Target Spaces of Non-Geometric String Backgrounds

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Zusammenfassung

Diese Doktorarbeit diskutiert zwei Aspekte von nicht-geometrischen Stringhintergründen, welche durch links-rechts asymmetrische konforme Feldtheorien definiert sind.

Mithilfe von simple currents werden vierdimensionale links-rechts asymmetrische Typ IIB Gepnermodelle mit $\mathcal{N} = 1$ Supersymmetrie konstruiert. Die Analyse der masselosen Vektormultiplets weist die Kombinatorik einer zugrundeliegenden complete intersection Calabi-Yau-Mannigfaltigkeit auf. Diese Beobachtung legt nahe, dass diese asymmetrischen Gepnermodelle einem voll rückgekoppelten $\mathcal{N} = 1$ Vakuum in der geeichten $\mathcal{N} = 2$ Supergravitationstheorie entsprechen, welche man durch eine Flusskompaktifizierung von Typ IIB auf der zugrundeliegenden Calabi-Yau-Mannigfaltigkeit erhält. Wir verifizieren diese Hypothese, indem wir das Spektrum von mehreren asymmetrischen Gepnermodellen mit den notwendigen Bedingungen vergleichen, welche beim Brechen von $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ in der geeichten Supergravitationstheorie existieren. Um weitere Beweise für die vermutete Korrespondenz zwischen asymmetrischen konformen Feldtheorien und geeichten Supergravitationstheorien zu finden, werden mithilfe einer stochastischen Computersuche sämtliche links-rechts asymmetrische Gepnermodelle in vier, sechs und acht Dimensionen mit mindestens acht Superladungen klassifiziert. Alle Modelle können wenigen verschiedenen Klassen zugeordnet werden, welche sich durch dimensionale Reduktion, asymmetrische Orbifolds mit $(-1)^{F_L}$ Divisor, Eicherweiterung und, am wichtigsten, den super-Higgs Effekt erklären lassen. Die Gepnermodelle in letzterer Klasse zeigen dabei die charakteristischen Merkmale einer gebrochenen geeichten Supergravitationstheorie.

Der zweite Teil der Doktorarbeit widmet sich dem nicht-assoziativen Sternprodukt, welches für nicht-geometrische R-Fluss Hintergründe auftreten soll. Da das Sternprodukt im Sektor der geschlossenen Strings auftritt, gab es in letzter Zeit Bemühungen, die Diffeomorphismensymmetrie der Stringtheorie mit dem Sternprodukt in Einklang zu bringen. In der mathematischen Literatur wurden schon erfolgreich Sternverallgemeinerungen von Tensoren, Zusammenhängen und der Riemann Krümmung konstruiert. Wir führen diese Bemühungen fort, indem wir mehrere Lücken füllen und zum Beispiel die Torsion, eine Metrik und ihr Sterninverses definieren. Wenn es darum geht, eine Metrik zu definieren, begegnen wir großen Schwierigkeiten, welche man allesamt auf die nicht-Assoziativität des Sternproduktes zurückführen kann.

Diese Doktorarbeit beruht hauptsächlich auf den Publikationen [1, 2, 3] sowie dem Review [4] des Autors. Kleinere Abschnitte basieren auf [5, 6] und dem Review [7].

Abstract

This PhD thesis will discuss two aspects of non-geometric string backgrounds defined by left-right asymmetric conformal field theories.

Using simple currents we construct four-dimensional left-right asymmetric Gepner models of type IIB with $\mathcal{N} = 1$ spacetime supersymmetry. An analysis of the massless vector multiplets reveals the combinatorics of an underlying complete intersection Calabi-Yau. This observation suggests that these asymmetric Gepner models might correspond to a fully backreacted $\mathcal{N} = 1$ vacuum in the $\mathcal{N} = 2$ gauged supergravity, which is the effective action of the type IIB flux compactification on the underlying Calabi-Yau manifold. We check this conjecture by comparing the spectrum of several asymmetric Gepner models with the necessary conditions for an $\mathcal{N} = 2 \to \mathcal{N} = 1$ breaking in the associated gauged supergravity. To provide further evidence for the conjectured correspondence between asymmetric CFTs and gauged supergravities, we classify left-right asymmetric Gepner models in four, six and eight dimensions with at least eight supercharges using a stochastic computer search. We can sort all models into few different classes that can be explained as dimensional reductions, asymmetric orbifolds with an $(-1)^{F_L}$ factor, gauge enhancement and, most importantly, the super-Higgs effect. The asymmetric Gepner models in the latter class exhibit the characteristic features that are expected from a gauged supergravity.

The second part of this thesis deals with the non-associative star product that is supposed to appear in presence of a non-geometric R-flux. Since the star product appears in the closed string sector there have been recent attempts to reconcile the diffeomorphism symmetry of string theory with the star product. In the existing mathematical literature were able to construct star generalizations to tensors, connections and the Riemann curvature. We continue these efforts by filling several gaps like introducing a torsion, a metric and its star inverse. When it comes to the metric we encounter major obstacles that can be traced back to the non-associativity of the star product.

This thesis is based mainly on the publications [1, 2, 3] as well as the review [4] of the author. Minor parts are taken from [5, 6] and the review [7].

Chapter 1

Introduction

For centuries physics have followed a simple but extremely successful pattern: Empiric experiments yielded ever new results, which then had to be explained and put into a consistent mathematical theory. However, since the 20th century mathematical knowledge has increased to such an extent that the traditional order, theory follows experiment, has been turned around in not just a few cases. One of the major examples for such a turnaround is probably Albert Einstein, who found his theory of general relativity by relying solely on gedankenexperiments and mathematical reasoning, rather than specific experiments. Only afterwards his theory was experimentally confirmed to be superior to Newton's law of gravity. Another famous example is the anticipation of the antiparticle of the electron by Paul Dirac, found four years later or the prediction of the neutrino by Wolfgang Pauli. Nowadays particle physicists have agreed on a certain quantum field theory that is called the standard model. This name "standard model" clearly reflects its enormous success. The standard model for instance correctly predicted the $W^{\pm}$, Z^0 and Higgs bosons long before they were discovered. In case of the Higgs boson the detection needed the gigantic LHC to finally find it over 50 years after its first appearance in a theory paper.

Nevertheless there are experiments that point towards physics beyond the standard model, like the non-zero mass of the neutrinos, the existence of dark matter or the dominance of matter over antimatter. But these results can most likely be explained by other quantum field theories that modify the standard model slightly, for instance by adding more particles. Physicists have not agreed on such a modified standard model so far simply because we are lacking an experimental "smoking gun" that singles out a particular quantum field theory to be the correct one.

But even when the standard model is adjusted accordingly to some future experimental results, the common theories have problems that suggest the existence of a more fundamental theory. The first problem is that realistic quantum field theories are not "UV-complete". This means that at a certain (usually very high) energy scale the theory predicts its own breakdown since the perturbative expansion becomes divergent. Furthermore when treating Einstein gravity as quantum field theory there are clear signals that gravity is only the effective theory of an unknown microscopic theory that governs the physics beyond roughly the Planck scale. Since the energies where the known theories break down are far from experimentally reachable in a laboratory, the only hope to find experimental hints for the unknown physics lies in cosmological observations, e.g. from the cosmic microwave background, dark energy or black holes.

But right now, surprising experimental data is neither available nor expected soon. Consequently, current theoretical physicists cannot follow the old pattern but can just stick to the second approach, thus mathematical arguments or educated guesses, to find a more fundamental theory. Although originally formulated as possible theory of the strong interactions, string theory can in hindsight be justified exactly as this: It is a plausible new ansatz not to explain experiments, but rather to solve the problems that the common theories have. Since string theory has turned out to be much more than just a plausible ansatz, it is still, 50 years after its beginning, a highly active research area and it has become the prime candidate for a unified theory of all forces and matter.

Let us explain the basic concept of string theory: The UV divergences of quantum field theory appear at high energies, thus small distances, and come mathematically from point-like interactions. A possible solution to avoid divergences is therefore to smear out the particles and interactions by giving all particles a volume. To start easily, we smear the particles only in one dimension such that they become strings or, from the spacetime viewpoint, the former worldlines become worldsheets. Since the action of a particle in spacetime is simply the length of its worldline, a possible action for a string is the area of its worldsheet

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'}\int_\Sigma dA\,.$$

This action is called Nambu-Goto action. Here, $\frac{1}{2\pi\alpha'} \sim M_s^2$ is a constant that encodes the energy of the string per length unit and Σ is the worldsheet. This action is hard to quantize since it contains a square root when choosing a particular parametrization of the worldsheet. One therefore rather uses the classically equivalent Polyakov action, that in conformal gauge reads

$$S_{\text{P}} = -\frac{1}{4\pi\alpha'}\int_\Sigma d\sigma dt\ \eta_{\mu\nu}\,\gamma^{ab}\,\partial_a X^\mu \partial_b X^\nu\,.$$

The variables σ and t parametrize the worldsheet Σ equipped with a flat Minkowski metric γ_{ab}. The field $X^\mu = X^\mu(\sigma,\tau)$ embeds the worldsheet into the flat target space that has the metric $\eta_{\mu\nu}$. When introducing left- and right-moving coordinates $z = t + i\sigma$ and $\overline{z} = t - i\sigma$, the action becomes

$$S \sim \int_\Sigma dz d\overline{z}\ \eta_{\mu\nu}\,\partial X^\mu \overline{\partial} X^\mu\,.$$

Later it will be very important that this action is manifestly invariant under $z \leftrightarrow \overline{z}$ and therefore left-right symmetric.

The Polyakov action in the above form is invariant under the angle preserving conformal transformations. In particular, rescalings are part of the conformal group such that any energy scale is forbidden. When quantizing the action one gets a conformal field theory that, since the conformal algebra has an infinite number of generators, is fully solvable. Furthermore, also stringy loop corrections from worldsheets with higher genus turn out to be UV finite, such that string theory is well defined at any energy scale in contrast to most quantum field theories.

When looking at the Hilbert space of string theory one has to distinguish two sectors: The first one describes open strings while the other one describes closed strings. In the closed sector there appear three massless fields, the symmetric traceless $h_{\mu\nu}$, the antisymmetric Kalb-Ramond field $B_{\mu\nu}$ and the scalar ϕ, called the dilaton. When computing the effective action for the $h_{\mu\nu}$ field one finds the usual Einstein-Hilbert action with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Therefore $h_{\mu\nu}$ is the graviton that measures fluctuations of the metric. Although from now on we will restrict ourselves to the closed strings, let us shortly mention what happens in the open string sector. There one finds a field A_μ that behaves like a massless spin one field. To sum up, string theory turns out to be a UV finite, unified theory of quantum gravity and gauge fields.

Of course there are phenomenological problems with string theory. For instance, the theory is only anomaly free in 26 dimensions and the field content is clearly far from being realistic. For this reason one usually investigates the supersymmetric extension of the Polyakov action. The resulting superstring lives in ten dimensions and also has spacetime fermions. Huge efforts are put into the question on how to compactify the 10D theory to get a realistic four-dimensional model. Since we do not address these issues in this thesis, we will not discuss them any further here.

Let us instead turn to a peculiarity of string theory. Being extended objects, the way strings probe the spacetime is intriguingly different from the way usual point particles do. To give an example take one direction of the target space to be a circle of radius R. As extended objects closed strings might wind around this circle and cannot be shrunk to one point. Consequently, apart from the quantized momentum, closed strings need another quantum number that counts how often the string is wound around the circle. Clearly, this is not necessary for point particles that cannot wind around a circle. There is another major difference in the way point particles and strings probe the circle. For particles, every choice of the radius R leads to a different behavior. This is not the case for string theory. It turns out that strings cannot distinguish between a circle of radius R and $R' = \frac{\alpha'}{R}$. One says that string theory on radius R is T(arget space)-dual to string theory on radius $R' = \frac{\alpha'}{R}$ since both theories can be mapped onto each other by interchanging the momentum and the winding number of the string. To realize T-duality from the worldsheet point of view, recall that one can introduce left- and right-moving coordinates $z/\overline{z} = t \pm i\sigma$ under which the string embedding function can be split as $X^\mu(z,\overline{z}) = X^\mu_L(z) + X^\mu_R(\overline{z})$. T-duality is then simply the sign inversion of the right-moving part $X^\mu_R(\overline{z}) \to -X^\mu_R(\overline{z})$ Therefore the stringy T-duality is a manifestly left-right asymmetric operation.

Related to the left-right asymmetric T-duality there is another peculiarity with strings: They can actually probe spaces that do not have a geometric description and are therefore called non-geometric backgrounds. Since these non-geometric backgrounds will be the main topic of this thesis, let us introduce them in more detail in the following.

So far we have only spoken about strings in a flat spacetime. By simply replacing the flat metric in the Polyakov action by a curved one $\eta_{\mu\nu} \to G_{\mu\nu}(X)$ one can let strings propagate in a curved spacetime. Furthermore we can also allow for non-trivial background values of the other massless fields of string theory. E.g. to have a non-trivial background Kalb-Ramond field we need to add

$$S_B = \frac{1}{4\pi\alpha'} \int_\Sigma d\sigma dt\, \epsilon^{ab} B_{\mu\nu}(X)\, \partial_a X^\mu \partial_b X^\nu$$

to the action. All the additional terms are, like the original Polyakov action, left-right symmetric and conformally invariant. The full action is a non-linear sigma-model and appeared first to describe sigma particles.

Right now only the first quantization of this action is known. This means that we can only quantize the string, thus X^μ, while all the background fields $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ etc. are fixed and classical, and from the sigma model viewpoint they are coupling constants. To have an anomaly free quantum theory one has to ensure that the β functions of these coupling constants are vanishing, since otherwise there would be an energy scale that breaks conformal invariance. In case the background fields are chosen correctly, the action defines a particular conformal field theory that describes strings moving in the given background. This suggests that there is a clear correspondence between conformal field theories and backgrounds. Unfortunately, this is not the case. Matching backgrounds and conformal field theories was successful only in rather few cases. There are fundamental and technical reasons for this:

For the superstring the major limitation is that conformal field theories cannot incorporate non-trivial background of the fermionic Ramond fields. The technical reason lies in the fact that the vertex operators for the potentials of the R fields are lacking. The usual superstring sigma models can therefore only describe how the superstring couples to the purely bosonic NS-NS background fields and consequently only NS-NS backgrounds have a corresponding conformal field theory.

Another obstacle consists in the difficulty that, given a usually highly abstract conformal field theory, identifying the background is a very hard task. Furthermore the metric and therefore the sigma model of some of the most interesting backgrounds like the Calabi-Yau compactifications cannot be explicitly written down. An exact agreement between a background and a conformal field theory has therefore been established in rather few cases and relies on usually indirect arguments. The best understood case is the flat space, toroidal compactifications, and orbifolds thereof. All of them correspond to (quotients of) free conformal field theories. More non-trivially Gepner models are the CFTs to certain hypersurface Calabi-Yau compactifications, and strings on group manifolds are described by Wess-Zumino-Witten models.

There is furthermore the large class of left-right asymmetric conformal field theories for which we do not have a target space description. Recall that the above string Polyakov action couples the background fields left-right symmetrically to the string. Starting from the Polyakov action we can therefore only obtain left-right symmetric conformal field theories. Nevertheless also genuinely left-right asymmetric conformal field theories do exist and they define valid string theory backgrounds. Being left-right asymmetric, the corresponding background cannot be the one of just a metric and maybe additionally a B-field. Lacking a metric that defines the geometry of the target space, these backgrounds are called *non-geometric* backgrounds. The attentive reader will have noticed that we encountered a left-right asymmetry before when discussing T-duality. Indeed, one way to think about non-geometric background is that they are the T-dual backgrounds to a geometric background. This means that an un-T-dualized string can propagate in its T-dual background. Since T-duality is not defined for point particles, such non-geometric backgrounds are intrinsically stringy. In this light it is questionable whether there exists an associated effective action for the string excitations since this effective action is usually derived in the point particle limit. These non-geometric string backgrounds are the main topic of this thesis. Let us explain the content and the logic of this thesis:

In the first part of the thesis we will argue that we can identify a geometric remnant in many asymmetric CFTs. This remnant is a geometric background that upon a suitable non-geometric perturbation adjusts itself such that it becomes the non-geometric background.

The idea for this identification stems from the observation that there is a connection between asymmetric toroidal orbifolds and perturbed, thus gauged, maximal supergravity theories obtained by torus compactifications. The 4D supergravity theory of string theory on a six-dimensional torus has a large global symmetry group from the isometries of the six-torus. Parts of this global symmetry can be gauged by turning a subgroup into a local symmetry. From the compactification viewpoint the gaugings correspond to a perturbation of the background geometry with additional background fluxes. But as it turns out, there are more gaugings than geometric background fluxes to turn on. These additional gaugings correspond to non-geometric background fluxes that are the T-dual fluxes to the known geometric fluxes. It is well-established that fluxes around a cycle in the internal geometry always yield a non-trivial monodromy around this circle. If we identify the monodromy of the non-geometric flux, we can build the corresponding conformal field theories by imposing the monodromy via an orbifold of the known torus conformal field theory. For non-geometric fluxes this orbifold will be genuinely left-right asymmetric.

It is natural to ask if one can relate more involved asymmetric conformal field theories to a gauged supergravity. In the publication [2] we therefore asked the question whether we can identify a corresponding gauged supergravity for four-dimensional Gepner models with left-right asymmetric simple current extensions that have $\mathcal{N} = 1$ supersymmetry. These asymmetric Gepner models should be related to an $\mathcal{N} = 2$ gauged supergravity, since pure Gepner models usually describe strings moving in a Calabi-Yau background. The identification of a geometry in Gepner models is a highly non-trivial task. We therefore dedicate a whole chapter to a detailed review of pure Gepner models with an emphasis on the geometry behind Gepner models. Having this knowledge, a close inspection of the spectra of the asymmetric Gepner models reveals an underlying Calabi-Yau structure. We review the supersymmetry breaking process from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ in gauged supergravity and deduce bounds on the spectrum of the $\mathcal{N} = 1$ vacua. We find that the spectra of the asymmetric

$\mathcal{N}=1$ Gepner models fit into the bounds one expects for the broken $\mathcal{N}=2$ supergravity of the Calabi-Yau we associated to the particular Gepner model. This motivates us to conjecture that in general asymmetric conformal field theories correspond to fully backreacted gauged supergravities. The phrase "fully backreacted" reflects that gauged supergravities are perturbed supergravities in which the backreaction of the perturbating fields is neglected. But conformal field theories represent strings moving in a fully consistent background that therefore must already have fully adjusted according to the perturbation. To a given asymmetric conformal field theory we have therefore identified a geometric remnant that under a suitable perturbation backreacts towards the non-geometric background.

The perfect setting to provide further evidence for the conjecture is asymmetric Gepner models with more supersymmetry. They should correspond to supergravity theories that, constrained by the large amount of supersymmetry, only allow for definite supersymmetry breaking patterns. In the publication [3] we therefore classify all asymmetric Gepner models in four, six and eight dimensions with more than eight supercharges. Indeed we find all models either satisfy the constraints from gauged supergravity or can be explained by other mechanisms.

Having only an indirect geometric interpretation of non-geometric backgrounds there is still the question of what the intrinsic geometry of non-geometric backgrounds is. There are hints that the non-geometric nature manifests itself in the presence of the non-associative algebra

$$[x^i, x^j] = \frac{i}{3\hbar}\ell_s^4\, R^{ijk}\, p_k \qquad \text{and} \qquad [x^i, p_j] = i\hbar\, \delta^i{}_j\,.$$

where l_s is the string length and $R^{\mu\nu\sigma}$ is a constant antisymmetric tensor called the R-flux that is the fully T-dual flux to the known $H=dB$ flux. The algebra is non-associative and violates the Jacobi identity, both by a defect term $\sim R^{ijk}$. To quantize the algebra we will use deformation quantization algebras where a given algebra is realized by introducing a star product. For the above algebra the star product reads

$$f\star g = f\cdot g + \tfrac{1}{2}i\hbar(\partial_i f\,\tilde{\partial}_p^i g - \tilde{\partial}_p^i f\,\partial_i g) + \tfrac{il_s^4}{6\hbar}R^{ijk}p_k\,\partial_i f\,\partial_j g + \dots\ ,$$

and indeed by inserting x and p we recover the above algebra. The appearance of a non-associative star product is intriguing since quantum mechanics usually forbids non-associativity by assumption and the conformal field theories of string theory are ordinary quantum theories. But so far in all calculations the contradiction could be resolved and the non-associative defects vanish by momentum conservation, the strong constraint of double field theory, or by only giving boundary terms without contribution to observables.

The second main part of this thesis is based on the publication [1] and will elaborate whether it is possible to formulate a gravity theory on a space that obeys the above commutation relations between the coordinates. A special emphasis will be put on the question whether there is a mechanism that preserves the usual diffeomorphism symmetry of closed strings. Since the graviton vertex operator for the non-geometric backgrounds is unknown, we cannot construct the star gravity theory as effective action of string theory. Instead we follow the usual steps in constructing Einstein's gravity and adjust every ingredient accordingly to the star product.

In a first step a star generalization to the diffeomorphism has to be found. To reconcile star products with a symmetry there are two approaches. The first introduces a star symmetry by changing the way the symmetry acts on fields. Since star symmetries do only exist for associative star products and the symmetry group $U(N)$, we have to stick to the second approach, called twisted symmetries. In twisted symmetries, instead of changing the symmetry, the Leibniz rule is changed, or twisted, to guarantee covariance between the symmetry and the star product. Since twisted diffeomorphisms exist and appear naturally for non-associative star products, we will take this approach. We review the known procedure to define twisted tensors, twisted connections and a twisted Riemann curvature tensor. We fill existing gaps like the introduction of covariant derivatives of vectors or the introduction of the torsion tensor. But in doing so we find that the non-associative nature of the algebra leads to severe problems when introducing a metric. Let us mention two of them: Since the placing of the brackets between the three objects in a scalar product matters, there is no unambiguous generalization of a scalar product. Furthermore we can show that an inverse can only exist in very special cases. Ignoring these fundamental problems, we nevertheless try to find hints on how to reconcile the star gravity with the expectations of string theory. We find no obvious mechanism how to restore the usual diffeomorphism symmetry of string theory and associativity.

This thesis is structured as follows: In the second chapter we will review Gepner models that describe strings moving in certain Calabi-Yau backgrounds. In the course of this chapter we will introduce all necessary techniques to understand the following chapters. The third chapter starts with a review of the relationship of asymmetric toroidal orbifolds and gauged maximal supergravity. Then, based on [2], we explain how to deduce the geometric interpretation of four-dimensional asymmetric $\mathcal{N} = 1$ Gepner models and compare their spectrum with $\mathcal{N} = 2$ gauged supergravity. In chapter four we will present the results of [3] and classify all asymmetric Gepner models in $D = 4, 6, 8$ with more than eight supercharges. Finally, in the last chapter, we will try to formulate a gravity theory on the non-geometric R-flux space following [1].

Chapter 2

Conformal field theories of Calabi-Yau compactifications

This chapter will review conformal field theories that describe superstring motion in a spacetime $\mathcal{M}_{10D} = \mathbb{R}^{3,1} \otimes Y_3$ where $\mathbb{R}^{3,1}$ is a flat Minkowski spacetime and Y_3 is a Calabi-Yau manifold defined as a hypersurface of a weighted projective space. This is one of the few string vacua where one knows both a supergravity description and the full classical string solution in terms of a conformal field theory. These conformal field theories were found by Doron Gepner and are accordingly called Gepner models. The aim of this section is to understand Gepner's construction and especially how these highly abstract CFTs can be given a geometric interpretation in terms of a Calabi-Yau manifold. We will introduce the essential techniques such as the usage of simple currents, the computation of Hodge numbers or reading off the monomials from a CFT spectrum. All these techniques are employed later in the following main chapters and therefore the key to understand them.

The chapter starts with a short recap of Calabi-Yau compactifications from the supergravity viewpoint while the rest of the chapter will be dedicated to conformal field theories. After introducing general features of the partition function we present certain conformal field theories that will be used as building blocks in Gepner's construction. Afterwards we will turn to simple currents, a systematic method to construct modular invariant partition functions. In the fourth section we will review $N = 2$ superconformal field theories and why they necessarily appear in supersymmetric compactification of string theory. Subsequently we will introduce a class of $N = 2$ superconformal field theories, the Landau-Ginzburg models, that also turn out to be the minimal series of $N = 2$ superconformal field theories. Then we will see how hypersurfaces in weighted projective spaces, thus Calabi-Yau manifolds, are actually described by Landau-Ginzburg orbifolds. After a short intermezzo about the computation of the Hodge data of these Calabi-Yau manifolds we will take all the building blocks and put them together into Gepner models.

While we try to introduce all essential concepts it is helpful to have some knowledge of Calabi-Yau compactifications and conformal field theories when reading this chapter (see e.g. the books [8, 9]).

2.1 IIB string theory on Calabi-Yau threefolds

When compactifying type IIB string theory on a Calabi-Yau threefold Y_3 we get $\mathcal{N} = 2$ target space supersymmetry in four dimensions. This chapter is meant as a reminder of known facts about the Kaluza-Klein compactification on a compact Calabi-Yau manifold and to introduce notation we need later.

