Focusing resonance cones

B. Levine, G. J. Greene, and R. W. Gould

California Institute of Technology, Pasadena, California 91125
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The potential for an oscillating ring source immersed in cold, magnetized, collisionless plasma in the resonance cone regime ($K_{\parallel}/K_{\perp}<0$) is evaluated exactly and asymptotically, giving insight into the gross spatial behavior of the focusing resonance cones. The nature of the singularity in potential is clarified by the introduction of a noninfinitesimal collision frequency. Thermal effects are considered numerically, revealing an interesting interference structure in potential as well as a density depression near the focus of the cone due to the ponderomotive force.

I. INTRODUCTION

Resonance cones[1,2] emanating from ring sources have recently been studied in experiments with the objectives of focusing in order to observe ponderomotive force effects[3,4] and exciting a single axial wavenumber by means of a phased array of these sources.[5] In this paper an exact expression for the potential of an oscillating ring charge in a highly-magnetized, cold, collisionless plasma in the resonance cone regime ($K_{\parallel}/K_{\perp}<0$) is derived. The character of the singularity in the potential is clarified by introduction of a noninfinitesimal collision frequency; likewise, the focusing nature of the cones becomes apparent. First-order warm plasma effects are considered in numerical evaluation of the electric potential which displays the thermal interference structure. Lastly, an approximation to the ponderomotive force-induced density modification for weak fields is evaluated. Ion terms and sheath effects around the probe are neglected, and the plasma is assumed to be spatially uniform.

II. THE POTENTIAL

A. Collisionless theory

The volume charge density

$$\rho_{ext}(\mathbf{r},t)=(S/2\pi\rho)\delta(\rho-\rho_0)\delta(z)e^{-i\omega t} \tag{1}$$

describes an oscillating ring charge of radius ρ_0 centered in the $z=0$ plane (Fig. 1) having total charge magnitude S and oscillation frequency ω. The Fourier transform with respect to the spatial variables being taken, Eq. (1) yields

$$\hat{\rho}_{ext}(\mathbf{k},t)=Se^{-i\omega t}J_0(k_{\perp}\rho_0). \tag{2}$$

Since the quasi-static approximation $\mathbf{E}=-\nabla\phi$ is appropriate for determination of the near-field solution, there results

$$\rho_{ext}=\nabla\cdot\mathbf{D}=\nabla\cdot\epsilon_0\mathbf{K}\cdot\mathbf{E}=-\epsilon_0\nabla\cdot\mathbf{K}\cdot\nabla\phi\ , \tag{3}$$

where $\mathbf{K}$, the plasma dielectric tensor, has components $K_{xx}=K_{yy}=K_{\perp}$, $K_{xy}=-K_{yx}=K_H$, $K_{yz}=K_{zy}=0$, and $K_{zz}=K_{\parallel}$. Fourier transforming Eq. (3) with respect to the spatial variables and utilizing Eq. (2) yields

$$\hat{\phi}(\mathbf{k},t)=(S/\epsilon_0)e^{-i\omega t}[J_0(k_{\perp}\rho_0)/(k_{\perp}^2K_{\perp}+k_{\parallel}^2K_{\parallel})]. \tag{4}$$

It is readily verified (Ref. 6, p. 679) that evaluation of the inverse Fourier transform of Eq. (4) gives

$$\phi(\mathbf{r},t)=\frac{Se^{-i\omega t}}{2\pi^2\epsilon_0K_{\perp}}\int_{k_{\parallel}=0}^{\infty}dk_{\parallel}\cos k_{\parallel}z\begin{cases}I_0[\rho k_{\parallel}(K_{\parallel}/K_{\perp})^{1/2}]K_0[\rho_0k_{\parallel}(K_{\parallel}/K_{\perp})^{1/2}], & \rho<\rho_0\\ I_0[\rho_0k_{\parallel}(K_{\parallel}/K_{\perp})^{1/2}]K_0[\rho k_{\parallel}(K_{\parallel}/K_{\perp})^{1/2}], & \rho>\rho_0\end{cases} \tag{5}$$

