$$

$SU_2 \times SU_2$:
$$4 = (2,2),$$
$$\psi = \mathrm{diag}(a,a,a,a)\exp(i\delta), \tag{92}$$
$$\alpha_4 = \tfrac{1}{4}, \quad \alpha_6 = \tfrac{1}{16}, \quad \alpha_4' = \exp(4i\delta)/16;$$

$SU_2 \times Z_2 \times Z_2$:
$$4 = 1 + 1 + 2,$$
$$\psi = \mathrm{diag}(a,b,0,0)\exp(i\delta); \tag{93}$$

$U_1 \times U_1 \times Z_2$:
$$4 = 1[1,0] + 1[-1,0] + 1[0,1] + 1[0,-1],$$
$$\psi = \mathrm{diag}(a,a,b,b)\exp(i\delta); \tag{94}$$

$SO_3 \times Z_2$:
$$4 = 1 + 3,$$
$$\psi = \mathrm{diag}(a,b,b,b)\exp(i\delta); \tag{95}$$

$U_1 \times Z_2 \times Z_2$:
$$4 = 1[0] + 1[0] + 1[1] + 1[-1],$$
$$\psi = \mathrm{diag}(a,b,c,c)\exp(i\delta). \tag{96}$$

The cross section of the orbit space at any angle δ is identical, except for different scale factors, to the orbit space of the SO_8 adjoint representation (Fig. 6) with $[SU_3 \times Z_2]$ at **P1**, $[SU_2 \times U_1 \times Z_2]$ at **P2**, $[SU_2 \times SU_2]$ at $\pm$**P3**, $[SU_2 \times Z_2 \times Z_2]$ at **L1**, $[U_1 \times U_1 \times Z_2]$ at **L2**, $[SO_3 \times Z_2]$ at **C3**, and $[U_1 \times Z_2 \times Z_2]$ on the surfaces.

2. Symmetric traceless tensors of SO(N)

Symmetric traceless tensors ψ_{ij} of SO_N can be diagonalized through a group transformation, $\psi'_{ij} = O_{ik}(g)O_{jl}(g)\psi_{kl}$, where $O_{ij}(g)$ is a real orthogonal matrix representing a group element. We abbreviate the diagonal elements as $\psi_{ij} = \mathrm{diag}(\psi_1,\psi_2,\ldots,\psi_N)$ with ψ_i real. The traceless condition is given by $\delta_{ij}\psi_{ij} = \psi_1 + \psi_2 + \cdots + \psi_N = 0$. Thus there are $N-1$ independent basic invariants. They are given by

$$I_2 = \psi_{ij}\psi_{ij}, \quad I_3 = \psi_{ij}\psi_{jk}\psi_{ki}, \quad I_4 = \psi_{ij}\psi_{jk}\psi_{kl}\psi_{li}, \cdots. \tag{97}$$

We immediately see that the orbit space of an SO_N symmetric traceless tensor is identical to that of the SU_N adjoint representation.

SO(3) symmetric traceless tensor 5: The orbit space is identical to that of SU_3 adjoint:

$U_1 \times Z_2$ $[3 = 1(0) + 1(1) + 1(-1)]$, $\alpha_3 = \pm 1/\sqrt{6}$,

$Z_2 \times Z_2$ occupying the interior.

SO(5) symmetric traceless tensor 14: Without going into details we identify various portions of the orbit space (Fig. 4) as follows:

$SU_2 \times SU_2 \times Z_2$ $[5 = (1,1) + (2,2)]$ at $\pm$**P1**,

$SO_3 \times U_1 \times Z_2$ $[5 = 1(2) + 1(-2) + 3(0)]$ at $\pm$**P2**,

$SO_3 \times Z_2 \times Z_2$ $[5 = 1 + 1 + 3]$ at **C1**,

$U_1 \times U_1 \times Z_2 \times Z_2$ at **C2**,

$U_1 \times Z_2 \times Z_2 \times Z_2$ on the surfaces, and

$Z_2 \times Z_2 \times Z_2 \times Z_2$ occupying the interior.

Embedding of each subgroup is indicated by the branching rule given in the square bracket.

3. Symmetric tensors of Sp(2N)

Symmetric tensors ψ_{ij} of Sp_{2N} are adjoint representations.

B. Antisymmetric tensor representations

1. Antisymmetric tensors of SU(N)

Antisymmetric tensors, φ_{ij}, of SU_N can be skew-diagonalized through a group transformation, $\varphi'_{ij} = U_{ik}(g)U_{jl}(g)\varphi_{kl}$. Each diagonal element consists of a real number for odd N and it comes with an overall phase factor for even N. Thus there are $(N-1)/2$ for odd N ($N/2+1$ for even N) independent basic invariants. They are given by

$$I_2 = \varphi_{ij}\varphi^{ij}, \quad I_4 = \varphi_{ij}\varphi^{jk}\varphi_{kl}\varphi^{li}, \cdots, \quad \text{for odd } N, \tag{98a}$$
$$I_2 = \varphi_{ij}\varphi^{ij}, \quad I_4 = \varphi_{ij}\varphi^{jk}\varphi_{kl}\varphi^{li}, \cdots, \tag{98b}$$
$$I'_{N/2} = \epsilon^{ij\cdots kl}\varphi_{ij}\cdots\varphi_{kl},$$
$$I'^*_{N/2} = \epsilon_{ij\cdots kl}\varphi^{ij}\cdots\varphi^{kl}, \quad \text{for even } N.$$

We immediately see that (the cross section at any angle δ of) the orbit space of an SU_N antisymmetric tensor is identical to that of the SO_N adjoint for $N > 4$. In the following we match various portions of each pair of orbit spaces.

SU(5) antisymmetric tensor $10 + \overline{10}$: The orbit space is identical to that of the SO_5 adjoint, with

$SU_2 \times SU_3$ $[5 = (2,1) + (1,3)]$ replacing $SO_3 \times U_1$,

Sp_4 $[5 = 1 + 4]$ replacing $SU_2 \times U_1$, and

$SU_2 \times SU_2$ $[5 = (1,1) + (2,1) + (1,2)]$ replacing $U_1 \times U_1$.

SU(6) antisymmetric tensor $15 + \overline{15}$: The cross section of the orbit space at any angle δ is identical to that of the SO_6 adjoint, with

SP_6 $[6=6]$ at $\pm \mathbf{P}_1$,

$SU_2 \times SU_4$ $[6=(2,1)+(1,4)]$ at **P2**,

$SU_2 \times Sp_4$ $[6=(2,1)+(1,4)]$ at **C1**, and

$SU_2 \times SU_2 \times SU_2$ $[6=(2,1,1)+(1,2,1)+(1,1,2)]$ occupying the interior.

SU(7) antisymmetric tensor $21+\overline{21}$: The orbit space is identical to that of the SO_7 adjoint (Fig. 2), with

$SU_2 \times SU_5$ $[7=(2,1)+(1,5)]$ at **P1**,

$SU_3 \times Sp_4$ $[7=(3,1)+(1,4)]$ at **P2**,

Sp_6 $[7=1+6]$ at **P3**,

$SU_3 \times SU_2 \times SU_2$ $[7=(3,1,1)+(1,2,1)+(1,1,2)]$ at **L1**,

$SU_2 \times Sp_4$ $[7=(1,1)+(2,1)+(1,4)]$ at **C2**, and

$SU_2 \times SU_2 \times SU_2$ $[7=(1,1,1)+(2,1,1)+(1,2,1)+(1,1,2)]$ occupying the interior.