To get an intuition about the Kaluza-Klein compactification let us take a massless 10D scalar ϕ and see what happens when splitting the spacetime as $\mathcal{M}_{10D} = \mathbb{R}^{3,1} \times Y_3$. In particular the equations of motion will split

$$\Box_{10D}\phi = \Box_{4D}\phi + \Delta_{Y_3}\phi = 0\,. \tag{2.1.1}$$

To get an effective four-dimensional description, ϕ is expanded in terms of eigenfunctions Y_n of the internal Laplacian Δ_{Y_3}

$$\phi(x,y) = \sum_n \phi_n(x) Y_n(y)\,. \tag{2.1.2}$$

The internal coordinates are called y and the external ones x. When writing the eigenvalue equation as $\Delta_{Y_3} Y_n = -m_n^2 Y_n$ the equation of motion becomes

$$(\Box_{4D} - m_n^2)\phi_n(x) = 0\,. \tag{2.1.3}$$

This reveals that ϕ_n is a scalar with mass m_n in four dimensions. When repeating the analysis for p-forms one obtains a qualitatively similar result. In energy regions where $E < m_n$ we can truncate the spectrum to only the massless modes. To do so the internal part of any form has to be harmonic, thus eliminated by Δ_{Y_3}. Due to

$$\Delta_{Y_3} = dd^* + d^*d\,, \tag{2.1.4}$$

where $d^* = \pm \star d \star$, a p-form ω_p is harmonic if and only if ω_p is closed $d\omega_p = d^*\omega_p = 0$. Hodge's theorem states that any p-form ω_p has a unique decomposition

$$\omega_p = h_p + d\alpha_{p-1} + d^*\beta_{p+1}\,, \tag{2.1.5}$$

where h_p is a harmonic p-form. Therefore the harmonic forms are in the cohomology quotient

$$H^p(Y_3) = \frac{\{d\alpha_p = 0\}}{\{\alpha_p = d\beta_{p-1}\}}\,. \tag{2.1.6}$$

The cohomology is in one-to-one correspondence to the homology of the space that describes the non-contractible closed cycles of the manifold. Intuitively speaking the massless modes correspond to the constant Fourier mode around each non-trivial cycle. The number of non-trivially closed forms/non-contractible cycles of dimension p is counted by the Betti numbers $b^p = \dim(H^p(Y_3))$. In the following we are using that any Calabi-Yau manifold is a complex manifold and introduce holomorphic and antiholomorphic coordinates. Accordingly the cohomology is split as $H^{p,q}(Y_3)$ where p, q denotes the holomorphic/antiholomorphic dimension. The dimensions are denoted as Hodge numbers $h^{p,q} = \dim(H^{p,q}(Y_3))$.

We will now explicitly expand the 10D forms into the harmonic forms of the Calabi-Yau to obtain the massless 4D spectrum. The 4D indices will be Greek letters while the Calabi-Yau indices are named $i, j, \bar{\imath}, \bar{\jmath}$ and so on.

NS-NS Sector

There are two kinds of deformations of the metric that preserve the Ricci-flatness, the complex structure and the Kähler deformations. The complex structure deformations are controlled by the third cohomolgy group of the Calabi-Yau Y_3 since the complex structure is fixed by the choice of the holomorphic threeform Ω. In terms of a symplectic basis $\{\alpha_\Lambda, \beta^\Lambda\} \in H^3(Y_3)$ with $\Lambda = 0, \dots, h^{2,1}$, the holomorphic threeform Ω is

$$\Omega = X^\Lambda \, \alpha_\Lambda - F_\Lambda \, \beta^\Lambda = \big(\alpha_\Lambda \, , \, \beta^\Lambda\big) \cdot V_2 \, , \tag{2.1.7}$$

where $V_2^T = (X^\Lambda \, , \, -F_\Lambda)$. The periods X^Λ span projective coordinates on the complex structure moduli space. Fixing the projective freedom can be done by introducing $z^a = X^a/X^0$ where $a = 1, \dots, h^{1,2}$. Since the other coefficients F_Λ can be computed as derivatives of a prepotential $F_\Lambda = \partial F/\partial X^\Lambda$, the complex structure is completely characterized by the position on the manifold spanned by the z^a. Using the period matrix $\mathcal{N}$ (see e.g. the appendix of chapter 14 in [9]) one can build the matrix

$$\mathcal{M}_1 = \begin{pmatrix} \mathbb{1} & \mathrm{Re}\,\mathcal{N} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} -\mathrm{Im}\,\mathcal{N} & 0 \\ 0 & -\mathrm{Im}\,\mathcal{N}^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ \mathrm{Re}\,\mathcal{N} & \mathbb{1} \end{pmatrix} , \tag{2.1.8}$$

that we only state for later reference. For the even cohomology of the Calabi-Yau we introduce the basis

$$\{\omega_A\} = \{1, \, \omega_\mathsf{A}\} \quad \text{and} \quad \{\sigma^A\} = \big\{\tfrac{\sqrt{g}}{\mathcal{V}} \, dx^6, \, \sigma^\mathsf{A}\big\} \, , \qquad A = 0, \dots, h^{1,1} \tag{2.1.9}$$

where $\mathcal{V}$ is the volume of the Calabi-Yau. $\{\omega_\mathsf{A}\} \in H^{1,1}(Y_3)$ is a basis of the two-forms while $\{\sigma^\mathsf{A}\} \in H^{2,2}(Y_3)$ is a basis of the four-forms. Therefore the index A runs from $1, \dots, h^{1,1}$.

The second class of Ricci-flatness preserving deformations is called Kähler deformation. These deformations parametrize the size of the two-cycles and can therefore be collected into a two-form, the Kähler form J. We can expand J and the internal part of the Kalb-Ramond field $B_{i\bar{\jmath}}$ into the above basis

$$J = t^\mathsf{A} \omega_\mathsf{A} \, , \qquad\qquad B = b^\mathsf{A} \omega_\mathsf{A} \, . \tag{2.1.10}$$

Together they form the complexified Kähler form $\mathcal{J} = B + iJ$ appearing in

$$e^{B+iJ} = e^{\mathcal{J}} = \big(\omega_A \, , \, \sigma^A\big) \cdot V_1 \, , \tag{2.1.11}$$

where V_1 is

$$V_1 = \begin{pmatrix} 1 \\ \mathcal{J}^{\mathsf{A}} \\ \frac{1}{6}\kappa_{\mathsf{ABC}}\mathcal{J}^{\mathsf{A}}\mathcal{J}^{\mathsf{B}}\mathcal{J}^{\mathsf{C}} \\ \frac{1}{2}\kappa_{\mathsf{ABC}}\mathcal{J}^{\mathsf{B}}\mathcal{J}^{\mathsf{C}} \end{pmatrix} . \tag{2.1.12}$$

κ_{ABC} is the triple intersection number $\kappa_{\mathsf{ABC}} = \int_{Y_3} \omega_{\mathsf{A}} \wedge \omega_{\mathsf{B}} \wedge \omega_{\mathsf{C}}$. The complexified Kähler form $\mathcal{J}$ contributes complex scalars in 4D. Further NS-NS scalars are the dilaton ϕ and the Hodge dual $\tilde{\phi}$ of the Kalb-Ramond field $B_{\mu\nu}$.

Using the period matrix of the Kähler sector we can built a similar matrix as (2.1.8) that we call $\mathcal{M}_2$.

Ramond-Ramond sector

In the following we will discuss the massless 4D fields from the R-R-sector. Let us start with the scalars. There is a single scalar from the C_0 form and another scalar from the Hodge dual of $(C_2)_{\mu\nu}$. The two-form C_2 contributes further scalars from the components $(C_2)_{i\bar{j}}$. The self-dual four-form gives scalars by Hodge dualizing $(C_4^{\mathrm{SD}})_{\mu\nu i\bar{j}}$ into scalars. Having the dualization in mind one can summarize all the scalars from the R-R-forms as

$$\mathcal{C}\Big|_{\text{scal.}} = \tilde{\xi}_0 + \xi^{\mathsf{A}}\omega_{\mathsf{A}} + \tilde{\xi}_{\mathsf{A}}\sigma^{\mathsf{A}} + \xi^0\omega_0 = \left(\omega_A\,,\,\sigma^A\right)\cdot\Xi\,. \tag{2.1.13}$$

The democratic formulation $\mathcal{C} = C_0 + C_2 + C_4 + C_6$ is used here and the $(2h^{1,1}+2)$-dimensional vector Ξ contains the R-R axions $\Xi^T = (\xi^A\,, \tilde{\xi}_A)$.

The R-R-sector contributes vectors by putting the self-dual four-form C_4^{SD} on three-cycles $(C_4^{\mathrm{SD}})_{\mu ijk}$, $(C_4^{\mathrm{SD}})_{\mu ij\bar{k}}$, $(C_4^{\mathrm{SD}})_{\mu i\bar{j}\bar{k}}$ and $(C_4^{\mathrm{SD}})_{\mu\bar{i}\bar{j}\bar{k}}$. Expansion into the above basis yields

$$C_4 = A^\Lambda\alpha_\Lambda + \tilde{A}_\Lambda\beta^\Lambda\,, \tag{2.1.14}$$

where $(A, \tilde{A})$ are the four-dimensional vector fields. Only half of them are actually independent due to the self-duality of the four-form. As such there are only $h^{1,1}+1$ vectors from the R-R-sector.

Summary

To summarize, the bosonic massless spectrum of type IIB on a Calabi-Yau and their 10D origin is

4D \ 10D	C_0	C_2	C_4^{SD}	ϕ	B_2	g
g	-	-	-	-	-	1
2-Form/Scalar	-	1	$h^{1,1}$	-	1	-
Vector	-	-	$h^{1,2}+1$	-	-	-
Scalar	1	$h^{1,1}$	-	1	$h^{1,1}$	$(h^{1,1}_{\mathrm{K}} + 2h^{1,2}_{\mathrm{CS}})$

The $\mathcal{N}=2$ massless multiplets have the following particle content

$$\begin{aligned}
&\text{massless } \mathcal{N}=2 \text{ gravity} && \mathcal{G}_{(2)} = 1\cdot[2]+2\cdot[\tfrac{3}{2}]+1\cdot[1] = (2)_{\rm b}+(4)_{\rm f}+(2)_{\rm b}\,,\\
&\text{massless } \mathcal{N}=2 \text{ vector} && \mathcal{V}_{(2)} = 1\cdot[1]+2\cdot[\tfrac{1}{2}]+2\cdot[0] = (2)_{\rm b}+(4)_{\rm f}+(2)_{\rm b}\,,\\
&\text{massless } \mathcal{N}=2 \text{ hyper} && \mathcal{H}_{(2)} = 2\cdot[\tfrac{1}{2}]+4\cdot[0] \qquad\qquad = (4)_{\rm f}+(4)_{\rm b}\,.
\end{aligned} \tag{2.1.15}$$

The first equality states the number of particles of each spin, while the second equality counts the fermionic and bosonic degrees of freedom of each spin. The massless IIB spectrum in 4D falls into the $\mathcal{N}=2$ multiplets as follows

$$\begin{aligned}
&\text{massless } \mathcal{N}=2 \text{ gravity} && \mathcal{G}_{(2)} \supset \left(g_{\mu\nu}\,,\ A^0\right),\\
&\text{massless } \mathcal{N}=2 \text{ vector} && \mathcal{V}_{(2)} \supset \left(A^a\,,\ z^a\right),\\
&\text{massless } \mathcal{N}=2 \text{ hyper} && \mathcal{H}_{(2)} \supset \left(\mathcal{J}^{\mathsf{A}}\,,\ \xi^{\mathsf{A}}\,,\ \tilde{\xi}_{\mathsf{A}}\right),\\
& && \mathcal{H}^{\text{univ.}}_{(2)} \supset \left(\phi\,,\ \tilde{\phi}\,,\ \xi^0\,,\ \tilde{\xi}_0\right).
\end{aligned} \tag{2.1.16}$$

The 4D IIB spectrum has $N_V = h^{1,2}$ vector multiplets and $N_H = h^{1,1}+1$ hypermultiplets next to the obligatory supergravity multiplet.

The scalars of the vector multiplets z^a are the coordinates on a special Kähler manifold. The $4(h^{1,1}+1)$ scalars of the hypermultiplets are the coordinates of a special quaternionic Kähler manifold that has a fibration structure in which the base is spanned by the complex Kähler moduli $\mathcal{J}^{\mathsf{A}}$ that form a special Kähler manifold.

2.2 Conformal field theory

This section starts with a small review about partition functions and related topics like characters or modular invariance. Then we will shortly recap Kac-Moody algebras and present selected conformal field theories that we need later as building blocks for Gepner models.

2.2.1 Partition functions and related topics

The one-loop diagram of a closed string theory is a worldsheet with one hole, a torus. Computing quantum corrections is therefore done by looking at conformal field theories on a torus. Similar to quantum field theory also in string theory every particle can run in the loop. The probability for a particle to run in a loop is given by a Boltzmann factor. This notion gives rise to the partition function

$$\mathcal{Z}(\tau,\bar{\tau}) = \mathrm{Tr}_{\mathcal{H}}\left(e^{-2\pi H\,\mathrm{Im}(\tau)}\ e^{+2\pi P\,\mathrm{Re}\tau}\right) = \mathrm{Tr}_{\mathcal{H}}\left(q^{L_0-\frac{c}{24}}\ \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right). \tag{2.2.1}$$

Since in closed string theory the target space fields are combinations of left- and right-movers, the partition function connects left- and right-movers. The trace runs over the whole Hilbert-space of the conformal field theory reflecting that all particles can run in a loop. L_0 are the zero modes of the Virasoro-algebra

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12}(m^3 - m)\, \delta_{m+n,0}\,, \tag{2.2.2}$$

that generate conformal transformations. The zero modes $L_0 \pm \bar{L}_0$ generate translations in time and space and the eigenvalue of L_0 is the conformal dimension h of the state. The energy is $H = L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24}$ where the central charge term $c, \bar{c}$ is a normal ordering constant. In contrast to the quantum field theory case also a term with the momentum $-iP = L_0 - \bar{L}_0$ appears in the exponent of the partition function since the string worldsheet is spread over both time and space. When expanding the partition function according to the energy, one can read off the number of states at each energy level E by looking at the coefficient of the exponential $e^{-2\pi E\, \mathrm{Im}(\tau)}$. In particular, massless states must have $h = \frac{c}{24}$ and appear as constant piece in the expansion. Usually one expands only sectors of the partition function, since bosons and fermions contribute with a different sign to the partition function such that in target space supersymmetric theories $\mathcal{Z} = 0$.

Modular invariance

The complex parameter τ with $\mathrm{Im}(\tau) > 0$ appearing in $q = e^{2\pi i\tau}$ parametrizes the complex structure of the torus $\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$. To be well defined in string theory the partition function (2.2.1) has to be invariant under the global diffeomorphism of the torus that are the modular transformations

$$SL(2,\mathbb{Z})/\mathbb{Z}_2: \quad \tau \to \frac{a\tau + b}{c\tau + d}\,, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})/\mathbb{Z}_2\,. \tag{2.2.3}$$

The generators of this group are the modular transformations

$$S: \tau \to -\frac{1}{\tau} \qquad \text{and} \qquad T: \tau \to 1 + \tau\,. \tag{2.2.4}$$

The torus partition function and therefore the possibilities to combine left- and right-movers is highly constrained by modular invariance. Finding and classifying non-trivial partition functions is a hard task and has only been achieved in special cases.

In quantum field theory a momentum variable parametrizes the loop and to get the amplitude one has to integrate over the momentum. In string theory the parameter τ takes a similar role as the momentum such that for a full amplitude we have to integrate over τ. For instance the vacuum amplitude is $\int_T \frac{d^2\tau}{\mathrm{Im}(\tau)} \mathcal{Z}(\tau, \bar{\tau})$ where the $\mathrm{Im}\tau$ factor guarantees modular invariance of the measure and T is the space of inequivalent τ which is the Teichmüller space. Since T excludes the unit circle around the origin, this integral is finite reflecting the UV-finiteness of string theory. Furthermore, notice that the integration over $\mathrm{Re}\tau$ yields a delta function enforcing level matching $L_0 - \bar{L}_0 = 0$.

Highest weights and conformal families

The spectrum of a conformal field theory can be characterized by stating the highest weights $|\phi_i\rangle$ of the theory. These states are annihilated by every single annihilation operators

$$L_n|\phi_i\rangle = 0, \qquad \text{for} \qquad n > 0\,, \qquad\qquad L_0|h\rangle = h_i|\phi_i\rangle\,. \tag{2.2.5}$$

The vacuum is always a highest weight and usually denoted by $i = 0$. All other states can be constructed from the highest weight by applying the creation operator of the symmetry algebra, e.g.

$$|\phi_i\rangle\,, \quad L_{-1}|\phi_i\rangle\,, \quad L_{-2}|\phi_i\rangle\,, \quad L_{-1}L_{-1}|\phi_i\rangle, \ldots\,. \tag{2.2.6}$$

These states form the conformal family $[\phi_i]$. By the operator-state correspondence the descendants of ϕ_i correspond to the field ϕ_i, its derivatives and normal ordered products of ϕ_i together with the energy-momentum tensor.

Characters, the modular S-matrix and the Verlinde formula

Since the highest weights define separate sectors of the theory one can split the partition function into the contribution of the different highest weights. The contribution of a highest weight defines the character

$$\chi_i = \mathrm{tr}_{\mathcal{H}_i}\ \left(q^{L_0 - \frac{c}{24}}\right)\,. \tag{2.2.7}$$

The trace runs only over the left-moving Hilbert space $\mathcal{H}_i$ that is associated to the highest weight with index i, therefore the whole conformal family $[\phi_i]$. Under a modular T-transformation the character is multiplied with a phase $e^{2\pi i\left(h_i - \frac{c}{24}\right)}$ while under a modular S-transformation the characters of the different highest weights mix. This mixing can be parametrized by the modular S-matrix in the following form

$$\chi_i\left(-\tfrac{1}{\tau}\right) = S_{ij}\chi_j(\tau)\,. \tag{2.2.8}$$

The modular S-matrix is usually symmetric, unitary and squares to the charge conjugation matrix.

When computing the OPE of a field in $[\phi_i]$ with a field in $[\phi_j]$ one finds that only the fields of some conformal families appear. This gives rise to the fusion algebra

$$[\phi_i] \times [\phi_j] = N_{ij}^{k}\ \ [\phi_k]\,, \tag{2.2.9}$$

that schematically captures which conformal families appear in each others OPEs. One of the deepest results of conformal field theory is the Verlinde formula. It states that the fusion coefficients N_{ij}^k can be expressed in terms of the modular S-matrix as

$$N_{ij}^{k} = \sum_m \frac{S_{im}S_{jm}S^*_{mk}}{S_{0m}}\,, \tag{2.2.10}$$

where the sum runs over all highest weights and 0 labels the vacuum highest weight. This formula connects tree-level (sphere) and one-loop (torus) amplitudes in an intriguing manner.

In terms of the characters (2.2.7) the partition function (2.2.1) can be written as

$$\mathcal{Z}(\tau,\overline{\tau}) = \chi_i(\tau)\, M_{ij}\, \overline{\chi}_j(\bar{\tau})\,. \tag{2.2.11}$$

The matrix M collects how the highest weights from the left and the right are combined to target space fields. To guarantee modular invariance we infer $[M,S] \stackrel{!}{=} 0$. Finding a non-trivial $M \neq 1$ with $[M,S]$=0 is a hard task. Invariance under the modular T-transformation is guaranteed if

$$h_i - \overline{h}_j \in \mathbb{Z} \tag{2.2.12}$$

together with $c = \overline{c}$.

2.2.2 Kac-Moody algebra

Most theories we will review in the following have a Kac-Moody symmetry. These algebras capture, that the algebra of the currents J^a generating the symmetry may have a central extension from normal ordering. The modes j^a_n of the Kac-Moody algebra $\hat{\mathfrak{g}}_k$ at level k obey

$$[j^a_m, j^b_n] = i f^{abc}\, j^c_{m+n} + k\, m\, \delta_{m+n,0}\, \delta^{ab}\,, \tag{2.2.13}$$

where the f^{abc} are the structure constants of the underlying Lie algebra $\mathfrak{g}$. One can define an energy momentum tensor according to Sugawara's construction

$$T(z) = \frac{1}{2(k+C_{\mathfrak{g}})} N(J^a J^a)(z)\,, \tag{2.2.14}$$

where $N(\dots)$ denotes the normal ordering and $C_{\mathfrak{g}}$ is the dual coxeter number of $\mathfrak{g}$. One can expand $T(z)$ into

$$T(z) = \sum_n z^{-n-2} L_n\,. \tag{2.2.15}$$

where the modes L_n are

$$L_n = \frac{1}{2(k+C_{\mathfrak{g}})} \left(\sum_{l\leq -1} j^a_l j^a_{n-l} + \sum_{l>-1} j_{n-l} j_l \right)\,, \tag{2.2.16}$$

since the normal ordering places annhihilation operators to the right. Using the Kac-Moody algebra of the currents one can show that the modes L_n obey the Virasoro-algebra (2.2.2). Furthermore the commutator

$$[L_n, j^a_m] = -m\, j^a_{m+n} \tag{2.2.17}$$

reveals that the currents have conformal dimension one. For the central charge of the CFT one finds

$$c = \frac{k \dim \mathfrak{g}}{k + C_{\mathfrak{g}}}\,, \tag{2.2.18}$$

where $\dim \mathfrak{g}$ is the dimension of the Lie algebra $\mathfrak{g}$.

2.2.3 $\widehat{\mathfrak{so}}(N)_1$ and the bosonic string map

When looking at string theory on a space with $10 - D$ compact internal directions, the remaining external directions have a $SO(D-1,1)$ Lorentz symmetry. The usage of light-cone gauge effectively eliminates two dimensions and the massless states transform under the little group $SO(D-2)$. To describe the external directions from the worldsheet point of view we need $N = D - 2$ free bosons and their superpartners, $N = D - 2$ free fermions transforming in the vector representation of $SO(N)$. The CFT for the N fermions is the $\widehat{\mathfrak{so}}(N)_1$ CFT that must have central charge $c = N/2$ due to the field content. This central charge is consistent with (2.2.18) when inserting $C_{\mathfrak{g}} = N - 2$ and $\dim \mathfrak{g} = \frac{1}{2}N(N-1)$.

Having fermions we have two separate sectors, the NS and the R sector. Both, the R and the NS sector can further be split into states with even and odd fermion number. The resulting four sectors correspond to the four highest weights of the theory. The first NS highest weight is the true vacuum whose sector we name O. The second NS highest weight is $\Psi^i_{-1/2}|0\rangle$. Due to its transformation behavior this sector is called vector V. In the R sector we have zero modes satisfying the Clifford algebra such that the groundstate is degenerate and transforms as a spinor. The fermion number translates into the chirality that distinguishes the two subsectors of the Ramond sector. We name these two sectors the spinor S and antispinor C.

In the following we only have $D = 4, 6, 8$ non-compact directions thus even $N = 2n$. Let us present a collection of the relevant properties of $\widehat{\mathfrak{so}}(2n)_1$.

character	h	q mod 2	degeneracy
$O = \frac{1}{2}\Big(\big(\frac{\theta_3}{\eta}\big)^n + \big(\frac{\theta_4}{\eta}\big)^n\Big)$	0	0	0
$V = \frac{1}{2}\Big(\big(\frac{\theta_3}{\eta}\big)^n - \big(\frac{\theta_4}{\eta}\big)^n\Big)$	$\frac{1}{2}$	1	$2n$
$S = \frac{1}{2}\Big(\big(\frac{\theta_2}{\eta}\big)^n + \big(\frac{\theta_1}{\eta}\big)^n\Big)$	$\frac{n}{8}$	$\frac{n}{2}$	2^{n-1}
$C = \frac{1}{2}\Big(\big(\frac{\theta_2}{\eta}\big)^n - \big(\frac{\theta_1}{\eta}\big)^n\Big)$	$\frac{n}{8}$	$\frac{n}{2} - 1$	2^{n-1}.

(2.2.19)

Here O, V, S, C denote the characters of the four highest weights and θ are the Jacobi theta functions. The modular S-matrix of the $\widehat{\mathfrak{so}}(2n)_1$ theory is

$$S^{\widehat{\mathfrak{so}}(2n)_1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i^{-n} & -i^{-n} \\ 1 & -1 & -i^{-n} & i^{-n} \end{pmatrix} . \tag{2.2.20}$$

Using the Verlinde formula (2.2.10) the fusion rules are

n odd	o	v	s	c
o	o	v	s	c
v	v	o	c	s
s	s	c	v	o
c	c	s	o	v

n even	o	v	s	c
o	o	v	s	c
v	v	o	c	s
s	s	c	o	v
c	c	s	v	o.

(2.2.21)

The bosonic string map

$\widehat{\mathfrak{so}}(2n)_1$ describes $2n$ fermions that we can bosonize into n free bosons on a circle of radius $R = \sqrt{2\alpha'}$. This suggests that we can take a partition function of a fermionic string and turn it into a bosonic string partition function.

There are two major problems. The first one is the mismatch in the central charge. A conformal field theory in bosonic string theory has $c = 24$ while a conformal field theory for a fermionic string has $c = 12$. We have to find an additional ingredient that accounts for the $\Delta c = 12$. The second problem is that the mapping has to preserve modular invariance.