provided

$$\mathrm{Re}(K_{\parallel}/K_{\perp})^{1/2}>0. \tag{6}$$

The inequality given by Eq. (6) holds upon inclusion of a small, nonzero collision frequency ν into the cold, magnetized plasma dielectric tensor expression. The integral in Eq. (5) is expressible in terms of a Legendre function of the second kind (Ref. 6, p.732), a so-called conical function, to wit

$$\phi(\mathbf{r},t)=\frac{Se^{-i\omega t}}{4\pi^2\epsilon_0(\rho\rho_0K_{\perp}K_{\parallel})^{1/2}}Q_{-1/2}\left[\frac{z^2+(\rho^2+\rho_0^2)K_{\parallel}/K_{\perp}}{2\rho\rho_0K_{\parallel}/K_{\perp}}\right], \tag{7}$$

again, provided Eq. (6) holds and $z\neq0$. $Q_{-1/2}(\zeta)$ is singular for arguments $\zeta=\pm1$, hence the surfaces on which the potential is singular are

$$z=\pm(\rho-\rho_0)(-K_{\parallel}/K_{\perp})^{1/2},\quad z=\pm(\rho+\rho_0)(-K_{\parallel}/K_{\perp})^{1/2}. \tag{8}$$

These are two cones of half-angle $\mathrm{arccot}(-K_{\parallel}/K_{\perp})^{1/2}$ which intersect on the source ring and whose vertices are given by $z_c=\pm\rho_0(-K_{\parallel}/K_{\perp})^{1/2}$. There are no singularities for the case $K_{\parallel}/K_{\perp}>0$.

The nature of the singularity for $\arg Q_{-1/2}=1$ (the discussion for $\arg Q_{-1/2}=-1$ is similar) can be made manifest by consideration of the identity[7]

$$Q_n(\mu)=\frac{1}{2}\left(\log\frac{\mu+1}{\mu-1}\right)F[-n,n+1;\ 1;\ (1-\mu)/2]$$

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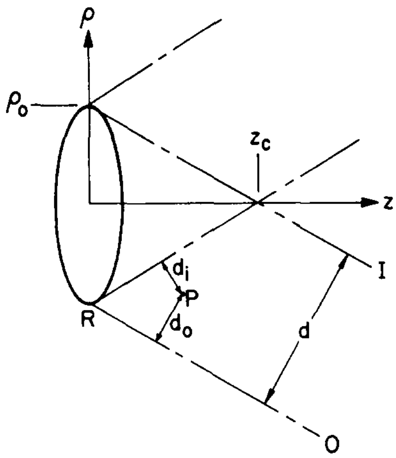

FIG. 1. Geometry of the problem. R denotes the ring source in the $z=0$ plane, centered on $\rho=0$. I and O denote the inner and outer cones of cold plasma theory, d_i and d_o are the distances from an arbitrary point P to the inner and outer cones, and d is the distance from a vertex to the outer cone. The static magnetic field is along the z axis, and the vertex of the inner cone is at z_c.

$$+\sum_{p=0}^{\infty}\frac{(n+p)!}{(n-p)!}\frac{[(\mu-1)/2]^p}{(p!)^2}[\psi(p)-\psi(n)], \qquad (9)$$

where n is an arbitrary real number, $\psi(z)=d/dz \times \log\Gamma(z+1)$, and it is implicit that $\log(1+\mu/1-\mu)$ is taken for $\mu<1$. For $n=-1/2$ and $\mu=1+\epsilon$, $\epsilon \ll 1$, there results

$$Q_{-1/2}(1+\epsilon)\sim\frac{1}{2}\log\frac{2}{|\epsilon|} \text{ as } \epsilon\to 0, \qquad (10)$$

where

$$\epsilon=\frac{(\rho-\rho_0)^2(K_\parallel/K_\perp)+z^2}{2\rho\rho_0(K_\parallel/K_\perp)} . \qquad (11)$$

