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LETTER TO THE EDITOR

Some ideas on the freely decaying Navier–Stokes turbulence

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Abstract. In this letter, we formulate a new approach to the Navier-Stokes turbulence. By analogy with Wilson's theory of the critical phenomena, we try to identify the upper critical dimension of the freely decaying Navier-Stokes turbulence. We point out the possibility of many fixed points. We also point out possible future developments.

This is the second of a series of articles investigating the Navier-Stokes turbulence. The objective is to find out what the relationships are, if any, between properties of the fluid turbulence and statistical dynamics of the Navier-Stokes equation. The major contribution of this letter is to present a new formulation in calculating the statistical properties of the freely decaying Navier-Stokes turbulence.

First, I have to contrast this approach with previous attempts on the Navier-Stokes turbulence. The best reference on the subject of applying the field-theoretic method to the Navier-Stokes turbulence is the paper by De Dominicis and Martin [1]. They considered the Navier-Stokes equation with a Gaussian-distributed random force. In fact, the majority of the work in applying the renormalisation-group technique to the Navier-Stokes turbulence starts with a Gaussian-distributed random force. As it turns out, the operator dimensionality of the velocity field is determined by the random force [1]. The scaling behaviour of the velocity-velocity correlation function obtained by De Dominicis and Martin is thus at a fixed point dominated by the random force, instead of the Navier-Stokes equation itself. Using the random phase approximation, Edwards actually obtained similar results before the advent of the renormalisation group technique in the critical phenomena [2]. The reason that a Gaussian-distributed random force was introduced is due to the fact that we do not know the probability distribution of the velocity field in the fully developed turbulence. By specifying the distribution of the random force, we indirectly obtain the probability distribution of the velocity field through the equation of motion, i.e. the Navier-Stokes equation with the random force.

In this paper, I will pursue a very different approach. We start with the incompressible Navier-Stokes equation (with $\rho = 1$):

$$\begin{aligned} &\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla P + \nu\nabla^2\boldsymbol{v} \qquad \text{for } t \geq 0 \\ &\nabla\cdot\boldsymbol{v} = 0 \qquad \text{and} \qquad \lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{v}(\boldsymbol{x}, t) = 0 \\ &\boldsymbol{v}(\boldsymbol{x}, t=0) = \boldsymbol{A}(\boldsymbol{x}). \end{aligned} \tag{1}$$

We would like to compute the energy spectrum of the freely decaying isotropic and homogeneous Navier–Stokes turbulence. The only assumption we make is the existence and uniqueness of solutions to equation (1) for all time in three dimensions in the limit of vanishing viscosity.

The formal solution of equation (1), with initial condition $A(x)$, is given by

$$v(x,t)=\int_0^t [\nu\nabla^2 v-\nabla P-(v\cdot\nabla)v]\,\mathrm{d}\tau+A(x). \tag{2}$$

The generating functional of n-point velocity correlation functions, averaged over the initial-velocity profile distribution function $\mu(A(x))$, is (summation convention)

$$Z[l]=\int\prod_{\alpha=1}^{d}[\mathscr{D}v_\alpha(x,t)]\,\mathrm{d}\mu(A(x))\exp\left\{\int_a \mathrm{d}^d x\int_0^\infty \mathrm{d}t\, l_\alpha(x,t)v_\alpha(x,t)\right\}$$
$$\times\prod_x\prod_{t\geqslant 0}\prod_{\alpha=1}^{d}\delta\left\{v_\alpha(x,t)-\int_0^t[\nu\nabla^2 v_\alpha-\partial_\alpha P-(v\cdot\nabla)v_\alpha]\,\mathrm{d}\tau-A_\alpha(x)\right\}. \tag{3}$$

The philosophy behind our approach is the following. We take as granted that the incompressible Navier–Stokes equation is the underlying evolution equation appropriate for the statistical description of the fully developed turbulence in the limit of infinite Reynolds number. Hence, we must restrict the velocity field configuration to those satisfying the Navier–Stokes equation. This explains the presence of the δ-function at every spacetime point.