When looking at the modular S-matrix of $\widehat{\mathfrak{so}}(2n)_1$ in (2.2.20) we see that it is invariant under $n \to n + 4$. This suggests that we map the characters of $\widehat{\mathfrak{so}}(D-2)_1$ onto the characters of $\widehat{\mathfrak{so}}(D+22)_1$. In this way both the central charge is taken care of and modular invariance is preserved. The only obstacle is that the conformal dimension[1] of the R states shifts by $3/2$ under the mapping $\widehat{\mathfrak{so}}(D-2)_1 \leftrightarrow \widehat{\mathfrak{so}}(D+22)_1$ while the NS states keep their conformal dimension. The level matching condition of modular T-invariance $h_L - h_R \in \mathbb{Z}$ is therefore broken for $\mathrm{R} \otimes \overline{\mathrm{NS}}$ and $\mathrm{NS} \otimes \overline{\mathrm{R}}$ states. This can be cured by mapping the characters O and V onto each other, since their conformal dimension differs by $1/2$. When interchanging O and V the S-matrix is only invariant if we simultaneously interchange the characters (S, C) with either $(-C, -S)$ or $(-S, -C)$. Notice that both choices $(-C, -S)$ or $(-S, -C)$ are connected by charge conjugation. Since a charge conjugation interchanges IIA and IIB, starting with a bosonic partition function we can arrive both at a IIA and a IIB partition function. For IIB we take the same convention for both sides while for IIA we have to use different choices for the left- and right-movers. Since a charge conjugation is easy to apply in the CFTs we are looking at, we can stick to one choice and apply charge

[1] see (2.2.19)

conjugation afterwards. In the following we will take the choice $(S, C) \leftrightarrow (-C, -S)$. A practical reason for this is that we assign a number $s_0 = (0, 2, -1, 1)$ to each of the four characters $(0, V, S, C)$.The mapping then amounts to $s_0 \to s_0 + 2 \bmod 4$ which is easy to realize in a computer program.

To summarize, we have a map called *bosonic string map* which maps a fermionic string partition function onto a bosonic string partition function via

$$\phi_{\text{bsm}}(O, V, S, C) \in \widehat{\mathfrak{so}}(D-2)_1 \to (V, O, -C, -S) \otimes 1 \in \widehat{\mathfrak{so}}(D+22)_1 \,. \tag{2.2.22}$$

where O, V, S, C denote the characters (2.2.19). When applying the bosonic string map onto only say the right-movers we get a heterotic string partition function that descents from the $SO(32)$ heterotic string.

By the same arguments as above, a similarly looking map maps the characters of $\widehat{\mathfrak{so}}(D-2)_1 \leftrightarrow \widehat{\mathfrak{so}}(D+6)_1$ onto each other without spoiling modular invariance. But then we have to find another explanation where the remaining 8 units of central charge are coming from. The solution lies in the other heterotic string theory, the $E_8 \times E_8$ heterotic string. Noticing that $SO(D+6)$ is a subgroup of E_8, the missing $c = 8$ can be attributed to the 8 free bosons on the root lattice of the other E_8 factor in the $E_8 \times E_8$ heterotic string. This is consistent since $(\hat{\mathfrak{e}}_8)_1$ has only a singlet representation with conformal dimension 0. Therefore the additional $(\hat{\mathfrak{e}}_8)_1$ factor does not change the way the states are mapped onto each other but can cure the mismatch in the central charges. The second bosonic string map is

$$\phi_{\text{bsm}}(O, V, S, C) \in \widehat{\mathfrak{so}}(D-2)_1 \to (V, O, -C, -S) \otimes 1 \in \widehat{\mathfrak{so}}(D+6)_1 \otimes (\hat{\mathfrak{e}}_8)_1 \,. \tag{2.2.23}$$

There are therefore two consistent bosonic string maps. The choice of bosonic string map is only relevant when looking at massive states or when constructing heterotic partition functions. Since we only consider the massless type IIB spectrum we in principle do not have to specify which one we take. In our computer programs we decided to work with the $\widehat{\mathfrak{so}}(D-2)_1 \to \widehat{\mathfrak{so}}(D+6)_1 \otimes (\hat{\mathfrak{e}}_8)_1$ bosonic string map.

Target space interpretation

The $\widehat{\mathfrak{so}}(D-2)_1$ theory is the CFT of the uncompactified directions. Being the little group, the transformation behavior under this symmetry tells us about the particle interpretation from the D-dimensional viewpoint. In closed string theory every state combines a left- and right-moving $\widehat{\mathfrak{so}}(D-2)_1$ state. In the following we will state which target space particle appears for every combination of the $\widehat{\mathfrak{so}}(D-2)_1$ highest weights. We will restrict ourselves to massless modes. Universal in all dimensions is

$$\begin{aligned} V \otimes \overline{V} \quad &\to \quad g_{\mu\nu} + B_{\mu\nu} + D \,, \\ V \otimes \overline{O} \quad &\to \quad A_\mu \,, \\ O \otimes \overline{O} \quad &\to \quad \phi \,, \end{aligned} \tag{2.2.24}$$

where g is the metric, B is the Kalb-Ramond field, D is the dilaton. ϕ is a real scalar while A_μ is a vector. Using (2.2.19) one can check that the number of degrees of freedom is correct. All other tensor products involve the spinorial degrees of freedom and their decomposition depends on the number of non-compact directions. The target space fields can be recovered either by group theory or more easily by counting the degrees of freedom and following the fusion rules (2.2.21).

In case of four non-compact dimensions $D=4$ we find

$$D=4: \qquad \begin{array}{rcl} S\otimes\overline{C} & \to & \phi\,, \\ S\otimes\overline{S},\ C\otimes\overline{C} & \to & \tfrac{1}{2}V\,, \end{array} \tag{2.2.25}$$

where $\frac{1}{2}V$ denotes one of the two degrees of freedom of a massless vector in four dimensions. In six non-compact dimensions $D=6$ one finds

$$D=6: \qquad \begin{array}{rcl} S\otimes\overline{C} & \to & A_\mu \\ S\otimes\overline{S},\ C\otimes\overline{C} & \to & \phi+T^{\rm (A)SD}_{\mu\nu}\,, \end{array} \tag{2.2.26}$$

where $T^{\rm (A)SD}_{\mu\nu}$ denotes a self-dual or anti-self-dual two-form with three degrees of freedom characteristic for chiral 6D theories. An important fact is that in chiral theories the tensors that are part of tensor multiplets must either all be self-dual or all be anti-self-dual. Accordingly the B field from (2.2.24) has to be split into a self-dual and an anti-self-dual part. The part that has the same chirality as the remaining tensors forms another tensor multiplet, while the other part will be part of the supergravity multiplet. Finally in 8D we have

$$D=8: \qquad \begin{array}{rcl} S\otimes\overline{C} & \to & \phi+C_{\mu\nu}\,, \\ S\otimes\overline{S},\ C\otimes\overline{C} & \to & A_\mu+\tfrac{1}{2}C_{\mu\nu\rho}\,. \end{array} \tag{2.2.27}$$

2.2.4 A free boson on a circle of radius $R=\sqrt{2k}$

The partition function of a free boson on a circle of radius R is

$$\mathcal{Z}_{\rm circ.}(\tau,\overline{\tau}) = \frac{1}{|\eta(\tau)|^2}\sum_{m,n\in\mathbb{Z}} q^{\frac{1}{2}\left(\frac{m}{R}+\frac{Rn}{2}\right)^2}\,\overline{q}^{\frac{1}{2}\left(\frac{m}{R}-\frac{Rn}{2}\right)^2}\,. \tag{2.2.28}$$

m is the quantized momentum along the circle, n is the winding number, $q=e^{2\pi i\tau}$ and $\eta(\tau)$ is the Dedekind η function. For the special values $R=\sqrt{2k}$ this theory is called $\widehat{\mathfrak{u}}(1)_k$.[2] As it turns out, for these special values of the radius the partition function can be written in terms of characters $\chi^{(k)}_m$

$$\mathcal{Z}_{\widehat{\mathfrak{u}}(1)_k}(\tau,\overline{\tau}) = \sum_{m=-k+1}^{k}\left|\chi^{(k)}_m\right|^2 \qquad \text{with} \qquad \chi^{(k)}_m = \frac{1}{\eta(\tau)}\,\Theta_{m,k}(q) \tag{2.2.29}$$

[2]Speaking of a level does not make sense for abelian algebras since the level can be chosen arbitrarily by a rescaling of the generators. One should therefore rather see the name as a definition of the theory.

labeled by an integer charge

$$m = -k+1, \dots, k\,. \tag{2.2.30}$$

The capital Θ functions are

$$\Theta_{m,k}(\tau) = \sum_{n \in \mathbb{Z}+\frac{m}{2k}} q^{k\,n^2}\ . \tag{2.2.31}$$

The highest weight of each character has conformal dimension

$$h_m^{(k)} = \frac{m^2}{4k}\,, \tag{2.2.32}$$

which can be seen by expanding $\eta(\tau) \approx q^{-\frac{1}{24}} + \dots$ and comparing this with the definition of a character (2.2.7) due to the central charge being $c = 1$. The modular S-matrix for these characters is found to be

$$S_{m,m'} = \frac{1}{\sqrt{2k}} \exp\left(-\pi i \frac{mm'}{k}\right)\ . \tag{2.2.33}$$

2.2.5 The $\widehat{\mathfrak{su}}(2)_k$ Kac-Moody algebra

Another important building block we need later is the $\widehat{\mathfrak{su}}(2)_k$ theory. The structure coefficents in (2.2.13) are the ones of $\mathfrak{su}(2)$, $f^{abc} = \sqrt{2}\,\epsilon^{abc}$ where ϵ is the purely antisymmetric tensor in three dimensions, therefore $\dim \mathfrak{g} = 3$ and the dual Coxeter number is $C_{\mathfrak{g}} = 2$. The central charge is according to (2.2.18)

$$c_{\widehat{\mathfrak{su}}(2)_k} = \frac{3k}{k+2}\,. \tag{2.2.34}$$

Let us look at the highest weight states of this CFT. The Cartan subalgebra consists of L_0 and j_0^3. A highest weight has therefore two quantum numbers $|h,q\rangle$, the conformal dimension h and the charge defined by $\hat{j}_0^3|h,q\rangle = q|h,q\rangle$. For convenience we defined $\hat{j}_0^i = \frac{1}{\sqrt{2}} j_0^i$. The $\hat{j}_0^i$ form an $\mathfrak{su}(2)$ subalgebra whose representation theory is well known from quantum mechanics. Unitary highest weights representations with a certain conformal dimension have a spin $\frac{l}{2}$ labeled by a positive integer $l \in \mathbb{N}_0$. The spin in the z direction is labeled by another number $\alpha \in 0, \dots, l$ stating how often we went down the ladder from the highest state. With every step in the ladder the charges decreases by two such that the charge of a state is $q = l - 2\alpha$. For the conformal dimension one finds

$$h = \frac{l(l+2)}{4(k+2)}\,. \tag{2.2.35}$$

The values of k and l are further constrained since there are more $\mathfrak{su}(2)$ subalgebras hidden in the Kac-Moody algebra. For them to have integer eigenvalues and to guarantee non-negative norm states one has to demand

$$k \in \mathbb{Z}^+ \qquad \text{and} \qquad 0 \le l \le k\,. \tag{2.2.36}$$

Having the highest weights let us state their characters. It is useful to define generalized characters that sort each state not only with respect to their conformal dimension but also according to their charge

$$\chi_l^{(k)}(\tau, z) = \mathrm{tr}_{\mathcal{H}_l} \left(q^{L_0 - \frac{c}{24}} \, e^{-2\pi i z \hat{j}_0^3} \right) . \tag{2.2.37}$$

The result can be computed using the Weyl-Kac character formula [10]

$$\chi_l^{(k)}(\tau, z) = \frac{\Theta_{l+1,k+2}(\tau, z) - \Theta_{-l-1,k+2}(\tau, z)}{\Theta_{1,2}(\tau, z) - \Theta_{-1,2}(\tau, z)} , \tag{2.2.38}$$

where the generalized Θ functions are defined by

$$\Theta_{l,k}(\tau, z) = \sum_{n \in \mathbb{Z} + \frac{l}{2k}} q^{k\, n^2} \, e^{-2\pi i n k z} . \tag{2.2.39}$$

In the definition (2.2.37) one can infer that for $z = 0$ the generalized Θ functions become the usual Θ functions (2.2.31) and one should recover the usual characters appearing in the partition function. But this is not possible in (2.2.38) which would render the fraction ill defined. Therefore, one rather has to take the limit $z \to 0$. The form (2.2.37) is in particular useful to determine the modular S-matrix for which one finds

$$S_{ll'}^{(k)} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2} \, (l+1)(l'+1) \right) . \tag{2.2.40}$$

Using the Verlinde formula the fusion rules are

$$\begin{aligned} &N_{l_1 l_2}^{l_3} = 1 \text{ for } |l_1 - l_2| \leq l_3 \leq \min(l_1 + l_2, 2k - l_1 - l_2) \text{ and } l_1 + l_2 + l_3 \text{ even} \\ &N_{l_1 l_2}^{l_3} = 0 \text{ else} . \end{aligned} \tag{2.2.41}$$

These coefficients are consistent with spin addition. It is remarkable that for $\widehat{\mathfrak{su}}(2)_k$ a full classification of modular invariant partition functions exists. The classification follows an ADE scheme. An explanation will be provided later in 2.6.2 when we connect this classification to the ADE classification of singularities, which can in turn be connected to the ADE classification of Lie-groups by the IIA/heterotic duality. The classification is

Level	$\mathcal{Z}$	Name
$k=n$	$\sum_{l=0}^{n} \|\chi_l\|^2$	$A_{n+1}, n \geq 1$
$k=4n$	$\sum_{l=0}^{n-1} \|\chi_{2l}+\chi_{k-2l}\|^2 + 2\|\chi_{k/2}\|^2$	$D_{2n+2}, n \geq 1$
$k=4n+2$	$\sum_{l=0}^{k/2} \|\chi_{2l}\|^2 + \sum_{l=0}^{2n-2} \chi_{2l+1}\overline{\chi}_{k-2l-1}$	$D_{2n+1}, n \geq 2$
$k=10$	$\|\chi_0+\chi_6\|^2 + \|\chi_3+\chi_7\|^2 + \|\chi_4+\chi_{10}\|^2$	E_6
$k=16$	$\sum_{l=0,4,6} \|\chi_l+\chi_{16-l}\|^2 + (\chi_2+\chi_{14})\overline{\chi}_8 + \chi_8(\overline{\chi}_2+\overline{\chi}_{14}) + \|\chi_8\|^2$	E_7
$k=28$	$\|\sum_{l=0,10,18,28} \chi_l\|^2 + \|\sum_{l=6,12,16,22} \chi_l\|^2$	E_8

(2.2.42)

Let us establish a relationship between the $\widehat{\mathfrak{su}}(2)_1$ and the $\widehat{\mathfrak{u}}(1)_k$ theories from section 2.2.4. Taking a free boson ϕ on a circle of radius $R = \frac{1}{\sqrt{2}}$, the vertex operators $V_\pm = e^{\pm i\sqrt{2}X}$ are currents since they have conformal dimension one. Together with the usual current $j = i\,\partial\phi$ they form an $\widehat{\mathfrak{su}}(2)_1$ Kac-Moody algebra. On the other hand this theory can be T-dualized to radius $R' = \sqrt{2}$ which is the $\widehat{\mathfrak{u}}(1)_1$ theory. The equivalence of the $\widehat{\mathfrak{su}}(2)_1$ and $\widehat{\mathfrak{u}}(1)_1$ can be checked by comparing the spectrum and indeed, both have exactly two highest weight states of conformal dimension 0 and $\frac{1}{4}$.

Computing the partition function of this theory would be hard in the $\widehat{\mathfrak{su}}(2)_1$ theory due to the complicated Weyl-Kac character formula (2.2.37). But for the dual $\widehat{\mathfrak{u}}(1)_1$ the more tractable formula (2.2.29) tells us

$$\mathcal{Z}(\tau,\overline{\tau}) = \frac{1}{|\eta(\tau)|^2}\left(|\Theta_{0,1}|^2 + |\Theta_{1,1}|^2\right) . \tag{2.2.43}$$

As it turns out we can in general express the character (2.2.37) of the $\widehat{\mathfrak{su}}(2)_k$ theory in terms of the Θ functions. Indeed, there is an expansion [11]

$$\chi_l^{(k)}(\tau) = \sum_{m=-k+1,\ldots,k \text{ and } l+m \text{ even}} C_{l,m}^{(k)}(\tau)\, \Theta_{m,k}(\tau)\,, \tag{2.2.44}$$

where the $C_{l,m}^{(k)}(\tau)$ are called string functions. In this expansion the string functions $C_{l,m}^{(k)}(\tau)$ encode how the $\widehat{\mathfrak{u}}(1)_k$ theory is embedded into the $\widehat{\mathfrak{su}}(2)_k$ theory. In the above example (2.2.43) they were constant but for higher k they are not. They are explicitly computed in [11].

2.3 Simple currents

Let us introduce a systematic method to find off-diagonal modular invariant partition functions using special fields called *simple currents*. This technique was developed by

Schellekens and Yankielowicz in [12]. Simple currents J_a are defined by their particular simple fusion rules. The fusion product of a simple current with any other primary must result in exactly one primary

$$[J_a] \times [\phi_i] = [\phi_{J(i)}] \,. \tag{2.3.1}$$

Actually, most theories have simple currents. For instance looking back at the fusion rules of the $\widehat{\mathfrak{so}}(2n)_1$ theory in (2.2.21) we see that any highest weight of this theory is in fact a simple current. By associativity, the product of two simple currents is a simple current as well. Assuming a finite amount of primaries in our theory there must exist a number $\mathcal{N}_a$ for which $J_a^{\mathcal{N}_a} = 1$. We will denote the number $\mathcal{N}_a$ as the length of the simple current. The simple currents $\{J_a, J_a^2, \ldots, J_a^{\mathcal{N}_a}\}$ form a finite abelian subgroup $\mathbb{Z}_{\mathcal{N}_a}$ of the fusion algebra and there exists a fusion inverse of every simple current $(J_a^n)^{-1} = J_a^{\mathcal{N}_a - n}$. Consequently, every simple current groups the conformal primaries into orbits of length $\mathcal{N}_a^i$

$$\{\phi_i, J_a \times \phi_i, J_a^2 \times \phi_i, \ldots, J_a^{\mathcal{N}_a^i - 1} \times \phi_i\} \,. \tag{2.3.2}$$

Since $J_a^{\mathcal{N}_a} = 1$, the length of the i-th orbit $\mathcal{N}_a^i$ must be a divisor of $\mathcal{N}_a$. Translating the fusion rule (2.3.1) of a simple current into an OPE results in

$$J_a(z)\phi_i(w) \propto \frac{\phi_{J(i)}(w)}{(z-w)^{Q_i^{(a)}}} + \text{desc.} \,, \tag{2.3.3}$$

where $Q_i^{(a)}$ is called *monodromy charge*. The name is chosen since $Q_i^{(a)}$ mod 1 denotes the monodromy $e^{-2\pi i Q_i^{(a)}}$ when circling the simple current around the primary $(z-w) \rightarrow e^{2\pi i}(z-w)$. By applying $J_a(z)$ onto the last formula until $J_a^{\mathcal{N}_a} = 1$ appears shows

$$Q_i^{(a)} = \frac{t_a^i}{\mathcal{N}_a} \bmod 1 \,, \qquad t_a^i \in \mathbb{Z} \,, \tag{2.3.4}$$

since the monodromy of the identity is $Q_0^{(a)} = 0$ mod 1. Probably the easiest way to compute the monodromy charge uses the formula

$$Q_i^{(a)} = h(\phi_i) + h(J_a) - h(J_a \times \phi_i) \bmod 1 \,. \tag{2.3.5}$$

Repeating the same steps that lead to (2.3.4) for the OPE of simple currents one finds

$$J_a(z)J_a(w) \propto \frac{J_a^2(w)}{(z-w)^{\frac{r_a}{\mathcal{N}_a}}} + \text{desc.} \,. \tag{2.3.6}$$

For simple currents we give the parameter t_a^i the special letter r_a and its own name, the *monodromy parameter*. To compute the monodromy parameter r_a we deduce from (2.3.6)

$$2h_{J_a} = h_{J_a^2} + \frac{r_a}{\mathcal{N}_a} \bmod 1 \,, \tag{2.3.7}$$

and by iteration

$$h_{J_a^n} = r_a \frac{n(n-1)}{2\mathcal{N}_a} + n h_{J_a} \bmod 1\,. \tag{2.3.8}$$

Using $h_{J_a^{\mathcal{N}_a}} = h_1 = 0 \bmod 1$ results in

$$h_{J_a} = r_a \frac{\mathcal{N}_a - 1}{2\mathcal{N}_a} \bmod 1\,. \tag{2.3.9}$$

This relation was used in our computer programs to compute the monodromy parameter r_a for a simple current. Notice that r_a must be defined modulo $\mathcal{N}_a$ for odd $\mathcal{N}_a$ while for even $\mathcal{N}_a$ we must have r_a modulo $2\mathcal{N}_a$ to cover all cases. When r_a is known, one can also deduce a relation for the monodromy charge of the field $J_a^n \phi_i$ that appears in the OPE $J_a(z_1)\dots J_a(z_n)\phi_i(w)$. Recall that the monodromy charge denotes the monodromy when going around the primary of interest with the simple current, therefore for $J_a(z_1)\dots J_a(z_n)\phi_i(w)$ a monodromy encircling all z_i and w. By deforming the contour the full monodromy must be the sum of each of the individual monodromies. Using this we compute

$$Q^{(a)}(J_a^n \phi_i) = Q^{(a)}(\phi_i) + n\, Q^{(a)}(J_a) = \frac{t_i + r_a n}{\mathcal{N}_a} \bmod 1\,. \tag{2.3.10}$$

As a corollary we find that if the simple current has integral conformal dimension, thus $r_a = 0$, every primary in an orbit has the same monodromy charge. Such simple currents are called *orbit simple currents*.

With all this machinery established let us come to the key observation by looking at a generic OPE between two primaries

$$\phi_i(z)\phi_j(w) = \sum_k c_{ij}^k (z-w)^{h_k - h_i - h_j} \phi_k(w) + \text{desc.}\,. \tag{2.3.11}$$

Taking a monodromy with respect to a simple current J_a on both sides one finds

$$Q^{(a)}(\phi_i) + Q^{(a)}(\phi_j) = Q^{(a)}(\phi_k)\,. \tag{2.3.12}$$

Therefore the monodromy charge is actually a conserved quantity. When modding out this symmetry by an orbifold-like procedure with twist $e^{2\pi i (Q_L^{(a)} + Q_R^{(a)})}$, we get a new partition function. In the seminal paper [12] the authors use this twist very explicitly. Here we directly state the matrix M appearing in the partition function (2.2.11)

$$\mathcal{Z}(\tau, \overline{\tau}) = \chi_i(\tau) M_{ij}\, \overline{\chi}_j(\overline{\tau}) \tag{2.3.13}$$

that for a simple current J_a reads

$$(M(J_a))_{kl} = \sum_{p=1}^{\mathcal{N}_a} \delta(\phi_k, J_a^p \times \phi_l)\ \ \delta^{(1)}\big(\hat{Q}^{(a)}(\phi_k) + \hat{Q}^{(a)}(J_a^p \times \phi_l)\big)\,, \tag{2.3.14}$$

where $\hat{Q}^{(a)}(\phi_i)$ is half the monodromy charge

$$\hat{Q}^{(a)}(J^p\phi_i) = \frac{t_i + rp}{2\mathcal{N}_a} \bmod 1\,, \tag{2.3.15}$$

and $\delta^{(1)}$ is a delta function modulo one. The delta function can also be written as

$$\delta\left(Q(\phi_k) + \frac{rp}{2\mathcal{N}_a}\right)\,, \tag{2.3.16}$$

where p denotes the power of the simple currents acting on ϕ_k. An important obstruction is that the proof of modular invariance of the partition function relies on r_a being even. Since r_a is defined modulo $\mathcal{N}_a$ or $2\mathcal{N}_a$ one can choose r_a to be even except when $\mathcal{N}_a$ is even and r_a is odd.

Roughly speaking, the matrix M_a in (2.3.14) induces a partition function that connects a left-moving primary to its whole right-moving orbit if the monodromy charges add up to an integer. For instance, for orbit simple currents with $h_{J_a} = 0$ where $r_a = 0$ the partition function is of the form

$$\mathcal{Z} = \sum_{\text{Orbits}} |\chi_\phi + \chi_{J\phi} + \cdots + \chi_{J^{\mathcal{N}_a - 1}\phi}|^2\,. \tag{2.3.17}$$

The sum is understood to run over all orbits in which the monodromy charge is an integer. The partition function of orbit simple currents contains only perfect squares such that it is left-right symmetric.

Modular invariance is not spoiled when using more than one simple current in a partition function

$$\mathcal{Z} \sim \vec{\chi}^T(\tau) M(J_1) M(J_2) \ldots \vec{\chi}(\bar{\tau})\,, \tag{2.3.18}$$

where we neglected a normalization constant to have a unique vacuum. In general the matrices of different simple currents do not commute making the order of the matrices very important. But there is a criterion to guarantee commutativity. The matrices $M(J_1)$ and $M(J_2)$ commute if and only if the simple currents J_1 and J_2 are relatively local. This property can be translated into a vanishing relative monodromy charge $Q^1(J_2) = Q^2(J_1) = 0$.

Examples

Looking back at the classification of modular invariants of $\widehat{\mathfrak{su}}(2)_k$ (2.2.42) one recognizes in (2.3.17) the form of the D_{2n+2} invariant. In the following we will recover the D partition function of $\widehat{\mathfrak{su}}(2)_k$ with even k using the simple current technique. In a first step let us classify the simple currents of $\widehat{\mathfrak{su}}(2)_k$. Playing around with the fusion coefficients of $\widehat{\mathfrak{su}}(2)_k$, (2.2.41), one sees that the inequality

$$|l_1 - l_2| \leq l_3 \leq \min(l_1 + l_2, 2k - l_1 - l_2) \tag{2.3.19}$$

becomes $l_3 = k - l_2$ for $l_1 = k$ since $l \leq k$. For $l_1, l_2 \neq k$ the inequality is satisfied for more than one value of l_3. Therefore, there is one non-trivial simple current $J = (k)$ in the $\widehat{\mathfrak{su}}(2)_k$ theory besides the trivial one with $l = 0$ being the identity. The simple current J has length $\mathcal{N} = 2$ and using (2.2.35) the conformal dimension is $h_J = \frac{k}{4}$.