If one defines d_i, d_o, and d as the respective distances from (ρ, z) to the inner cone [for $|z|\le\rho_0(-K_\parallel/K_\perp)^{1/2}$], from (ρ, z) to the outer cone, and from either vertex to the outer cone, then

$$d_{i,o}=\left|\frac{z\pm(\rho-\rho_0)(-K_\parallel/K_\perp)^{1/2}}{(1-K_\parallel/K_\perp)^{1/2}}\right|, \quad d=\frac{2\rho_0(-K_\parallel/K_\perp)^{1/2}}{(1-K_\parallel/K_\perp)^{1/2}}, \qquad (12)$$

and thus

$$\frac{2}{|\epsilon|}=\frac{(\rho/\rho_0)}{(d_i/d)(d_o/d)} . \qquad (13)$$

Hence, $d_i\to 0$ or $d_o\to 0$ implies $\epsilon\to 0$, unless d_i/ρ or d_o/ρ is kept constant as the limit is taken; that is, the vertices themselves are not in the domain of validity of Eq. (10). Allowing, with no loss of generality, $d_i\to 0$, the asymptotic expression for the potential on the inner cone becomes

$$\phi(\mathbf{r},t)\sim\frac{S e^{-i\omega t}}{4\pi^2\epsilon_0\rho_0(K_\parallel K_\perp)^{1/2}} \times\left[\frac{\log(\rho/\rho_0)^{1/2}-\log(d_i/d)^{1/2}}{(\rho/\rho_0)^{1/2}}\right] \text{ as } d_i\to 0, \qquad (14)$$

where $\mathbf{r}=(\rho, z)$ and $0<|z|<\rho_0(-K_\parallel/K_\perp)^{1/2}$.

B. Collisional effects

The $\log(d_i/d)^{1/2}$ term speciously dominates expression (14), since d_i may be made arbitrarily small, whereas ρ is constrained to be positive by exclusion of the vertices from the domain of validity. However, inclusion of a small, nonzero collision frequency ν makes it impossible for the expression for d_i given by Eq. (12) to vanish, making it mandatory to retain the $\log(\rho/\rho_0)^{1/2}$ term, particularly near the vertices. For a highly magnetized, cold plasma, neglecting terms of order (m_e/m_i) and following Allis *et al.*,[8]

$$K_\perp\simeq 1, \quad K_\parallel\simeq-\frac{\omega_{pe}^2}{\omega^2}\left(1-i\frac{\nu}{\omega}\right), \qquad (15)$$

where m_e and m_i are the electron and ion masses, respectively, and ω_{pe} is the electron plasma frequency.

Simple algebra shows that on the surface $z+(\rho-\rho_0)\times(\omega_{pe}/\omega)=0$

$$d_i/d\simeq\tfrac{1}{4}(1-\rho/\rho_0)(\nu/\omega)\exp(i\pi/2). \qquad (16)$$

The $\log(\rho/\rho_0)^{1/2}$ term is the sole term retained when

$$\rho/\rho_0\ll\nu/4\omega \text{ or } |z-\rho_0(\omega_{pe}/\omega)|\ll\tfrac{1}{4}(\nu/\omega)(\rho_0\omega_{pe}/\omega). \qquad (17)$$

Thus, there exists $\delta>0$ such that in two limiting cases

$$\phi(\mathbf{r},t)\sim\frac{-S\exp[-i(\omega t\pm\pi/2)]}{8\pi^2\epsilon_0\rho_0(\omega_{pe}/\omega)}\frac{\log[(1-\rho/\rho_0)(\nu/\omega)]}{(\rho/\rho_0)^{1/2}}, \quad \text{for } 0<|z|<\rho_0(\omega_{pe}/\omega)-\delta, \qquad (18a)$$

$$\phi(\mathbf{r},t)\sim\frac{S\exp[-i(\omega t\pm\pi/2)]}{8\pi^2\epsilon_0\rho_0(\omega_{pe}/\omega)}\frac{\log(\rho/\rho_0)}{(\rho/\rho_0)^{1/2}}, \quad \text{for } (1-\tfrac{1}{4}\nu/\omega)\rho_0(\omega_{pe}/\omega)\ll|z|<\rho_0(\omega_{pe}/\omega), \qquad (18b)$$

where $z\sim(\rho-\rho_0)(\omega_{pe}/\omega)$. For all intents and purposes, Eq. (18a) characterizes the singularity. The focusing nature of the cones is readily seen in either case.

