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WEIGHTS AND $L \log L$

A. CARBERY, S.-Y. A. CHANG AND J. GARNETT

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Let $\omega(x)$ be a positive locally integrable weight on $[0,1]$. Discussed are conditions on ω necessary and sufficient for the (dyadic) Hardy-Littlewood maximal function to map $L \log L(w\,dx)$ into $L^1(\omega\,dx)$ or into weak L^1.

1. Introduction. Let Mf denote the (dyadic) Hardy-Littlewood maximal function of f, for f locally integrable on $\mathbf{R}^n$. That is,

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f|,$$

the sup being taken over all dyadic cubes in $\mathbf{R}^n$ containing x. It is well-known that M is a bounded operator from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ when $p > 1$, takes L^1 to weak L^1, and for functions f supported in a dyadic cube Q_0 satisfies

$$\int_{Q_0} Mf \leq C \int_{Q_0} |f| \log^+ |f| + C|Q_0|.$$

More recently, Muckenhoupt and others have studied the behaviour of M when the L^p spaces with respect to Lebesgue measure are replaced by those with respect to the measure $\omega(x)\,dx$, $\omega \in L^1_{\text{loc}}$. A nonnegative locally integrable function ω is said to be in Muckenhoupt's (dyadic) A_p class for $1 \leq p < \infty$ if

$$\sup_{Q \text{ dyadic}} \left\| \frac{\omega_Q}{\omega(x)} \right\|_{L^q(\omega\,dx/\omega(Q))} = A_p(\omega) = A_p < \infty. \tag{1.1}$$

Here $1/p + 1/q = 1$, ω_Q is the average $(1/|Q|)\int_Q \omega$ of ω over Q and $\omega(Q) = \int_Q \omega$. (More generally, if E is a measurable set in $\mathbf{R}^n$ we denote $\int_E \omega$ by $\omega(E)$.) In [3], Muckenhoupt proved that given $\omega \in L^1_{\text{loc}}$ there exists a constant $C = C_{p,\omega,n}$ such that

$$\int |Mf|^p \omega \leq C \int |f|^p \omega \tag{1.2}$$

if and only if ω is in the A_p class, and that $\omega\{Mf > \lambda\} \leq (C/\lambda)\int |f|\omega$ if and only if ω is in the A_1 class. It would therefore seem reasonable to expect that a necessary and sufficient condition on $\omega \in L^1_{\mathrm{loc}}$ for the existence of a C such that

$$\int_{Q_0} Mf\omega \le C\int_{Q_0} |f|\log^+|f|\omega + C\omega(Q_0) \tag{1.3}$$

should hold whenever $\operatorname{supp} f \subseteq Q_0$ would be obtained by taking the "exponential limit" as $q \to \infty$ in (1.1). That is, ω should satisfy what we shall call the A^* condition:

(A^*) There exists an $\varepsilon > 0$ and a $C > 1$ such that

$$\sup_{Q \text{ dyadic}} \int_Q \exp\left\{\frac{\varepsilon\omega_Q}{\omega(x)}\right\}\frac{\omega(x)\,dx}{\omega(Q)} \le C.$$

Unfortunately, while A^* is necessary for (1.3) to hold, it is not sufficient (as we shall see in §4). In §3 we give a necessary and sufficient condition A^{**} that (1.3) hold. This condition may be realized as the "exponential limit" as $q \to \infty$ of certain other expressions, which, for each $p > 1$ are equivalent with the A_p condition. The equivalence breaks down in the limit, however, and A^* turns out to be necessary and sufficient only for M to take $L \log L(\omega\, dx)$ to weak $L^1(\omega\, dx)$. In §4 we prove this fact, and we compare our rather complicated condition A^{**} to growth conditions on $A_p(\omega)$ as $p \downarrow 1$, and see that none of these conditions is adequate to describe A^{**}. It would be of interest to find a more concise, easy-to-verify form of A^{**}.

Central to our proof of the equivalence of A^{**} with (1.3) is a new proof of the weighted L^p theorem which does not rely upon interpolation or upon the step "$\omega \in A_p \Rightarrow \omega \in A_{p-\varepsilon}$ for some $\varepsilon > 0$", used both by Muckenhoupt and Coifman and C. Fefferman, [**2**]. E. Sawyer [**4**] and M. Christ and R. Fefferman [**1**] have recently given other such proofs, but neither proof has a counterpart in the $L \log L$ setting. Section 2 is devoted to this new proof.

