HYPER VERSION
RECEIVED: *November 12, 2001*, ACCEPTED: *February 12, 2002*

Non-abelian Wilson surfaces

Iouri Chepelev

Department of Physics, California Institute of Technology
Pasadena, CA 91125, USA, and
C.N. Yang Institute for Theoretical Physics
SUNY at Stony Brook, NY 11794, USA
Email: chepelev@theory.caltech.edu

ABSTRACT: A definition of non-abelian genus zero open Wilson surfaces is proposed. The ambiguity in surface-ordering is compensated by the gauge transformations.

KEYWORDS: p-branes, Gauge Symmetry, Differential and Algebraic Geometry.

JHEP02(2002)013

Contents

1. Introduction

A higher dimensional generalization of the non-abelian Wilson line is not known. Only recently the notion of a connection on a non-abelian 1-gerbe was introduced in the work of Breen and Messing [1].

A motivation for defining the Non-abelian Wilson Surfaces comes from the string theory. NWS are relevant to six dimensional theories on the world volumes of coincident five branes [2].

The main problem in defining NWS is the lack of a natural order on a 2-dimensional surface. A naive guess for the NWS is

$$P \exp\left(\int_{\Sigma} B\right), \tag{1.1}$$

where B is a non-abelian 2-form. The choice of a surface-ordering P involves a time-slicing of the 2-surface Σ. A no-go theorem of Teitelboim [3] states that no such a choice is compatible with the reparametrization invariance.

Let us recall the notion of a connection on a non-abelian 1-gerbe [1]. A connection on a principal bundle (0-gerbe) can be thought of as follows. Let x_0 and x_1 be two infinitesimally close points. The fibers S_{x_0} and S_{x_1} over these points are sets and the connection is a function

$$f_{01} : S_{x_1} \to S_{x_0}\,. \tag{1.2}$$

The connection on a non-abelian 1-gerbe is defined by analogy with the 0-gerbe case [1]. The fibers are categories C_{x_0} and C_{x_1}, and the connection is a functor

$$\varepsilon_{01} : C_{x_1} \to C_{x_0}\,. \tag{1.3}$$

Let x_0, x_1 and x_2 be three infinitesimally close points. A diagram of functors and natural transformations is shown in figure 1. Let $\mathrm{Aut}(G)$ be the group of automorphisms of a non-abelian group G. Let $\mathrm{Lie}(G)$ be the Lie algebra of G. It is shown in [1] that 2-arrow K, 1-arrow κ and 1-arrow ε in the diagram correspond to a $\mathrm{Lie}(G)$-valued 2-form B, a $\mathrm{Lie}(\mathrm{Aut}(G))$-valued 2-form ν and a $\mathrm{Lie}(\mathrm{Aut}(G))$-valued 1-form μ respectively.

The paper is organized as follows. In section 2 a definition of NWS is proposed. Section 3 is devoted to gauge transformations. Some comments are listed in section 4.

JHEP02(2002)013

2. Definition

We interpret the infinitesimal 2-simplex in figure 1 as a transmuted form of an infinitesimal Wilson surface expressed in the language of category theory. The fibered category in the formulation of [1] can be thought of as an 'internal symmetry space' of a non-abelian string. Let Σ be a 2-dimensional surface with the disk topology. Let C be a clockwise oriented boundary of Σ and P a marked point on it (see figure 2). We associate group elements

$$W[\Sigma, C, P] \in G$$

and

$$V[\Sigma, C, P] \in \mathrm{Aut}(G)$$

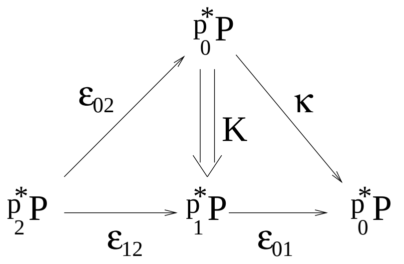

Figure 1: ε_{ij} is a cartesian functor from the fibered category p_j^*P to p_i^*P, κ is a cartesian functor from p_0^*P to p_0^*P, and K is a 2-arrow from $\kappa \circ \varepsilon_{02}$ to $\varepsilon_{01} \circ \varepsilon_{12}$.

