TABLE IV
EQUATIONS AND CONDITIONS USED TO PROVE (36)

Case	Equation number	Condition
$j_3 < j_1 < j_2$*	(19-α) , (18-β)	$p_2 > 0$, $q_1 < 0$
$j_3 < j_2 < j_1$	(18-α) , (21-β)	$p_1 > 0$, $q_4 > 0$
$j_1 < j_2 < j_3$	(21-α) , (18-β)	$p_4 < 0$, $q_1 < 0$
$j_2 < j_1 < j_3$	(18-α) , (19-β)	$p_1 > 0$, $q_2 < 0$

and

$$j_{i_1} \equiv j_{i_2} + 1 \equiv j_{i_3} + 1 \equiv j_4 - 1 \qquad \text{mod } 3.$$

That is, it is known that these conditions are equivalent to those given by (II)-(iv), (II)-(v), and (II)-(vi) in Table I when $m_k = 3$. Thus the restriction on $j_{\max}(J)$ does not change.

Moreover the set of a_i obtained from $a_{i_1} = (r-1)a_{i_2} = a_{i_3} = -(r-1)a_4$ also satisfies (37). Thus the situation as previously described may happen also for $A(r,2)$. However, we have from (38)

$$j_{i_1} + 1 \equiv j_{i_2} + 1 \equiv j_{i_3} \equiv j_4 \qquad \text{mod } 2,$$

which are equivalent to one of the congruences in (II)-(iv), (II)-(v), or (II)-(vi). Thus no new restriction on $j_{\max}(J)$ is needed here.

Except for the case of $a_{i_1} = (r-1)a_{i_2} = a_{i_3} = -(r-1)a_4$, we can find several sets of a_i satisfying (37). However, we cannot find those sets of a_i in Table I. This fact means that under those conditions J cannot be divided by an A that is composed of three or more $A(r,m_k)$, even if one of them is $A(r,2)$. Therefore this discussion does not impose any more stringent restriction on $j_{\max}(J)$.

Case (III)

(III)-(ii): This case has the same condition on j_i as that considered by Kondratyev and Trofimov [1] for the binary case. It follows from the results obtained there that (13) is a sufficient condition for $A \nmid J$.

Finally we must consider the cases where $w_r(J) < 4$. However, the details for these cases are omitted here, because they can be discussed in a similar and even simpler way than that in the case of $w_r(J) = 4$. The result obtained is that looser restrictions than (5) and (13) will do.

From all that has been discussed previously and the inequalities

$$\min_{I_1,I_2} \left(\prod_{k \in I_1} m_k + \prod_{k \in I_2} m_k \right) < \prod_{k \in I} m_k - 2 < \prod_{k \in I} m_k - 1$$

we can conclude that the following theorem is valid.

Theorem 2: A radix-r AN code generated by $A = \prod_{k \in I} A(r,m_k)$ has distance not less than five under the three conditions stated in Theorem 1.

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A Note on the Griesmer Bound

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***Abstract*—Griesmer's lower bound for the word length n of a linear code of dimension k and minimum distance d is shown to be sharp for fixed k, when d is sufficiently large. For $k \leq 6$ and all d the minimum word length is determined.**

I. INTRODUCTION

Denote by $n(k,d)$ the smallest integer n such that there exists an (n,k) binary linear code with minimum distance at least d. In 1960 Griesmer [1] proved that[1]

$$n(k,d) \geq \sum_{i=0}^{k-1} \lceil d/2^i \rceil \tag{1.1}$$

and showed that for certain values of k and d the inequality (1.1) was in fact an equality. In 1965 Solomon and Stiffler [2] simplified Griesmer's proof of (1.1) and at the same time generalized it to linear codes over an arbitrary finite field $GF(q)$, where it takes the form[1]

$$n(k,d) \geq \sum_{i=0}^{k-1} \lceil d/q^i \rceil. \tag{1.2}$$

More important, however, Solomon and Stiffler introduced the notion of "puncturing" a $(q^k - 1, k)$ maximal-length shift-register code and showed that for many more values of k and d equality holds in (1.2).

In this correspondence we shall use the technique of puncturing to show that for fixed k, when d is sufficiently large, the Griesmer bound (1.2) is sharp. That is, we will show that for each k there exists an integer $D(k)$ such that if $d \geq D(k)$, then

$$n(k,d) = \sum_{i=0}^{k-1} \lceil d/q^i \rceil.$$

As a matter of fact we will only prove this for $q = 2$, the extension to general q being easy but notationally awkward.

We shall use the notation

$$g(k,d) = \sum_{i=0}^{k-1} \lceil d/2^i \rceil$$

in the rest of the paper.

II. THE THEOREM OF SOLOMON–STIFFLER

Let V_k denote a k-dimensional vector space over $GF(2)$. Let $S_1, S_2, \cdots, S_t$ be subspaces of V_k of dimensions $k_1, k_2, \cdots, k_t$ such

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[1] Actually these bounds were obtained in the form

$$n(k,d) \geq \sum_{i=0}^{k-1} d_i,$$

where $d_0 = d$ and $d_i = \lceil d_{i-1}/q \rceil$. It is easy to see, however, that $d_i = \lceil d/q^i \rceil$.

that no element (except 0) of V_k is contained in more than h of the S_i. Then Solomon and Stiffler showed that there exists an (n,k) binary linear code with minimum distance d, where[2]

$$n = h(2^k - 1) - \sum_{i=1}^{t} (2^{k_i} - 1)$$

$$d \geq h2^{k-1} - \sum_{i=1}^{t} 2^{k_i-1} = d'.$$

Furthermore if the k_i are *distinct*, $n = g(k,d')$ and so the code is length optimal; i.e., $n(k,d) = g(k,d)$. Finally they showed that a sufficient condition for the existence of such subspaces S_i is that $\sum k_i \leq kh$.