For simplicity, we choose to work with the dyadic Hardy-Littlewood maximal operator in this note. However, our results can easily be extended to those for the full maximal operator by standard techniques. Finally, C denotes a constant depending only possibly on the dimension (but not necessarily the same at each occurrence), and dependence of constants upon other quantities is indicated by subscripts.

2. The weighted L^p theorem. In this section we give a new proof of the weighted L^p theorem of Muckenhoupt. We shall need the insights it provides when we treat the case of $L \log L$ in §3.

We first prove the elementary fact that if $\omega \in A_p$, then $(1/\omega)^{q-1}$ satisfies the doubling condition.

LEMMA 2.1. *Suppose $\omega \in A_p$. Then there exists an $a < 1$ such that whenever E is a measurable subset of a cube Q and $|E| \leq \frac{1}{2}|Q|$, we have*

$$\int_E \left(\frac{\omega_Q}{\omega}\right)^q \frac{\omega\, dx}{\omega(Q)} \leq a \int_Q \left(\frac{\omega_Q}{\omega}\right)^q \frac{\omega\, dx}{\omega(Q)}.$$

Proof. By Hölder's inequality we have

$$\omega(Q)\frac{|Q-E|}{|Q|} = \int_{Q-E} \omega_Q \leq \left(\int_{Q-E}\left(\frac{\omega_Q}{\omega}\right)^q \omega\, dx\right)^{1/q}\left(\int_Q \omega\, dx\right)^{1/p},$$

that is,

$$\omega(Q)^{1/q}\frac{|Q-E|}{|Q|} \leq \left(\int_{Q-E}\left(\frac{\omega_Q}{\omega}\right)^q \omega\, dx\right)^{1/q}.$$

Thus if $\omega \in A_p$ and $|E| \leq \frac{1}{2}|Q|$ we obtain

$$\frac{1}{2^q A_p^q}\int_Q \left(\frac{\omega_Q}{\omega}\right)^q \omega\, dx \leq \left(\frac{1}{2}\right)^q \omega(Q) \leq \int_{Q-E}\left(\frac{\omega_Q}{\omega}\right)^q \omega\, dx.$$

Hence

$$\int_E \left(\frac{\omega_Q}{\omega}\right)^q \omega\, dx \leq a\int_Q \left(\frac{\omega_Q}{\omega}\right)^q \omega\, dx,$$

with $a = 1 - (2A_p)^{-q}$. □

THEOREM 2.2. *Let $p > 1$. Then $\omega \in A_p$ if and only if there exists a constant $C_{p,\omega}$ such that*

$$\int |Mf|^p \omega \leq C_{p,\omega}\int |f|^p \omega \quad \textit{for all } f \in L^p(\omega).$$

Proof. As in [3], setting $f = \chi_Q \omega^{-1/p-1}$, we see that (1.2) implies that $\omega \in A_p$. To see that A_p is sufficient for (1.2), we apply a Calderón-Zygmund decomposition to Mf, and choosing $R_k = 2^{k(n+1)}$, we write $D_k = \{Mf > R_k\} = \bigcup_j Q_j^k$, where the Q_j^k are the maximal dyadic cubes satisfying

$$2^n R_k \geq \frac{1}{|Q_j^k|}\int_{Q_j^k} |f| > R_k.$$

Notice that $|Q_j^k \cap D_{k+1}| \leq \frac{1}{2}|Q_j^k|$ by our choice of R_k. Let $E_k = D_k - D_{k+1}$. Then

$$[1 - 2^{-(n+1)p}]\sum_k R_k^p \omega(D_k) \leq \int |Mf|^p \omega \leq 2^{(n+1)p} \sum_k R_k^p \omega(D_k)$$

$$\leq 2^{(n+1)p} \sum_k R_k^{p-1} \sum_j \frac{1}{|Q_j^k|} \int_{Q_j^k} |f| \omega(Q_j^k)$$

$$= 2^{(n+1)p} \sum_k R_k^{p-1} \sum_j \sum_{l=0}^{\infty} \int_{Q_j^k \cap E_{k+l}} |f| \omega_{Q_j^k}$$

$$\leq 2^{(n+1)p} \sum_k R_k^{p-1} \sum_j \sum_{l=0}^{\infty} \left(\int_{Q_j^k \cap E_{k+l}} |f|^p \omega \, dx \right)^{1/p}$$

$$\cdot \left(\int_{Q_j^k \cap E_{k+l}} \left(\frac{\omega_{Q_j^k}}{\omega} \right)^q \omega \, dx \right)^{1/q}.$$