with the data (Σ, C, P). We write $W[\Sigma]$ and $V[\Sigma]$ when the omitted arguments are obvious from the context. With an open curve C we associate an element of $\mathrm{Aut}(G)$:

$$M[C] \in \mathrm{Aut}(G)\,. \tag{2.1}$$

Let $C = C_2 \circ C_1$ be a composition of curves C_2 and C_1. We assume

$$M[C] = M[C_2 \circ C_1] = M[C_2]M[C_1]\,. \tag{2.2}$$

We now propose an equation relating $M[C]$, $W[\Sigma, C]$ and $V[\Sigma, C]$. For a group element $g \in G$ we denote by i_g the inner automorphism

$$i_g(h) = ghg^{-1}\,, \qquad \forall h \in G\,. \tag{2.3}$$

The conjectural equation reads

$$M[C] = i_{W[\Sigma]}V[\Sigma]\,. \tag{2.4}$$

An infinitesimal version of this equation was first derived in [1] from the requirement that K in figure 1 is a natural transformation. We regard eq. (2.4) as a fundamental equation relating bulk and boundary of the non-abelian string world-sheet.

Eq. (2.4) can be used to find a composition rule for two NWS. Consider the 2-surfaces in figure 2. The identity

$$\begin{aligned} i_{W[\Sigma_2 \circ \Sigma_1, P_1]}V[\Sigma_2 \circ \Sigma_1] &= M[C \circ C_4 \circ C_3] \\ &= M[C]M[C_4 \circ C_5^{-1}]M[C^{-1}]M[C \circ C_5 \circ C_3] \\ &= M[C]i_{W[\Sigma_2,P_2]}V[\Sigma_2, P_2]M[C^{-1}]i_{W[\Sigma_1,P_1]}V[\Sigma_1, P_1] \end{aligned} \tag{2.5}$$

suggests the following composition rule for Wilson surfaces:

$$\begin{aligned} W[\Sigma_2 \circ \Sigma_1] &= M[C](W[\Sigma_2])M[C]V[\Sigma_2]M[C^{-1}](W[\Sigma_1])\,, \\ V[\Sigma_2 \circ \Sigma_1] &= M[C]V[\Sigma_2]M[C^{-1}]V[\Sigma_1]\,. \end{aligned} \tag{2.6}$$

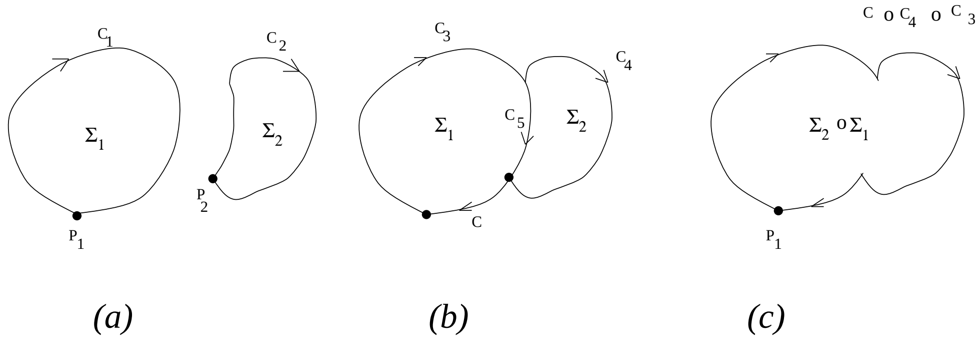

Figure 2: Composition of surfaces with the disk topology. (a) Surfaces Σ_i with the marked points P_i and the clockwise oriented boundaries C_i. (b) Surfaces are joined along the common boundary segment C_5. (c) The resulting surface $\Sigma_2 \circ \Sigma_1$ with the marked point P_1 and the clockwise oriented boundary $C \circ C_4 \circ C_3$.

An infinitesimal version of eq. (2.6) appeared implicitly in the category-theoretic definition of the curvature in [1].