Let us start with $k = 4n$ since then $h_J = 1 = 0 \bmod 1$ and therefore $r = 0$. The orbits are $\{\phi_l, \phi_{k-l}\}$ and the monodromy charge is

$$Q(l) = \frac{l}{2}\,, \tag{2.3.20}$$

thus $Q = 0 \bmod 1$ for even l and $Q = \frac{1}{2} \bmod 1$ for odd l. Due to (2.3.16) we may only have integer monodromy charge and every primary in an orbit is connected to any other primary in that orbit. The partition function is therefore

$$\mathcal{Z} = \sum_{l=0}^{n-1} |\chi_{2l} + \chi_{k-2l}|^2 + 2|\chi_{k/2}|^2\,. \tag{2.3.21}$$

Notice that the $\frac{k}{2}$ orbit is $\{\chi_{k/2}, \chi_{k/2}\}$ since the simple current acts as identity on the primary with $l = \frac{k}{2}$. Summing over the fields in the $\frac{k}{2}$ orbit produces the same character $\chi_{k/2}$ twice. To capture the factor of two, the $\chi_{k/2}$ character is excluded from the sum. We found the partition function of the D_{2n+2} series (2.2.42).

Next we look at $k = 4n-2$ where the conformal dimension of the simple current $J = (k)$ is $h_J = n - \frac{1}{2} = \frac{1}{2} \bmod 1$. Using (2.3.9) the monodromy parameter is now $r = 2$ while the monodromy charge is still $Q(l) = \frac{l}{2}$. Recall (2.3.16)

$$\delta\left(Q(\phi_k) + \frac{rp}{2\mathcal{N}_a}\right)\,, \tag{2.3.22}$$

where p denotes the amount of simple currents acting on the primary, thus $p = 0, 1$. For even l where $Q_l = 0 \bmod 1$ the delta function vanishes unless $p = 0$. In words, a field with even l can only be coupled to itself and not to the other field in the orbit for which $p = 1$. In contrast for odd l where $Q_l = 1/2$ we need $p = 1$. Therefore an odd field can be coupled only to its partner in the orbit. The partition function is

$$\mathcal{Z} = \sum_{l=0}^{k/2} |\chi_{2l}|^2 + \sum_{l=0}^{2n-2} \chi_{2l+1}\overline{\chi}_{k-2l-1}\,, \tag{2.3.23}$$

This is the D_{2n+1} invariant (2.2.42).

Let us see what happens for odd k. There $h_J = \frac{1+2n}{4}$ giving an odd $r = 1 + 2n$. Since $\mathcal{N}$ is even we cannot use the modulo relation of r which is $r = r \bmod 4$ to get an even r. The simple current technique does therefore not lead to a new modular invariant. To conclude, for even k the only non-trivial simple current $l = k$ reproduces the D invariant.

2.4 $N = 2$ superconformal field theories

This section will review the salient features of $N = 2$ superconformal field theories (SCFTs) [13, 14, 15]. Since this is textbook standard ([8, 9]) we will be brief and stick to those features relevant later.

Algebra

The conformal enhancement of the supergravity multiplet consists of the energy momentum tensor $T(z)$ with its modes L_m, two fermionic supercurrents $G^{\pm}(z)$ with their modes $G^{\pm}_{m\pm a}$, where the real paramater a is $a \in \mathbb{Z}$ for the R sector while $a \in \mathbb{Z} + \frac{1}{2}$ for the NS sector, and a current $J(z)$ with modes j_m. The modes satisfy the $N = 2$ superconformal algebra

$$\begin{aligned}
[L_m, L_n] &= (m-n)\, L_{m+n} + \frac{c}{12}(m^3 - m)\, \delta_{m+n,0}\,, \\
[L_m, j_n] &= -n\, j_{m+n}\,, \\
[L_m, G^{\pm}_{n\pm a}] &= \left(\frac{m}{2} - n \mp a\right) G^{\pm}_{m+n\pm a}\,, \\
[j_m, j_n] &= \frac{c}{3}\, m\, \delta_{m+n,0}\,, \\
[j_m, G^{\pm}_{n\pm a}] &= \pm G^{\pm}_{m+n\pm a}\,, \\
\{G^{+}_{n+a}, G^{-}_{m-a}\} &= 2\, L_{m+n} + (m-n+2a)\, j_{m+n} + \frac{c}{3}\left((m+a)^2 - \frac{1}{4}\right)\delta_{m+n,0}\,, \\
\{G^{\pm}, G^{\pm}\} &= 0\,.
\end{aligned} \tag{2.4.1}$$

We infer that the quasi-primary $T(z)$ has conformal dimension 2, the primaries $G^{\pm}(z)$ have conformal dimension 3/2 and the primary $J(z)$ has conformal dimension 1. The current $J(z)$ satisfies a $\widehat{\mathfrak{u}}(1)$ Kac-Moody algebra which we recognize as R-symmetry, since the supercurrents $G^{\pm}$ have charge ± 1. In the following, states $|h, q\rangle$ are labeled by their conformal dimension h and their $\mathfrak{u}(1)$ charge q that are the eigenvalues of $|h, q\rangle$ with respect to the cartan subalgebra L_0 and j_0. The definition of highest weight states (2.2.5) must be adjusted to the enlarged symmetry algebra as follows:

$$G^{\pm}_n |h, q\rangle = 0\,, \quad \text{for } n > 0\,, \tag{2.4.2}$$

from which similar statements for L_n and j_n follow by use of the algebra (2.4.1). Using unitarity, $0 \leq |G^{\pm}_0 |h, q\rangle|^2$, the bound

$$h_R \geq \frac{c}{24} \tag{2.4.3}$$

can be derived for states in the Ramond sector. We recognize states saturating this bound as the Ramond groundstates.

In a similar fashion we can derive from $0 \leq |G^{\pm}_{-1/2} |h, q\rangle|^2$ the bound

$$2h \geq |q| \tag{2.4.4}$$

for states in the NS sector. States saturate this bound if and only if they are annihilated by the supercurrent $G^{\pm}_{-1/2}$

$$G^{\pm}_{-1/2}\,|h = \pm q/2\rangle = 0\,. \tag{2.4.5}$$

Clearly the vacuum satisfies this equation. This equation and the bound $2h \geq |q|$ suggest that we found the conformal analogue of short multiplets, the BPS states. We will call states chiral and antichiral depending on whether they are annihilated by $G^{+}_{-1/2}$ or $G^{-}_{-1/2}$. Using $0 \leq |G^{\pm}_{-3/2}|h,q\rangle|^2$ one finds another bound for NS chiral primaries

$$6h \leq c\,. \tag{2.4.6}$$

The (anti)chiral fields close under fusion, and their OPE is non-singular as can be seen by dimensional analysis. As such the (anti)chiral fields define a multiplicative ring, called chiral ring or antichiral ring. When left- and right-movers are combined there are in total four rings that we call the (a,a), (c,a), (a,c) and the (c,c) ring.

For any state in the NS sector there is a unique orthogonal decomposition

$$|\phi\rangle = |\phi_0\rangle + G^{+}_{-1/2}|\phi_1\rangle + G^{-}_{-1/2}|\phi_2\rangle\,, \tag{2.4.7}$$

where $|\phi_0\rangle$ is a chiral primary while $|\phi_1\rangle$ and $|\phi_2\rangle$ are not. This relation can be interpreted as the worldsheet analogue of the Hodge decomposition of forms (2.1.5), when writing the Hodge decomposition in complex coordinates. The $G^{+}_{-1/2}$ and $G^{-}_{-1/2}$ are the analogues of the holomorphic derivatives ∂, ∂^* from the worldsheet point of view. For the right-moving counterparts we identify (oppositely) $\overline{G}^{-}_{-1/2}$ and $\overline{G}^{+}_{-1/2}$ with the antiholomorphic derivatives $\overline{\partial}, \overline{\partial}^*$. The anticommutator $\{G^{+}_{-1/2}, G^{-}_{1/2}\}$ is then related to the internal Laplacian $\Delta_\partial = \Delta_{\overline{\partial}} = \Delta$ (2.1.4). Most importantly the chiral fields are on equal footing as harmonic forms and are therefore directly related to the topology, the non-trivially closed cycles.

In case of a Calabi-Yau threefold Y_3 we find the following relation: The Kähler moduli, being elements of $H^{1,1}(Y_3)$, correspond to the (a,c) and (c,a) primaries while the complex structure moduli, being elements of $H^{1,2}(Y_3)$, correspond to the (c,c) and the (a,a) primaries.

The chiral ring, spectral flow and spacetime supersymmetry

In any $N = 2$ SCFT there exists an automorphism of the algebra induced by the generators U_η called spectral flow operator [16, 17] where $\eta \in \mathbb{R}$ is a real parameter. U_η acts on an operator as $L^{\eta}_m = U_\eta L_m U^{-1}_\eta$ and on states as $|h_\eta, q_\eta\rangle = U_\eta|h,q\rangle$. The transformed operators are

$$\begin{aligned} L^{\eta}_m &= L_m + \eta\, j_m + \frac{c}{6}\eta^2\,\delta_{m,0}\,, \\ G^{\eta\,\pm}_{m\pm a} &= G^{\pm}_{m\pm(a+\eta)}\,, \\ j^{\eta}_m &= j_m + \frac{c}{3}\,\eta\,\delta_{m,0}\,, \end{aligned} \tag{2.4.8}$$

and still satisfy the $N = 2$ superconformal algebra. As such, any choice of a and therefore also the R and the NS sector are connected by the spectral flow. Since the Cartan subalgebra generated by j_0 and L_0 maps onto each other without changing the moding, we can express the transformed quantities q_η and h_η in terms of the original ones

$$q_\eta = q - \frac{c}{3}\eta\,, \qquad h_\eta = h - \eta\, q + \frac{\eta^2}{6}c\,. \tag{2.4.9}$$

In contrast, notice that in general it does not make sense to act with an untransformed $G^\pm$ on a transformed state $|h_\eta, q_\eta\rangle$ for general η since the moding of $G^\pm$ is changed by spectral flow.

By acting with $U_{\frac{1}{2}}$ onto (anti)chiral primaries from the NS sector with $2h = q$ one sees that every (anti)chiral primary is mapped onto a Ramond groundstate saturating $24h = c$. As such the R groundstate is degenerate and the degeneracy is counted by the (anti)-chiral primaries in the NS sector.

For $\eta = 1$ the NS sector is mapped onto the NS sector itself. A closer inspection reveals that the chiral and the antichiral fields are exchanged and therefore equivalent. This can also be deduced from the fact that both, chiral and antichiral fields, can be mapped to the R groundstates.

For $\eta = -1$ the chiral sector is mapped onto itself. In particular the true vacuum $|0, 0\rangle$ is mapped onto a state with $6h = c$ and vice versa. Recalling that any chiral primary has to obey $6h \leq c$ together with the uniqueness of the vacuum implies

$$\text{Exactly one chiral NS field saturates the inequality } 6h \leq c \tag{2.4.10}$$

From the knowledge of the chiral ring one can therefore directly read off the central charge of the theory.

Let us find a representation of U_η. First, bosonize the $U(1)$ current as

$$J(z) = \sqrt{\frac{c}{3}}\partial\phi\,, \tag{2.4.11}$$

such that any state $\Phi_{q,\overline{q}}$ with charge $(q, \overline{q})$ has the form

$$\Phi_{q,\overline{q}} = \exp\left(i\sqrt{\frac{3}{c}}\,(q\phi - \overline{q}\overline{\phi})\right)\cdot\mathcal{O}\,, \tag{2.4.12}$$

where $\mathcal{O}$ is some operator with zero charge. In this representation the left- and right-moving spectral flow operators are

$$U_\eta = e^{i\eta\sqrt{\frac{c}{3}}\phi}\,, \qquad \overline{U}_{\overline{\eta}} = e^{-i\overline{\eta}\sqrt{\frac{\overline{c}}{3}}\overline{\phi}}\,. \tag{2.4.13}$$

Let us discuss the relation of the spectral flow automorphism to spacetime supersymmetry. Spacetime bosons arise from the $(\mathrm{R}, \overline{\mathrm{R}})$ and the $(\mathrm{NS}, \overline{\mathrm{NS}})$ sectors and spacetime fermions arise from the $(\mathrm{R}, \overline{\mathrm{NS}})$ and the $(\mathrm{NS}, \overline{\mathrm{R}})$ sector. To be supersymmetric we need the same amount of spacetime fermions and bosons and a mapping between the superpartners. This is guaranteed if there is a bijective map on the worldsheet

$$(\mathrm{R}, \overline{\mathrm{R}}) \leftrightarrow (\mathrm{NS}, \overline{\mathrm{R}}) \leftrightarrow (\mathrm{NS}, \overline{\mathrm{NS}}) \leftrightarrow (\mathrm{R}, \overline{\mathrm{NS}})\,. \tag{2.4.14}$$

The spectral flow generated by $U_{1/2}$ and $\overline{U}_{\overline{1/2}}$ is tailor-made for such a map since $\eta \in \mathbb{Z} + \frac{1}{2}$ shifts the boundary conditions of the supercurrent by $\frac{1}{2}$ and therefore maps the R and the NS sector onto each other. But notice that

$$U_\eta(z)\, \Phi_{q,\overline{q}}(w, \overline{w}) \sim (z - w)^{q\eta}\, \Phi_{q+\frac{c}{3}\eta,\overline{q}}(w, \overline{w}) \tag{2.4.15}$$

is multivalued for $\eta = \frac{1}{2}$ except for $q \in \mathbb{Z}$ where an (for spinors) unimportant sign ambiguity might exist. In this case the spectral flow operator is well defined but only semi-local. Notice that only if $q \in \mathbb{Z}$ the fermion number operator $(-1)^{F_L} = e^{i\pi J_0}$ is really well defined. When repeating these arguments for the right-movers we obtain $\overline{q} \in \mathbb{Z}$ and $(-1)^{F_R} = e^{-i\pi J_0}$.

With well defined operators $(-1)^{F_L}$ and $(-1)^{F_R}$ we can implement the GSO projection by further demanding $q \in 2\mathbb{Z}$ and similarly for the right-movers. In this case the spectral flow operator is fully local which, as shown in [18, 19, 20], is a necessary requirement for spacetime supersymmetry. As a corollary, the central charge needs to be a multiple of three since otherwise we have chiral fields with non-integral charge by spectral flow.

Examples

The easiest example for an $N = 2$ SCFT is given by two free bosons and two free fermions transforming as vectors under $SO(2)$. In string theory this content describes two extended directions. The characters of the fermionic can be found in (2.2.19) while the two free bosons add a factor of $|\eta(\tau)|^{-4}$ to the partition function. The central charge $c = 1 + 1 + \frac{1}{2} + \frac{1}{2} = 3$ of the theory is the sum of the central charges of the individual theories of bosons and fermions $c = 1 + 1 + \frac{1}{2} + \frac{1}{2} = 3$.

Let us verify the central charge $c = 3$ using the theorem (2.4.10). Since we look for NS chiral primaries, let us look at the highest weights of the NS sector. The only candidates are the vacuum and the highest weight V. The vacuum is always a chiral primary, so let us look at the vector highest weight state $\Psi_{-1/2}|0\rangle$ where $\Psi_r = \psi_r^{(1)} + i\psi_r^{(2)}$ is the complexification of the two fermions $\psi^{(1)}$ and $\psi^{(2)}$. Using the usual commutation relations of a single fermion one gets the anticommutation relations for the complex fermion

$$\{\Psi_r, \Psi_s\} = 0 \qquad \text{and} \qquad \{\Psi_r, \overline{\Psi}_s\} = \delta_{r+s,0}\,, \tag{2.4.16}$$

where $\overline{\Psi} = \psi_r^{(1)} - i\psi_r^{(2)}$. To test whether $\Psi_{-1/2}|0\rangle$ is a chiral primary we need the supercurrents. They should have conformal dimension 3/2 which, using the particle content of the theory, can only be built by combining the current $J_{\mathrm{bos}} \sim \partial\Phi$ of the complexified boson Φ with the complex fermion Ψ as

$$G^+ \sim N(J_{\mathrm{bos}}\Psi) \qquad \text{and} \qquad G^- \sim N(J_{\mathrm{bos}}\overline{\Psi})\,, \tag{2.4.17}$$

where N denotes the normal ordering. Using (2.4.16) it is clear that any commutator of G^+ with Ψ must vanish. In particular this results in

$$G^+_{-1/2}\Psi_{-1/2}|0\rangle = \{G^+_{-1/2}, \Psi_{-1/2}\}|0\rangle = 0\,. \tag{2.4.18}$$

Therefore next to the vacuum also the highest weight $\Psi_{-1/2}|0\rangle$ is a chiral primary in the NS sector. By consulting (2.2.21) one sees that O and V close under fusion as they should. The NS chiral field with the highest conformal dimension is V whose conformal dimension is $h = 1/2$ such that we again deduce with help of (2.4.10) the central charge to be $c = 6h = 3$.

The spectral flow operator U_{-1} mapping the vacuum and V onto each other is V. Also the spectral flow $U_{\pm 1/2}$ mapping the R onto the NS sector can be found. From the fusion rules (2.2.21) $U_{\pm 1/2}$ must be S/C. Indeed the counting matches and every chiral NS primary is mapped onto a different Ramond groundstate.

2.5 Landau-Ginzburg model

In the following section we will first review $N = (2,2)$ Landau-Ginzburg (LG) models in two dimensions. Their conformal fix points will turn out to be the minimal superconformal series with $c < 3$. The second part will present more details about this minimal series by providing a description in terms of a coset.

2.5.1 Landau-Ginzburg models

We will use holomorphic coordinates $z, \overline{z}$ which in superspace are accompanied by Grassmanian coordinates $\theta^\pm, \overline{\theta}^\pm$. As in $z, \overline{z}$, the bar in $\theta^\pm, \overline{\theta}^\pm$ denotes the left- and right-moving piece while the superscript $\pm$ denotes the chirality of the spinor. The supersymmetry generators can be represented by

$$D_\pm = \frac{\partial}{\partial\theta^\pm} + \theta^\mp \frac{\partial}{\partial z}\,, \qquad \overline{D}_\pm = \frac{\partial}{\partial\overline{\theta}^\pm} + \overline{\theta}^\mp \frac{\partial}{\partial\overline{z}}\,. \tag{2.5.1}$$

We consider a chiral scalar field $\Phi = \Phi(z, \overline{z}, \theta^\pm, \overline{\theta}^\pm)$ obeying

$$D_+\Phi = \overline{D}_+\Phi = 0\,, \tag{2.5.2}$$

whose dynamics is governed by an action with a kinetic term and a potential term. In superspace language this reads

$$S = \int d^2z\ \ d\theta^+ d\theta^- d\overline{\theta}^+ d\overline{\theta}^-\ \ K(\Phi, \Phi^*) + \left(\int d^2z\ \ d\theta^- d\overline{\theta}^-\ \ W(\Phi) + c.c.\right)\,. \tag{2.5.3}$$

The integrand K of the D-term called Kähler potential must be real and leads to the kinetic terms. The integrand W of the F-term called superpotential must be a holomorphic function. To concretely see the purpose of each term we first solve the constraint (2.5.2). This is only easy when using the bosonic combination of the coordinates $y^{\pm} = z \mp \theta^+\theta^-$ and their right-moving counterpart since $y^-, \overline{y}^-$ and $\theta^-, \overline{\theta}^-$ are annihilated by D_+ and $\overline{D}_+$ while $y^+, \overline{y}^+, \theta^+, \overline{\theta}^+$ are not. Using these coordinates the constraint (2.5.2) is solved by

$$\begin{aligned}\Phi &= \Phi(y^-, \overline{y}^-, \theta^-, \overline{\theta}^-) \\ &= \phi(y^-, \overline{y}^-) + F(y^-, \overline{y}^-) \cdot \theta^-\overline{\theta}^- + \text{Fermionic parts}\,.\end{aligned} \tag{2.5.4}$$

Expanding y shows $F(y^-, \overline{y}^-) = F(z, \overline{z})$ and

$$\phi(y^-, \overline{y}^-) = \phi(z, \overline{z}) - \partial\phi \cdot \theta^+\theta^- - \overline{\partial}\phi \cdot \overline{\theta}^+\overline{\theta}^- + \tfrac{1}{2}\partial\overline{\partial}\phi \cdot \theta^+\theta^-\overline{\theta}^+\overline{\theta}^- \,. \tag{2.5.5}$$

Recalling that integration in superspace is the same as differentiation $\int d\theta = \frac{\partial}{\partial\theta}$ we see that the first integral in (2.5.3) picks out the $\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-$ term while the second integral picks out the $\theta^-\overline{\theta}^-$ term. Using this we can perform the integral to see which terms appear in a usual action $S = \int d^dx\mathcal{L}$. Inspection reveals the following bosonic terms:

- In K, we have a term proportional to $\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-$ when a $\overline{\partial}\phi \cdot \overline{\theta}^+\overline{\theta}^-$ term is multiplied with a $\partial\phi \cdot \theta^+\theta^-$ term or when $\phi(z, \overline{z})$ is multiplied with $\frac{1}{2}\partial\overline{\partial}\phi \cdot \theta^+\theta^-\overline{\theta}^+\overline{\theta}^-$. Both yield kinetic contributions $\sim \partial\phi\overline{\partial}\phi^*$ after an appropriate partial integration. The remaining terms $\sim \partial_\Phi\partial_{\overline{\Phi}}K$ may then not contribute further Grassmanians and can therefore only consists of more ϕ. The first class of terms is in total

$$\mathcal{L}_{\text{kin}} \sim \partial_\Phi\partial_{\overline{\Phi}}K|_{\vec{\theta}=0} \cdot \partial\phi\overline{\partial}\phi^* \,. \tag{2.5.6}$$

  For $K = \Phi\Phi^*$ we get the usual canonical kinetic terms. Notice that there are no kinetic terms for F which we therefore recognize as auxiliary fields.

- In K, there is a further term proportional to $\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-$ when two F meet each other in FF^*. Then the other terms may not contribute further Grassmanians and can yield more powers of ϕ. The terms are therefore

$$\sim |F|^2 \cdot \partial_\Phi\partial_{\overline{\Phi}}K|_{\vec{\theta}=0} \,. \tag{2.5.7}$$

- To have a non-vanishing contribution from the superpotential we need exactly one F and no further Grassmannian from the rest. The terms are therefore

$$\sim F \cdot \partial_\Phi W|_{\vec{\theta}=0} \tag{2.5.8}$$

Putting (2.5.8) together with (2.5.7) we find the equation of motion $F \sim \frac{\partial_\Phi W}{\partial_\Phi\overline{\partial}_{\Phi^*}K}|_{\vec{\theta}=0}$. Integrating out the auxiliary field F results in a potential

$$\mathcal{L}_{\text{pot}} \sim \left|\frac{\partial_\Phi W}{\partial_\Phi\overline{\partial}_{\Phi^*}K}|_{\vec{\theta}=0}\right|^2 \,. \tag{2.5.9}$$

For the canonical Kähler potential $K \sim \Phi\Phi^*$ and the soon to be justified $W \sim \Phi^{k+2}$ we have the bosonic terms

$$S_{\text{bos}} \sim \int d^2z\, \partial\phi\, \overline{\partial}\phi^* + g|\phi^{k+1}|^2 \tag{2.5.10}$$

in the action.