SU(8) antisymmetric tensor $28+\overline{28}$: The cross section of the orbit space at any angle δ is identical to that of the SO_8 adjoint (Fig. 6), with

$SU_2 \times SU_6$ $[8=(2,1)+(1,6)]$ at **P1**,

$Sp_4 \times SU_4$ $[8=(4,1)+(1,4)]$ at **P2**,

Sp_8 $[8=8]$ at $\pm$ **P3**,

$SU_2 \times SU_2 \times SU_4$ $[8=(2,1,1)+(1,2,1)+(1,1,4)]$ at **L1**,

$Sp_4 \times Sp_4$ $[8=(4,1)+(1,4)]$ at **L2**,

$SU_2 \times Sp_6$ $[8=(2,1)+(1,6)]$ at **C3**,

$SU_2 \times SU_2 \times Sp_4$ $[8=(2,1,1)+(1,2,1)+(1,1,4)]$ on the surfaces, and

$SU_2 \times SU_2 \times SU_2 \times SU_2$ $[8=(2,1,1,1)+(1,2,1,1)+(1,1,2,1)+(1,1,1,2)]$ occupying the interior.

SU(9) antisymmetric tensor $36+\overline{36}$: The orbit space is identical to that of the SO_9 adjoint (Fig. 5), with

$SU_2 \times SU_7$ $[9=(2,1)+(1,7)]$ at **P1**,

$Sp_4 \times SU_5$ $[9=(4,1)+(1,5)]$ at **P2**,

$Sp_6 \times SU_3$ $[9=(6,1)+(1,3)]$ at **P3**,

Sp_8 $[9=1+8]$ at **P4**,

$SU_2 \times SU_2 \times SU_5$ $[9=(2,1,1)+(1,2,1)+(1,1,5)]$ at **C1**,

$Sp_6 \times SU_2$ $[9=(6,1)+(1,2)+(1,1)]$ at **C2**,

$Sp_4 \times Sp_4$ $[9=(4,1)+(1,4)+(1,1)]$ at **C3**,

$Sp_4 \times SU_2 \times SU_3$ $[9=(4,1,1)+(1,2,1)+(1,1,3)]$ at **C4**,

$SU_2 \times SU_2 \times SU_2 \times SU_3$ $[9=(2,1,1,1)+(1,2,1,1)+(1,1,2,1)+(1,1,1,3)]$ occupying the warped triangular surface **P1**–**P2**–**P3** bounded by **C1** and **C2**,

$Sp_4 \times SU_2 \times SU_2$ $[9=(4,1,1)+(1,2,1)+(1,1,2)]$ closing the rest of the boundary of the generic stratum,

$SU_2 \times SU_2 \times SU_2 \times SU_2$ $[9=(1,1,1,1)+(2,1,1,1)+(1,2,1,1)+(1,1,2,1)+(1,1,1,2)]$ occupying the interior.

2. Antisymmetric tensors of SO(N)

Antisymmetric tensors φ_{ij} of SO_N are adjoint representations.

3. Antisymmetric traceless tensors of Sp(2N)

Antisymmetric traceless tensors φ_{ij} of Sp_{2N} are skew-diagonalized through a group transformation, $\varphi'_{ij} = S_{ik}(g)S_{jl}(g)\varphi_{kl}$, where $S_{ij}(g)$ is a simplectic matrix satisfying $S_{ki}S_{lj}f_{kl} = f_{ij}$. We abbreviate the skew-diagonal elements as $\varphi_{ij} = \text{skew-diag}(\varphi_1,\varphi_2,\dots,\varphi_N)$ with φ_i real. In this notation $f_{ij} = \text{skew-diag}(1,1,\dots,1)$. The traceless condition is given by $f_{ij}\varphi_{ij} = 2(\varphi_1+\varphi_2+\cdots+\varphi_N)=0$. Thus there are $N-1$ independent basic invariants:

$$I_2 = f_{ik}f_{jl}\varphi_{ij}\varphi_{kl}\,, \quad I_3 = f_{il}f_{jm}f_{kn}\,\varphi_{im}\varphi_{jn}\varphi_{kl},\cdots. \tag{99}$$

We shall see that the orbit space of an antisymmetric traceless tensor of Sp_{2N} is identical, except for different scale factors, to that of the SU_N adjoint representation.

Sp(6) antisymmetric tensor 14: The generic stratum is invariant under $[Sp_2 \times Sp_2 \times Sp_2]$. The orbit parameter is defined as follows:

$$\alpha_3 = (2a^3+2b^3-2(a+b)^3)/(2a^2+2b^2+2(a+b)^2)^{3/2}\,. \tag{100}$$

Each stratum and its little group are represented as follows:

$Sp_2 \times Sp_4$:

$$\begin{aligned} &6=(2,1)+(1,4)\,,\\ &\varphi = \text{skew-diag}(a,a,-2a)\,,\\ &\alpha_3 = \pm 1/2\sqrt{3}\,. \end{aligned} \tag{101}$$

The orbit space is identical, except for different scale factors, to that of the SU_3 adjoint.

Sp(8) antisymmetric traceless tensor 27: The generic stratum is invariant under $[Sp_2 \times Sp_2 \times Sp_2 \times Sp_2]$. The orbit parameters are defined as follows:

$$\begin{aligned} \alpha_3 &= (2a^3+2b^3+2c^3-2(a+b+c)^3)/(2a^2+2b^2+2c^2+2(a+b+c)^2)^{3/2}\,,\\ \alpha_4 &= (2a^4+2b^4+2c^4+2(a+b+c)^4)/(2a^2+2b^2+2c^2+2(a+b+c)^2)^2\,. \end{aligned} \tag{102}$$

Each stratum and its little group are represented as follows:

$Sp_2 \times Sp_6$:

$$\begin{aligned} &8=(2,1)+(1,6)\,,\\ &\varphi = \text{skew-diag}(a,a,a,-3a)\,,\\ &\alpha_3 = \pm 1/\sqrt{6}\,, \quad \alpha_4 = 7/24\,; \end{aligned} \tag{103}$$

$Sp_4 \times Sp_4$:

$$\begin{aligned} &8=(4,1)+(1,4)\,,\\ &\varphi = \text{skew-diag}(a,a,-a,-a)\,,\\ &\alpha_3 = 0\,, \quad \alpha_4 = 1/8\,; \end{aligned} \tag{104}$$

$Sp_2 \times Sp_2 \times Sp_4$:

$$\begin{aligned} &8=(2,1,1)+(1,2,1)+(1,1,4)\,,\\ &\varphi = \text{skew-diag}(a,a,b,-2a-b)\,. \end{aligned} \tag{105}$$

The orbit space is identical, except for different scale factors, to that of the SU_4 adjoint (Fig. 1). Identifications are:

$[Sp_2 \times Sp_6]$ at $\pm$ **P1**, $[Sp_4 \times Sp_4]$ at **P2**, and $[Sp_2 \times Sp_2 \times Sp_4]$ at **C1**.