Eq. (2.6) can be understood as follows. When the curve C is absent, i.e. when the marked points of Σ_1 and Σ_2 coincide, eq. (2.6) simplifies to

$$W[\Sigma_2 \circ \Sigma_1] = W[\Sigma_2]V[\Sigma_2](W[\Sigma_1])\,,$$
$$V[\Sigma_2 \circ \Sigma_1] = V[\Sigma_2]V[\Sigma_1]\,. \tag{2.7}$$

Thus when the marked points of the two surfaces coincide, the Wilson surfaces are composed as in eq. (2.7). If we think of $V[\Sigma, P]$ as an operator which acts on the objects with the marked point P and assume that only the objects with the same marked points can be multiplied, then the meaning of eq. (2.6) becomes clear. The role of $M[C]$ in eq. (2.6) is to transform the objects with the marked point P_2 to the objects with the marked point P_1.

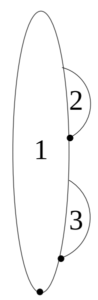

Figure 3: $\Sigma_3 \circ (\Sigma_2 \circ \Sigma_1) \neq \Sigma_2 \circ (\Sigma_3 \circ \Sigma_1)$.

Composition of three or more surfaces is in general ambiguous. Consider figure 3. Using the composition rule (2.6) it can be shown that

$$W[\Sigma_3 \circ (\Sigma_2 \circ \Sigma_1)] \neq W[\Sigma_2 \circ (\Sigma_3 \circ \Sigma_1)]\,,$$
$$V[\Sigma_3 \circ (\Sigma_2 \circ \Sigma_1)] \neq V[\Sigma_2 \circ (\Sigma_3 \circ \Sigma_1)]\,. \tag{2.8}$$

Given

$$V[\delta\Sigma] \approx 1 + v[P] \equiv 1 + v_{\mu\nu}[P]\sigma^{\mu\nu} \tag{2.9}$$

for an infinitesimal surface $\delta\Sigma$ with the area element $\sigma^{\mu\nu}$, we want to find $V[\Sigma]$ for a finite-size surface Σ. This can be done using a trick similar to the one used in the context of the non-abelian Stokes formula [4]. Consider the contour C' in figure 4. From the relation

$$M[C'] = M[C_P^{-1}]M[\delta C]M[C_P]M[C] \tag{2.10}$$

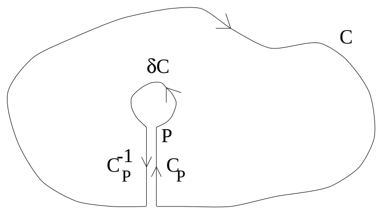

Figure 4: Contour $C' = C_P^{-1} \circ \delta C \circ C_P \circ C$.

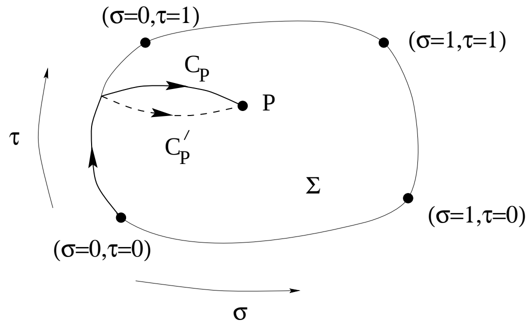

Figure 5: A parametrized surface Σ. The path C_P consists of two segments: the first segment ($\sigma = 0 = \text{const.}, \tau$) is from $\tau = 0$ to τ and the second segment ($\sigma, \tau = \text{const.}$) is from $\sigma = 0$ to σ.