As a scalar is dimensionless in two dimensions the superpotential is always a relevant operator while the kinetic term is irrelevant. Consequently the superpotential is driving and fully determining the renormalization group (RG) flow. But remembering that loop diagrams contribute only integrals over the whole superspace $\int d^2zd^4\theta\ f_{\text{loop}}$ we see that just K is renormalized. So W is not only dictating the flow but is also an invariant. Now assume that we can choose K in such a way that K is infinitesimally small at the RG fixed point. Then not only the flow but also the theory at the fixed point is universally specified by the superpotential. We will not go into further detail about the hard non-perturbative question whether there exists a fixed point. Instead we will derive a simple criteria on the superpotential that must be fulfilled to have an RG fixed point. This discussion will follow [21]. At the conformal fixpoint the theory should be invariant under holomorphic dilations $z \to z' = \lambda^2 z$ inducing $\theta^{\pm\prime} = \lambda\theta^{\pm}$. The scale factor of this transformation can be read off from the transformation of the metric $g' = \lambda^2 g$. Recalling that $d\theta$ scales with the inverse of λ the measure in front of W transforms as $\int d^2zd\theta^+d\overline{\theta}^+ = \int d^2z'd\theta^{+\prime}d\overline{\theta}^{+\prime} \cdot \lambda^{-1}$. To be invariant, the factor λ^{-1} must be absorbed by a redefinition of the fields Φ which is only possible if the superpotential is quasi-homogeneous

$$\lambda\, W(\Phi) = W(\lambda^w\, \Phi)\,. \tag{2.5.11}$$

$w = 2h_\Phi$ is twice the conformal dimension h_Φ of Φ since the conformal transformation is $z' = \lambda^2 z$. For a single LG model this means that we have a single term in W which we write for convenience as

$$W = g\ \Phi^{k+2}\,. \tag{2.5.12}$$

Then w in (2.5.11) is $w = \frac{1}{k+2}$ and therefore $h_\Phi = \frac{1/2}{k+2}$. Along the same lines we can obtain $\overline{h}_\Phi = \frac{1/2}{k+2}$. Let us determine the (c,c) ring of the CFT at the fixed point. Next to the identity operator also Φ is a (c,c) primary by definition (2.5.2). Since the chiral primaries of the (c,c) ring close under fusion, also Φ^n defined by the fusion $[\Phi]^n$ is a chiral field. For $K \to 0$ the equation of motion is $\Phi^{k+1} = 0$. The (c,c) ring is therefore

$$\mathcal{R}_{(c,c)} = \frac{C[\Phi]}{\partial_\Phi W(\Phi)} = \{1, \Phi, \ldots, \Phi^k\} \tag{2.5.13}$$

The conformal dimension of an element Φ^n in the chiral ring can be deduced from $h_\Phi = \frac{1/2}{k+2}$ to be

$$h_{\Phi^n} = \frac{n/2}{k+2}. \tag{2.5.14}$$

As consistency check for the known relation $h = 2q$ let us deduce the charge from another viewpoint. From the superconformal algebra (2.4.1) one can read off that the $D^\pm = G^\pm$ have charge ± 1. Looking at (2.5.1) this is only possible if the θ have charge ± 1 as well. To be invariant, the integral $\int d^2z d\theta^- d\overline{\theta}^- \, W$ must have zero charge which is only the case if the superpotential has charge $(q, \overline{q}) = (1,1)$ and therefore $(q_{\Phi^{k+2}}, \overline{q}_{\Phi^{k+2}}) = (1,1)$. By charge conservation of the OPE one therefore has $q_{\Phi^n} = \overline{q}_{\Phi^n} = \frac{n}{k+2}$. Indeed everything turns out to be consistent and we have $h = \overline{h} = 2q = 2\overline{q}$ as expected for a field in the (C,C) ring.

The same analysis can be performed with the conjugate $\overline{\Phi}$. The conformal dimension is the same while the charge has a minus sign in front. Therefore the (a,a) ring is

$$\mathcal{R}_{(a,a)} = \frac{C[\overline{\Phi}]}{\partial_{\overline{\Phi}} W(\overline{\Phi})} = \{1, \overline{\Phi}, \ldots, \overline{\Phi}^k\} \tag{2.5.15}$$

Notice that we could just identify the elements of the (a,a) and the (c,c) rings. The reason that we could not find the (a,c) and (c,a) ring is that these rings are empty for pure LG models.

Let us turn to the central charge. We identify the state Φ^k as the chiral primary with the highest conformal dimension. Then according to theorem (2.4.10) the central charge of the CFT is

$$c = \frac{3k}{k+2} \,. \tag{2.5.16}$$

Therefore $c < 3$ which is the regime where one would expect the minimal rational unitary $N = 2$ SCFTs. Indeed, when computing the Kac-determinant to exclude nullstates and further excluding non-unitary theories one finds the same allowed conformal dimensions [22, 23, 24, 25, 26]. Due to more and more convincing evidence, e.g. from the comparison of the elliptic genus [27], it is well established that the minimal series of $N = 2$ SCFTs is realized by the conformal fixed points of LG models.

2.5.2 The minimal $N = 2$ SCFTs as coset

As it will turn out, the formulation of the minimal series as an LG model will be very handy to deduce the geometric interpretation of the Gepner models, but for a concrete description of the whole (non-chiral) spectrum a realization of the minimal models in terms of a coset is more suitable.[3] A good starting point is the $\widehat{\mathfrak{su}}(2)_k$ theory that already has the right central charge, see (2.2.34). But since $\widehat{\mathfrak{su}}(2)_k$ is not an $N = 2$ SCFT it cannot be the full answer. More hints about the coset can be deduced from the known chiral spectrum. Recall (2.5.14) that the chiral fields are labeled by one integer $l = 0, \ldots, k$ with $h = \frac{l/2}{k+2}$. Comparison with the conformal dimension of states in the $\widehat{\mathfrak{su}}(2)_k$ theory (2.2.35) shows that we have to subtract $\frac{l^2}{4(k+2)}$ to get the right conformal dimension for the chiral primaries.

[3] For an introduction to cosets consult e.g. [8, 9].

This is the conformal dimension of a state with quantum number l in the $\widehat{\mathfrak{u}}(1)_{k+2}$ theory, see (2.2.32). We therefore have to divide $\widehat{\mathfrak{su}}(2)_k$ by $\widehat{\mathfrak{u}}(1)_{k+2}$. But the levels in the numerator and the denominator do not match and the central charge subtraction by the $\widehat{\mathfrak{u}}(1)_{k+2}$ factor must be compensated. Both can be cured by putting an additional $\widehat{\mathfrak{u}}(1)_2$ into the coset. The correct coset for the $N = 2$ minimal SCFTs is therefore [28]

$$\frac{\widehat{\mathfrak{su}}(2)_k \times \widehat{\mathfrak{u}}(1)_2}{\widehat{\mathfrak{u}}(1)_{k+2}}\,. \tag{2.5.17}$$

Tracing the branching rules results in an indirect definition of the characters of the coset

$$\chi_l^{\widehat{su}(2)_k}(\tau)\;\chi_s^{\widehat{u}(1)_2}(\tau) = \sum_{m=-k-1}^{k+2} \chi_m^{\widehat{u}(1)_{k+2}}(\tau)\;\chi_{m,s}^l(\tau)\,, \tag{2.5.18}$$

from which one can show that

$$\chi_{m,s}^l(\tau) = \sum_{j=1}^{k+2} C^{(k)}_{l,m-4j-s}(\tau)\;\Theta_{-2m+(4j+s)(k+2)}(\tau)\,. \tag{2.5.19}$$

Here the string functions from the decomposition (2.2.44) of the $\widehat{\mathfrak{su}}(2)_k$ characters in terms of the $\widehat{\mathfrak{u}}(1)_k$ characters appear. Recall from the individual theories (2.2.36), (2.2.30) that the range of the indices is

$$0 \le l \le k, \qquad -k-1 \le m \le k+2, \qquad s = -1, 0, 1, 2\,. \tag{2.5.20}$$

By a closer inspections of (2.5.19) one can get more information about these quantum numbers:

- To have $\chi_{m,s}^l(\tau) \neq 0$ one must have

$$l + m + s \in 2\mathbb{Z}\,. \tag{2.5.21}$$

- There is an invariance under the shifts

$$s \to s + 4\,, \qquad m \to m + 2(k+2)\,, \tag{2.5.22}$$

  which amounts to defining s, m modulo 4 and $2(k + 2)$, respectively.

- (2.5.19) is invariant under

$$l \to k - l\,, \qquad m \to m + (k+2)\,, \qquad s \to s + 2\,, \tag{2.5.23}$$

  showing that the range (2.5.20) is actually a double cover of the spectrum.

In the following we will denote a state in the $N=2$ minimal SCFT (k) by the triplet

$$(l, m, s) \in (k) \tag{2.5.24}$$

subject to the above restrictions.

Applying a modular S-transformation onto (2.5.18) we get the S-matrix

$$\begin{aligned} S_{(l,m,s),(l',m',s')} &= (S^{\widehat{\mathfrak{u}}(1)_{k+2}})^{-1}_{m,m'} S^{\widehat{\mathfrak{su}}(2)_k}_{l,l'} S^{\widehat{\mathfrak{u}}(1)_2}_{s,s'} \\ &= \frac{1}{2k+4} \sin\left(\frac{\pi}{k+2}(l+2)(l'+2)\right) e^{-\pi i \left(\frac{ss'}{2} - \frac{mm'}{k+2}\right)}, \end{aligned} \tag{2.5.25}$$

where we plugged in (2.2.33) and (2.2.40) in the second line. Since the three indices (l, m, s) are independent, we can use the known individual partition functions of each constituent for the partition functions of the $N=2$ SCFTs. This means that the ADE classification of the modular invariants of the $\widehat{\mathfrak{su}}(2)_k$ theory classifies all modular invariants of the $n=2$ supersymmetric minimal models as well.

The conformal dimension of the highest weight states of the characters (2.5.19) in the coset is

$$h^l_{m,s} = h^{\widehat{\mathfrak{su}}(2)_k}_l + h^{\widehat{\mathfrak{u}}(1)_2}_s - h^{\widehat{\mathfrak{u}}(2)_{k+2}}_m = \frac{l(l+2)}{4(k+2)} + \frac{s^2}{8} - \frac{m^2}{4(k+2)}, \tag{2.5.26}$$

and for the charge with respect to the current of the $N=2$ superconformal algebra one finds

$$q_{m,s} = -\frac{m}{k+2} + \frac{s}{2}. \tag{2.5.27}$$

Actually the formulas for the conformal dimension and the charge are only correct if by the identifications (2.5.23) and (2.5.22) the state (l, m, s) is brought into the range

$$0 \leq |m - s| \leq l \qquad \text{and} \qquad s \in -1, 0, 1, 2\,. \tag{2.5.28}$$

But this is not always possible and there are two exceptions. In the NS sector there is the state $(l, -l, 2)$ with $l > 0$ for which the above formulas apply when using $s = -2$. The state $(0, 0, 2)$ has $h = \frac{3}{2}$ and $q = 1$ since $(0, 0, 2) = G^+_{-3/2}|0\rangle$.

From the discussion of the form of the coset or by demanding $2h = \pm q$ or by comparison with (2.5.14) it is clear that the chiral fields are

$$\begin{aligned} \text{chiral fields}: &\quad (l, l, 0) \\ \text{antichiral fields}: &\quad (l, -l, 0)\,. \end{aligned} \tag{2.5.29}$$

Further special states are, for instance, the spectral flow operators $U_{\pm 1/2} = (0, \mp 1, \mp 1)$ which mediate between NS and R sectors. The fusion rules are deduced from (2.5.25) using the Verlinde formula (2.2.10)

$$(l_1, m_1, s_1) \times (l_2, m_2, s_2) = \sum_{l_3 = |l_1 - l_2|}^{\min(l_1+l_2, 2k-l_1-l_2)} (l_3, m_1 + m_2, s_1 + s_2)\,. \tag{2.5.30}$$

Simple currents can be read off from this formula. Since the range (2.5.20) is a double cover we only need simple currents of the form

$$(0, l, m) \quad \text{where} \quad l + m \in 2\mathbb{Z}\,. \tag{2.5.31}$$

In particular the simple current which implements the D invariant is

$$J_D = (k, 0, 0) \widehat{=} (0, k+2, 2)\,. \tag{2.5.32}$$

The abelian algebra formed by the simple currents is [29]

$$\mathbb{Z}_{4(k+2)} \quad (k \;\text{ odd})\,, \qquad \mathbb{Z}_{2(k+2)} \times \mathbb{Z}_2 \quad (k \;\text{ even})\,, \tag{2.5.33}$$

where the elements of $\mathbb{Z}_{4(k+2)}$ and $\mathbb{Z}_{2(k+2)}$ are given by the simple currents $(0, i, i)$ and $i \in 0, \ldots, 4(k+2)$ or $i \in 0, \ldots, 2(k+2)$ respectively. For even k one can see that there are simple currents that are not of the form $(0, i, i)$. In case $k = 8$, the simple current implementing the D-invariant is $(0, 10, 10) \widehat{=} (0, 10, 2) = J_D$. But for $k = 6$, $J_D = (0, 8, 2) \widehat{=} (0, 8, 10)$ is not of the form $(0, i, i)$. The missing piece is provided by the additional $\mathbb{Z}_2$ factor that stands for the simple current $(k, k+2, 0)$. For $k = 6$ the simple current of the D-invariant J_D is the product $J_D = (6, 8, 0) \times (0, 8, 8) = (6, 16, 8) \widehat{=} (6, 0, 0)$.

2.6 The geometry of Landau-Ginzburg models

This section deals with the geometric interpretation of the LG models. This section will follow mainly [30] and the nice review [31] but uses a more coherent convention than the literature.

We will analyze LG models with r chiral superfields which we denote by $(X_1, \ldots X_r)$. A necessary requirement to have a conformal fixed point is that the superpotential must be quasi-homogeneous

$$W(\lambda^{w_1} X_1, \ldots, \lambda^{w_r} X_r) = \lambda^d W(X_1, \ldots X_r)\,. \tag{2.6.1}$$

This ensures that conformal rescalings can be absorbed into a redefinition of the fields X_i. The equation (2.6.1) suggests to regard the X_i as coordinates of the complex weighted projective space $W\mathbb{CP}^{(r-1)}_{w_1,\ldots,w_r}$ defined by the equivalence relation

$$(X_1, \ldots, X_r) \sim (\lambda^{w_1} X_1, \ldots, \lambda^{w_r} X_r)\,, \tag{2.6.2}$$

where $\lambda \in \mathbb{C}^* = \mathbb{C}/\{0\}$. Here $(r-1)$ is the complex dimension and the w_i are the weights of the coordinates X_i. In this light (2.6.1) forces the superpotential W to be a quasi-homogeneous polynomial of degree d in the weighted projective space $W\mathbb{CP}^{(r-1)}_{w_1,\ldots,w_r}$.

The following calculations are relying only on (2.6.1) without referring to a concrete realization of the superpotential. Nevertheless let us state the form of the superpotential that will appear later in Gepner models. Since we will take the direct tensor product of minimal models, we have a Fermat polynomial as the superpotential

$$W = \sum_{j=1}^{r} X_j^{k_j+2} \,. \tag{2.6.3}$$

For this choice we have $d = w_i(k_i+2)\ \forall i$ which is satisfied if d is the least common multiple of all (k_i+2).

Coming back to the weighted projective space, recall that the point $(0,\dots,0)$ is excluded from $W\mathbb{CP}^{r-1}_{w_1,\dots,w_r}$. The X_i are therefore not local coordinates. Instead one finds r patches defined by $X_j \neq 0$ such that the remaining X_i with $i \neq j$ can take values in $\mathbb{C}^{r-1}$. Let us go to the patch $j=1$ where $X_1 \neq 0$ and introduce new coordinates ξ_i by

$$\xi_1^{\frac{w_1}{d}} = X_1 \,, \qquad\qquad \xi_i = \frac{X_i}{\xi_1^{\frac{w_i}{d}}} \,. \tag{2.6.4}$$

The reason for this special coordinate transformation lies in the particularly simple transformation of the superpotential which is

$$W(X_1, X_2, \dots, X_r) = \xi_1 \cdot W(1, \xi_2, \dots, \xi_r) \,. \tag{2.6.5}$$

We plug this into the path integral in the limit $K \to 0$

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}X_1 \dots \mathcal{D}X_r \;\; e^{i\int d^2z\, d^2\theta^- W(X_1,\dots,X_r)+c.c.} \\ &= \int \mathcal{D}\xi_1 \dots \mathcal{D}\xi_r \;\; J \;\; e^{i\int d^2z\, d^2\theta^- \xi_1 \cdot W(1,\xi_2,\dots,X_r)+c.c.} \,, \end{aligned} \tag{2.6.6}$$

where the Jacobi determinant J is computed to be

$$J = \frac{w_1}{d} \cdot \xi_1^{\left(\sum_i \frac{w_i}{d} - 1\right)} \,. \tag{2.6.7}$$

When this Jacobi determinant is constant we can integrate over ξ_1 to get a $\delta(\int W)$ that enforces $W=0$. Therefore

$$\sum_i w_i = d \qquad \Rightarrow \qquad W = 0 \,. \tag{2.6.8}$$

For a non-zero Kähler potential this picture does not change severely and instead of the delta function we get e.g. a Gaussian peak if the canonical Kähler potential is selected (see the appendix of [30]). We see that the quantum amplitude receives only contributions if the equation $W=0$ is satisfied such that the fields X_i can only propagate on the $(r-2)$-dimensional hypersurface $W=0$ of degree d. Therefore the conformal fixed point of the above LG model describes a hypersurface of degree d in a weighted projective space which we write as

$$W\mathbb{CP}^{r-1}_{w_1,\dots,w_r}[d] \,. \tag{2.6.9}$$

It is well known that the first Chern class of such a hypersurface is vanishing precisely upon imposing (2.6.8) making the hypersurface Calabi-Yau. Let us make some comments:

The freedom to choose the defining polynomial of the Calabi-Yau corresponds to the choice of complex structure. The deformations of the complex structure preserving the Ricci-flatness translate into small deformations of the superpotential $W' = W + \epsilon \cdot [d]$ where $[d]$ is a polynomial of degree d. Since the charge and the conformal dimension of the superpotential are $q_W = (1,1)$ and $h_W = (\frac{1}{2}, \frac{1}{2})$, the complex structure deformations are therefore in 1:1 correspondence to the chiral fields with $h = (\frac{1}{2}, \frac{1}{2})$ on the CFT side.[4] Now, $h = (\frac{1}{2}, \frac{1}{2})$ is precisely the condition for a field in the NS-NS sector of string theory to be massless. As discussed in section 2.1, the complex structure deformations should be massless from the compactification viewpoint as well. To summarize when using the above construction as internal CFT in a string theory, the massless modes are precisely the ones that are expected from geometric compactification on the associated Calabi-Yau. It is natural to conjecture that using an LG model as internal CFT describes strings propagating on the corresponding Calabi-Yau manifold defined by $W = 0$.

A very important detail that has been disregarded so far: The coordinate change from the X to the ξ is actually not one-to-one since

$$X_i = \xi_i \cdot \xi_1^{\frac{w_i}{d}} = \xi_i \cdot (\xi_1 e^{2\pi i})^{\frac{w_i}{d}} = e^{2\pi i \frac{w_i}{d}} \cdot \xi_i \cdot (\xi_1)^{\frac{w_i}{d}} = e^{2\pi i \frac{w_i}{d}} \cdot X_i \,. \tag{2.6.10}$$

Recalling $\frac{w_i}{d} = \frac{1}{k_i+2} = q_{X_i}$ one sees that ξ and X are only one-to-one for states with integer charge. For the above discussion to hold one needs to restrict to integer charge. Thus only Landau Ginzburg orbifolds where one divides out by $e^{2\pi i q}$ have a geometric interpretation. That such an orbifold is always possible in LG models was shown and explicitly computed in [32]. The attentive reader will have noticed that the $q \in \mathbb{Z}$ condition has already appeared in the discussion around (2.4.15). There this condition ensures an unambiguous spectral flow operator $U_{1/2}$ between the NS and the R sector. It is very intriguing that this relation gets an interpretation in LG models.

Another important point about the LG orbifold is that the twisted sectors of the $e^{2\pi i q}$ orbifold add states to the (a,c) ring which is empty in pure LG models. Following the discussion after (2.4.7), the (a,c) ring corresponds to the Kähler moduli of the Calabi-Yau.

2.6.1 The Calabi-Yau condition in LG models

One can use the LG orbifold of the last section as the internal conformal field theory in a string CFT. The CFT should then correspond to strings propagating on the Calabi-Yau manifold that is associated to that LG orbifold. That this works depends crucially on the Calabi-Yau condition (2.6.8). Let us test in several examples whether (2.6.8) can be satisfied.

In order to have a proper type II theory, the total central charge has to add up to 12. Assuming D non-compact directions the LG orbifold must have central charge $(10 - D) \cdot \frac{3}{2}$.

[4] A thing to notice here: In LG models there are no chiral fields representing $X_i^{k_i+1}$ and $X_i^{k_i+2}$. The geometric counterpart for this is that these deformations can be eliminated by a coordinate redefinition.

The central charge of an LG model with fields X_i is the sum of all individual central charges such that

$$\sum_i c_i = (10 - D)\,\frac{3}{2}\,. \tag{2.6.11}$$

where $c_i = \frac{3k_i}{(k_i+2)}$. In the following we will specialize to the most interesting case with $D = 4$ and $\sum_{\min} c_i = 9$. In a first example let us tensor five $k = 3$ minimal models which we write as $k_i = (3,3,3,3,3)$. When taking the direct tensor product the superpotential of the LG model is the Fermat-polynomial

$$W_{3^5} = X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5\,. \tag{2.6.12}$$

The weights are $w_i = 1$ and $d = 5$ such that indeed $\sum w_i = d$ as demanded by the Calabi-Yau condition (2.6.8). The associated Calabi Yau is $\mathbb{P}_{1,1,1,1,1}[5]$, the famous quintic. Another example is $k_i = (1,4,4,4,4)$ with superpotential

$$W_{1,4^4} = X_1^3 + X_2^6 + X_3^6 + X_4^6 + X_5^6\,. \tag{2.6.13}$$

We read off $w_i = (2,1,1,1,1)$ and $d = 6$ such that again $\sum w_i = d$ and the Calabi-Yau is $W\mathbb{CP}_{2,1,1,1,1}[6]$. Similar for $k_i = 2,2,2,6,6$ the superpotential is

$$W_{2^3,6^2} = X_1^4 + X_2^4 + X_3^4 + X_4^8 + X_5^8\,. \tag{2.6.14}$$

The weights are $2,2,2,1,1$ and $d = 8$ as it should. The corresponding Calabi-Yau is $W\mathbb{CP}_{2,2,2,1,1}[8]$.

As one can see, the condition that the central charges add up to 9 guarantees that the weights and the degree of the polynomial obey

$$\sum_{i=1}^{5} w_i = d\,. \tag{2.6.15}$$

This means that we always have an interpretation as a Calabi-Yau when tensoring five minimal models with total central charge 9.

All examples consisted of 5 tensored minimal models, so let us also check what happens for 4 minimal models. A first example is $k_i = (6,6,6,6)$ which directly translates into $w_i = (1,1,1,1)$ but $d = 8$. As one can see, $\sum w_i \neq d$, and both sides differ by four. Exactly the same happens for $k_i = (2,10,10,10)$. Here $w_i = 2,1,1,1$ and $d = 12$ so that now the difference between $\sum w_i$ and d is 6. The solution to this puzzle lies in the observation, that in both cases

$$\sum_{i=1}^{4} w_i + \frac{d}{2} = d\,. \tag{2.6.16}$$

This can be interpreted as an additional fifth minimal model with weight $w_5 = \frac{d}{2}$ thus $k_5 = 0$. Since its chiral sector is trivial and by the equation of motion $X_5 = 0$, it does not change anything geometrically. The superpotentials are therefore

$$\begin{aligned} W_{6^4} &= X_1^8 + X_2^8 + X_3^8 + X_4^8 + X_5^2 \,, \\ W_{2,10^3} &= X_1^4 + X_2^{12} + X_3^{12} + X_4^{12} + X_5^2 \,. \end{aligned} \tag{2.6.17}$$

The need of a fifth variable in models with four tensored minimal models can also be seen by dimensional analysis. To have a hypersurface Calabi-Yau with three complex dimensions we need to have a $W\mathbb{CP}$ with five coordinates thus five tensored minimal models. These five dimensions get reduced to the three of the Calabi-Yau by the projective freedom and by the polynomial constraint. It follows that we have only a geometric interpretation as hyperplane Calabi-Yau compactification for four or five tensored minimal models. Less than four models cannot add up to $c = 9$ and more do not have the correct dimension.

2.6.2 ADE invariants in LG models

The following section deals with the question how to translate the ADE classification of the partition functions of the minimal models (2.2.42) into the LG language. For this recall another ADE classification in the context of $SU(2)$: Defining $\xi_k = e^{\frac{2\pi i}{k}}$ and $w_k = \mathrm{diag}(\xi_k, (\xi_k)^{-1})$ the discrete subgroups of $SU(2)$ are classified as follows (see e.g. appendix C of [33]):

- A_{k+1} has the generators w_{k+2} and is the cyclic group of order $k+2$.
- D_p, with $p \geq 4$ and generators w_{2p} plus an additional $\mathbb{Z}_2$ generator τ that can be expressed in terms of the second Pauli matrix $\tau = i\sigma_2$. It is the binary dihedral group and of order $4p - 8$.
- E_6 with generators w_4 and

$$\kappa_6 = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_8^7 & \xi_8^7 \\ \xi_8^5 & \xi_8 \end{pmatrix} . \tag{2.6.18}$$

  It forms the binary tetrahedral group of order 24.
- E_7 with generators w_4, κ_6 as for E_6 but an additional w_8 forming the binary octahedral group of order 48
- E_8 with generators $\alpha = -\mathrm{diag}(\xi_5^3, \xi_5^2)$ and

$$\kappa_4 = \frac{1}{\xi_5^2 - \xi_5^3} \begin{pmatrix} \xi_5 + \xi_5^4 & 1 \\ 1 & -\xi_5 - \xi_5^4 \end{pmatrix} \tag{2.6.19}$$

  named the binary icosahedral group of order 120.