By repeated application of Lemma 2.1, we see that

$$\int_{Q_j^k \cap D_{k+l}} \frac{1}{\omega^q} \omega \, dx \leq a^l \int_{Q_j^k} \frac{1}{\omega^q} \omega \, dx.$$

Consequently,

$$\int |Mf|^p \omega \leq 2^{(n+1)p} \sum_k R_k^{p-1} \sum_j \sum_{l=0}^{\infty} \left(\int_{Q_j^k \cap E_{k+l}} |f|^p \omega \, dx \right)^{1/p} \left(a^l A_p^q \omega(q_j^k) \right)^{1/q}$$

$$\leq 2^{(n+1)p} \sum_k R_k^{p-1} \sum_{l=0}^{\infty} \left(a^l A_p^q \right)^{1/q} \left(\int_{E_{k+l}} |f|^p \omega \right)^{1/p} (\omega(D_k))^{1/q}$$

$$= 2^{(n+1)p} A_p \sum_{l=0}^{\infty} a^{l/q} \sum_k \left(\int_{E_{k+l}} |f|^p \omega \right)^{1/p} (R_k^p \omega(D_k))^{1/q}$$

$$\leq 2^{(n+1)p} A_p \sum_{l=0}^{\infty} a^{l/q} \left(\int_{\mathbf{R}^n} |f|^p \omega \right)^{1/p} \left(\sum_k R_k^p \omega(D_k) \right)^{1/q}.$$

We therefore obtain

$$\int |Mf|^p \omega \leq C_{p,\omega} \int |f|^p \omega$$

with

$$C_{p,\omega} = C_p A_p^p (1 - a^{1/q})^{-p} \leq C_p A_p^{p(q+1)}. \qquad \square$$

3. The L log L **theorem.** In this section we give a necessary and sufficient condition on ω for the maximal function to be bounded from L log $L(\omega)$ to $L^1(\omega)$ locally. By a Calderón-Zygmund decomposition of a dyadic cube Q we shall mean a collection of dyadic subcubes $\{Q_j^k\}_{k \geq 0}$ of

Q such that (a) in $Q_j^k \cap \operatorname{int} Q_l^k = \varnothing$ if $j \neq l$, (b) each Q_j^{k+1} is contained in some Q_l^k, $k \geq 0$, and (c) the 0-th generation of cubes consists of the single cube Q. In such a situation, we write $D_k = \bigcup_j Q_j^k$ and $E_k = D_k - D_{k+1}$.

THEOREM 3.1. *The following conditions on a locally integrable ω are equivalent*:

(i) *There exists a constant C_ω such that whenever f is supported in a dyadic cube Q, then*

$$\int_Q Mf\omega \leq C_\omega \int_Q |f| \log^+ |f| \omega + C_\omega \omega(Q).$$

(ii) (A^{**}) *There exists an $\varepsilon > 0$ and a C_ω such that whenever Q is a dyadic cube and $\{Q_j^k\}$ is a Calderón-Zygmund decomposition of Q, then*

$$\int_Q \exp\left\{ \varepsilon \sum_{k,j} \frac{\omega\left(Q_j^k \cap E_k\right)}{\omega(x)|Q_j^k|} \chi_{Q_j^k}(x) \right\} \frac{\omega\, dx}{\omega(Q)} \leq C_\omega.$$

(iii) *There exists an $\varepsilon > 0$ and a C_ω such that whenever T is a positive linear operator satisfying $|Tf(x)| \leq Mf(x)$, and Q is a dyadic cube, then*

$$\int_Q \exp\left\{ \varepsilon \frac{T^*(\chi_Q \omega)(x)}{\omega(x)} \right\} \frac{\omega\, dx}{\omega(Q)} \leq C_\omega.$$

(*Here, T^* is the adjoint of T with respect to $L^2(dx)$, i.e. $\int (Tf) g\, dx = \int f(T^* g)\, dx$.*)

Proof. We shall show (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iii). Suppose Q is a dyadic cube and T is a positive linear operator with $|Tf(x)| \leq Mf(x)$. Let $E = \{x \in Q | T^*(\chi_Q \omega)(x) > \omega(x)\}$. Let

$$g = e^{\varepsilon(T^*(\chi_Q \omega)/\omega)} \cdot \frac{\omega}{T^*(\chi_Q \omega)} \chi_E,$$

for a small ε to be determined later. Then

$$\int_E \exp\left\{ \varepsilon \frac{T^*(\chi_Q \omega)}{\omega} \right\} \omega\, dx$$

$$= \int g \frac{T^*(\chi_Q \omega)}{\omega} \omega\, dx = \int_Q Tg\omega\, dx \leq \int_Q Mg\omega\, dx$$

$$\leq C_\omega \int_Q g \log^+ g\omega\, dx + C_\omega \omega(Q) \quad \text{(by (i))}$$

$$\leq C_\omega \int_E e^{\varepsilon(T^*(\chi_Q \omega)/1)} \frac{\omega}{T^*(\chi_Q \omega)} \varepsilon \frac{T^*(\chi_Q \omega)}{\omega} \omega\, dx + C_\omega \omega(Q).$$