and eq. (2.4) one finds

$$V[\Sigma'] = M[C_P^{-1}]V^{-1}[\delta\Sigma]M[C_P]V[\Sigma]\,. \tag{2.11}$$

Thus we have

$$\delta V[\Sigma] = M[C_P^{-1}]v[P]M[C_P]V[\Sigma]\,. \tag{2.12}$$

A solution of this equation involves a choice of ordering and it is given by

$$V[\Sigma] = \hat{P}_\tau \exp\left(\int_\Sigma M[C_P^{-1}]v[P]M[C_P]\right), \tag{2.13}$$

where $\hat{P}_\tau$ is the ordering in τ and the curve C_P is defined in figure 5. Note that the expression eq. (2.13) depends on the parametrization $x^\mu = x^\mu(\sigma, \tau)$ of the surface Σ. For example a boundary-preserving reparametrization will change C_P to a C'_P (see figure 5). Thus $V[\Sigma]$ and $W[\Sigma]$ depend on the parametrization of Σ:

$$V = V[\Sigma, x^\mu(\sigma, \tau)]\,, \qquad W = W[\Sigma, x^\mu(\sigma, \tau)]\,. \tag{2.14}$$

In section 3 we will see that if (σ, τ) and $(\tilde{\sigma}, \tilde{\tau})$ are two different parametrizations of a surface Σ, then

$$(V[\Sigma, x^\mu(\sigma, \tau)], W[\Sigma, x^\mu(\sigma, \tau)])$$

and

$$(V[\Sigma, x^\mu(\tilde{\sigma}, \tilde{\tau})], W[\Sigma, x^\mu(\tilde{\sigma}, \tilde{\tau})])$$

are related by the gauge transformation. In other words, the non-abelian internal symmetry and the reparametrization symmetry mix.

3. Gauge transformations

In this section we introduce the gauge transformations which compensate the ambiguity in the composition of NWS. Suppose that a surface Σ is composed out of three or more smaller surfaces. Let $(W[\Sigma], V[\Sigma])$ and $(\tilde{W}[\Sigma], \tilde{V}[\Sigma])$ correspond to two different compositions resulting in the surface Σ. We have

$$M[C] = i_{W[\Sigma]} V[\Sigma] = i_{\tilde{W}[\Sigma]} \tilde{V}[\Sigma] \,. \tag{3.1}$$

Since W and $\tilde{W}$ are elements of a group G, there is a group element $R[\Sigma] \in G$ such that

$$\tilde{W}[\Sigma] = W[\Sigma](R[\Sigma])^{-1} \,. \tag{3.2}$$

Let us decompose W and $\tilde{W}$ into the abelian and non-abelian factors:

$$W = W_{\text{ab}} \cdot W_{\text{nonab}} \,, \qquad \tilde{W} = \tilde{W}_{\text{ab}} \cdot \tilde{W}_{\text{nonab}} \,. \tag{3.3}$$

It is clear that the ambiguity in the composition does not affect the abelian part. Thus we have

$$\tilde{W}_{\text{ab}}[\Sigma] = W_{\text{ab}}[\Sigma] \,. \tag{3.4}$$

Combining this equation with eq. (3.2) we find

$$\tilde{W}_{\text{nonab}}[\Sigma] = W_{\text{nonab}}[\Sigma](R[\Sigma])^{-1}. \tag{3.5}$$

We propose that eq. (3.4) and eq. (3.5) define the gauge transformation of W. In order for this gauge transformation of W to be compatible with eq. (3.1), V should transform as

$$\tilde{V}[\Sigma] = i_{R[\Sigma]} V[\Sigma]. \tag{3.6}$$

It can be checked that the gauge transformations (3.4)–(3.6) are compatible with the composition rule (2.6) provided that the composition rule for R is the same as that of W, namely

$$R[\Sigma_2 \circ \Sigma_1] = M[C](R[\Sigma_2]) M[C] V[\Sigma_2] M[C^{-1}](R[\Sigma_1]) \,. \tag{3.7}$$

More generally, consider a surface Σ divided into n smaller surfaces $\Sigma_1, \ldots, \Sigma_n$. Let C be the boundary of Σ. Repeating the reasoning leading to eq. (2.6) we have

$$M[C] = M[\mathcal{C}_1] i_{W[\Sigma_1]} V[\Sigma_1] M[\mathcal{C}_2] i_{W[\Sigma_2]} V[\Sigma_2] M[\mathcal{C}_3] \cdots \tag{3.8}$$

for some curves $\mathcal{C}_1, \mathcal{C}_2, \ldots$. From this equation we find

$$\begin{aligned} W[\Sigma] &= M[\mathcal{C}_1](W[\Sigma_1]) M[\mathcal{C}_1] V[\Sigma_1] M[\mathcal{C}_2](W[\Sigma_2]) \cdots \,, \\ V[\Sigma] &= M[\mathcal{C}_1] V[\Sigma_1] M[\mathcal{C}_2] V[\Sigma_2] M[\mathcal{C}_3] \cdots \,. \end{aligned} \tag{3.9}$$