Sp(10) antisymmetric traceless tensor 44: The generic stratum is invariant under $[Sp_2 \times Sp_2 \times Sp_2 \times Sp_2 \times Sp_2]$. The orbit parameters are defined as follows:

$$\alpha_3 = \frac{(2a^3 + 2b^3 + 2c^3 + 2d^3 - 2(a+b+c+d)^3)}{(2a^2 + 2b^2 + 2c^2 + 2d^2 + 2(a+b+c+d)^2)^{3/2}},$$
$$\alpha_4 = \frac{(2a^4 + 2b^4 + 2c^4 + 2d^4 + 2(a+b+c+d)^4)}{(2a^2 + 2b^2 + 2c^2 + 2d^2 + 2(a+b+c+d)^2)^2}, \quad (106)$$
$$\alpha_5 = \frac{(2a^5 + 2b^5 + 2c^5 + 2d^5 - 2(a+b+c+d)^5)}{(2a^2 + 2b^2 + 2c^2 + 2d^2 + 2(a+b+c+d)^2)^{5/2}}.$$

Each stratum and its little group are represented as follows:

$Sp_2 \times Sp_8$:

$$10 = (2,1) + (1,8),$$
$$\varphi = \text{skew-diag}(a,a,a,a,-4a), \quad (107)$$
$$\alpha_3 = \pm 3/2\sqrt{10}, \quad \alpha_4 = 13/40, \quad \alpha_5 = \pm 51/80\sqrt{10};$$

$Sp_4 \times Sp_6$:

$$10 = (4,1) + (1,6),$$
$$\varphi = \text{skew-diag}(2a,2a,2a,-3a,-3a), \quad (108)$$
$$\alpha_3 = \pm 1/2\sqrt{15}, \quad \alpha_4 = 7/60, \quad \alpha_5 = \pm 13/120\sqrt{15};$$

$Sp_2 \times Sp_2 \times Sp_6$:

$$10 = (2,1,1) + (1,2,1) + (1,1,6),$$
$$\varphi = \text{skew-diag}(a,a,a,b,-3a-b); \quad (109)$$

$Sp_2 \times Sp_4 \times Sp_4$:

$$10 = (2,1,1) + (1,4,1) + (1,1,4),$$
$$\varphi = \text{skew-diag}(a,a,b,b,-2a-2b); \quad (110)$$

$Sp_2 \times Sp_2 \times Sp_2 \times Sp_4$:

$$10 = (2,1,1,1) + (1,2,1,1) + (1,1,2,1) + (1,1,1,4),$$
$$\varphi = \text{skew-diag}(a,a,b,c,-2a-b-c). \quad (111)$$

The orbit space is identical, except for different scale factors, to that of the SU_5 adjoint (Fig. 4). Identifications are: $[Sp_2 \times Sp_8]$ at $\pm$ **P1**, $[Sp_4 \times Sp_6]$ at $\pm$ **P2**, $[Sp_2 \times Sp_2 \times Sp_6]$ at **C1**, $[Sp_2 \times Sp_4 \times Sp_4]$ at **C2**, and $[Sp_2 \times Sp_2 \times Sp_2 \times Sp_4]$ on the surfaces.

C. Other low-dimensional irreducible representations

The remaining irreps that allow less than four-dimensional orbit spaces (or the cross sections at arbitrary phase angles) are the defining representations of various groups, spinor representations of SO_N, and SO_3 representations.

The defining representations of classical Lie groups and G_2 yield single quadratic invariants only and their orbit spaces are trivial. Their little groups are

SU_{N-1} $(N \geqslant 2)$ for $N + \bar{N}$ of SU_N,

SO_{N-1} $(N \geqslant 3)$ for N of SO_N,

Sp_{2N-2} $(N \geqslant 3)$ for $2N + 2N$ of Sp_{2N}, and

SU_3 for 7 of G_2.

The spinor representations of SO_N for low N (<10) also yield single quadratic invariants only. Their little groups are

Sp_2 for $4+4$ of SO_5,

SU_3 for $4+\bar{4}$ of SO_6,

G_2 for 8 of SO_7,

SO_7 for 8 of SO_8,

SO_7 for 16 of SO_9.

The spinor representation of SO_{10}, the defining representations of F_4 and E_6, and low-dimensional (less than 8) SO_3 representations yield nontrivial low-dimensional orbit spaces.

Spinor representation of SO(10) 16 + $\overline{16}$

The 16 component complex spinor of SO_{10} is left invariant under SU_4 and is reduced, through an SO_{10} transformation generated by the 30 non-SU_4 generators made out of 45 σ-matrices σ_{ij},[23] to two real components, say, ψ_4 and ψ_6. Two independent basic invariant polynomials exist:

$$I_2 = \bar{\psi}\psi = 2(\psi_4^*\psi_4 + \psi_6^*\psi_6),$$
$$I_4 = \bar{\psi}\gamma_i\psi\bar{\psi}\gamma_i\psi = \bar{\psi}\gamma_5\psi\bar{\psi}\gamma_5\psi + \bar{\psi}\gamma_{10}\psi\bar{\psi}\gamma_{10}\psi \quad (112)$$
$$= 16\psi_4^*\psi_4\psi_6^*\psi_6,$$

where we left the complex conjugate intact to show the contraction of ψ's between γ matrices.

Each stratum and its little group are represented as follows:

SU_5:

$$16 = 1 + \bar{5} + 10, \quad (113)$$
$$\psi_4 = a, \quad \psi_6 = 0, \quad \alpha_4 = 0;$$

SO_7:

$$16 = 1 + 7 + 8, \quad (114)$$
$$\psi_4 = a, \quad \psi_6 = a, \quad \alpha_4 = 1.$$

Defining representation of F(4) 26

The representation spaces of exceptional groups are naturally described on the octonionic basis. We refer the reader to Refs. 45 and 46 for further details.

The 26-dimensional defining representation of F_4 is represented by a 3×3 real, symmetric, and traceless matrix over octonions:

$$\varphi = \begin{pmatrix} a & \gamma & \bar{\beta} \\ \bar{\gamma} & b & \alpha \\ \beta & \bar{\alpha} & c \end{pmatrix}, \quad (115)$$

where a,b,c are real numbers satisfying $a+b+c=0$ and α, β, γ are real octonions (the bar denotes octonionic conjugation). It is left invariant under an SO_8 transformation $(26 = 1 + 1 + 8_v + 8_s + 8_c)$. The dimension formula (83) yields $2 = 26 - 52 + 28$, which is the number of independent basic invariants. They are given by

$$I_2 = \tfrac{1}{2}\,\mathrm{Tr}(\varphi^\dagger \cdot \varphi), \quad I_3 = \tfrac{1}{3}\,\mathrm{Tr}[(\varphi \times \varphi) \cdot \varphi], \quad (116)$$

where the dot represents the Jordan product (half the anticommutator) and $\varphi\times\varphi$ is the Freudenthal product:

$$\varphi\times\varphi=\varphi(\varphi-\mathrm{Tr}\,\varphi)-\tfrac{1}{2}\mathrm{Tr}[\varphi(\varphi-\mathrm{Tr}\,\varphi)]\,.$$

An F_4 transformation involving those generators not included in SO_8 reduces φ to a diagonal matrix containing two real parameters:

$$\varphi=\begin{pmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & -(a+b)\end{pmatrix}\equiv[a,b,-(a+b)]\,. \quad (117)$$