If we now choose ε such that $C_\omega \varepsilon < 1$ we obtain (iii).

(iii) $\Rightarrow$ (ii). If Q is a dyadic cube and $\{Q_j^k\}$ is a Calderón-Zygmund decomposition of Q, let

$$Tf(x) = \sum_k \sum_j \frac{1}{|Q_j^k|} \int_{Q_j^k} f(t)\, dt\, \chi_{Q_j^k \cap E_k}(x).$$

Then clearly $|Tf(x)| \leq Mf(x)$ and

$$T^*(\chi_Q \omega)(x) = \sum_{k,j} \frac{\omega\left(Q_j^k \cap E_k\right)}{|Q_j^k|} \chi_{Q_j^k}(x).$$

Thus (ii) is a special case of (iii).

(ii) $\Rightarrow$ (i). We assume that f is supported inside a dyadic cube Q and proceed as in the positive part of Theorem 2.2 with the same Calderón-Zygmund decomposition used there. Then the collection $\{Q_j^k \cap Q\}_{k \geq 1}$ together with Q forms a Calderón-Zygmund decomposition of Q in the sense above. So we have

$$\begin{aligned}
\int_Q Mf\omega\, dx &\leq C \sum_{k=1}^{\infty} R_k \omega(E_k \cap Q) + C\omega(Q) \\
&\leq C \sum_{k=1}^{\infty} \sum_j \frac{1}{|Q_j^k|} \int_{Q_j^k} |f| \omega\left(Q_j^k \cap Q \cap E_k\right) + C\omega(Q) \\
&\leq C \int_Q |f(x)| \left\{ \sum_{k=1}^{\infty} \sum_j \frac{\omega\left(Q_j^k \cap Q \cap Q_k\right)}{|Q_j^k \cap Q|} \chi_{Q_j^k \cap Q}(x) \right\} dx + C\omega(Q) \\
&\leq C \int_Q \frac{|f|}{\varepsilon} \log^+ \frac{|f|}{\varepsilon} \omega\, dx \\
&\quad + C \int_Q \exp\left\{ \varepsilon \sum_{k,j} \frac{\omega\left(Q_j^k \cap Q \cap E_k\right)}{\omega(x) |Q_j^k \cap Q|} \chi_{Q_j^k \cap Q}(x) \right\} \omega\, dx + C\omega(Q) \\
&\leq C_\omega \int_Q |f| \log^+ |f| \omega\, dx + C_\omega \omega(Q),
\end{aligned}$$

by Young's inequality and (ii). We have finished the proof of the theorem. $\square$

REMARK. We may re-work the above proof in the case of L^p to obtain a condition similar to A^{**} (involving Calderón-Zygmund decompositions of cubes, but with the exponential replaced by an L^q-norm) which would necessarily be equivalent to A_p. As noted in the introduction, the equivalence fails in the $L \log L$ case and A^{**} turns out to be strictly stronger than A^*. See the following section.

4. A comparison of A^{**} with A^* and related conditions. While it is clear that $A_1 \Rightarrow A^{**} \Rightarrow A^*$, we give in this section some examples to show that the reverse implications do not hold. We show the equivalence between A^* and the growth condition $A_p = O(1/(p-1))$, and show that no growth condition $A_p = O((1/(p-1))^\beta)$ with $\beta > 0$ is sufficient to imply A^{**}. Finally, although an inequality of the form $\|Mf\|_{M^p(\omega)} \leq (C/(p-1))\|f\|_{L^p(\omega)}$ is sufficient for ω to be in A^{**} by Yano's theorem [5], we give an example to show that such an inequality is not necessary.

If $0 < \alpha < \infty$ and any of the equivalent conditions of Proposition 4.1 below is satisfied, we say that ω belongs to A_α^*.