It is easy to see that the gauge transformations (3.4)–(3.6) are compatible with eq. (3.9) provided that $R[\Sigma]$ is composed out of $R[\Sigma_i]$ as follows:

$$R[\Sigma] = M[\mathcal{C}_1](R[\Sigma_1]) M[\mathcal{C}_1] V[\Sigma_1] M[\mathcal{C}_2](R[\Sigma_2]) \cdots \,. \tag{3.10}$$

Thus R should be composed by the rule of composition of W.

We now introduce new gauge transformations. These are the transformations of M, V and W compatible with eq. (2.4).

Let $\Lambda[P]$ be an $\mathrm{Aut}(G)$-valued function of point P. Let C be a directed path from P_1 to P_2. The gauge transformation of $M[C]$ reads

$$\tilde{M}[C] = \Lambda[P_2]M[C]\Lambda[P_1]^{-1}\,. \tag{3.11}$$

When $P_1 = P_2 = P$ this equation becomes

$$\tilde{M}[C] = \Lambda[P]M[C]\Lambda[P]^{-1}\,. \tag{3.12}$$

From this equation and

$$\tilde{M}[C] = i_{\tilde{W}}\tilde{V} \tag{3.13}$$

one finds

$$i_W V = \Lambda^{-1} i_{\tilde{W}}\tilde{V}\Lambda = i_{\Lambda^{-1}(\tilde{W})}\Lambda^{-1}\tilde{V}\Lambda\,. \tag{3.14}$$

Thus we propose the gauge transformations:

$$\begin{aligned} \tilde{V}[\Sigma, P] &= \Lambda[P]V[\Sigma, P]\Lambda[P]^{-1}\,, \\ \tilde{W}[\Sigma, P] &= \Lambda[P](W[\Sigma, P])\,. \end{aligned} \tag{3.15}$$

We now consider a new gauge transformation which is a finite generalization of the infinitesimal transformation considered in [1]. The transformation reads

$$\tilde{M}[C] = i_{\mathcal{Z}[C]}M[C]\,, \tag{3.16}$$

where $\mathcal{Z}[C]$ is a G-valued functional of C. The composition rule for $\mathcal{Z}$ can be inferred from the following chain of equations:

$$\begin{aligned} i_{\mathcal{Z}[C_2\circ C_1]}M[C_2\circ C_1] &= \tilde{M}[C_2\circ C_1] \\ &= \tilde{M}[C_2]\tilde{M}[C_1] \\ &= i_{\mathcal{Z}[C_2]}M[C_2]i_{\mathcal{Z}[C_1]}M[C_1] \\ &= i_{\mathcal{Z}[C_2]}i_{M[C_2](\mathcal{Z}[C_1])}M[C_2\circ C_1]\,. \end{aligned} \tag{3.17}$$

This equation suggests the following composition rule for $\mathcal{Z}$:

$$\mathcal{Z}[C_2\circ C_1] = \mathcal{Z}[C_2]M[C_2](\mathcal{Z}[C_1])\,. \tag{3.18}$$

If a $\mathrm{Lie}(G)$-valued 1-form ζ is given, $\mathcal{Z}[C]$ for an open path C can be constructed as follows. Let us divide C into n small subpaths as in figure 6a. Applying eq. (3.18) we find

$$\begin{aligned} \mathcal{Z}[C] &= \mathcal{Z}[C_n]\cdot M[C_n](Z[C_{n-1}])\cdot M[C_n\circ C_{n-1}](\mathcal{Z}[C_{n-2}])\times\cdots \\ &\quad \times M[C_n\circ C_{n-1}\cdots C_2](\mathcal{Z}[C_1]) \\ &\approx (1+\zeta_\mu[P_n]dx^\mu)(1+M[C_n](\zeta_\mu[P_{n-1}])dx^\mu)\times\cdots \\ &\quad \times(1+M[C_n\circ C_{n-1}\cdots C_2](\zeta_\mu[P_1])dx^\mu)\,. \end{aligned} \tag{3.19}$$