C. Warm plasma effects

Inclusion of nonzero temperature gives rise to an interference structure on the inside of the resonance cone emanating from an oscillating point charge.[2] Hence, for a ring source an interference structure would be expected to appear outside the cone for $z<z_c$ and both inside and outside for $z>z_c$. (Observation of the interference structure within the cone for the case $z>z_c$ has been reported in Ref. 3, p. 111). To ascertain the main features of this structure, we have evaluated Eq. (5) numerically using the warm plasma dielectric functions

$$K_\parallel\simeq 1-\frac{\omega_{pe}^2}{k_\parallel^2 v_{th}^2}Z'\left(\frac{\omega}{k_\parallel v_{th}}\right) \text{ and } K_\perp\simeq 1 . \qquad (19)$$

Z' is the derivative of the plasma dispersion function and $v_{th}^2=2\kappa T_e/m_e$, where κ is Boltzmann's constant and T_e is the electron temperature. Here, we have assumed $\omega_{ce}^2>\omega^2$, ω_{pe}^2 where ω_{ce} is the electron cyclotron frequency.

The integrand of Eq. (5) has a logarithmic singularity at $k_\parallel=0$. For purposes of numerical evaluation, it is

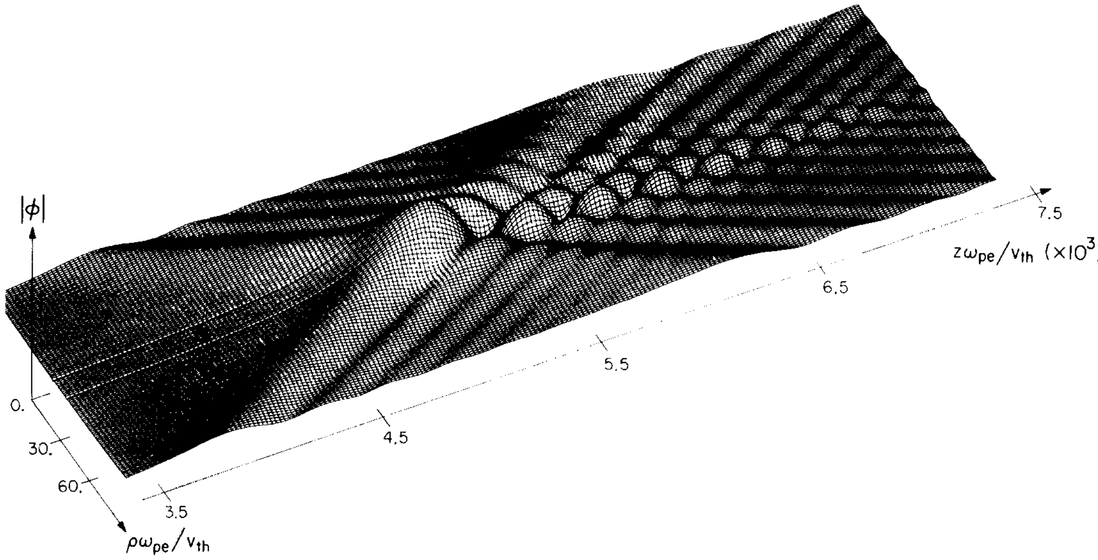

FIG. 2. Magnitude of the complex potential for a part of the inner cone, displaying the interference structure due to thermal effects. The ring source is located as in Fig. 1; the dimensionless parameters are $\omega/\omega_{pe}=0.02$ and $\rho_0\omega_{pe}/v_{th}=270$. The resonance cone focus occurs at $z\omega_{pe}/v_{th}=4.93\times10^3$; the vertex given by cold plasma theory is at $z_c\omega_{pe}/v_{th}=4.83\times10^3$.