PROPOSITION 4.1. *Let $0 < \alpha < \infty$. Then the following conditions on a locally integrable ω are equivalent*:

(i) *There exists an $\varepsilon > 0$, and a C_ω such that*

$$\sup_{Q \text{ dyadic}} \int_Q \exp\left(\frac{\varepsilon\omega_Q}{\omega}\right)^\alpha \frac{\omega\, dx}{\omega(Q)} = C_\omega < \infty.$$

(ii) *There exists a C_ω such that*

$$\omega\{Mf > \lambda\} \leq C_\omega \int \frac{|f|}{\lambda}\left[1 + \log^+\left(\frac{|f|}{\lambda}\right)\right]^{1/\alpha} \omega.$$

(iii) $A_p(\omega) = O((1/(p-1))^{1/\alpha})$ *as $p \downarrow 1$.*

Proof. We show the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and, for simplicity, treat only the case $\alpha = 1$.

(i) $\Rightarrow$ (ii). Suppose $\varepsilon > 0$ is such that

$$\sup_Q \int_Q \exp\left(\frac{\varepsilon\omega_Q}{\omega}\right)\frac{\omega\, x}{\omega(Q)} \leq C_\omega.$$

By Young's inequality, we see that

$$(4.1)\quad \frac{1}{|Q|}\int_Q |f| = \int_Q \frac{\omega_Q}{\omega}|f|\frac{\omega\, dx}{\omega(Q)}$$
$$\leq C\int_Q \exp\left(\frac{\varepsilon\omega_Q}{\omega}\right)\frac{\omega\, dx}{\omega(Q)} + C_\varepsilon\int_Q |f|(1 + \log^+|f|)\frac{\omega\, dx}{\omega(Q)}.$$

By homogeneity it suffices to show that

$$\omega\{Mf > \lambda_0\} \le \int |f|(1 + \log^+|f|)\,\omega\,dx$$

for some fixed λ_0, which we take to be $C_\omega C + C_\varepsilon$. Now, by (4.1), $\{Mf > \lambda_0\} = \bigcup Q_j$, where the Q_j are disjoint dyadic cubes satisfying $\int_{Q_j} |f|(1 + \log^+|f|)\,\omega\,dx/\omega(Q) > 1$. Hence

$$\omega\{Mf > \lambda_0\} = \omega\Big(\bigcup Q_j\Big) \le \int |f|(1 + \log^+|f|)\,\omega\,dx.$$

(ii) $\Rightarrow$ (iii). Using the inequality $a(1 + \log^+ a) \le (C/(p-1))a^p$ (valid when $a \ge 1/2$ and $p > 1$) and combining it with the fact that if $\omega\{Mf > \lambda\} \le (A/\lambda^p)\int |f|^p\omega$, then $A_p(\omega) \le C^p A^{1/p}$, we see the result immediately.

(iii) $\Rightarrow$ (i). Expanding out the exponential,

$$\begin{aligned}\int_Q \exp\left(\frac{\varepsilon\omega_Q}{\omega}\right)\frac{\omega\,dx}{\omega(Q)} &= \int_Q \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\varepsilon\omega_Q}{\omega}\right)^k\frac{\omega\,dx}{\omega(Q)}\\ &\le 1 + \varepsilon + \sum_{k=2}^{\infty}\frac{\varepsilon^k}{k!}A_{k'}(\omega)^k\\ &\le 1 + \varepsilon + \sum_{k=2}^{\infty}\frac{\varepsilon^k}{k!}C_\omega^k k^k \le C \quad \text{if } \varepsilon \le (2eC_\omega)^{-1}. \qquad \square\end{aligned}$$

We remark here that it can also be shown that the condition A_α^* is equivalent to the seemingly stronger condition

$$\sup_Q \int_Q \exp\left\{\frac{\varepsilon M(\omega\chi_Q)}{\omega}\right\}^\alpha \frac{\omega\,dx}{\omega(Q)} < \infty.$$

For $j \in \mathbf{Z}$, let $I_j = [2^{-j-1}, 2^{-j})$, and from now on we consider weights of the form

$$\omega(x) = \sum_{j\in\mathbf{Z}} \omega_j \chi_{I_j}(x).$$

We shall restrict ourselves to sequences satisfying $\omega_k = \omega_0$, $k \le 0$, $\omega_j \downarrow 0$ as $j \to \infty$, and $\omega_j \le C\omega_{j+1}$ for all j.