In the large n limit we thus find

$$\mathcal{Z}[C] = \hat{P}\,\exp\left(\int_C M[C''](\zeta_\mu[P])dx^\mu\right), \tag{3.20}$$

JHEP02(2002)013

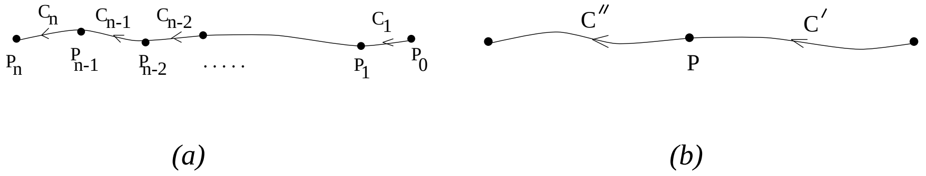

Figure 6: (a) The path C is divided into n small subpaths: $C = C_n \circ C_{n-1} \cdots \circ C_1$. ($b$) The point P divides $C = C'' \circ C'$.

where C'' and P are as in figure 6b, and $\hat{P}$ is the path ordering operator.

A choice of transformation of V and W compatible with eq. (2.4) and eq. (3.16) is

$$\begin{aligned} \tilde{V}[\Sigma, C] &= V[\Sigma, C]\,, \\ \tilde{W}[\Sigma, C] &= \mathcal{Z}[C] W[\Sigma, C]\,. \end{aligned} \tag{3.21}$$

Infinitesimal versions of these transformations agree with the transformations that can be derived from [1]. Let us consider an infinitesimal surface $\delta\Sigma$ with the area element $\sigma^{\mu\nu}$. Assume that $M[C] \in \mathrm{Aut}(G)$ is an inner automorphism given by

$$\begin{aligned} M[C](g) &= \hat{P}\mathrm{exp}\left(\int_C \mu\right) g\, \hat{P}\mathrm{exp}\left(-\int_C \mu\right) \\ &= \hat{P}\mathrm{exp}\left(\int_C \mu_{\mathrm{adjoint}}\right)(g)\,, \qquad \forall g \in G\,, \end{aligned} \tag{3.22}$$

where μ is a Lie(G)-valued 1-form. From eq. (3.21) and

$$W[\delta\Sigma] \approx 1 + B_{\mu\nu}\sigma^{\mu\nu}\,, \tag{3.23}$$

one can find the transformation of the 2-form B:

$$\tilde{B} = B + d\zeta - \frac{1}{2}[\zeta, \zeta] - [\mu, \zeta]\,. \tag{3.24}$$

The transformation of B corresponding to eqs.(3.4,3.5) reads

$$\tilde{B}_{\mathrm{ab}} = B_{\mathrm{ab}}\,, \qquad \tilde{B}_{\mathrm{nonab}} = B_{\mathrm{nonab}} - \rho\,, \tag{3.25}$$

where ρ is a Lie(G)-valued 2-form defined in

$$R[\delta\Sigma] \approx 1 + \rho_{\mu\nu}\sigma^{\mu\nu}\,. \tag{3.26}$$

Eq. (3.25) agrees with the transformations that can be derived from [1].

Unlike the gauge transformations (3.4)–(3.6), (3.15), the transformation (3.21) is not compatible with the composition rule (2.6). To find the correct transformation, $\mathcal{Z}[C]$ in eq. (3.21) should be 'smeared' over the surface Σ. We give an explicit formula for the gauge transformation of $V[\Sigma]$. It reads

$$\tilde{V}[\Sigma] = \hat{P}_\tau \exp\left(\int_\Sigma i_{\mathcal{Z}[C_P]} M[C_P] v[P] M[C_P^{-1}] i_{\mathcal{Z}[C_P]^{-1}}\right). \tag{3.27}$$

4. Comments

• We found three kinds of gauge transformations of M, V and W. These are $\Lambda[P]$-transformations (3.11), (3.15), $R[\Sigma]$-transformations (3.4)–(3.6) and $\mathcal{Z}[C]$-transformations (3.16), (3.21). Eq. (3.21) is valid only for infinitesimal surfaces and should be replaced by a 'smeared' version eq. (3.27).