Written in terms of these two real parameters, the invariant polynomials are

$$I_2=\tfrac{1}{2}(a^2+b^2+(a+b)^2),\quad I_3=\pm ab(a+b)\,. \quad (118)$$

Each stratum and its little group are represented as follows:

SO_9:

$$26=1+9+16\,,$$
$$\psi=[a,a,-2a]\,,\quad \alpha_3=\pm 2/3\sqrt{3}\,. \quad (119)$$

Defining representation of E(6) 27 + $\overline{27}$

The 27-dimensional complex defining representation of E_6 is represented by a 3×3 complex, Hermitian, octonionic matrix

$$\varphi=\begin{pmatrix} a & \gamma & \bar\beta\\ \bar\gamma & b & \alpha\\ \beta & \bar\alpha & c\end{pmatrix}, \quad (120)$$

where a,b,c are complex numbers and α,β,γ are complex octonions. It is left invariant under an SO_8 transformation ($27=1+1+1+8_v+8_s+8_c$). The dimension formula (83) yields $4=27+27-78+28$, which is the number of independent basic invariants. They are given by[46]

$$I_2=\tfrac{1}{2}\,\mathrm{Tr}(\varphi^\dagger\cdot\varphi),\quad I_4=\tfrac{1}{2}\,\mathrm{Tr}[(\varphi\times\varphi)^\dagger\cdot(\varphi\times\varphi)]\,, \quad (121)$$
$$I_3=\tfrac{1}{3}\,\mathrm{Tr}[(\varphi\times\varphi)\cdot\varphi]\,,\quad I_3^*=\text{complex conjugate of } I_3\,.$$

An E_6 transformation involving those generators not included in SO_8 reduces φ to a diagonal matrix containing four real parameters:

$$\varphi=\exp(i\delta)\begin{pmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{pmatrix}\equiv\exp(i\delta)[a,b,c]\,. \quad (122)$$

Written in terms of these four real parameters the invariant polynomials are

$$I_2=\tfrac{1}{2}(a^2+b^2+c^2),\quad I_4=\tfrac{1}{2}(a^2b^2+b^2c^2+c^2a^2)\,,$$
$$I_3=\exp(3i\delta)abc\,. \quad (123)$$

Each stratum and its little group are represented as follows:

F_4:

$$27=1+26\,,$$
$$\varphi=\exp(i\delta)[a,a,a]\,, \quad (124)$$
$$\alpha_4=\tfrac{2}{3},\quad \alpha_3=\exp(3i\delta)(2/3)^{3/2}\,;$$

SO_{10}:

$$27=1+10+16\,,$$
$$\varphi=\exp(i\delta)[a,0,0]\,, \quad (125)$$
$$\alpha_4=0,\quad \alpha_3=0\,;$$

SO_9:

$$27=1+1+9+8+8\,,$$
$$\varphi=\exp(i\delta)[a,b,b]\,. \quad (126)$$

The cross section of the orbit space at an arbitrary phase angle δ is shown in Fig. 8. It is again a warped triangle.

The generic strata of SU_2 representations 4 + 4 and 6 + 6 have trivial little groups and thus their orbit spaces are four- and eight-dimensional. The SO_3 seven-dimensional representation (totally symmetric traceless third-rank tensor) is non-coregular and has five invariants of degree 2, 4, 6, 10, and 15 in the integrity basis and a syzygy. They are listed in Ref. 47. Its maximal little groups[12,48] are U_1, T, D_3, and D_2.

D. Comments

Our observations for single irreps are summarized as follows:

(1) The orbit spaces for the adjoint representations of Lie groups of the same rank all have similar geometrical shapes, namely, straight line for groups of rank two, triangle for groups of rank three, tetrahedron for groups of rank four, and so on. (This pattern was evident in the examples of Ref. 16.)

This implies that there is an interesting relationship between the degrees of polynomial invariants, the number of maximal little groups, and the shape of the orbit space. For example, the SU_5 adjoint has only two maximal little groups but odd degree invariants such as I_3 and I_5 duplicate the cusps providing the third and fourth cusps needed to build a tetrahedron. For the adjoint representations of all the other groups of rank four there are four maximal little groups and their invariants are of even degree yielding only four cusps, again just enough to build a tetrahedron.

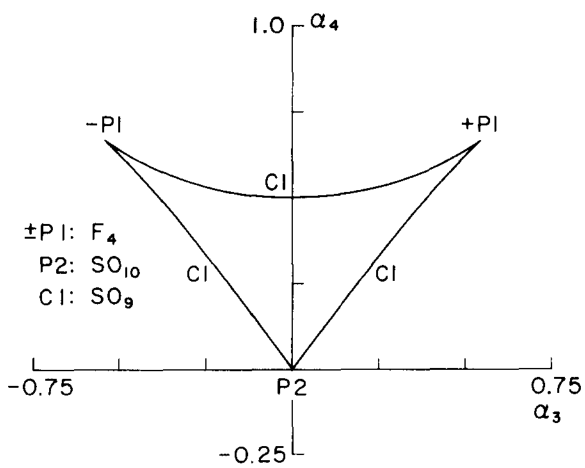

FIG. 8. The orbit space of 27 + $\overline{27}$ of E_6.

However, this relationship does not seem to hold in the case of non-coregular representations. The SO_3 seven-dimensional representation has four maximal little groups. If its orbit space is built out of polynomials of degree 2, 4, 6, and 10, then it is a warped tetrahedron. If it is built out of 2nd, 4th, 6th, and 15th degree polynomials, then some cusps may be duplicated and the orbit space may even become a warped octahedron. It will be interesting to see if the orbit space built out of all five polynomials and the syzygy has indeed a tetragonal shape.

(2) The orbit spaces (or the cross sections at an arbitrary phase angle) of symmetric and antisymmetric tensor representations are identical, up to scale factors, to those of adjoint representations.

Gürsey suggested that this similarity among the orbit spaces of adjoints, symmetric and antisymmetric second-rank tensors results from deeper mathematical roots.[49] These representations all have definite exchange symmetries among the tensor indices. Even the $27 + \overline{27}$ of E_6 has such symmetry: it is a Hermitian matrix (over octonions).

Michel[12] pointed out that the orbit spaces of the vector representations of Weyl groups are, up to scale factors, identical to those of corresponding adjoint representations. We quote two examples given in Ref. 50. The orbit spaces for the vector representations of the tetrahedral groups T and T_d are identical to that of SO_6 adjoint. The orbit spaces for the vector representations of the tetrahedral group T_h and the octahedral groups O and O_h is identical to that of SO_7 adjoint.

(3) Lower-dimensional strata of higher symmetries form the boundaries of higher-dimensional strata of lower symmetries in an orderly way. The hierarchy of protrusiveness on the orbit space boundary is not a global property (a poorly defined concept in any case) but a local property, which is shown by the saddle-shaped surfaces in most three-dimensional orbit spaces.

The last observation is not what we like because it may lead to a counterexample to the minimal symmetry breaking principle. However, none of the cases we have considered makes a counterexample.

In our formalism we take the singlet form for a given subgroup as the definition of a stratum. Its equation is obtained by putting the singlet form into the invariant polynomials and is thus parametric. In order to obtain the singlet form, which is the minimum information needed to specify an extremum point in any case, we have to find the matrix elements of group generators over the given representation and require that the subgroup generators annihilate the representation vector.