Dividing $\mathbb{C}^2$ by one of the above discrete subgroups $\Gamma \subset SU(2)$ we get a space $\mathbb{C}^2/\Gamma$ with a singularity at the origin. To see how we can characterize this singularity take the example $\mathbb{C}^2/A_{k+1}$. Let us denote the coordinates of $\mathbb{C}^2$ as X, Y. The A_{k+1} invariant polynoms are X^{k+2}, XY and Y^{k+2} which we use to define new variables $W_1 = X^{k+2}$, $W_2 = Y^{k+2}$ and $Z = XY$ subject to the identification $Z^{k+2} = W_1W_2$. After the coordinate transformation $-W_1 = A + iB$ and $W_2 = A - iB$ we get the identification

$$0 = A^2 + B^2 + Z^{k+2} \tag{2.6.20}$$

We can summarize what we have done in

$$\frac{\mathbb{C}[X,Y]}{A_{k+2}} = \mathbb{C}[XY, X^{k+2}, Y^{k+2}] := \frac{\mathbb{C}[Z, W_1, W_2]}{Z - XY} = \frac{\mathbb{C}[Z, A, B]}{A^2 + B^2 + Z^{k+2}} \tag{2.6.21}$$

This construction embeds $\mathbb{C}^2/A_{k+1}$ into the space $\mathbb{C}^3$ with the constraining equation (2.6.20). Using similar steps for the other discrete subgroups of $SU(2)$ one gets the identifications

- A_{k+1}: $0 = y^2 + x^2 + z^{k+2}$
- D_p: $0 = y^2 + x^2 z + z^{p-1}$
- E_6: $0 = y^2 + x^3 + z^4$
- E_7: $0 = y^2 + x^3 + xz^3$
- E_8: $0 = y^2 + x^3 + z^5$

The above polynomials fully characterize the space $\mathbb{C}^2/\Gamma$ and most importantly the singularity at the origin. We have therefore an ADE classification of singularities widely used e.g. in F-theory.

To connect this classification to LG models the above relations are interpreted as superpotentials of an LG model [21]. Having more than one variable in the polynomial means that we have tensored LG models. But e.g. in the A_{k+1} polynomial the y^2 and the x^2 correspond to LG models with trivial chiral ring. They are therefore also geometrically trivial and we can forget them. Therefore a LG model with superpotential z^{k+2} describes a minimal model with the A invariant partition function. Similarly a minimal model with D partition function is a LG model with superpotential $W_D = z^{p-1} + x^2 z$. This form of the D invariant can be understood another viewpoint when comparing the A and D superpotentials. Under the coordinate transformation

$$z_A = z_D^{\frac{1}{2}} \qquad \text{and} \qquad x_A = x_D \cdot z_D^{\frac{1}{2}} \tag{2.6.22}$$

the polynomials of the A and D series are mapped onto each other [34]. This transformation is not bijective unless we identify $x_A \sim -x_A$ and $z_A \sim -z_A$.[5] Imposing the identification in

[5]This $\mathbb{Z}_2$ identification reminds us of the additional $\mathbb{Z}_2$ generator τ of the D_p subgroup of $SU(2)$ and that the simple current implementing the D invariant has length two.

an A model we get twisted states and the D invariant is reproduced. From the polynomial for the D invariant the weights can be read off as $w_1 = \frac{2d}{k+2}$ and $w_2 = \frac{d}{2} - \frac{d}{k+2}$ with $d = k+2$. Notice that the D invariant changes the geometric interpretation severely.

2.6.3 Summary

In the last two sections we exemplified how to translate hypersurface Calabi-Yau manifolds into the LG language and vice versa. We looked at prototype examples that should give an idea how to treat more general models. A detailed dictionary can be found in [35] where all cases are brought together.

2.7 Hodge numbers of Calabi-Yau manifolds

This section explains how to compute the Hodge numbers of Calabi-Yaus and exemplify this in the examples we need later. This discussion will mostly follow [35]. Although we do not use toric resolutions, a background in toric geometry is recommended, see e.g. [31], the appendix of [36] or [37]. The aim is to compute the Hodge data of a Calabi-Yau constructed as hypersurface or complete intersection Calabi-Yau in a product of weighted projective spaces.

First we generalize the notation of (2.6.9),

$$W\mathbb{CP}^{n-1}_{w_1,\ldots,w_n}[d]\,, \tag{2.7.1}$$

that denoted a hypersurface Calabi-Yau defined by a polynomial of degree d in a weighted projective space $W\mathbb{CP}^{n-1}_{w_1,\ldots,w_n}$. Instead of using a single polynomial one can also define a Calabi-Yau through more polynomials. The Calabi-Yau is then the intersection of several hypersurfaces defined by polynomials and therefore called complete intersection Calabi-Yau. The notation captures this by writing $[d_1\ d_2\ldots]$ where d_i is the degree of the i-th polynomial. One can further generalize this by taking tensor products of weighted projective spaces. To see what the polynomials look like in this case we refer to the later example around (2.7.14). We write the different $W\mathbb{CP}$ in different lines such that we end up with the notation

$$\begin{matrix} W\mathbb{CP}^{n_1}_{w_{11},w_{12},w_{13},w_{14},\ldots} \\ W\mathbb{CP}^{n_2}_{w_{21},w_{22},\ldots} \\ \vdots \end{matrix} \begin{bmatrix} d_{11} & d_{12} & \ldots \\ d_{21} & d_{22} & \\ \vdots & & \ddots \end{bmatrix}. \tag{2.7.2}$$

Being agnostic about possible singularities one can compute the Euler number of this Calabi-Yau by expanding

$$\frac{\prod_i \prod_j (1 + w_{ij} J_i) \times \prod_j \sum_i d_{ij} J_i}{\prod_j (1 + \sum_i d_{ij} J_i) \times \prod_i \prod_j w_{ij}}\,, \tag{2.7.3}$$

and looking at the coefficient of $\prod_i J_i^{n_i}$. Here J_i is the hyperplane (1,1) form of every weighted projective space. Usually the ambient space contributes as many Kähler moduli as there are $W\mathbb{CP}$s. For instance for $M = \mathbb{P}_{1,1,1,1,1}[5]$ we find

$$\frac{(1+J)^5 \times 5J}{(1+5J)} = 5J + 50J^3 - 200J^4 + \dots\,, \tag{2.7.4}$$

from which we read off $\chi(M) = -200$. Since the Calabi-Yau inherits one Kähler modulus from the ambient space we have $h^{1,1} = 1$ and due to

$$\chi(M) = 2(h^{1,1} - h^{1,2}) \tag{2.7.5}$$

we find $h^{1,2} = 101$. This is the famous quintic.

Now imagine two weights have a common divisor, e.g. the two weights $w_5 = w_6 = 2$ in $\mathbb{P}_{1,1,1,1,2,2}$. In this example the projective rescaling by $\lambda = -1$ leaves the two variables w_5 and w_6 invariant. As such when setting $X_1 = \dots X_4 = 0$ we get a subspace of the projective space which is $\mathbb{Z}_2$ singular since it is a fixed point of the $\lambda = -1$ rescaling. If this singularity furthermore intersects the hyperplane or the intersection that defines the Calabi-Yau, also the Calabi-Yau is singular in a point or curve D. This is cured by excising the singular part D and replacing it with a non-singular space. When doing so for the case of a $\mathbb{Z}_N$ singularity the Euler number changes as

$$\chi(M) = \chi_{\text{sing}} - \frac{1}{N}\chi(D) + N\,\chi(D)\,. \tag{2.7.6}$$

Let us look at a complete intersection Calabi-Yau in the singular ambient space of above, $M = \mathbb{P}_{1,1,1,1,2,2}[5\ 3]$. Without resolution the Euler number is $\chi_{\text{sing}} = -165$ which is clearly not an even number thus not viable according to (2.7.5). Let us take a look at the polynomials defining the Calabi-Yau to see whether the Calabi-Yau inherits the $\mathbb{Z}_2$ singularity from the ambient space. We name the variables with weight one X_i and the variables with weight two Y_i. Using the notation $[X^d]$ for a polynomial of power d in the variables X_i the polynomials defining the Calabi-Yau are schematically

$$\begin{aligned} 0 &= [X^5] + [X^3]\cdot[Y] + [X]\cdot[Y^2]\,,\\ 0 &= [X^3] + [X]\cdot[Y]\,. \end{aligned} \tag{2.7.7}$$

When going to the location of the singularity where $X_i = 0$ both polynomial constraints are automatically satisfied. This shows that the Calabi-Yau manifold shares the whole singularity of the ambient space. The singularity spans a $\mathbb{P}_{1,1}$ whose Euler number is $\chi(\mathbb{Z}_2) = 2$. When excising this singularity with (2.7.6) the total Euler number is $\chi(M) = -162$.

When resolving a singularity one excises the singular parts and inserts new curves whose size is controlled by $(1,1)$ forms. Let us list the additional forms from the blow-ups

- Resolving a singular curve of order N adds $(N-1)$ (1,1)-forms.

- Resolving a singular point of order N adds $\frac{1}{2}(N-1)$ (1,1)-forms.
- If on a singular curve of order N there are singular points of order $N \cdot M$, for each point one gets $\frac{1}{2}(N-1)$ (1,1)-forms.

Therefore in the above example we get one additional (1,1)-form and the full Hodge data is

$$M = \mathbb{P}^5_{1,1,1,1,2,2}[5\ 3]^{h^{1,1}=2, h^{1,2}=83} . \tag{2.7.8}$$

This result agrees with the known data in the literature [38] (see also [39, 40]).

Often, two singularities are on top of each other as one can infer in the example $M = \mathbb{P}_{1,1,2,4,4}[12]$. First M has a $\mathbb{Z}_2$ singularity and secondly a $\mathbb{Z}_4$ singularity. When setting $X_1 = X_2 = 0$ we are on top of the $\mathbb{Z}_2$ singularity. To see how much of the ambient space singularity intersects the Calabi-Yau we look at the polynomial on the singularity

$$0 = [X_3^6] + [X_4^3] + [X_5^3] , \tag{2.7.9}$$

where we only wrote down the Fermat polynomials. This equation defines the space $C_2 = \mathbb{P}_{1,2,2}[6]$. But actually, the singular space C_2 again has a $\mathbb{Z}_2$ singularity along $C_4 = \mathbb{P}_{1,1}[3]$ that is the $\mathbb{Z}_4$ singularity in $M = \mathbb{P}_{1,1,2,4,4}[12]$. Therefore both singularities lie on top of each other. For the Euler numbers we find using (2.7.3)

$$\chi_M^{\text{sing}} = -\frac{795}{4} , \qquad \chi(C_2) = -\frac{3}{2} , \qquad \chi(C_4) = 3 . \tag{2.7.10}$$

To not excise the $\mathbb{Z}_4$ twice, we cut the $\mathbb{Z}_4$ out of the $\mathbb{Z}_2$

$$\chi\left(\frac{C_2}{C_4}\right) = \chi(C_2) - \frac{1}{2} \cdot \chi(C_4) = -3 . \tag{2.7.11}$$

The full Euler number is

$$\begin{aligned} \chi(M) &= \chi^{\text{sing}}(M) - \frac{1}{2}\chi\left(\frac{C_2}{C_4}\right) - \frac{1}{4}\chi(C_4) + 2 \cdot \chi\left(\frac{C_2}{C_4}\right) + 4 \cdot \chi(C_4) \\ &= -192 . \end{aligned} \tag{2.7.12}$$

The resolution of the singularities added four Käehler moduli to the existing one such that we have $h^{1,1} = 5$ and $h^{1,2} = 101$.

The next example is $M = \mathbb{P}_{1,1,1,2,3}[8]$ which at first glance has a $\mathbb{Z}_2$ and a $\mathbb{Z}_3$ singularity. Since three does not divide 8, the weight three variable is always accompanied by another variable in the polynomial of degree eight. Therefore as in (2.7.7) the full singularity is on top of the Calabi-Yau. The singular is space is $\mathcal{C}_3 = \mathbb{P}_1$ which is a point with $\chi(\mathcal{C}_3) = 1$. When looking at the $\mathbb{Z}_2$ singularity we set all other variables to zero to be left with the polynomial $X_4^4 = 0$. This is only satisfied for $X_4 = 0$ but the point where all $X_i = 0$ is not in the ambient space. Therefore there the $\mathbb{Z}_2$ singularity is not shared by the Calabi-Yau.

Using (2.7.3) we find $\chi^{\text{sing}}(M) = -\frac{632}{3}$ and the Euler number of the resolved Calabi-Yau is

$$\chi(M) = \chi^{\text{sing}}(M) - \frac{1}{3}\chi(\mathcal{C}_3) + 3\chi(\mathcal{C}_3) = -208\,. \tag{2.7.13}$$

Since we excised a point, we get one additional Kähler modulus and in total $h^{1,1} = 2, h^{1,2} = 106$.

Let us come to another example,

$$\mathcal{M} = \begin{matrix}\mathbb{P}^4_{1,1,2,3,4}\\ \mathbb{P}^1_{1,1}\end{matrix}\begin{bmatrix}7 & 4\\ 1 & 1\end{bmatrix}\,. \tag{2.7.14}$$

There are three singularities, a $\mathbb{Z}_2$, $\mathbb{Z}_3$ and a $\mathbb{Z}_4$. Similar in the last example, the $\mathbb{Z}_4$ is on top of the $\mathbb{Z}_2$ singularity. Let us denote the coordinates of the $\mathbb{P}^4_{1,1,2,3,4}$ as X_i and the coordinates of the $\mathbb{P}^1_{1,1}$ as Y_i.

We start with the $\mathbb{Z}_4$ singularity. The weight 4 does not divide the degree 7 giving a similar polyonomial as (2.7.7). Setting all coordinates except for the one with weight 4 to zero therefore sets the polynomial of degree 7 to zero. The second equation is

$$X_5Y_1 + X_5Y_2 = 0 \tag{2.7.15}$$

This equation is satisfied for $Y_1 = -Y_2$ which is a point in $\mathbb{P}^1$. Therefore the $\mathbb{Z}_4$ singularity intersects the Calabi-Yau at a point $\mathcal{P}_4$ whose Euler number is $\chi(P_4) = 1$

Next the $\mathbb{Z}_3$ singularity. The weight three does neither divide seven nor four, so in every polynomial X_4 comes with other variables. Setting them to zero to get to the locus of the singularity sets both polynomials to zero. As such the Calabi-Yau shares the full singularity of the ambient space, which is a $\mathcal{C}_3 = \mathbb{P}^1$. The Euler number is $\chi(C_3) = 2$.

Lastly the $\mathbb{Z}_2$ singularity: Both 2 and 4 do not divide 7, so the first polynomial is automatically zero. When going to the singularity $X_1 = X_2 = X_4 = 0$ the second polynomial becomes

$$0 = X_3^2Y_i + X_4Y_i\,, \tag{2.7.16}$$

which is the defining relation of the space

$$\mathcal{C}_2 = \begin{matrix}\mathbb{P}_{1,2}\\ \mathbb{P}_{1,1}\end{matrix}\begin{bmatrix}2\\ 1\end{bmatrix}\,. \tag{2.7.17}$$

Let us collect the relevant Euler numbers

$$\begin{gathered}\chi(\mathcal{C}_3) = 2\,, \qquad \chi(\mathcal{P}_4) = 1\,,\\ \chi(\mathcal{C}_2/\mathcal{P}_4) = \chi_F(\mathcal{C}_2) - \tfrac{1}{2}\chi(\mathcal{P}_4) = 1\,,\end{gathered} \tag{2.7.18}$$

where we excised the $\mathbb{Z}_4$ out of the $\mathbb{Z}_2$ since they lie on top of each other. Furthermore

$$\chi^{\text{sing}}(\mathcal{M}) = -\frac{1471}{12}\,. \tag{2.7.19}$$

The resolved Calabi-Yau has the Euler numbers

$$\begin{aligned}\chi(\mathcal{M}) = \chi^{\text{sing}}(\mathcal{M}) - \tfrac{1}{2}\chi(\mathcal{C}_2/\mathcal{P}_4) - \tfrac{1}{3}\chi(\mathcal{C}_3) - \tfrac{1}{4}\chi(\mathcal{P}_4)\\ + 2\chi(\mathcal{C}_2/\mathcal{P}_4) + 3\chi(\mathcal{C}_3) + 4\chi(\mathcal{P}_4)\\ = -112\,.\end{aligned} \tag{2.7.20}$$

The resolution adds the following extra Kähler moduli

$$\mathbb{Z}_2 : h^{1,1} = 1\,, \qquad \mathbb{Z}_3 : h^{1,1} = 2\,, \qquad \mathbb{Z}_4 : h^{1,1} = 1\,. \tag{2.7.21}$$

The full data of the resolved Calabi-Yau is $(h^{2,1}, h^{1,1}) = (62, 6)$.

2.8 Gepner models

Having gathered all ingredients, it is only a small step towards constructing the CFT describing string motion in a Calabi-Yau space.

In lightcone gauge we have effectively $D-2$ non-compact directions that, from the worldsheet point of view, are described by $D-2$ free bosons ϕ_i accompanied by their fermionic partner. These fermions transform in the vector representation of $SO(D-2)$ such that we need a $\widehat{\mathfrak{so}}(D-2)_1$ theory. Following the discussion about the geometry of Landau-Ginzburg models we use tensored minimal models to describe the internal space. The complete theory is

$$\bigotimes_{i=1}^{r} k_i \otimes \widehat{\mathfrak{so}}(D-2)_1 \otimes (D-2)\,, \tag{2.8.1}$$

with the restriction

$$\sum_{i=1}^{r} c_i = \sum_{i=1}^{r} \frac{3k_i}{k_i+2} = (10-D)\cdot\frac{3}{2}\,, \tag{2.8.2}$$

to have a total central charge

$$c_{\text{tot}} = \sum_{i=1}^{r} c_i + c_{\widehat{\mathfrak{so}}(D-2)_1} + c + (D-2) = (10-D)\cdot\frac{3}{2} + \frac{D-2}{2} + (D-2) = 12 \tag{2.8.3}$$

as required for type II. Neglecting the bosonic part we can label the highest weights of this theory by

$$(l_1, m_1, s_1)\ldots(l_r, m_r, s_r)(s_0) \in \bigotimes_{i=1}^{r} k_i \otimes \widehat{\mathfrak{so}}(D-2)_1\,. \tag{2.8.4}$$

The total conformal dimension and the $U(1)$ charge of the highest weights are simply the sum of the individual ones. Stating a theory with the right central charge is easy but actually building a string vacuum is a hard task and the great achievement of Gepner in his paper [41]. Notice that his work was prior to any knowledge of the geometric background of LG models and a systematic understanding of $N = 2$ SCFTs. In [42] the connection to Calabi-Yau manifolds was found by Gepner as well shortly after his first paper. There he also gave more evidence for the connection by computing Yukawa couplings.

In the following we will not use his β-vector method to implement projections on the spectrum but project using simple currents. We will first construct a bosonic partition function and then map it to a superstring partition function using the bosonic string map from page 18. Recall that it maps the characters $(O, V, S, C)_{D-2}$ on $(V, O, -C, -S)_{D+6}$ such that modular invariance and level matching is preserved.

The first requirement is to have odd charges. As discussed after (2.4.15) this will project onto a supersymmetric spectrum with states that have conformal dimension $h = 1/2, 3/2, \ldots$. Since the bosonic string map shifts all dimensions by 1/2, the charges get shifted by 1, such that the bosonic partition function we construct first needs even charges.

Let us implement this projection using a simple current. For simplicity, we first try whether the easiest class of simple currents, the orbit simple currents that have integer monodromy charge, thus $r = 0$, works out. Recalling (2.3.16) we see that we need a simple current that assigns the monodromy charge $Q = \frac{q}{2} \bmod 1$ to every state. Plugging this into the defining relation of the monodromy charge (2.3.5) together with the form of simple currents in minimal models (2.5.31) results in

$$J_{\text{GSO}} = (0, 1, 1)^r, (S)_{D+6} \,. \tag{2.8.5}$$

The second requirement is that there must be an overall $N = 1$ SCFT to have a proper type II string theory. We choose the total supercurrent to be the sum of the individual supercurrents

$$G_{\text{tot}} = \sum_{j=1}^{r} G_j + :\partial_z X^\mu \, \psi_\mu: \ . \tag{2.8.6}$$

The total R(NS) sector is then the tensor product of all individual R(NS) sectors. As such states which mix the NS and R sector are not allowed. Projecting on pure NS/R states can be achieved via an orbit simple current as well. We will exploit that the sector of the $\widehat{\mathfrak{so}}(D+6)_1$ factor must match sector of every minimal model. One can therefore split the projection in r individual projections that ensure that the r'th minimal factor has the same sector as the $\widehat{\mathfrak{so}}(D+6)_1$. In minimal models the label s determines whether a state is in the R or NS sector. Therefore the monodromy charge should depend only on s. This is easily seen to be the case if the simple current has $l = m = 0$. Playing around with the definition (2.3.5) one sees that the right simple current is

$$J_i = (0, 0, 0) \ldots (0, 0, 2)_i \ldots (0, 0, 0), (S)_{D+6} \ . \tag{2.8.7}$$

It correctly projects onto pure $\text{NS}_i \otimes \text{NS}$ and $\text{R}_i \otimes \text{R}$ states between the i'th minimal factor and the $\widehat{\mathfrak{so}}(D+6)_1$ factor. Then using one such simple current for each minimal factor gives pure $\text{NS}^{\otimes r} \otimes \text{NS}$ and $\text{R}^{\otimes r} \otimes \text{R}$ sectors as we wish.

The bosonic partition function is in total

$$Z_{\text{Bos}}(\tau,\overline{\tau}) \sim \vec{\chi}^T(\tau)\, M(J_{\text{GSO}}) \prod_{r=i}^{r} M(J_i)\, \vec{\chi}(\overline{\tau})\,, \tag{2.8.8}$$

when neglecting the η functions and an overall normalization constant. Applying the inverse of the bosonic string map on the left-moving characters only gives a heterotic partition function, applying it on both, the left- and right-moving characters, results in the IIB partition function

$$Z_{\text{IIBGepner}}(\tau,\overline{\tau}) \sim \vec{\chi}^T(\tau)\, M(J_{\text{GSO}}) \prod_{i=1}^{r} M(J_i)\, \vec{\chi}(\overline{\tau})\Big|_{\phi^{-1}_{\text{bsm}}}\,. \tag{2.8.9}$$

Since all simple currents in the partition function are orbit simple currents the partition function and therefore Gepner models are left-right symmetric as discussed after (2.3.17).

The IIA partition function is reached by adding the matrix $M(J_{IIA})$ associated to the simple current

$$J_{IIA} = (0,1,1)^r, (C)_{D+6} \tag{2.8.10}$$

to the partition function. As one can check, $M(J_{IIA})$ acts as charge conjugation on the right-movers. Recalling IIA $=$ IIB$/(-1)^{F_L}$ we see that a simple current can implement a $(-1)^{F_L}$ orbifold.

Gepner models solve the non-linear sigma model of string theory where the background is $\mathbb{R}^{3,1} \times Y^3$.[6] The internal space Y^3 is a Calabi-Yau defined as hypersurfaces in weighted projective spaces that can be deduced from the rules provided in the former sections or by consulting [35]. Gepner models use direct tensor products of minimal models leading to a Fermat-type polynomial (2.6.3) that, by setting it to zero, defines the Calabi-Yau. The form and coefficients of the polynomial specify the point where we are in complex structure moduli space. Unfortunately, the Fermat-point is far away from the physically relevant regime. This can be seen by expanding the partition functions in powers of $q,\overline{q}$. The powers that appear are all of the form $q^{n/2}$ where $n \in \mathbb{N}_0^+$ such that the mass of every massive state scales with the string scale. Therefore all cycles are at the self-dual radius. Having this in mind it is no surprise to find gauge symmetry enhancements in Gepner models.

Examples: The Quintic

Let us state the most common Gepner model, the quintic $\mathbb{P}_{1,1,1,1,1}[5]$ therefore $k = 3^5$. The quintic is a threefold such that we have $D = 4$ non-compact directions. We will only state the massless modes with $h_{L/R} = \frac{1}{2}$ and omit the subscript in $(s_0)_{D-2}$ as we are only

[6] This is remarkable given the fact that we do not know the metric on the Calabi-Yau manifold and therefore cannot write down the non-linear sigma model explicitly.

interested in the IIB string. Always special is the vacuum sector for which we find the orbit

$$\begin{aligned}&\big[(0,0,0)^5(V)+(0,1,1)^5(C)+(0,-1,-1)^5(S)\big]_L\\ &\qquad\otimes\overline{\big[(0,0,0)^5(V)+(0,1,1)^5(C)+(0,-1,-1)^5(S)\big]}_R\,.\end{aligned}\tag{2.8.11}$$

With help of (2.2.25) one can deduce the spectrum from the target space perspective as

- a massless $\mathcal{N}=2$ gravity-multiplet consisting of $g+2\psi_{3/2}+V_\mu$
- and a massless $\mathcal{N}=2$ hypermultiplet consisting of $4\phi+2\psi_{1/2}$. This is the universal hypermultiplet.

For the other charged orbits we use a notation where we collect the contribution of the minimal models into the form (h,q). The first type of orbits comes from the (c,c) sector and looks like

$$\big[(\tfrac{1}{2},-1)(O)+(\tfrac{3}{8},\tfrac{1}{2})(S)\big]_L\otimes\overline{\big[(\tfrac{1}{2},-1)(O)+(\tfrac{3}{8},\tfrac{1}{2})(S)\big]}_R\,.\tag{2.8.12}$$

The (c,c) states are always accompanied by their charge conjugate orbit from the (a,a) ring

$$\big[(\tfrac{1}{2},+1)(O)+(\tfrac{3}{8},-\tfrac{1}{2})(C)\big]_L\otimes\overline{\big[(\tfrac{1}{2},+1)(O)+(\tfrac{3}{8},-\tfrac{1}{2})(C)\big]}_R\,.\tag{2.8.13}$$

Combined they yield the field content of an $\mathcal{N}=2$ vector multiplet with its field content $V_\mu+2\psi_{\frac{1}{2}}+2\phi$.