convenient to treat the singularity separately. Let the integrand in Eq. (5) be defined as $f(k_\parallel)$. Then, the potential may be written

$$\phi(\mathbf{r},t)=\frac{Se^{-i\omega t}}{2\pi^2\epsilon_0}\Bigg(\int_0^\eta dk_\parallel\Bigg\{f(k_\parallel)+\cos(k_\parallel z)\times\log\left[\frac{1}{2}\rho_0 k_\parallel\left(\frac{\omega_{pe}^2}{\omega^2}-1\right)^{1/2}\right]\Bigg\}-\int_0^\eta dk_\parallel\cos(k_\parallel z)\times\log\left[\frac{1}{2}\rho_0 k_\parallel\left(\frac{\omega_{pe}^2}{\omega^2}-1\right)^{1/2}\right]+\int_\eta^\infty dk_\parallel f(k_\parallel)\Bigg) \quad (20)$$

for arbitrary η. The singularity now appears only in the second integral which may be evaluated exactly in terms of the sine integral $\mathrm{Si}(x)$. For an appropriate choice of $\eta\{$i.e., $\eta<1/[\rho(\omega_{pe}^2/\omega^2-1)^{1/2}]$, $1/z$, $\omega/v_{th}\}$, the second integrand is a slowly varying function of $k_\parallel$, and the integral may be evaluated by Simpson's rule, while the last integral may now be computed using the fast Fourier transform algorithm.[9]

Dimensionless variables appropriate for the evaluation of Eq. (5) are: frequency, $f\equiv\omega/\omega_{pe}$; distance along the ρ direction, $\mathcal{R}\equiv(\omega_{pe}/v_{th})\rho$; radius of the ring source, $\mathcal{R}_0\equiv(\omega_{pe}/v_{th})\rho_0$; and distance along the z direction, $\mathfrak{z}\equiv(\omega_{pe}/v_{th})z$. The potential was evaluated for typical

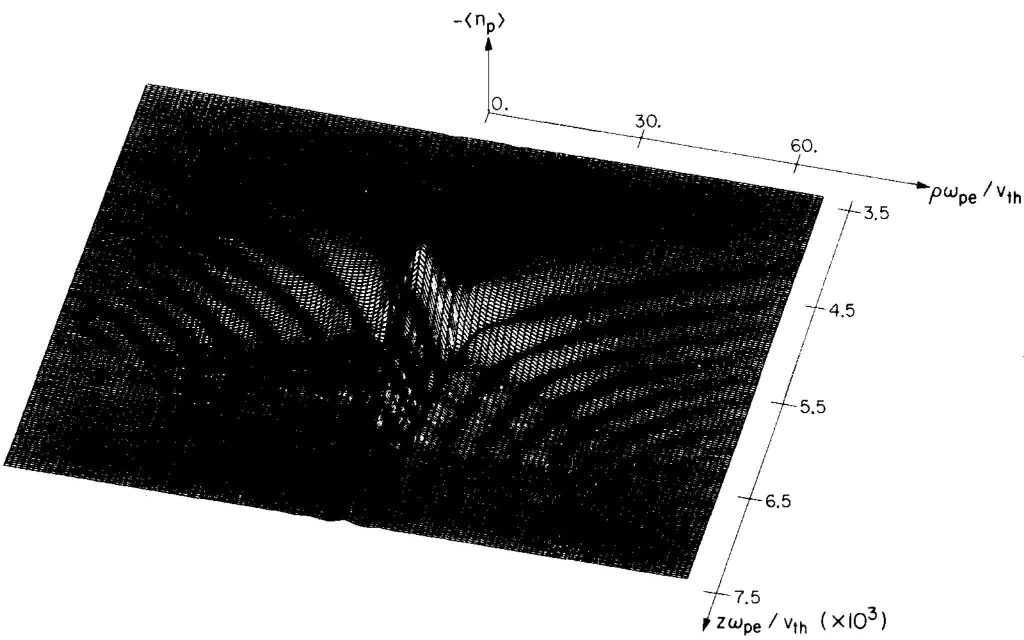

FIG. 3. First-order density depression due to the pondermotive force for $\omega/\omega_{ce}=0.02$. The location of the ring source and the dimensionless parameters are as in Fig. 2; note change in orientation of axes. The peak density depression occurs at $z\omega_{pe}/v_{th}=5.15\times10^3$.

parameters $f=0.056$ and $\mathcal{R}_0=270$, which may correspond, for example, to a laboratory plasma with $n_e=10^{10}$ cm^{-3}, $T_e=2$ eV, $\rho_0=4$ cm, and $\omega/2\pi=50$ MHz.