It is convenient to have nonparametric equations for strata. Since there are fewer independent parameters than basic invariants for all the strata except for the generic stratum, we should have some identities among the basic invariants on these strata. Like syzygies they are polynomials. It is not easy, though possible in principle, to derive these identities from our parametric equations. Abud and Sartori[29] devised a general method for finding nonparametric equations of orbits. It is a good tractable method usable also for the projected orbit space associated with a Higgs potential. It requires only the knowledge of invariant polynomials. One can obtain the singlet forms (though not the little groups) from the nonparametric orbit equations by solving high degree algebraic equations. This is as difficult as minimizing a Higgs potential using a conventional method. The method outlined in the previous paragraph is the only tractable way for finding the little groups and singlet forms, as far as we know. Jarić[50] devised another elegant nonparametric method for representations of finite groups which can be used for adjoint representations of low rank compact Lie groups. He provided both singlet forms and nonparametric equations for orbits. However, its applicability seems to be limited to only a small number of representations. It will be interesting to see if his method can be extended to more complicated cases.

In a Higgs potential there appear invariant polynomials only up to fourth degree. Thus we deal with a projected orbit space. Due to the projection some cusps are buried inside the projected space, as shown in SO_N and Sp_{2N} examples. These buried cusps cannot yield the absolute minimum, though they correspond to maximal little groups. A similar phenomenon was noticed earlier in the examples of SO_N adjoint + vector representations.[17] As a matter of fact, it was observed much earlier by Li.[23] This implies that, in unification theories, simple-minded classification of possible symmetry-breaking directions based on maximal or maximaximal little groups is not enough. One should check if the symmetry breaking really occurs in the desired direction.

5. TWO IRREDUCIBLE REPRESENTATIONS

The orbit spaces of two irreducible representations are normally high-dimensional because after one of the representations is simplified only a small number of group parameters are left for further simplification of the other representation. We have found two cases where the orbit space is three-dimensional, SU_3 adjoint + vector and SO_5 adjoint + vector.

A. SU(3) adjoint + vector representations

Using the same notation as in Ref. 15, the orbit parameters are

$$\alpha_3 = \frac{\Sigma\varphi_i^3}{(\Sigma\varphi_i^2)^{3/2}}, \tag{127}$$

$$\beta_1 = \frac{\Sigma\chi_i^*\varphi_i\chi_i}{(\Sigma\varphi_i^2)^{1/2}(\Sigma\chi_i^*\chi_i)}, \quad \beta_2 = \frac{\Sigma\chi_i^*\varphi_i^2\chi_i}{(\Sigma\varphi_i^2)(\Sigma\chi_i^*\chi_i)}. \tag{128}$$

The stratum of each little group is represented as follows:

SU_2:

$$\begin{gathered} 8 = 1 + 2 + 2 + 3, \quad 3 = 1 + 2, \\ \varphi = [a, a, -2a], \quad \chi = [0,0,c], \\ \alpha_3 = \pm 1/\sqrt{6}, \quad \beta_1 = \pm 2/\sqrt{6}, \quad \beta_2 = 2/3; \end{gathered} \tag{129}$$

U_1:

$$\varphi = [a,b,-a-b]\,,\quad \chi = [0,0,c]\,,$$
$$\alpha_3 = (a^3 + b^3 - (a+b)^3)/(a^2 + b^2 + (a+b)^2)^{3/2}\,, \tag{130}$$
$$\beta_1 = -(a+b)/(a^2 + b^2 + (a+b)^2)^{1/2}\,,$$
$$\beta_2 = (a+b)^2/(a^2 + b^2 + (a+b)^2)\,.$$

The generic stratum is represented by Eqs. (127)–(128) and its little group is the null group. Can a curve confine a three-dimensional volume? The answer is no, and thus the stratum of the null group must confine itself. The volume is extremized when either χ_1 or χ_2 is equal to zero with all the other components nonzero. The orbit space is shown in Fig. 9. The strata of SU_2, namely, the cusps, are the most protrudent as we might guess from the fact that they satisfy the most singular boundary conditions. The stratum of U_1, namely, the curve, is the next most singular. This may lead us to expect that such a hierarchical relationship would be a prominent feature of the orbit space of two irreps. But, as we shall see in the next example, the strata of a lower level little group can be as singular as the higher level ones.

B. SO(5) adjoint + vector representations

Using the same notation as in Ref. 17, the orbit parameters are

$$\alpha_4 = \frac{\Sigma\varphi_i^4}{2(\Sigma\varphi_i^2)^2}\,, \tag{131}$$

$$\beta_2 = \frac{\Sigma\chi_i\varphi_i^2\chi_i}{(2\Sigma\varphi_i^2)(\Sigma\chi_i\chi_i + \chi_3\chi_3)}\,, \tag{132}$$

$$\beta_4 = \frac{\Sigma\chi_i\varphi_i^4\chi_i}{(2\Sigma\varphi_i^2)^2(\Sigma\chi_i\chi_i + \chi_3\chi_3)}\,,$$

where i runs from 1 to 2. The stratum of each little group is represented as follows:

SO_3:

$$10 = 1 + 3 + 3 + 3\,,\quad 5 = 1 + 1 + 3\,,$$
$$\varphi = [a,0]\,,\quad \chi = [c,0,0]\,, \tag{133}$$
$$\alpha_4 = \tfrac{1}{2}\,,\quad \beta_2 = \tfrac{1}{2}\,,\quad \beta_4 = \tfrac{1}{4}\,;$$

$SU_2 \times U_1$:

$$10 = 1(0) + 1(2) + 1(-2) + 3(0) + 2(1) + 2(-1)\,,$$
$$5 = 1(0) + 2(1) + 2(-1)\,,$$
$$\varphi = [a,a]\,,\quad \chi = [0,0,c]\,, \tag{134}$$
$$\alpha_4 = \tfrac{1}{4}\,,\quad \beta_2 = 0\,,\quad \beta_4 = 0\,;$$

$U_1 \times U_1$:

$$\varphi = [a,b]\,,\quad \chi = [0,0,c]\,,$$
$$\alpha_4 = (2a^4 + 2b^4)/(2a^2 + 2b^2)^2\,,\quad \beta_2 = 0\,,\quad \beta_4 = 0\,; \tag{135}$$