Before we proceed further, we make two comments here. First of all, we have introduced a spatial lattice cut-off of order 'a' in the spatial integral. The reason is as follows. The Navier–Stokes equation is a hydrodynamic equation, thus a coarse-grained equation. The lattice cut-off 'a' is roughly the scale below which the hydrodynamic equation ceases to be valid, i.e. one has to use the molecular dynamics to describe the system. Secondly, we have constrained the velocity field to satisfy the Navier–Stokes equation by using $\delta\{v_\alpha-\int_0^t[\nu\nabla^2 v_\alpha-\partial_\alpha P-(v\cdot\nabla)v_\alpha]\,\mathrm{d}\tau-A_\alpha\}$ instead of $\delta\{(\partial v_\alpha/\partial t)-\nu\nabla^2 v_\alpha+\partial_\alpha P+(v\cdot\nabla)v_\alpha\}$. There are two advantages. The dependence on the initial condition is explicitly shown and we do not have to worry about the determinant: $\det[(\partial/\partial t)+(\delta F/\delta v)]$, where $F[v]=-\nu\nabla^2 v+\nabla P+(v\cdot\nabla)v$.

As pointed out in the previous article [3], the operator dimensionality of the velocity field is indeterminate as written in equation (3). To proceed further with the current approach, we use the following trick. Due to the presence of the δ-function, we can insert an arbitrary functional of v inside the functional integral of $Z[l]$ such that the value of the inserted functional is 1 whenever the δ-function is satisfied. Then, with the Fourier integral representation of the δ-function, $Z[l]$ can be written as

$$Z[l]=\int\prod_{\alpha=1}^{d}[\mathscr{D}v_\alpha(x,t)][\mathscr{D}\psi_\alpha(x,t)]\,\mathrm{d}\mu(A(x))\exp\left\{\int_a \mathrm{d}^d x\int_0^\infty \mathrm{d}t\, l_\alpha(x,t)v_\alpha(x,t)\right\}$$
$$\times\exp\left\{-b\int_a \mathrm{d}^d x\int_0^\infty \mathrm{d}t\left(\frac{\partial v_\alpha}{\partial t}+(v\cdot\nabla)v_\alpha+\partial_\alpha P-\nu\nabla^2 v_\alpha\right)\left(\nabla^{2n}v_\alpha\right)\right\}$$
$$\times\exp\left\{\int_a \mathrm{d}^d x\int_0^\infty \mathrm{d}t\,\mathrm{i}\psi_\alpha\left(v_\alpha-\int_0^t[\nu\nabla^2 v_\alpha-\partial_\alpha P-(v\cdot\nabla)v_\alpha]\,\mathrm{d}\tau-A_\alpha(x)\right)\right\} \tag{4}$$

where b is an arbitrary positive number and $n=0,1,2,3,\ldots$. Equation (4) is identical to equation (3) for any choice of n, because the second line of equation (4) equals 1 whenever v_α satisfies the Navier–Stokes equation.

We are mainly interested in the fluctuating turbulence flow field. If there is any mean flow in the system, we would go to the reference frame where the mean flow is nearly zero. That is, we only average over those initial conditions which give rise to zero mean flow field: $\langle v\rangle = 0$. This is reminiscent of the critical region where $\langle s\rangle \simeq 0$.

As mentioned earlier, the lattice cut-off 'a' is the scale below which the hydrodynamic equation ceases to be valid. Obviously, the 'viscous cut-off' is much larger than 'a'. We, in turn, are interested in the scaling behaviour in the region where $k \ll k_\nu$, with k_ν the viscous cut-off in the momentum space. Hence, we are interested in the *infrared* behaviour with respect to the lattice cut-off.

We can now perform a naive dimensional analysis to identify the upper critical dimension of this system. We first notice that, by the incompressibility condition, we have the following identity: $\nabla \cdot [(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}] = -\nabla^2 P$. Hence, the operator dimensionality of ∇P is the same as that of $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$. By analogy with the procedure employed in the critical phenomena [4], we rescale the length and velocity magnitude:

$$\begin{aligned} x &= \Lambda x' \qquad & t &= \Lambda^2 t' \\ v &= \xi v' \qquad & \psi &= \eta \psi' \end{aligned} \tag{5}$$