• The ambiguity in surface-ordering necessitates the introduction of gauge transformations which compensate the ambiguity. Locally this amounts to the transformation eq. (3.25). The number of gauge degrees of freedom present in a NWS is enormous. Thus NWS may be relevant to a topological string theory describing topological sectors of the non-abelian string of [2].

• Infinitesimal version of eq. (2.6) can be derived from the composition rule for the natural transformation K in figure 1.

• We defined NWS on a local trivial patch. To define NWS globally one should cover the manifold with an atlas $\{U_\alpha\}$ and introduce $W_\alpha, V_\alpha, M_\alpha$ for each patch U_α. As usual the quantities on the overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$ are related by the gauge transformations. An analysis of global issues will be carried out elsewhere.

• We defined NWS with the disk topology. A generalization to higher-genus surfaces will be discussed elsewhere.

Note added. After submitting the original version of this paper to hep-th, the work [5] was brought to our attention. In [5] an equation similar to eq. (2.13) was taken as a definition of Wilson surface. The case considered in [5] corresponds, in our notation, to the C-independent $M[C]$. The surface-ordering ambiguities are absent in this case. For a list of miscellaneous work on non-abelian 2-form theories, see [6].

Acknowledgments

This work was supported by the DOE under grant No. DE-FG03-92ER40701, by the Research Foundation under NSF grant PHY-9722101 and by CRDF Award RP1-2108.

References

[1] L. Breen and W. Messing, *Differential geometry of Gerbes*, math.AG/0106083.

[2] A. Strominger, *Open p-branes*, *Phys. Lett.* **B 383** (1996) 44 [hep-th/9512059];
O.J. Ganor, *Six-dimensional tensionless strings in the large-N limit*, *Nucl. Phys.* **B 489** (1997) 95 [hep-th/9605201].

[3] C. Teitelboim, *Gauge invariance for extended objects*, *Phys. Lett.* **B 167** (1986) 63.

[4] M.B. Halpern, *Field strength and dual variable formulations of gauge theory*, *Phys. Rev.* **D 19** (1979) 517;
I.Ya. Aref'eva, *Non-abelian Stokes formula*, *Theor. Math. Phys.* **43** (1980) 353;
P.M. Fishbane, S. Gasiorowicz and P. Kaus, *Stokes' theorems for nonabelian fields*, *Phys. Rev.* **D 24** (1981) 2324;

R.L. Karp, F. Mansouri and J.S. Rno, *Product integral formalism and non-abelian Stokes theorem*, *J. Math. Phys.* **40** (1999) 6033 [hep-th/9910173];
Product integral representations of wilson lines and Wilson loops and non-abelian Stokes theorem, *Turk. J. Phys.* **24** (2000) 365 [hep-th/9903221].

[5] O. Alvarez, L.A. Ferreira and J. Sanchez Guillen, *A new approach to integrable theories in any dimension*, *Nucl. Phys.* **B 529** (1998) 689 [hep-th/9710147].

[6] P.G.O. Freund and R.I. Nepomechie, *Unified geometry of antisymmetric tensor gauge fields and gravity*, *Nucl. Phys.* **B 199** (1982) 482;
P. Orland, *Duality for nonabelian lattice fields*, *Nucl. Phys.* **B 163** (1980) 275;
D.Z. Freedman and P.K. Townsend, *Antisymmetric tensor gauge theories and nonlinear sigma models*, *Nucl. Phys.* **B 177** (1981) 282;
A. Lahiri, *Gauge transformations of the non-abelian two-form*, hep-th/0107104; *Local symmetries of the non-abelian two-form*, hep-th/0109220.