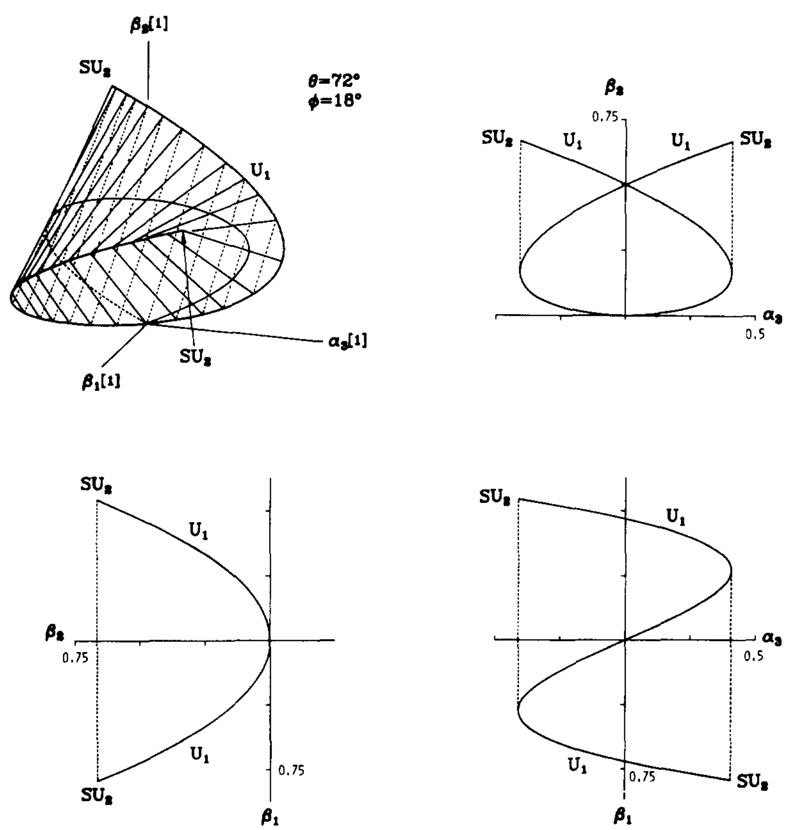

FIG. 9. The complete orbit space of the SU_3 adjoint + vector. Shown at the upper left corner is a view from the direction oriented 18° from the β_1 axis and 72° from the β_2 axis. The dotted lines are hidden lines. The numbers in the square brackets are the relative ratios of scale. Each projection is a view from the positive direction of the axis not shown in the picture. Here the dotted lines are portions of the boundary belonging to the null group stratum. Thus the hidden curves are drawn solidly.

U_1:

$$\varphi = [a,b]\,,\quad \chi = [0,c,d]\,,$$
$$\alpha_4 = (2a^4 + 2b^4)/(2a^2 + 2b^2)^2\,,$$
$$\beta_2 = (b^2c^2)/(2a^2 + 2b^2)(c^2 + d^2)\,, \tag{136}$$
$$\beta_4 = (b^4c^2)/(2a^2 + 2b^2)^2(c^2 + d^2)\,.$$

The generic stratum is represented by Eqs. (131)–(132) and its little group is the null group. The stratum of U_1 is two-dimensional and thus has a chance to enclose the whole volume. The U_1 stratum occupies the surfaces represented by dotted lines in Fig. 10, but the surface represented by solid lines is a part of the generic stratum. This is in contrast to the case of one irrep where there was no mixture of this kind. That is, equally singular surfaces consist of both the stratum of a maximaximal little group and a lower level one. Though the portion of the surface belonging to the null group is more singular than the interior, there is no way to distinguish them because there is no more subgroup left. The volume is extremized when either χ_2 is equal to zero (U_1) or χ_3 is zero (the null group) with all the other components nonzero.

C. Comments

Contrary to the case of one irrep where the strata of successively lower level little groups occupy successively higher-dimensional and less singular (locally less protrudent) surfaces on the orbit space boundary, the orbit space boundary of two irreps is more complex and things are pretty much mixed. Whereas orbit parameters associated with each irrep tend to form warped concave boundary surfaces, orbit parameters associated with both irreps tend to destroy such behavior. With the representation vector of one irrep fixed (consequently, orbit parameters associated with that irrep fixed), one can rotate the vector of the other irrep creating a volume traced by pencils.

It is notable that the generic strata in both examples are not totally open as in an irrep case. They close themselves partially. The same is true for lower-dimensional strata.

In the case of SU_3 adjoint + vector (Fig. 9) we find that the maximaximal little groups, SU_2 and U_1, occupy the most protrudent portions of the boundary. But in the case of SO_5 adjoint + vector (Fig. 10) we find that the U_1 stratum occupies the boundary planes indicated by the dotted lines and the stratum of the null group occupies the boundary plane indicated by the solid lines. That is, there is no sharp distinction between the maximaximal little group U_1 and the lower level little group, the null group, in terms of dimensionality and concavity.

Another interesting point is that the little groups alone cannot distinguish the fine structure of the orbit space. In both of the above-mentioned examples we see that the null group strata consist of two-dimensional surface *and* three-dimensional volume. In the SO_5 case the strata of U_1 consists of an edge curve *and* two-dimensional surfaces. This indistinguishability comes from the fact that, whereas for a given group there are only a finite number of subgroups, there is no limit to the dimension of a representation. As we see from Eq. (83) the orbit space dimension can be arbitrarily high. On

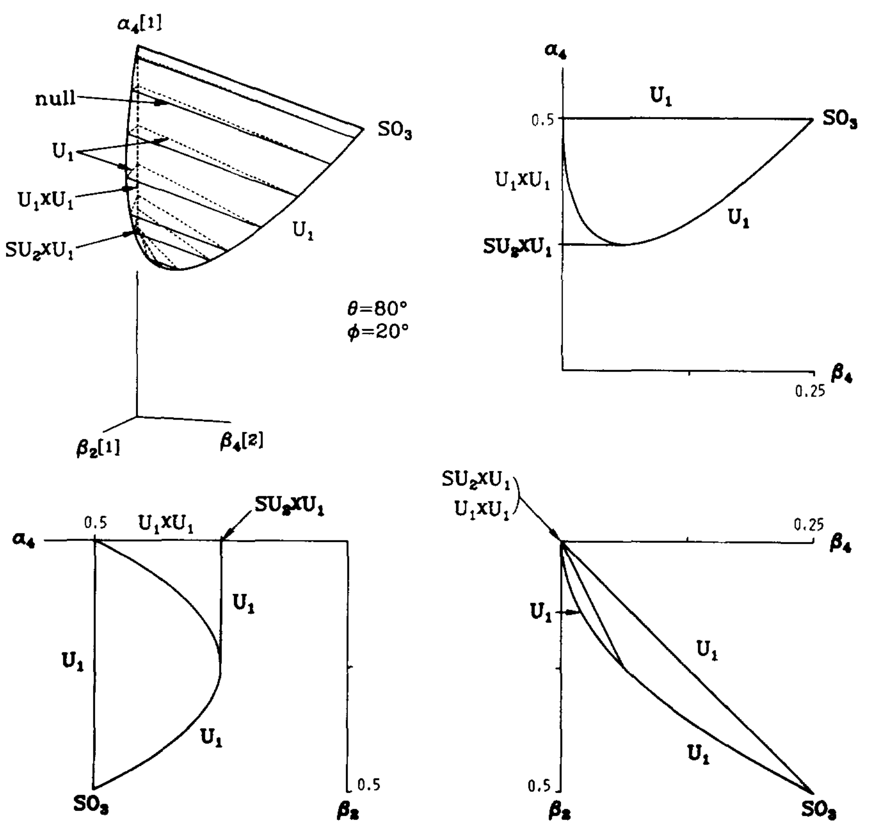

FIG. 10. The complete orbit space of the SO_5 adjoint + vector. Shown at the upper left corner is a view from the direction oriented 20° from the β_2 axis and 80° from the α_4 axis. The dotted lines are hidden lines. The numbers in the square brackets are the relative ratios of scale. Each projection is a view from the positive direction of the axis not shown in the picture. Here the hidden curves and lines are drawn solidly.

the other hand, the number of subgroups is too small to classify all the dimensions of the orbit space. Thus the indistinguishability is inevitable in both cases.