LEMMA 4.2. *Let $\omega(x)$ be a weight of the above form.*
(a) $\omega \in A_\alpha^* \Leftrightarrow \exists\, \varepsilon > 0$, C_ω *such that*

$$\sum_{k=j}^{\infty} \exp\left\{\varepsilon\left(\frac{\omega_j}{\omega_k}\right)\right\}^\alpha \frac{\omega_k}{2^k} \le C_\omega \frac{\omega_j}{2^j}$$

(b) $\omega \in A^{**} \Rightarrow \exists\, \varepsilon > 0$, C_ω *such that*

$$\sum_{k=0}^{\infty} \exp\left\{ \frac{\varepsilon}{\omega_k} \sum_{l=0}^{k} \omega_l \right\} \frac{\omega_k}{2^k} \le C_\omega .$$

Proof. (a) We need only check the A_α^* condition on dyadic intervals of the form $[0, 2^{-m})$, $m \in \mathbf{Z}$, since ω is constant on all others. Also, because of the form of ω, only the positive integers are relevant. On such an interval $I = [0, 2^{-m})$, $m \ge 0$, $\omega(I) \approx \omega_m / 2^m$ and $\omega_I \approx \omega_m$, and the expression for A_α^* follows.

(b) Since $(1/x)\int_0^x |f(t)|\, dt \le CMf(x)$, $\omega \in A^{**}$ implies that

$$\int_0^1 \frac{1}{x} \int_0^x |f(t)| dt\, \omega(x)\, dx = \int_0^1 \left(\int_t^1 \frac{\omega(x)}{x} dx \right) |f(t)| dt$$
$$\le C_\omega \int_0^1 |f| \log^+ |f| \omega + C_\omega$$

whenever $\operatorname{supp} f \subseteq [0, 1]$. By duality, we obtain

$$\int_0^1 \exp\left\{ \frac{\varepsilon}{\omega(t)} \int_t^1 \frac{\omega(x)}{x} dx \right\} \omega(t)\, dt \le C_\omega$$

which reduces to the above expression. □

PROPOSITION 4.3. *Let* $\omega_j = (j+1)^{-\beta}$ *where* $0 \le \beta < \infty$.
(a) $\omega \in A_\alpha^* \Leftrightarrow \alpha\beta \le 1$
(b) $\omega \in A^{**} \Leftrightarrow \beta < 1$
(c) $\omega \in A_1 \Leftrightarrow \beta = 0$.
Consequently no two of the conditions A_1, A^{**}, $A_1^* = A^*$ *coincide.*

Proof. (c) is obvious; so is (a) and the forward implication of (b) once we have applied the previous lemma. To prove the reverse implication of (b), we reverse the steps of part (b) of the lemma. Let

$$M_0 f(x) = \sup_{\substack{x \in I \\ |I| < x}} \frac{1}{|I|} \int_I |f(t)| dt,$$

and for $n \ge 1$,

$$M_n f(x) = \sup_{\substack{x \in I \\ 2^{n-1} x \le |I| < 2^n x}} \frac{1}{|I|} \int_I |f(t)| dt.$$

(Here, we are assuming that $\operatorname{supp} f \subseteq [0, 2^{-m})$ for some integer m, ω being constant on all other dyadic cubes.) Then $Mf(x) \leq \sum_{n=0}^{\infty} M_n f(x)$. Now

$$\int_0^{2^{-m}} M_0 f(x)\omega(x)\,dx = \sum_{k=m}^{\infty} \omega_k \int_{I_k} M_0 f(x)\,dx$$
$$\leq C \sum_{k=m}^{\infty} \omega_k \left(\int_{I_k} |f| \log^+ |f|\,dx + 2^{-k} \right)$$
$$\leq C \int_0^{2^{-m}} |f| \log^+ |f| \omega\,dx + C\omega([0, 2^{-m})),$$

by the unweighted $L \log L$ theorem and the property $\omega_{j+1} \leq \omega_j \leq C\omega_{j+1}$ of ω. Now let $\tilde{M}f(x) = \sum_{n=1}^{\infty} M_n f(x)$; notice that

$$M_n f(x) \leq \frac{1}{2^{n-1}x} \int_0^{2^{n+1}x} |f(t)|\,dt.$$

Hence,

$$\int_0^{2^{-m}} \tilde{M}f(x)\omega(x)\,dx \leq \int_0^{2^{-m}} \left(\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \int_{t/2^{n+1}}^{2^{-m}} \frac{\omega(x)}{x}\,dx \right) |f(t)|\,dt$$
$$\leq C_\varepsilon \int_0^{2^{-m}} |f| \log^+ |f| \omega\,dx$$
$$+ C_\varepsilon \int_0^{2^{-m}} \exp\left\{ \frac{\varepsilon}{\omega(t)} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \int_{t/2^{n+1}}^{2^{-m}} \frac{\omega(x)}{x}\,dx \right\} \omega(t)\,dt.$$

With a little calculation, the reader may verify that the last term is dominated by $\omega(0, 2^{-m})$ in the case $\omega_j = (j+1)^{-\beta}$, $0 < \beta < 1$. □

EXAMPLE 4.4. For each α, $1 < \alpha < \infty$, there exists $\omega \in A_\alpha^* - A^{**}$.