III. Main Result

Theorem: For each k there exists an integer $D(k)$ such that

$$n(k,d) = g(k,d), \qquad \text{if } d \geq D(k).$$

Proof: We show that $D(k) = \lceil (k-1)/2 \rceil 2^{k-1}$ will do. Write $d = d_0 + (h-1)2^{k-1}$, where $1 \leq d_0 \leq 2^{k-1}$. Then if $d \geq \lceil (k-1)/2 \rceil 2^{k-1}$ it follows that $h \geq \lceil (k-1)/2 \rceil$. Next we write $2^{k-1} - d_0$ in its binary expansion

$$2^{k-1} - d_0 = \sum_{i=1}^{t} 2^{k_i-1}, \qquad 0 < k_1 < k_2 < \cdots < k_t < k.$$

Then

$$\sum_{i=1}^{t} k_i \leq 1 + 2 + \cdots + k - 1 = k(k-1)/2 \leq k \cdot h$$

and so by the results of Solomon–Stiffler quoted in Section II, $n(k,d) = g(k,d)$.

IV. Numerical Results

We have been able to calculate the exact values of $n(k,d)$ for $k \leq 6$ and all d. It turns out that the value $D(k) = \lceil (k-1)/2 \rceil \cdot 2^{k-1}$ given in our theorem is extremely conservative; for example, for $k = 6$ our theorem only guarantees that if $d \geq 96$, $n(6,d) = g(6,d)$, while $d \geq 20$ would do. Much of this disparity arises from our use of the very weak sufficient condition $\sum k_i \leq kh$ for the existence of subspaces $S_1, S_2, \cdots, S_t$.

Thus consider the example $k = 6$, $d = 35$. Examining the proof in Section III, we write $35 = 3 + 1 \cdot 32$ $(h = 2)$, and $32 - 3 = 29 = 2^4 + 2^3 + 2^2 + 2^0$. Thus we need to find subspaces of V_6 of dimensions 5, 4, 3, and 1 that cover each nonzero vector of V_6 at most twice. Since $5 + 4 + 3 + 1 = 13 > 6 \cdot 2$, the condition of Solomon–Stiffler does not apply. However, if the vectors of V_6 are coordinatized $x = (x_1, x_2, \cdots, x_6)$, consider the following subspaces:

$S_1 = \{x: x_1 = 0\}$	dimension 5
$S_2 = \{x: x_2 = x_3 = 0\}$	dimension 4
$S_3 = \{x: x_4 = x_5 = x_6 = 0\}$	dimension 3
$S_4 = \{111111 \text{ and } 000000\}$	dimension 1.

These subspaces have the desired property of covering each nonzero vector at most twice and so $n(6,35) = g(6,35)$.

However, even if we knew exact necessary and sufficient conditions for the existence of the subspaces S_i, we would not always get the best possible code. For $k = 6$, $d = 17$ we would need subspaces of dimensions 4, 3, 2, and 1 that covered every nonzero element at most once; but it is easy to see that any two subspaces of dimensions 4 and 3 in V_6 must share at least one nonzero vector. Thus the Solomon–Stiffler results could not yield a (37,6) code with $d = 17$. However, in his original paper (Theorem 5) Griesmer gave a construction that yields such a code.

[2] It can be shown that $d = d'$ unless the dual subspaces $S_i\perp$ completely cover V_k.

TABLE I

k	d	$g(k,d)$	$n(k,d)$	Comments
5	3	8	9	HB; (9,5) = (15,11) Hamming shortened
5	5	12	13	search; (13,5) = (15,7) BCH shortened
6	3	9	10	HB; (10,6) = (15,11) Hamming shortened
6	5	13	14	$n(5,3)$; (14,6) = (15,7) BCH shortened
6	7	16	17	$n(5,4)$; (17,6) = (23,12) Golay shortened
6	9	21	22	$n(5,5)$; (22,6) found *ad hoc*[a]
6	11	24	25	$n(5,6)$; (25,6) found *ad hoc*[b]
6	13	28	29	search; (29,6) = (31,6) RM minus 2 columns
6	19	40	41	search; (41,6) = Solomon–Stiffler construction with dimensions 3,3,3,1 (h = 1)

[a]Take as columns in the generator matrix the 6-place binary expansions of: 2,3,4,6,8,9,11,12,16,17,20,21,26,32,33,38,44,51,58,61,62,63.
[b]Take as columns 1,1,2,4,6,8,10,13,16,18,21,27,28,31,32,34,37,43,45,46,53,54,57,58,60.

We conclude the paper with Table I, which shows those values of k and d with $k \leq 6$ for which $n(k,d) > g(k,d)$. The column titled "Comments" explains how we calculate $n(k,d)$. HB means that the Hamming bound forces $n(k,d) > g(k,d)$. "Search" means that a computer search found no codes of length $g(k,d)$. An entry like $n(5,3)$ refers to the bound, proved by Griesmer, that $n(k,d) \geq d + n(k-1, \lceil d/2 \rceil)$. Thus if $n(k-1, \lceil d/2 \rceil) > g(k-1, \lceil d/2 \rceil)$, then $n(k,d) > g(k,d)$ as well. We only list odd d because of the relationship $n(k,d) = n(k, d+1) - 1$ for odd d.

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A Note on One-Step Majority-Logic Decodable Codes

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***Abstract*—Construction of shortened geometric codes as shown here results in 1-step majority-logic decodable codes. The shortened codes retain the error-correction ability of the parent codes and the decoders for the shortened codes are much simpler than for the parent code. A table of shortened codes is given.**

I. Shortened Finite Geometry Codes

A shortened cyclic code retains at least the error-correcting capability of the parent full-length cyclic (n,k) code. In the case

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