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APPENDIX A

Some people might doubt that the minimization can be achieved so cheaply. To remove a possible doubt, we interpret our method of minimization in the orbit space in terms of conventional language. We show how to find all the extrema (in the representation space) of a smooth group-invariant function which is monotonic in the orbit parameters. We also explain how to tell the type of an extremum, i.e., a local minimum or an inflection point.

1. Single irreducible representation

Let us consider a group invariant smooth function, $P(\varphi)\equiv F(I_2,\lambda_1,\lambda_2,\cdots)$, which is a monotonic function of orbit parameters λ_a in the projected orbit space. In order to find an extremum of P in the representation space, we need to find the solution of the equation

$$\frac{\partial P}{\partial\varphi_i}=\frac{\partial r}{\partial\varphi_i}\frac{\partial F}{\partial r}+\frac{\partial\lambda_1}{\partial\varphi_i}\frac{\partial F}{\partial\lambda_1}+\frac{\partial\lambda_2}{\partial\varphi_i}\frac{\partial F}{\partial\lambda_2}+\cdots\equiv 0, \qquad \text{(A1)}$$

with $r=I_2^{1/2}$. Due to the assumed monotonicity, all $\partial F/\partial\lambda_a$ are nonzero in the projected orbit space.

Case (i): $\partial F/\partial r=0$

There are two ways to satisfy Eq. (A1):

$$\partial\lambda_a/\partial\varphi_i=0 \quad \text{for all } a \text{ and } i. \qquad \text{(A2)}$$

This is satisfied at all the cusps (including buried ones) corresponding to maximal little groups[28,29,16] with phase angles of complex invariants not counted

$$\left(\frac{\partial\lambda_1}{\partial\varphi_i},\frac{\partial\lambda_2}{\partial\varphi_i},\cdots\right)\perp\left(\frac{\partial F}{\partial\lambda_1},\frac{\partial F}{\partial\lambda_2},\cdots\right) \quad \text{for all } i. \qquad \text{(A3)}$$

This is satisfied when the contour of directional extrema contacts the orbit space boundary tangentially. Thus extrema of P with respect to φ_i may occur at points on the curves, two dimensional surfaces, etc. Notice that the tangential contact can occur.

It is laborious to check signs of a Hessian matrix in order to find the extremum type. However, once the orbit space is constructed, we can easily tell the type of an extremum from the way the contour of directional extremum meets the orbit space. For example, in the case of SO_8 adjoint (Fig. 11) there are only five extrema consisting of four cusps and one tangential contact point on the boundary curve. Two of the cusps are local minima because the contour has the lowest values at the cusps in their neighborhoods. (The contour touching the upper cusp passes through the lower right portion of the orbit space. However, the cusp is isolated from that portion.) The lower right one is the absolute minimum. On the other hand, the lower left one is a saddle point and the remaining two extrema are inflection points.

As we showed in Ref. 16, the contour of directional minima for a most general fourth-degree Higgs potential is flat or concave in the direction of increasing equipotential. Thus unless higher-dimensional strata are more protrudent than cusps, the absolute minimum will occur at cusps on the boundary of the projected orbit space. For a general smooth group-invariant function the contour may be convex in the direction of increasing equipotential. The cusps will still be the most likely points for the absolute minimum to occur. However, higher-dimensional strata will now have a better chance for becoming an absolute minimum.

Case (ii): $\partial F/\partial r\neq 0$

$$\left(\frac{\partial r}{\partial\varphi_i},\frac{\partial\lambda_1}{\partial\varphi_i},\frac{\partial\lambda_2}{\partial\varphi_i},\cdots\right)\perp\left(\frac{\partial F}{\partial r},\frac{\partial F}{\partial\lambda_1},\frac{\partial F}{\partial\lambda_2},\cdots\right) \quad \text{for all } i. \qquad \text{(A4)}$$

One of the equations cannot be satisfied, namely,

$$\epsilon_{ab\cdots}\frac{\partial\lambda_a}{\partial\varphi_i}\frac{\partial\lambda_b}{\partial\varphi_j}\cdots=\text{const (of } a \text{ and } i)\,\frac{\partial F}{\partial r}. \qquad \text{(A5)}$$

On the boundary the lhs of Eq. (A5) is identically zero. Thus Eq. (A5) can only be satisfied inside the projected orbit space, where the vectors, $(\partial\lambda_a/\partial\varphi_i,\partial\lambda_b/\partial\varphi_i,\cdots)$, are independent.

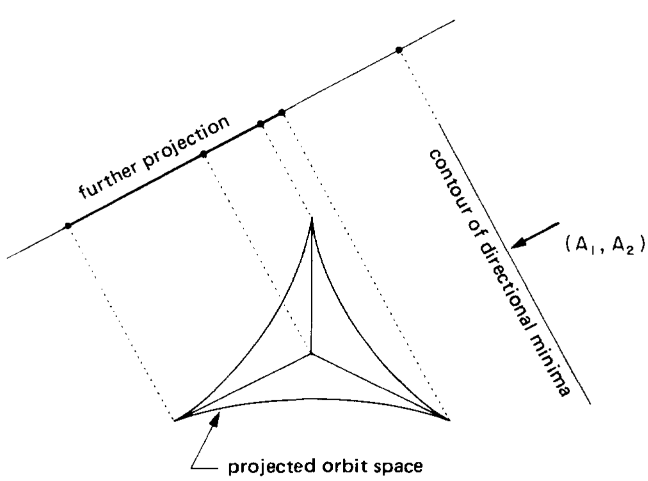

FIG. 11. The projected orbit space of the complete orbit space of the SO_8 adjoint is further projected onto a line with the contour of directional minima projected onto a point. Some cusps are projected onto the extreme boundary points; others into the interior.

The smallest number of independent vectors is the dimension of the projected orbit space. Now too many independent vectors have to be perpendicular to a fixed vector, the rhs of (A4). Therefore, Eq. (A4) cannot be satisfied at any point of the orbit space.

In fact, Michel[12] showed that a fourth degree Higgs potential cannot have an extremum in the generic stratum. We have extended his result to include more general cases. We have shown that a group-invariant function of a single irreducible representation monotonic in the projected orbit space can have an extremum only on the boundary of the projected orbit space with $\partial F/\partial r = 0$.

2. Two irreducible representations

Let us consider a group-invariant smooth function, $P(\varphi,\chi) \equiv F(I_2,\alpha_1,\alpha_2; J_2,\gamma_1,\gamma_2; \beta_1,\beta_2)$, which is a monotonic function of orbit parameters, $(\alpha_a, \gamma_c, \beta_b)$, in the projected orbit space. We have omitted further orbit parameters for the sake of saving space. It will not affect the generality of the following argument. In order to find an extremum of P in the representation space, we need to find the solution of the equation:

$$\frac{\partial P}{\partial \varphi_i} = \frac{\partial r}{\partial \varphi_i}\frac{\partial F}{\partial r} + \frac{\partial \alpha_1}{\partial \varphi_i}\frac{\partial F}{\partial \alpha_1} + \frac{\partial \alpha_2}{\partial \varphi_i}\frac{\partial F}{\partial \alpha_2} + \frac{\partial \beta_1}{\partial \varphi_i}\frac{\partial F}{\partial \beta_1} + \frac{\partial \beta_2}{\partial \varphi_i}\frac{\partial F}{\partial \beta_2} \equiv 0, \tag{A6a}$$

$$\frac{\partial P}{\partial \chi_j} = \frac{\partial s}{\partial \chi_j}\frac{\partial F}{\partial s} + \frac{\partial \gamma_1}{\partial \chi_j}\frac{\partial F}{\partial \gamma_1} + \frac{\partial \gamma_2}{\partial \chi_j}\frac{\partial F}{\partial \gamma_2} + \frac{\partial \beta_1}{\partial \chi_j}\frac{\partial F}{\partial \beta_1} + \frac{\partial \beta_2}{\partial \chi_j}\frac{\partial F}{\partial \beta_2} \equiv 0, \tag{A6b}$$

with $r = I_2^{1/2}$, $s = J_2^{1/2}$. Due to the assumed monotonicity, all $\partial F/\partial \alpha_a$, $\partial F/\partial \gamma_c$, $\partial F/\partial \beta_b$ are nonzero in the projected orbit space.