The second type of charged orbit mixes both the orbit and its charge conjugate. It contributes states from the (a,c) and $(c,c)/(a,a)$ ring

$$\begin{aligned}&\big[(\tfrac{1}{2},-1)(O)+(\tfrac{3}{8},\tfrac{1}{2})(S)+(\tfrac{1}{2},+1)(O)+(\tfrac{3}{8},-\tfrac{1}{2})(C)\big]_L\\ &\quad\otimes\overline{\big[(\tfrac{1}{2},-1)(O)+(\tfrac{3}{8},\tfrac{1}{2})(S)+(\tfrac{1}{2},+1)(O)+(\tfrac{3}{8},-\tfrac{1}{2})(C)\big]}_R\,,\end{aligned}\tag{2.8.14}$$

giving a vector and a hypermultiplet.

There is a clever strategy to count all multiplets in Gepner models arising in the charged sector. We take only the states $(\frac{1}{2},-1)(O)$ and compute their orbits. In the resulting list every S state will contribute a vector multiplet while every C will give rise to a hypermultiplet.

Let us take a look at the massless states in the 3^5 Gepner model. We will focus on the NS-NS states of the form $(\frac{1}{2},-1)(O)$ and keep the appearance of their charge conjugates $(\frac{1}{2},1)(O)$ and their superpartners in mind. The massless spectrum is

State	Polynomial	degeneracy
$(3,-3,0)(2,-2,0)(0,0,0)^3$	$X_i^3X_j^2$	20
$(3,-3,0)(1,-1,0)^2(0,0,0)^2$	$X_i^3X_jX_k$	30
$(2,-2,0)^2(1,-1,0)(0,0,0)$	$X_i^2X_j^2X_k$	30
$(2,-2,0)^2(1,-1,0)^3(0,0,0)$	$X_i^2X_jX_kX_l$	20
$(1,-1,0)^5$	$X_1X_2X_3X_4X_5$	1 .

Of these only the last one is in the (a, c) ring giving 101 vector multiplets and 1 hypermultiplet as such the Hodge numbers are $h^{1,1} = 1$ and $h^{1,2} = 101$. These number were computed geometrically around (2.7.4).

Example 2: The D-invariant

Let us state another example with a D-invariant, the model 6666_D. Following section 2.6.2 the corresponding superpotential should take the form

$$W = X_1^8 + X_2^8 + X_3^8 + Y^4 + YZ^2 = 0 \tag{2.8.15}$$

that defines the space $\mathbb{P}_{1,1,1,2,3}[8]$. We look at this particular example to see how the D invariant and its characteristic splitting into two variables Y, Z is reflected in the spectrum. We begin with the states that do not involve something non-trivial in the minimal model with the D-invariant

State	Polynomial	degeneracy
$(6,-6,0)(2,-2,0)(0,0,0)^2$	$X_i^6 X_j^2$	6
$(6,-6,0)(1,-1,0)^2(0,0,0)$	$X_i^6 X_j X_k$	3
$(5,-5,0)(3,-3,0)(0,0,0)^2$	$X_i^5 X_j^3$	6
$(5,-5,0)(2,-2,0)(1,-1,0)(0,0,0)$	$X_i^5 X_j^2 X_k$	6
$(4,-4,0)^2(0,0,0)^2$	$X_i^4 X_j^4$	3
$(4,-4,0)(3,-3,0)(1,-1,0)(0,0,0)$	$X_i^4 X_j^3 X_k$	6
$(4,-4,0)(2,-2,0)^2(0,0,0)$	$X_i^4 X_j^2 X_k^2$	3
$(3,-3,0)^2(2,-2,0)(0,0,0)$	$X_i^3 X_j^3 X_k^2$	3

For the states involving something non-trivial in the last factor we find

State	Polynomial	degeneracy
$(6,-6,0)(0,0,0)^2(2,-2,0)$	$X_i^6 Y$	3
$(5,-5,0)(1,-1,0)(0,0,0)(2,-2,0)$	$X_i^5 X_j Y$	6
$(4,-4,0)(2,-2,0)(0,0,0)(2,-2,0)$	$X_i^4 X_j^2 Y$	6
$(4,-4,0)(1,-1,0)(1,-1,0)(2,-2,0)$	$X_i^4 X_j X_k Y$	3
$(3,-3,0)^2(0,0,0)(2,-2,0)$	$X_i^3 X_j^3 Y$	3
$(3,-3,0)(2,-2,0)(1,-1,0)(2,-2,0)$	$X_i^3 X_j^2 X_k Y$	6
$(2,-2,0)^3(2,-2,0)$	$X_1^2 X_2^2 X_3^2 Y$	1
$(4,-4,0)(0,0,0)^2(4,-4,0)$	$X_i^4 Y^2$	3
$(3,-3,0)(1,-1,0)(0,0,0)(4,-4,0)$	$X_i^3 X_j Y^2$	6
$(2,-2,0)^2(0,0,0)(4,-4,0)$	$X_i^2 X_j^2 Y^2$	3
$(2,-2,0)(1,-1,0)^2(4,-4,0)$	$X_i^2 X_j X_k Y^2$	3

and

State	Polynomial	degeneracy
$(5,-5,0)(0,0,0)^2(3,-3,0)$	$X_i^5 Z$	3
$(4,-4,0)(1,-1,0)(0,0,0)(3,-3,0)$	$X_i^4 X_j Z$	6
$(3,-3,0)(2,-2,0)(0,0,0)(3,-3,0)$	$X_i^3 X_j^2 Z$	6
$(3,-3,0)(1,-1,0)^2(3,-3,0)$	$X_i^3 X_j X_k Z$	3
$(2,-2,0)^2(1,-1,0)(3,-3,0)$	$X_i^2 X_j^2 X_k Z$	3

as well as

State	Polynomial	degeneracy
$(2,-2,0)(0,0,0)^2(6,-6,0)$	$X_i^2 Y^3$	3
$(1,-1,0)^2(0,0,0)(6,-6,0)$	$X_i X_j Y^3$	3
$(1,-1,0)^3(5,-5,0)$	$X_1 X_2 X_3 Y Z$	1

They correctly add up to the Hodge numbers $h^{1,2} = 106$ and $h^{1,1} = 2$ that we computed after (2.7.13). Going through the list one sees that the simple currents erased nearly all odd l while the remaining even l turn into the variable Y. The $\frac{k}{2}$ state survived as well and represents the Z variable. Only one other odd state is left that corresponds to the $X_1X_2X_3YZ$ monomial that is always present if $\sum_i w_i = d$. As one can see through simple currents there might be holes in the spectrum and a single variable can split into several variables. This will reappear in the asymmetric Gepner models as well.

Furthermore notice that the simple current for the D-invariant completely changes the geometric interpretation. Without additional simple current, the model $k = 6,6,6,6$ corresponds to $\mathbb{P}_{1,1,1,1,4}[8]$ with $(h^{1,1}, h^{1,2}) = (1, 149)$. Clearly this is way different from the result we obtain when adding the simple current for the D-invariant.

Chapter 3

Conformal field theories for non-geometric backgrounds

Assuming a smooth ten-dimensional geometric space M equipped with a metric and possibly other background NS-NS fields, the non-linear sigma model that describes the motion of strings in M is the usual Polyakov action. Having the non-linear sigma model at hand one has to check whether the background is a consistent string theory background. In case the β functions are non-zero there is a mass scale such that conformal invariance is anomalous. If $\beta = 0$ the non-linear sigma model is anomaly free and defines a conformal field theory that is necessarily left-right symmetric, simply because the non-linear sigma model is manifestly left-right symmetric.

Take for instance $M = \mathbb{R}^{3,1} \times Y_3$ where Y_3 is a Calabi-Yau manifold. Although the metric of Y_3 is unknown, at least in principle one can write down the corresponding non-linear sigma model. M being Ricci-flat, at least at leading order the β functions vanish right away such that the non-linear sigma model defines a left-right symmetric conformal field theory. From the last chapter we know that Gepner models are the conformal field theories of certain Calabi-Yau manifolds and, indeed, Gepner models are left-right symmetric.

For a long time it is been known that there are also left-right asymmetric conformal field theories like, for instance, the asymmetric orbifolds [43]. They are perfectly well defined string backgrounds but, being left-right asymmetric, they have no underlying non-linear sigma model and at first sight a geometric interpretation seems to be impossible.

In the following chapter we will argue that one can associate a target space interpretation at least to certain left-right asymmetric conformal field theories. The starting point is the following observation: In an effective supergravity of a string compactification additional fluxes on the internal manifold gauge some subgroup of the global symmetry group. But as it turns out, the geometric fluxes are responsible for only half of all possible gaugings. The remaining gaugings in turn correspond to the T-duals of the geometric fluxes, named non-geometric fluxes. From the worldsheet point of view T-duality acts as left-right asymmetric reflection of the right-movers. This suggests a connection between left-right asymmetric conformal field theories and non-geometric fluxes. The following chapter will elaborate this connection and argue that certain left-right asymmetric conformal field theories are the string uplifts of fully backreacted vacua of gauged supergravity.

In the first part of the chapter we will review the existing literature about asymmetric toroidal orbifolds and their connection to gauged maximal supergravity. Then we take four-dimensional Gepner models and extend them with left-right asymmetric simple currents that break the right-moving supersymmetry such that we end up with $\mathcal{N} = 1$ supersymmtry. We show in several examples that one can identify an underlying Calabi-Yau geometry in these asymmetric Gepner models. This observation leads us to suspect that the asymmetric Gepner models could be related to $\mathcal{N} = 1$ vacua in the gauged supergravity of the identified Calabi-Yau manifold. To check this we review spontaneous partial supersymmetry breaking in $\mathcal{N} = 2$ gauged supergravity and derive bounds on the $\mathcal{N} = 1$ spectrum. We compare these bounds to the results from the asymmetric Gepner models and find a very good agreement. We therefore conjecture, that asymmetric Gepner models correspond to the fully backreacted $\mathcal{N} = 1$ vacua in a $\mathcal{N} = 2$ gauged supergravity.

3.1 Linking gauged supergravities and (asymmetric) toroidal orbifolds

This section outlines how we were motivated to work on asymmetric Gepner models. We start with general $\mathbb{T}^D$ compactifications, gauge the resulting supergravity and then review corresponding conformal field theories. The aim is to connect fluxes in the supergravity description with (asymmetric) conformal orbifolds.

3.1.1 Torus compactifications

The action of the NS-NS subsector of any superstring theory, the metric g_{10}, the Kalb-Ramond field B_{10} and the dilaton ϕ is

$$S = \int d^{10}x\sqrt{-g_{10}}\, e^{-\phi}\,\left(R_{10} + (\partial\phi)^2 - \tfrac{1}{12}H_{10}^2\right)\,, \tag{3.1.1}$$

where R_{10} is the ten-dimensional Ricci scalar and $H_{10} = dB_{10}$.

Compactifying this action on a $\mathbb{T}^D$ results in a theory with $O(D,D)$ global symmetry and an abelian $U(1)^{2D}$ gauge group. The $2D$ gauge bosons are $A^a_\mu = (g_{\mu i}, B_{\mu j})$, where μ is an index along the external, and i, j are indices along the $\mathbb{T}^D$ that we combine in $a \in 1, \dots 2D$. The dimensionally reduced action is [44]

$$\begin{aligned} S = \int d^{10-D}x\sqrt{-g}\, e^{-\phi}\Big(&R + (\partial\phi)^2 - \tfrac{1}{12}H^2 \\ &+ \tfrac{1}{8}L_{ab}\nabla_\mu K^{bc}L_{cd}\nabla^\mu K^{da} - \tfrac{1}{4}F^a_{\mu\nu}L_{ab}K^{bc}L_{cd}F^{d\mu\nu}\Big)\,. \end{aligned} \tag{3.1.2}$$

Here the Latin indices $a, b, c, \dots \in 1, \dots 2D$ label the fundamental representation of the global symmetry group $O(D,D)$ and $F^a = dA^a$ is the field strength for the gauge bosons A^a_μ.

The scalars are collected into a symmetric matrix K^{ab} with $\mathrm{Tr}(LK) = 0$ that takes values in the coset $O(D,D)/O(D) \times O(D)$. These scalars arise from, B_{ab} and g_{ab} and encode the size and shape of the $\mathbb{T}^D$. The matrix

$$L = \begin{pmatrix} 0 & 1_D \\ 1_D & 0 \end{pmatrix} \tag{3.1.3}$$

is the usual $O(D,D)$ metric. The reduced three-form field strength is

$$H_{\mu\nu\rho} = \partial_\mu B_{\mu\nu} - \tfrac{1}{2} A^a_\mu L_{ab} F^b_{\nu\rho} + \text{cycl.}\,. \tag{3.1.4}$$

The symmetry group acts as $A^a \to M^a{}_b A^b$ and $K^{ab} \to M^a{}_c M^a{}_d K^{cd}$ where $M \in O(D,D)$.

3.1.2 Gauged supergravity and flux compactifications

When gauging a subgroup of the global symmetry group by turning the subgroup into a local symmetry group one speaks of a gauged supergravity (GSUGRA) (see e.g. the review [45]). This leads to couplings between the fields and might additionally induce a scalar potential, possibly breaking supersymmetry.

From the string theory viewpoint these gaugings are reproduced when one perturbs a pure background with additional fluxes in the internal geometry. This is called a flux compactification. Notice conformal field theory is not able to have a non-trivial R-R background. The reason for this is that one cannot write down the appropriate sigma model due to the lack of vertex operators for the potentials. Since we will later compare the gauged supergravity with a conformal field theory, we restrict ourselves to NS-NS fluxes.

In case of the $\mathbb{T}^D$ compactification we gauge a subgroup $G \in O(D,D)$. The generators of G are denoted as $\mathcal{Z}_M$ and $\mathcal{X}_M$ depending on whether they arise from $g_{\mu i}$ or $B_{\mu i}$. They satisfy the gauge algebra [44, 46, 47]

$$\begin{aligned} [\mathcal{Z}_M, \mathcal{Z}_N] &= f^P_{MN} \mathcal{Z}_P + H_{MNP} \mathcal{X}^P, \\ [\mathcal{Z}_M, \mathcal{X}^N] &= \tilde{f}^N_{PM} \mathcal{X}^P + Q^{NP}_M \mathcal{Z}_P, \\ [\mathcal{X}^M, \mathcal{X}^N] &= \tilde{Q}^{MN}_P \mathcal{X}^P + R^{MNP} \mathcal{Z}_P\,. \end{aligned} \tag{3.1.5}$$

This algebra is also called Roytenberg-algebra [48].

The structure constants H correspond to the $H = dB$ fluxes and the geometric f-flux corresponds to fluxes of the metric. This interpretation is justified since the action of the gauged supergravity matches the action that is obtained from a Scherk Schwarz compactification on a twisted torus with additional H flux [49, 46]. Let us mention that in a Scherk-Schwarz compactification, the $\mathcal{Z}$ are the internal vielbein.

To interpret Q and R it is helpful to remember that the above gauge algebra naturally appears in the flux formulation of double field theory [50, 51]. There the $\mathcal{Z}$ are the internal vielbein and their T-dual winding vielbein are $\mathcal{X}$. Under T-duality $\mathcal{Z}$ and $\mathcal{X}$ are interchanged. Therefore Q and R are the T-dual fluxes to the geometric fluxes and correspond to winding fluxes. As such, the non-geometric fluxes cannot be obtained by a geometric compactification, but rather appear from a generalized Scherk-Schwarz reduction of double field theory [52, 53, 50, 51, 54].

There are attempts to find a geometric understanding of non-geometric backgrounds. Let us explain their general idea by first looking at a constant H-flux. Say we have such a flux around an S^1 with coordinate $x \sim x + 1$. To have a constant H-flux the B field must be linear in x, thus $B \sim x$. We infer that going around the circle amounts to an internal gauge transformation of the B field $B \to B + 2\pi n$ that from the $10-D$-dimensional viewpoint is a $O(D, D)$ transformation. Therefore, turning on an internal flux amounts to non-trivial monodromies in the global symmetry group.

This idea can be generalized to arbitrary monodromies in any duality or symmetry group, see e.g. the old work about U-folds [55], non-geometric K3 [56] or the more recent publication [57]. In case of the Q-flux background, Hull collected evidence that a Q-flux backround arises from compactifications on manifolds, whose transition functions incoorporate the non-geometric T-dualities. This is possible, since the isometry group of the torus $O(D, D)$ contains not only the geometric transformations but also T-dualities. A similar geometric notion seems impossible for R-flux backgrounds, where the local geometry seems to break down, see chapter 5.

3.1.3 (Asymmetric) Torus orbifolds

The monodromy picture is very useful to construct corresponding conformal field theories. Using orbifolds we can construct CFTs that have a certain monodromy. For instance we take the known CFT of a $\mathbb{T}^{D-1} \times S^1$ compactification and orbifold it with the twist

$$g(y) = \exp\left(\frac{My}{2\pi}\right) , \tag{3.1.6}$$

where $y \sim y + 2\pi$ is the coordinate along the S^1 and $M \in O(D-1, D-1)$. The orbifold imposes the monodromy e^M along the circle. As usual, orbifolding has two effects. First it will produce new states from the twisted sectors and secondly, some of the states are projected out.

The above twist does only depend on the coordinate $y = y_L + y_R$ but not on the dual coordinate $\tilde{y} = y_L - y_R$. Indeed, the above twist results in a flux configuration that only superficially contains non-geometric fluxes. But a closer look reveals that there is a duality frame in which only geometric fluxes appear. To capture truly non-geometric fluxes, we instead have to orbifold with a twist

$$g(y, \tilde{y}) = \exp\left(\frac{My}{2\pi}\right) \exp\left(\frac{\tilde{M}\tilde{y}}{2\pi}\right) \tag{3.1.7}$$

where $\tilde{M} \in O(D-1, D-1)$ denotes the dual monodromy. Since M and $\tilde{M}$ are independent the orbifold acts differently on the left- and right-movers. As such this is an asymmetric torodial orbifold. There is plenty of literature about these orbifolds [58, 59, 60, 61, 62, 63, 64], showing the relation between gaugings in maximal supergravity and (asymmetric) torus orbifolds. This consideration connects the two standard ways to stabilize moduli, fluxes and (asymmetric) orbifold. Going further, a natural conjecture is that both procedures are actually two sides of the same medal, namely the target space and worldsheet perspective of the same mechanism.

3.2 Asymmetric Gepner models with $\mathcal{N} = 1$ supersymmetry

Having looked at asymmetric toroidal orbifolds and observing the connection to gauged supergravities we asked ourselves how far this correspondence reaches. Might it be that asymmetric CFT are in general the full string solutions to gauged supergravities?

To check this conjecture in another setup, we left the framework of gauged maximal supergravities in favor of Gepner models. Instead of orbifolds we will use the simple current technique to construct left-right asymmetric Gepner models.[1] In this chapter we will always add a single simple current with non-trivial entries only in one minimal factor to keep the resulting spectrum as clear as possible. The partition function is

$$Z_{\mathrm{ACFT}}(\tau,\overline{\tau}) \sim \vec{\chi}^{T}(\tau)\, M(J_{\mathrm{ACFT}})\, M(J_{\mathrm{GSO}}) \prod_{r=1}^{5} M(J_i)\, \vec{\chi}(\overline{\tau})\Big|_{\phi_{\mathrm{bsm}}^{-1}} . \tag{3.2.1}$$

We choose the left-right asymmetric simple current J_{ACFT} in such a way that the resulting model has $\mathcal{N} = 1$ target space supersymmetry. The $\mathcal{N} = 1$ multiplets arise from the CFT perspective as follows. Orbits of the form

$$(h = 3/8)(s) \otimes \Big[(\overline{h} = 1/2, \overline{q} = 1)(o) + (\overline{h} = 3/8, \overline{q} = -1/2)(s)\Big] \tag{3.2.2}$$

lead to massless $\mathcal{N} = 1$ vector multiplets, combinations

$$(h = 3/8)(c) \otimes \Big[(\overline{h} = 1/2, \overline{q} = 1)(o) + (\overline{h} = 3/8, \overline{q} = -1/2)(s)\Big] \tag{3.2.3}$$

to massless R-R axion-like chiral multiplets and

$$(h = 1/2)(o) \otimes \Big[(\overline{h} = 1/2, \overline{q} = 1)(o) + (\overline{h} = 3/8, \overline{q} = -1/2)(s)\Big] \tag{3.2.4}$$

to massless NS-NS chiral multiplets.

Since pure Gepner models have $\mathcal{N} = 2$ target space supersymmetry our first guess is that these asymmetric Gepner models correspond to an $\mathcal{N} = 1$ Minkowski vacuum of an $\mathcal{N} = 2$ gauged supergravity of IIB reduced on a certain Calabi-Yau manifold. To compare the asymmetric Gepner models with a gauged $\mathcal{N} = 2$ supergravity it is clear, that in a

[1] We use only left-right asymmetric simple currents since in this case we can exclude a simple explanation in terms of an ungauged $\mathcal{N} = 2$ supergravity of a Calabi-Yau compactification.

first step we have to find the $\mathcal{N}=2$ supergravity. How to find these will be demonstrated in several examples in the following section in the following way: The remaining $\mathcal{N}=1$ spectrum still shows a remnant of a weighted projective space. Using this data we are able to reconstruct the Calabi-Yau of the $\mathcal{N}=2$ supergravity.

3.2.1 Odd level ACFT models

Let us start with a class of simple currents that essentially appeared in [65, 66, 67]. It requires a minimal factor with odd level. Taking the first minimal model to be odd the simple current is

$$J_{\text{ACFT}} = (0\ k_1+2\ 1)(0\ 0\ 0)^4\,(c)\,. \tag{3.2.5}$$

Notice that this simple current is in the R-sector and therefore mixes R and NS sector between the first minimal model and the others. It therefore does not commute with J_1 and in this way changes the right-moving target space supersymmetry.

The quintic example

In a first example let us extend the quintic Gepner model having $k=3^5$ with

$$J_{\text{ACFT}} = (0\ 5\ 1)(0\ 0\ 0)^4\,(c)\,, \tag{3.2.6}$$

resulting in a model with $\mathcal{N}=1$ target space supersymmetry. If we had chosen an s in the $\widehat{\mathfrak{so}}(2)_1$ factor, we would instead have gotten an enhancement, $\mathcal{N}_L=\mathcal{N}_R=2$. This shows that a simple current that mixes R and NS sector does not necessarily break the left-moving worldsheet supersymmetry to zero. As such we can still have a type II model where the $N=1$ supercurrent is hidden off-diagonally in the five minimal models.

The massless spectrum is

$$1\times(\phi,\varphi)\quad+\quad(N_V,N_{\text{ax}}\,;N_0)=(80,0\,;74)\,. \tag{3.2.7}$$

To get a clue about the complex structure moduli space that encodes the weighted projective space we summarized the vectors in 3.1. In the first four lines we see that the states in the last four minimal models are unaffected by the simple current as it acts only on the first factor. These four minimal models still behave like variables of weight one that are combined into monomials of degree five. We called them $\{x_2,x_3,x_4,x_5\}$ in the table. When demanding that the remaining states form monomials of degree five as well, we have to split the first minimal factor into two variables $\{y_0,y_1\}$ both of degree two to get the combinatorics correctly. It is natural to conjecture that the Calabi-Yau

$$\mathcal{M}_{\text{ACFT}} = \mathbb{P}_{1,1,1,1,2,2}[5\ 3] \tag{3.2.8}$$

state	polynom. rep.	deg.
$(0\,1\,1)(3\,4\,1)(2\,3\,1)(0\,1\,1)^2(s)$	$x_i^3\,x_j^2$	12
$(0\,1\,1)(3\,4\,1)(1\,2\,1)^2(0\,1\,1)(s)$	$x_i^3\,x_j\,x_k$	12
$(0\,1\,1)(2\,3\,1)^2(1\,2\,1)(0\,1\,1)(s)$	$x_i^2\,x_j^2\,x_k$	12
$(0\,1\,1)(2\,3\,1)(1\,2\,1)^3(s)$	$x_i^2\,x_j\,x_k\,x_l$	4
$(1\,2\,1)(3\,0\,0)(0\,0\,0)^3(s)+$ $(2\,3\,1)(3\,4\,1)(0\,1\,1)^3(s)$	$x_i^3\,y_m$	$2\times 4=8$
$(1\,2\,1)(2\,0\,0)(1\,0\,0)(0\,0\,0)^2(s)+$ $(2\,3\,1)(3\,4\,1)(0\,1\,1)^3(s)$	$x_i^2\,x_j\,y_m$	$2\times 12=24$
$(1\,2\,1)(1\,0\,0)^3(0\,0\,0)(s)+$ $(2\,3\,1)(1\,2\,1)^3(0\,1\,1)(s)$	$x_i\,x_j\,x_k\,y_m$	$2\times 4=8$

Table 3.1: Combinatorics of the $N_V=80$ massless vectors.

has something to do with this model.[2] While the weights and the polynomial constraint of degree five follow from the CFT spectrum, the other constraint with degree three is not visible in the spectrum, but follows from the Calabi-Yau condition(2.6.8). Notice that it is expected that parts of the spectrum of the full Calabi-Yau compactification cannot be visible in the broken $\mathcal{N}=1$ minimum since they become massive during the supersymmetry breaking.

This example is encouraging for two reasons. First, the $\mathcal{N}=1$ model still reveals the structure of a weighted projective space. Secondly, the Hodge numbers of the corresponding Calabi-Yau computed around (2.7.8) are $(h^{1,2},h^{1,1})=(83,2)$. Expecting that the supersymmetry breaking happens at least partly with a Stückelberg mechanism that eliminates vector/hyper pairs, the data of the Calabi-Yau fits the 80 vectors of the Gepner model intriguingly well. This is enough motivation to look for more examples. Let us first collect the Gepner models and then proceed with a more detailed comparison to the supergravity expectations.