A plot of the magnitude of the complex potential clearly shows the converging cone-like structure expected from cold plasma theory (Fig. 2). The focus of the cone ($\mathfrak{z}=4.93\times10^3$) is shifted slightly from the cold plasma vertex ($\mathfrak{z}_c=4.83\times10^3$). A simple interference structure is found outside the cone, while a complicated pattern appears inside the cone for $\mathfrak{z}>\mathfrak{z}_c$, resembling a series of diverging cones with vertices along the z axis. The magnitude of the potential is found to be approximately proportional to $\xi^{-1/2}$, where $\xi=[R^2+(\mathfrak{z}_c-\mathfrak{z})^2]^{1/2}$ represents the distance along the surface of the cone for $\mathfrak{z}<\mathfrak{z}_c$, provided we are not too close to the vertex ($\xi\gtrsim135$).

The ponderomotive force causes density modifications which, in turn, change ω_p and hence the structure of the cone. As a first-order approximation for small density perturbations we evaluate the ponderomotive force which arises from the fields obtained from linear theory.

The ponderomotive force may be derived by time-averaging the momentum transport fluid equations for each species over a period $2\pi/\omega$.[10] Let the density of a species be n_0+n_p, where n_p is the perturbation caused by the ponderomotive force, and assume $n_p\ll n_0$. The averaged fluid equation for one species in the quasistatic approximation is

$$\frac{\langle\nabla p\rangle}{n_0}-q(\mathbf{E}_a+\mathbf{v}_d\times\mathbf{B}_0)=-\frac{m}{2}\,\mathrm{Re}\{[(\boldsymbol{\mu}\cdot\mathbf{E})^*\cdot\nabla](\boldsymbol{\mu}\cdot\mathbf{E})\},\tag{21}$$

where p is the pressure, $\mathbf{E}_a$ is an ambipolar electric field, $\mathbf{v}_d$ is a drift velocity, $\boldsymbol{\mu}$ is the mobility tensor, and $\boldsymbol{\mu}\cdot\mathbf{E}$ will be evaluated using results from linear theory. Equation (21) represents a pressure balance, and the right-hand side may be identified as the ponderomotive force density. The particle density is assumed to reach an equilibrium where an equation of state for a species may be approximated by $\nabla p=\gamma\kappa T\,\nabla n_p$, where γ is the coefficient of adiabatic expansion. Neglecting thermal corrections to the mobility, assuming $T_i\ll T_e$ and $n_i\simeq n_e$, summing the z component of Eq. (21) over species and integrating, we find

$$\left\langle\frac{n_p}{n_0}\right\rangle\simeq\frac{-\pi\epsilon_0}{\gamma_e\kappa T_e}\sum_{\alpha=i,e}\left(\frac{q_\alpha^2}{m_\alpha}\right)\left(\frac{|\mathbf{E}_p|^2}{\omega^2-\omega_{c\alpha}^2}+\frac{|\mathbf{E}_z|^2}{\omega^2}\right),\tag{22}$$

where q_α is the charge of species α and the fields are those obtained from linear theory. Numerical evaluation for an argon plasma with $\omega/\omega_{ce}=0.02$, using the parameters given here, shows that the density depression is sharply peaked near the resonance cone focus (Fig. 3). The thermal effects cause an interference structure in density whose phase fronts lie roughly parallel to the surface of the cone for $\mathfrak{z}<\mathfrak{z}_c$ and parallel to the z axis for $\mathfrak{z}>\mathfrak{z}_c$.

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