Proof. Let $\lambda = 2^{-1/\alpha}$, and let $\omega_j = \lambda^r$ when $n_{r-1} < j \leq n_r$. Here, n_r is an increasing sequence of positive integers, $n_0 = 0$; let $\Delta_r = n_r - n_{r-1} = \#\{ j | \omega_j = \lambda^r \}$, so that $n_r = \Delta_1 + \cdots + \Delta_r$.

Let $0 = r_0 < r_1 < r_1 \cdots$ be an increasing sequence of positive integers ($r_j = j$ will do) and for $\sum_{j=1}^{l-1} r_j < r \leq \sum_{j=1}^{l} r_j$, let $\Delta_r = 2^{2r_1 + 2r_2 + \cdots + 2r_{l-1} + r_l + 2 - r}$. Thus $\{\Delta_r\}$ is a rearrangement of the geometric progression $\{2^r\}$, and clearly any set of j consecutive Δ_r's must contain one of size greater than or equal to 2^{j+1}; thus

$$n_{q+j} - n_q = \Delta_{q+1} + \cdots + \Delta_{q+j} \geq 2^{j+1} \quad \forall j, \forall q. \tag{4.2}$$

On the other hand,

$$\sum_{j=1}^{r_1 + \cdots + r_k} \Delta_j = 4(2^{r_1 + \cdots + r_k} - 1)$$

and therefore

$$\sum_{\substack{j=1 \\ j \neq r_1 + \cdots + r_k + 1}}^{r_1 + \cdots + r_{k+1}} \Delta_j = 4(2^{4_1 + \cdots + r_{k+1}} - 1) - 2^{r_1 + \cdots + r_{k+1} + 1}$$

$$= 2^{r_1 + \cdots + r_{k+1} + 1} - 4 = \Delta_{r_1 + \cdots + r_k + 1} - 4.$$

Therefore

$$\Delta_{r_1 + \cdots + r_k + 1} \geq \sum_{\substack{j=1 \\ j \neq r_1 + \cdots + r_k + 1}}^{r_1 + \cdots + r_{k+1}} \Delta_j \quad \forall k. \tag{4.3}$$

Now our sequence ω_j satisfies the conditions of Lemma 4.2; we shall use (4.2) to prove that $\omega \in A_\alpha^*$, and (4.3) to prove that $\omega \notin A^{**}$.

To see that $\omega \in A_\alpha^*$, we suppose that $n_{q-1} < j \leq n_q$ and compute the expression in Lemma 4.2.(a):

$$\sum_{k=j}^{\infty} \exp \varepsilon \left(\frac{\omega_j}{\omega_k} \right)^{\alpha} \frac{\omega_k}{2^k} \Big/ \frac{\omega_j}{2^j} = \sum_{k=j}^{n_q} 2^{j-k} e^{\varepsilon} + \frac{2^j}{\omega_j} \sum_{p=q+1}^{\infty} \sum_{k=n_{p-1}+1}^{n_p} e^{\varepsilon 2^{p-q}} \frac{\omega_k}{2^k}$$

$$\leq 2e^{\varepsilon} + 2^{n_q} \sum_{p=q+1}^{\infty} e^{\varepsilon 2^{p-q}} 2^{-n_{p-1}}$$

$$\leq 2e^{\varepsilon} + \sum_{j=0}^{\infty} 2^{n_q - n_{q+j}} e^{\varepsilon 2^{j+1}}$$

$$\leq C \quad \text{if } \varepsilon \leq \frac{\log 2}{2}, \qquad \text{by (4.2)}.$$