with $\Lambda \gg 1$. Because there is no cut-off in time, the rescaling of t is in fact arbitrary. The choice $t = \Lambda^2 t'$ makes all quadratic terms (ψv, vv) equally important. Since v and ψ are dummy functional integration variables, we are free to choose ξ and η. We choose $b\nu\Lambda^{d+2}\xi^2/\Lambda^{2n+2} = 1$ and $\Lambda^{d+2}\eta\xi = 1$ to give the theory a non-trivial free propagator. Then in the primed system, where the lattice cut-off has been shrunk to a/Λ, the 'action' becomes (we drop the prime notation)

$$\begin{aligned} \mathscr{S} \equiv & -\int_{a/\Lambda} \mathrm{d}^d\boldsymbol{x} \int_0^\infty \mathrm{d}t (\nabla^{2n} v_\alpha)\left(\frac{1}{\nu}\frac{\partial v_\alpha}{\partial t} - \nabla^2 v_\alpha\right) \\ & - \frac{\Lambda^{n-(d/2)+1}}{\nu\sqrt{b\nu}} \int_{a/\Lambda} \mathrm{d}^d\boldsymbol{x} \int_0^\infty \mathrm{d}t (\nabla^{2n} v_\alpha)[(\boldsymbol{v}\cdot\nabla)v_\alpha + \partial_\alpha P] \\ & + \int_{a/\Lambda} \mathrm{d}^d\boldsymbol{x} \int_0^\infty \mathrm{d}t\, \mathrm{i}\psi_\alpha\left(v_\alpha - \nu\int_0^t \nabla^2 v_\alpha\, \mathrm{d}\tau\right) \\ & + \frac{\Lambda^{n-(d/2)+1}}{\nu\sqrt{b\nu}} \int_{a/\Lambda} \mathrm{d}^d\boldsymbol{x} \int_0^\infty \mathrm{d}t\, \mathrm{i}\psi_\alpha \int_0^t [(\boldsymbol{v}\cdot\nabla)v_\alpha + \partial_\alpha P]\, \mathrm{d}\tau \\ & - \sqrt{b\nu}\Lambda^{(d/2)-n} \int_{a/\Lambda} \mathrm{d}^d\boldsymbol{x} \int_0^\infty \mathrm{d}t\, \mathrm{i}\psi_\alpha(\boldsymbol{x}, t) A_\alpha(\boldsymbol{x}). \end{aligned} \tag{6}$$

We see that the coefficient of the interaction term grows as $\Lambda^{n-(d/2)+1}$. When $n+1 \geqslant d/2$, the divergence of the perturbation series as $\Lambda \to \infty$ is cancelled by the diverging counterterm in order to make the renormalised theory finite [4]. In this framework, we can also see that higher-order terms do not alter the infrared scaling behaviour. Obviously, the viscous force contains terms besides $\nu\nabla^2\boldsymbol{v}$. In considering the requirement of analyticity and Galilean invariance [1], we conclude that the possible forms of the higher-order terms contributing to the viscous force are

$$\nabla \prod_{i=2}^{l} (\nabla_{\alpha_i} v_{\beta_i}) \qquad l \geqslant 2. \tag{7}$$

The coefficient then scales as

$$\Lambda^{d+2}\eta\frac{1}{\Lambda}\left(\frac{\xi}{\Lambda}\right)^l \Lambda^2 \propto [\Lambda^{n-(d/2)-1}]^{l-1}.$$

From the previous paragraph, we conclude that the upper critical dimension is $d_c = 2(n+1)$. We are puzzled by the fact that there are, seemingly, infinitely many d_c for different choices of n. We know that equation (4) is mathematically identical to equation (3) for any choice of n. How can there be infinitely many d_c? Are there many fixed points†? Which one corresponds to the fully developed turbulence? Could it be that there is no upper critical dimension for this system? Could the anomalous dimension change the conclusion of this analysis, which is based on the naive dimensionality of the field operator? Maybe there are other physical considerations that will enable us to pick out the correct fixed point corresponding to the fully developed turbulence. These questions will be addressed in forthcoming articles.

In summary, I have established a new framework for analytical computation of the statistical properties of the freely decaying Navier–Stokes turbulence. Detailed calculation of the energy-cascading exponent will be published soon.

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† The naive dimensional analysis applied to the classical Ising model gives two upper critical dimensions: $d_c = 4$ for Gaussian fixed point as discussed in [4], and $d_c = 0$ for high temperature fixed point (see [5]).