Case (i): $\partial F/\partial r = 0$ and $\partial F/\partial s = 0$

There are many ways to satisfy Eqs. (A6):

$$\frac{\partial \alpha_a}{\partial \varphi_i} = 0, \quad \frac{\partial \beta_b}{\partial \varphi_i} = 0, \quad \text{for all } a,b,i, \tag{A7a}$$

$$\frac{\partial \gamma_c}{\partial \chi_j} = 0, \quad \frac{\partial \beta_b}{\partial \chi_j} = 0, \quad \text{for all } c,b,j, \tag{A7b}$$

$$\left(\frac{\partial \alpha_1}{\partial \varphi_i}, \frac{\partial \alpha_2}{\partial \varphi_i}, \frac{\partial \beta_1}{\partial \varphi_i}, \frac{\partial \beta_2}{\partial \varphi_i}, 0, 0\right) \perp \left(\frac{\partial F}{\partial \alpha_1}, \frac{\partial F}{\partial \alpha_2}, \frac{\partial F}{\partial \beta_1}, \frac{\partial F}{\partial \beta_2}, \frac{\partial F}{\partial \gamma_1}, \frac{\partial F}{\partial \gamma_2}\right) \quad \text{for all } i, \tag{A8a}$$

$$\left(0, 0, \frac{\partial \beta_1}{\partial \chi_j}, \frac{\partial \beta_2}{\partial \chi_j}, \frac{\partial \gamma_1}{\partial \chi_j}, \frac{\partial \gamma_2}{\partial \chi_j}\right) \perp \left(\frac{\partial F}{\partial \alpha_1}, \frac{\partial F}{\partial \alpha_2}, \frac{\partial F}{\partial \beta_1}, \frac{\partial F}{\partial \beta_2}, \frac{\partial F}{\partial \gamma_1}, \frac{\partial F}{\partial \gamma_2}\right) \quad \text{for all } j. \tag{A8b}$$

Any combination of (AIa) and (AJb) with $I,J = 7,8$ will yield a solution to Eqs. (A6a) and (A6b). However, Eqs. (A7) are less frequently satisfied than in a single irrep case. The extremum conditions (A7) can also be satisfied partially in contrast to the single irrep case where all the orbit parameters are extremized simultaneously (with phase angles of complex invariants not counted). The points satisfying these conditions may be at cusps, on curves, on two dimensional surfaces, etc., on the orbit space boundary. They are all tangential contact points of the contour of directional extrema with the orbit space boundary. Again there are only finitely many extrema.

Since an orbit space for two irreps is formed from two independent spaces through the joint invariants, the Jacobian determinant[12,16] contains many zero elements. The dimension of a boundary portion is still given by the rank of the Jacobian.

Case (ii): $\partial F/\partial r \neq 0$ and/or $\partial F/\partial s \neq 0$

This condition again takes us into the projected orbit space and yields too many vectors to be perpendicular to a fixed vector.

Again we have shown that a group invariant function of two irreducible representations monotonic in the projected orbit space can have an extremum point only on the boundary of the projected orbit space with $\partial F/\partial r = 0$ and $\partial F/\partial s = 0$.

APPENDIX B

When the Higgs potential contains more than four independent invariant polynomials, it seems difficult to visually minimize the potential. We show how to find the absolute minima of these potentials. Let us consider a Higgs potential for a single irrep containing two third-degree invariants and three fourth-degree invariants. Call the associated orbit parameters, $\beta_1, \beta_2, \alpha_1, \alpha_2, \alpha_3$. Consider the following two-dimensional projected space of the five-dimensional orbit space:

$$\beta \equiv B_1\beta_1 + B_2\beta_2, \quad \alpha \equiv A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3, \tag{B1}$$

where B's and A's are the coupling coefficients of the corresponding invariant polynomials in the Higgs potential. Notice that β is proportional to the distance of the point (β_1, β_2) from the $\beta = 0$ line perpendicular to the vector (B_1, B_2), and that α is proportional to the distance of the point $(\alpha_1, \alpha_2, \alpha_3)$ from the $\alpha = 0$ plane perpendicular to the vector (A_1, A_2, A_3).

In the β–α space the absolute minimum of the potential can be found using the formula previously derived in Ref. 16. Now one necessarily asks whether we can uniquely determine (β_1, β_2) and $(\alpha_1, \alpha_2, \alpha_3)$ from a given set of (β, α). From the geometrical meaning of β and α, we see that there is a continuous range of orbit parameters satisfying Eq. (B1) for a given set of (β, α). However, the absolute minimum occurs at a unique point on the orbit space boundary, most protrudent to the direction of decreasing directional minimum. Thus we have a unique solution to Eq. (B1) at the absolute minimum. If the absolute minimum occurs on a concave portion of the orbit space boundary,[51] then there is a continuum of points satisfying Eq. (B1) and we have to stay in the five-dimensional orbit space. We have illustrated the mechanism in Fig. 11.

This raises the question: Is it safe to work in the projected orbit space which is dimensionally smaller than the representation vector space? Let us reconsider the significance of the boundary conditions:

$$\frac{\partial \lambda_a}{\partial \varphi_i} = 0 \quad \text{for all } a \text{ and } i, \tag{B2a}$$

$$\epsilon_{ab} \frac{\partial \lambda_a}{\partial \varphi_i} \frac{\partial \lambda_b}{\partial \varphi_j} = 0 \quad \text{for all } (a,b) \text{ and } (i,j), \cdots . \tag{B2b}$$

Equation (B2a) implies that, at a cusp corresponding to a null dimensional stratum, if we specify one orbit parameter, then all the other orbit parameters are determined. Equation (B2b) implies that, on a singular curve corresponding to a one-dimensional stratum, if we specify two orbit parameters, then all the other orbit parameters are determined, and so on.

The boundary conditions are strong enough to let us determine all the components of the scalar field (a vector in the representation space) at the absolute minimum from the knowledge of the norm and a small number of orbit parameters. The absolute minimum condition prompts the boundary condition, which in turn determines the whole vector.

Without further arguments we state that for an even degree Higgs potential of two irreps, we can safely work in the projected space, (α, β, γ), of the possibly high-dimensional projected orbit space of the complete orbit space.

However, in the process of further projection we lose the detailed extremum structure of the invariant function. This is evident in Fig. 11: we see five extrema before the projection and only two extrema afterwards. As far as we are looking for the absolute minimum only, the projection is harmless.

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