If ω were in A^{**}, Lemma 4.2.(b) would give the existence of C_ω and ε such that

$$\sum_{k=1}^{\infty} \frac{\omega_k}{2^k} \exp \left\{ \varepsilon \sum_{j=1}^{k} \frac{\omega_j}{\omega_k} \right\}$$

$$= \sum_{p=1}^{\infty} \sum_{k=n_{p-1}+1}^{n_p} \frac{\lambda^p}{2^k} e^{\varepsilon(k - n_{p-1})} \exp \left\{ \varepsilon \sum_{q=1}^{p-1} \Delta_q \lambda^{q-p} \right\}$$

$$= \sum_{p=1}^{\infty} \frac{\lambda^p}{2^{n_{p-1}}} e^{\varepsilon(\Delta_1 \lambda^{1-p} + \cdots + \Delta_{p-1} \lambda^{-1})} \sum_{k=n_{p-1}+1}^{n_p} \frac{e^{\varepsilon(k - n_{p-1})}}{2^{k - n_{p-1}}}$$

$$\leq C_\omega.$$

For $\varepsilon \leq (\log 2)/2$,

$$C_1 \leq \sum_{k=n_{p-1}+1}^{n_p} \frac{e^{\varepsilon(k - n_{p-1})}}{2^{k - n_{p-1}}} \leq C_2,$$

and so if ω were in A^{**}, we would have that for all ε sufficiently small

$$\alpha_p(\varepsilon) = \frac{\lambda^p}{2^{n_{p-1}}} e^{\varepsilon(\Delta_1 \lambda^{1-p} + \cdots + \Delta_{p-1}\lambda^{-1})}$$

satisfies $\alpha_p(\varepsilon) < 1$ for all but finitely many p. We shall now show that for arbitrarily small ε, $\log \alpha_p(\varepsilon)$ can be nonnegative for infinitely many p, and so ω cannot be in A^{**}. Observe that

$$\frac{\log \alpha_p(\varepsilon)}{\Delta_1 + \cdots + \Delta_{p-1}} = \frac{\varepsilon\left(\Delta_1 \lambda^{1-p} + \cdots + \Delta_{p-1}\lambda^{-1}\right)}{\Delta_1 + \cdots + \Delta_{p-1}} - \log 2 - \frac{p \log(1/\lambda)}{n_{p-1}},$$

and so if we choose $p = r_1 + r_2 + \cdots + r_{k+1} + 1$, (4.3) implies that

$$\frac{\log \alpha_p(\varepsilon)}{\Delta_1 + \cdots + \Delta_{p-1}} \geq \frac{\varepsilon\left(\Delta_1 \lambda^{1-p} + \cdots + \Delta_{p-1}\lambda^{-1}\right)}{2\Delta_{r_1 + \cdots + r_k + 1}} - \log 2 - \frac{p \log(1/\lambda)}{n_{p-1}}$$

$$\geq \frac{\varepsilon}{2}\lambda^{-r_{k+1}} - \log 2 - \frac{p\log(1/\lambda)}{n_{p-1}} \geq 0$$

if $k = k(\varepsilon)$ is chosen sufficiently large. This completes the example. □

EXAMPLE 4.5. Let $\omega_j = (j+1)^{-\beta}$, with $0 < \beta < 1$. While $\omega \in A^{**}$ by Proposition 4.3, ω does not satisfy $\int |Mf|^p \omega \leq C^p/(p-1)^p \int |f|^p \omega$ as $p \downarrow 1$.

Proof. Let $g(x) = (\log(1/x))^{\beta/q}\ (1/p + 1/q = 1)$. Then $\|g\|_{L^q(\omega)} \cong 1$, and

$$\|Mf\|_{L^p(\omega)} \geq C \sup\left\{\left|\int (Tf) g\, \omega\, dx\right| : \|g\|_{L^q(\omega)} \leq 1\right\}$$

where

$$Tf(x) = \frac{1}{x}\int_0^{2x} f(t)\, dt.$$

But

$$\int (Tf) g\, \omega\, dx = \int f(\tilde{T}g)\omega\, dx$$

where

$$\tilde{T}g(t) = \frac{1}{\omega(t)}\int_{t/2}^1 g(x)\omega(x)\frac{dx}{x} \geq Cq\left(\log\frac{2}{t}\right)^{\beta/q+1},$$

with our choice of g. Hence

$$(p-1)M_p \geq \frac{C}{q}\|T^*g\|_{L^q(\omega)} \geq C\left(\int_0^1 \left(\log\frac{2}{t}\right)^q dt\right)^{1/q},$$

(where $M_p = \sup\{\|Mf\|_{L^p(\omega)} | \|f\|_{L^p(\omega)} \leq 1\}$), and, since $\log(2/t) \notin L^\infty$, M_p is not $O(1/(p-1))$ as $p \downarrow 1$.

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CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CA 91125

AND

UNIVERSITY OF CALIFORNIA, LOS ANGELES
LOS ANGELES, CA 90024