General form

To generalize the first result we take an arbitrary Gepner model with an odd factor $k=(2l-1,k_2,k_3,k_4,k_5)$. When having only A-invariants the polynomial constraint is

$$x_1^{2l+1}+x_2^{k_2+2}+x_3^{k_3+2}+x_4^{k_4+2}+x_5^{k_5+2}=0 \tag{3.2.9}$$

in

$$\mathcal{M}_{\text{Gep}}=\mathbb{P}_{\frac{d}{(2l+1)},\frac{d}{k_2+2},\frac{d}{k_3+2},\frac{d}{k_4+2},\frac{d}{k_5+2}}[d]\,, \tag{3.2.10}$$

[2]Notice that the simple current drastically altered the geometric interpretation and the new Calabi-Yau has nothing to do with the quintic, in particular it is not a broken quintic model. Such a behaviour is known from the D-invariants as demonstrated the end of section 2.8

with $d = \mathrm{lcm}\{2l+1, k_2+2, k_3+2, k_4+2, k_5+2\}$. Adding the simple current (3.2.5) we find, similar to [66, 67], that the first variable gets split into two variables (y_0, y_1) of weight $w_0 = 2d/(2l+1)$ and $w_1 = ld/(2l+1)$. The corresponding broken $\mathcal{N} = 2$ supergravity should therefore be the one of type IIB on the Calabi-Yau

$$\mathcal{M}_{\mathrm{ACFT}} = \mathbb{P}_{\frac{2d}{(2l+1)}, \frac{ld}{(2l+1)}, \frac{d}{k_2+2}, \frac{d}{k_3+2}, \frac{d}{k_4+2}, \frac{d}{k_5+2}} \left[d \;\; \frac{d(l+1)}{(2l+1)} \right] . \tag{3.2.11}$$

To get more examples we can also add D-invariants in an even factor, say $k_4 = 2k$. Following section 2.6.2 we get

$$\mathcal{M}_{\mathrm{ACFT}} = \mathbb{P}_{\frac{2d}{(2l+1)}, \frac{ld}{(2l+1)}, \frac{d}{k_2+2}, \frac{d}{k_3+2}, \frac{d}{k+1}, \frac{dk}{2(k+1)}} \left[d \;\; \frac{d(l+1)}{(2l+1)} \right] , \tag{3.2.12}$$

where $d = \mathrm{lcm}\{2l+1, k_2+2, k_3+2, k+1\}$.

Using a small computer program we scanned the list of complete intersection Calabi-Yau [38] for examples that are of the above type. The full list is summarized in 3.2. As one can see we can only relate few ACFTs to that huge list of gauged supergravities. When trying to resolve other Calabi-Yau we extracted from other asymmetric Gepner models, the corresponding complete intersection Calabi-Yaus mostly turn out to not intersect transversally.

Gepner	$(N_V, N_{\mathrm{ax}}, N_0)$	CICY
$(3\,3\,3\,3\,3)$	$(80, 0, 74)$	$\mathbb{P}_{1,1,1,1,2,2}[5\ 3]_{(83,2)}$
$(5\,5\,5\,12_D)$	$(86, 2, 80)$	$\mathbb{P}_{1,1,1,2,3,3}[7\ 4]_{(89,3)}$
$(5\,5\,5\,12_A)$	$(86, 2, 80)$	$\mathbb{P}_{1,2,2,4,6,7}[14\ 8]_{(88,4)}$
$(7\,7\,7\,1\,1)$	$(74, 2, 70)$	$\mathbb{P}_{1,1,2,3,3,4}[9\ 5]_{(75,6)}$

Table 3.2: Examples with an asymmetric simple current in the first (odd) factor.

3.2.2 Level six ACFT Model

Next we consider models in which we add a simple current to a factor with even level. Unfortunately we could not find a similar rule as before so we can only present particular examples. We start with $(6_A, 6_A, 6_A, 6_D)$ extended by the simple current

$$J_{\mathrm{ACFT}} = (0\ 4\ 0)(0\ 0\ 0)^3\,(v) \,. \tag{3.2.13}$$

This simple current is essentially J_1^2 and indeed commutes with all the J_i being a pure NS-NS state. The simple current breaks supersymmetry by being non-local w.r.t. to J_{GSO} (2.8.5). Being local with respect to the J_i is reflected in the spectrum that showed no mixing between NS and R sectors such that the $N = 1$ supercurrent is still the sum of the individual supercurrents. Notice that this fact is unrelated to number of target space supercharges.

The resulting model has $\mathcal{N}=1$ target space supersymmetry and the massless spectrum

$$(N_V, N_{\rm ax}\,; N_0) = (60, 4\,; 64)\,. \tag{3.2.14}$$

Let us take a closer look at the spectrum in table 3.3 to see whether we can deduce a Calabi-Yau. The pure Gepner model corresponds to $\mathcal{M}_{\rm Gep} = \mathbb{P}_{1,1,1,2,3}[8]^{(106,2)}$. Since we add the simple current in the first factor, we expect the three weights from the other minimal models to be unaffected. Having this in mind we infer that these factors form monomials of degree seven. Forcing the other states to form monomials degree seven as well fixes the weight of the first variable to four. To correctly capture the twofold degeneracy

State	polynom. rep.	deg.
$(1\,2\,1)(6\,7\,1)(1\,2\,1)(0\,1\,1)(s)$	$s_\alpha\, x_i^6\, x_j$	$2\times 2=4$
$(1\,2\,1)(5\,6\,1)(2\,3\,1)(0\,1\,1)(s)$	$s_\alpha\, x_i^5\, x_j^2$	$2\times 2=4$
$(1\,2\,1)(4\,5\,1)(3\,4\,1)(0\,1\,1)(s)$	$s_\alpha\, x_i^4\, x_j^3$	$2\times 2=4$
$(1\,2\,1)(3\,4\,1)(0\,1\,1)(0\,1\,1)(s)$	$s_\alpha\, x_i^3\, z$	$2\times 2=4$
$(1\,2\,1)(2\,3\,1)(1\,2\,1)(0\,1\,1)(s)$	$s_\alpha\, x_i^2\, x_j\, z$	$2\times 2=4$
$(1\,2\,1)(5\,6\,1)(0\,1\,1)(2\,3\,1)(s)$	$s_\alpha\, x_i^5\, v$	$2\times 2=4$
$(1\,2\,1)(4\,5\,1)(1\,2\,1)(2\,3\,1)(s)$	$s_\alpha\, x_i^4\, x_j\, v$	$2\times 2=4$
$(1\,2\,1)(3\,5\,1)(2\,2\,1)(2\,3\,1)(s)$	$s_\alpha\, x_i^3\, x_j^2\, v$	$2\times 2=4$
$(1\,2\,1)(1\,2\,1)(0\,1\,1)(2\,3\,1)(s)$	$s_\alpha\, x_i\, v\, z$	$2\times 2=4$
$(1\,2\,1)(4\,5\,1)(0\,1\,1)(3\,4\,1)(s)$	$s_\alpha\, x_i^4\, w$	$2\times 2=4$
$(1\,2\,1)(3\,4\,1)(1\,2\,1)(3\,4\,1)(s)$	$s_\alpha\, x_i^3\, x_j\, w$	$2\times 2=4$
$(1\,2\,1)(2\,3\,1)(2\,3\,1)(3\,4\,1)(s)$	$s_\alpha\, x_1^2\, x_2^2\, w$	$2\times 1=2$
$(1\,2\,1)(0\,1\,1)(0\,1\,1)(3\,4\,1)(s)$	$s_\alpha\, w\, z$	$2\times 1=2$
$(1\,2\,1)(3\,4\,1)(0\,1\,1)(4\,5\,1)(s)$	$s_\alpha\, x_i^3\, v^2$	$2\times 2=4$
$(1\,2\,1)(2\,4\,1)(1\,2\,1)(4\,5\,1)(s)$	$s_\alpha\, x_i^2\, x_j\, v^2$	$2\times 2=4$
$(1\,2\,1)(1\,1\,1)(0\,1\,1)(6\,7\,1)(s)$	$s_\alpha\, x_i\, w^2 (\sim v^3)$	$2\times 2=4$

Table 3.3: Combinatorics of the $N_V = 60$ massless vectors

we introduce the factor s_α with $\alpha = 0, 1$ into the corresponding monomials. To conclude the proposal for the underlying (fluxed) Calabi-Yau threefold is

$$\mathcal{M}_{\rm ACFT} = \begin{matrix} \mathbb{P}_{1,1,2,3,4} \\ \mathbb{P}_{1,1} \end{matrix} \begin{bmatrix} 7 & 4 \\ 1 & 1 \end{bmatrix}\,, \tag{3.2.15}$$

i.e. a complete intersection in a product of weighted projective spaces. The Hodge numbers of the resolved CICY are computed in (2.7.14) to be $(h^{2,1}, h^{1,1}) = (62, 6)$.

3.2.3 Level ten ACFT model

The last example is the Gepner model with levels $(10, 10, 4, 4)$. The pure Gepner model describes IIB on the Calabi-Yau $\mathcal{M}_{\text{Gep}} = \mathbb{P}_{1,1,2,2,6}[12]$. We extend the model with the simple current

$$J_{\text{ACFT}} = (0\ 4\ 0)(0\ 0\ 0)^3\,(v) \qquad (3.2.16)$$

to get an $\mathcal{N} = 1$ Minkowski vacuum with the spectrum

$$(N_V, N_{\text{ax}}\,; N_0) = (59, 5\,; 68)\,. \qquad (3.2.17)$$

In 3.4 we list the degree of the three unfacted variables $\{x_2, x_3, x_4\}$ that have weight $\{1, 2, 2\}$ A closer inspection reveals two types of states. The first type of states $\{11, 7, 3\}$ is already

polynom. rep.	deg.
$p_{11}(x)$	18
$p_{10}(x)$	19
$p_7(x)$	$10 + 2$
$p_4(x)$	6
$p_3(x)$	3

Table 3.4: The $N_V = 59$ massless vectors sorted according to $p_d(x)$ where d is the degree from the three unaffected variables x_2, x_3, x_4.

present in the unmodified Gepner model while the second type of states $\{10, 7, 4\}$ arises as twisted sector of the simple current. We therefore introduce two variables of weight three and four. Furthermore, due to the maximal degree of the monomials, 11 and 10, we suggest as corresponding gauged supergravity the Calabi-Yau threefold

$$\mathcal{M}_{\text{ACFT}} = \mathbb{P}_{1,2,2,3,4,9}[11\ 10]\,. \qquad (3.2.18)$$

Employing the methods from section 2.7 or consulting the list [38], the Hodge numbers are $(h^{2,1}, h^{1,1}) = (66, 8)$.

3.3 Partial supersymmetry breaking of $\mathcal{N} = 2$ gauged supergravity

In the previous chapter we have examined certain Gepner models with a left-right asymmetric simple current extension. They had $\mathcal{N} = 1$ target supersymmetry and still displayed the structure of an underlying Calabi-Yau manifold. The idea is that the these CFTs correspond to $\mathcal{N} = 1$ Minkowski vacua of the $\mathcal{N} = 2$ supergravity from the corresponding Calabi-Yau compactification when NS-NS gaugings are turned on. To compare the CFT and the gauged supergravity side, the following section will review flux compactifications of IIB, thus $\mathcal{N} = 2$ gauged supergravities, and their partial supersymmetry breaking.

3.3.1 $\mathcal{N} = 2$ gauged supergravity

We start with gauging the $\mathcal{N} = 2$ supergravity of type IIB compactified on a Calabi-Yau threefold. We will use the notation of section 2.1 where we reviewed the ungauged spectrum. The NS-NS gaugings can be collected into a $(2h^{2,1} + 2) \times (2h^{1,1} + 2)$ matrix [68, 69, 70]

$$\mathcal{O} = \begin{pmatrix} q_\Lambda{}^A & f_{\Lambda A} \\ \tilde{q}^{\Lambda A} & \tilde{f}^\Lambda{}_A \end{pmatrix} . \tag{3.3.1}$$

The H-flux and its T-dual, the R-flux, are encoded in this matrix as

$$\begin{aligned} f_{\Lambda 0} &= h_\Lambda , & \tilde{f}^\Lambda{}_0 &= \tilde{h}^\Lambda , \\ q_\Lambda{}^0 &= r_\Lambda , & \tilde{q}^{\Lambda 0} &= \tilde{r}^\Lambda . \end{aligned} \tag{3.3.2}$$

Having gauged certain global symmetries along the $2h^{1,1}+3$ axionic directions $\{\xi^A, \tilde{\xi}_A, \tilde{\phi}\}$, the transformation under a gauge parameter λ being a $(2h^{2,1} + 2)$-dimensional vector is

$$\delta \begin{pmatrix} A \\ \tilde{A} \end{pmatrix} = d\lambda , \qquad \delta\Xi = -\mathcal{O}^T \cdot \lambda , \qquad \delta\tilde{\phi} = -\lambda^T \cdot C \cdot \tilde{\mathcal{O}} \cdot \Xi , \tag{3.3.3}$$

where

$$\tilde{\mathcal{O}} = C \cdot \mathcal{O} \cdot C^T , \qquad C = \begin{pmatrix} 0 & +\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} . \tag{3.3.4}$$

From another viewpoint [71, 68, 69] the fluxes arise as a defect of the former closed cycles to be closed after perturbing. From the nilpotency of the exterior derivative one can derive the Bianchi identities/quadratic constraints

$$\tilde{\mathcal{O}}^T \cdot \mathcal{O} = 0 , \qquad \mathcal{O} \cdot \tilde{\mathcal{O}}^T = 0 . \tag{3.3.5}$$

Due to gauging $n = \text{rank}(\mathcal{O}) + \Delta$ gauge fields become massive through the Stückelberg mechanism by eating up some of the axions. Δ is 0 or 1 depending on whether the NS-NS axion $\tilde{\phi}$ is gauged or not. After integrating out these n gauge fields, the scalar potential induced by the gaugings is [68, 69, 70, 72]

$$\begin{aligned} V = \ & \frac{1}{2} \Xi^T \cdot \tilde{\mathcal{O}}^T \cdot \mathcal{M}_1 \cdot \tilde{\mathcal{O}} \cdot \Xi \\ & + \frac{e^{-2\phi}}{2} V_1^T \cdot \tilde{\mathcal{O}}^T \cdot \mathcal{M}_1 \cdot \tilde{\mathcal{O}} \cdot \overline{V}_1 \\ & + \frac{e^{-2\phi}}{2} V_2^T \cdot \mathcal{O} \cdot \mathcal{M}_2 \cdot \mathcal{O}^T \cdot \overline{V}_2 \\ & - \frac{e^{-2\phi}}{4\mathcal{V}} V_2^T \cdot C \cdot \tilde{\mathcal{O}} \cdot \left(V_1 \times \overline{V}_1^T + \overline{V}_1 \times V_1^T \right) \cdot \tilde{\mathcal{O}}^T \cdot C^T \cdot \overline{V}_2 . \end{aligned} \tag{3.3.6}$$

Notice that the potential does not depend on the NS-NS axion $\tilde{\phi}$. Gauge invariance of this scalar potential is ensured by

$$\delta_\lambda\big(\tilde{\mathcal{O}} \cdot \Xi\big) = -\tilde{\mathcal{O}} \cdot \mathcal{O}^T \cdot \lambda = 0\,. \tag{3.3.7}$$

In addition to the n R-R axions and possibly the NS-NS axion ϕ the remaining axions can obtain a mass from the scalar potential (3.3.6). But since $\tilde{\mathcal{O}} \cdot \Xi$ the scalar potential can depend on at most $h^{1,1} + 1$ axions for $h^{1,2} > h^{1,1}$. Therefore at most $h^{1,1} + 1$ additional axions can become massive due to the scalar potential.

This scalar potential can be derived by dimensional reduction of double field theory [72] on a Calabi-Yau manifold when perturbing the compactification with fluxes. Another way is to extend the usual Calabi-Yau compactification with geometric fluxes with their mirror duals [68, 69]. This results in the notion of $SU(3) \times SU(3)$ compactifications.

3.3.2 Partial supersymmetry breaking

In the following chapter we will review the existence of $\mathcal{N} = 1$ minima in the superpotential (3.3.6). Actually, whether such a minimum exists was under heavy debate for a long time but finally settled in a series of papers [73, 74, 75]. Let us directly mention a very important result from their paper: A $\mathcal{N} = 1$ vacuum can only exist for simultaneous geometric and non-geometric gaugings.

Let us deduce quantitative bounds on the $\mathcal{N} = 1$ spectrum. For this recall the $\mathcal{N} = 1$ multiplets

$$\begin{array}{lll}
\text{massless } \mathcal{N}=1 \text{ gravity} & G_{(1)} = 1\cdot[2] + 1\cdot[\tfrac{3}{2}] & = (2)_\mathrm{b} + (2)_\mathrm{f}\,,\\
\text{massless } \mathcal{N}=1 \text{ vector} & V_{(1)} = 1\cdot[1] + 1\cdot[\tfrac{1}{2}] & = (2)_\mathrm{b} + (2)_\mathrm{f}\,,\\
\text{massless } \mathcal{N}=1 \text{ chiral} & C_{(1)} = 1\cdot[\tfrac{1}{2}] + 2\cdot[0] & = (2)_\mathrm{f} + (2)_\mathrm{b}\,,\\
\text{massive } \mathcal{N}=1 \text{ spin-3/2} & \overline{S}_{(1)} = 1\cdot[\tfrac{3}{2}] + 2\cdot[1] + 1\cdot[\tfrac{1}{2}] & = (4)_\mathrm{f} + (6)_\mathrm{b} + (2)_\mathrm{f}\,,\\
\text{massive } \mathcal{N}=1 \text{ vector} & \overline{V}_{(1)} = 1\cdot[1] + 2\cdot[\tfrac{1}{2}] + 1\cdot[0] & = (3)_\mathrm{b} + (4)_\mathrm{f} + (1)_\mathrm{b}\,,\\
\text{massive } \mathcal{N}=1 \text{ chiral} & \overline{C}_{(1)} = 1\cdot[\tfrac{1}{2}] + 2\cdot[0] & = (2)_\mathrm{f} + (2)_\mathrm{b}\,.
\end{array} \tag{3.3.8}$$

The breaking mechanism in [73, 74, 75] follows a two-step procedure. In a first step the actual supersymmetry breaking occurs and exactly one of the $\mathcal{N} = 2$ gravitinos becomes massive while the other one stays massless. Still having $\mathcal{N} = 1$ supersymmetry, the massive gravitino will go into an $\mathcal{N} = 1$ massive gravitino multiplet. Since this contains two massive vectors, two of the massless vectors have to become massive, too, by eating up two axions via the Stückelberg mechanism. Whether the NS-NS axion $\tilde{\phi}$ takes part in the breaking depends on the actual gaugings that induce the breaking. Additionally two axions that we call $\zeta_{1,2}$ are fixed by the complex valued relation

$$(\tilde{\xi}_A - G_{AB}\,\xi^B)\,D^A = 0\,, \tag{3.3.9}$$

where $G_{AB}=\partial_A\partial_B G$ is the metric computed from the prepotential G of the Kähler moduli. D^A is a constant complex vector that carries the information about the $\mathcal{N}=1$ minimum. An important aspect here is that, (3.3.9) being a quadratic, one can imagine a situation in which (3.3.9) does not fix two but only one additional axion. We therefore assume that $k_1 \in \{1,2\}$ axions are fixed. To summarize, we started with $h^{1,2}+1$ gauge fields and $2(h^{1,1}+1)+1$ real axions. After the first step we are left with $N_V = h^{1,2}-1$ and $N_{\rm ax}^{\rm real} = 2h^{1,1}+1-k_1$ massless real axions.

In the second step we take the effect of all those gaugings into account that did not directly participate in the breaking. Already being in an $\mathcal{N}=1$ minimum this computation occurs purely in $\mathcal{N}=1$ supergravity. The additional $n-2$ gaugings can make $n-2$ gauge fields massive by the Stückelberg mechanism. In course the axions that are eaten up by the gauge fields become massive, too.

Furthermore one gets an F- and a D-term potential where the F-term is independent of the axions. It is therefore enough to look at the D-terms. Since we search for a supersymmetric Minkowski vacuum the D-term has to vanish, giving $n-2$ real condition. This sets an upper bound on the number of axions k_2 that can become massive with this mechanism.

Since there is no further condition that the remaining axions have to satisfy we can summarize everything. We get $N_V = h^{1,2}-n+1$ remaining vectors and $N_{\rm ax}^{\rm real} = 2h^{1,1}-n+3-k_1-k_2$ massless axions. Now recall that we need at least two gaugings for supersymmetry breaking but have at most $h^{1,1}+1+\Delta$ gaugings, $\Delta=1$ if there is gauging along the NS-NS acion $\tilde{\phi}$ and $\Delta=0$ otherwise. Therefore

$$h^{2,1}-h^{1,1}-\Delta \leq N_V \leq h^{2,1}-1\,. \tag{3.3.10}$$

Similarly we get for the number of axions

$$2(h^{1,1}-h^{2,1}+N_V)+1 \leq N_{\rm ax}^{\rm real} \leq 2h^{1,1}-h^{2,1}+N_V+1\,. \tag{3.3.11}$$

In case the NS-NS axion is gauged the number of complex R-R axions is $N_{\rm ax} = N_{\rm ax}^{\rm real}/2$. If the NS-NS axion is remains massless there are $N_{\rm ax} = (N_{\rm ax}^{\rm real}-1)/2$ complex R-R axions. We find

$$N_V - N_{\rm ax} \leq h^{2,1}-h^{1,1}-\Delta\,, \tag{3.3.12}$$

and

$$N_V - 2N_{\rm ax} \geq h^{2,1}-2h^{1,1}-\Delta\,. \tag{3.3.13}$$

Notice that the NS-NS dilaton ϕ stays massless.

3.3.3 Validity of an $\mathcal{N}=2$ GSUGRA with non-geometric fluxes

This section comments on the validity of these considerations and whether we can really compare an $\mathcal{N}=2$ gauged supergravity with the findings in the asymmetric Gepner models.

While unambiguous from the pure supergravity viewpoint, these questions arise when the gauged supergravity is interpreted as the effective action of a string theory vacuum. From this perspective, the four-dimensional $\mathcal{N} = 2$ supergravity captures only the massless part of the full spectrum. Kaluza-Klein and higher spin modes are integrated out and α' corrections not taken into account. Furthermore, gaugings correspond to fluxes in the internal geometry whose backreaction is neglected in the effective supergravity theory. Let us discuss these issues.

Vectors being protected by symmetry, the relevant question for us regarding α' corrections is whether they can produce additional axionic mass terms that we miss in the above consideration. The hope is that axionic mass terms are constrained due to the axionic shift symmetry. This symmetry is inherited from the 10D gauge invariance of the p-forms and should therefore not be broken by α' corrections. While fluxes can break the shift symmetry, they do it in a controlled way as shown in [76, 77]. In fact there is still a remnant of the shift symmetry: Any shift in the axion can be compensated by an appropriate shift in the flux. Being a left over of the gauge invariance, also this modified shift symmetry should hold in any order in α'. Based on that, [76, 77] argues that the higher order corrections are functions of the tree level potential

$$V = V_{\text{tree}}(\phi_i) + V_{\alpha'}\big(V_{\text{tree}}(\phi_i)\big)\,. \tag{3.3.14}$$

This means that the tree level considerations are enough and no further mass terms are expected to come from α' corrections. How strictly this scheme applies to string theory is under current investigation [78, 79] but it is common sense that additional mass terms are rather constrained.

A second source for axionic mass terms for the axion could be non-perturbative corrections. For the R-R axions these would be D-brane instantons that are non-perturbative in the string coupling. They cannot be captured by the CFT that usually can only see non-perturbative effects in α', like the worldsheet instantons.

However remember that in Gepner models all masses scale with α' such that there is no hierarchy between the massive string modes, KK modes and the massive moduli. There could therefore be subtle effects e.g. by couplings to massive KK-modes that generate mass terms for R-R axions that are cannot be foreseen by the gauged supergravity.

Let us come to the next possible problem, the backreaction. Recall that the gaugings correspond to fluxes from the string theory viewpoint. More precisely one takes a Calabi-Yau compactification of string theory and perturbs it with fluxes. Every flux has an energy and therefore a backreaction onto the geometry that must be small to have a trustworthy perturbation. This is the case if the energy density of the fluxes appearing in Einstein's equation becomes negligible. The problem in realizing a small flux density is that the fluxes cannot be tuned arbitrarily small due to Diracs charge quantization fixing the fluxes to discrete values. To overcome this obstacle one usually tunes the cycle the flux is living on very large. In this way the energy stays constant but the energy density and therefore the backreaction decreases. This limit in which the perturbation can be trusted is called *dilute flux limit.*

Let us check whether there is a parameter regime in which the backreaction of the non-geometric fluxes becomes negligible. In the course of the computation we will also gain arguments why α' corrections are suppressed. We will follow the discussion of [5] also of the author. They analyze the phenomenology of type IIB orientifold compactifications with non-geometric fluxes and R-R F_3 flux. This is not exactly the setting we encounter here. We will nevertheless use this setup since the key result will be independent of the F_3 flux. The F_3 flux makes it possible to have a clear scaling structure and all scales come out as ratios of the fluxes. Therefore all results are very easily interpretable. We will only discuss the easiest example since the results do not change for more involved models.

The moduli of the example we look at are the axiodilaton S and a single Kähler modulus T where their saxionic parts $s = e^{-\phi}$ and $t \sim \mathcal{V}^{2/3}$ encode the dilaton and the volume. To stabilize the moduli we add H-, Q- and F_3 fluxes with flux quanta h, q, and f_3 respectively. In any minimum, an overall flux scaling dictates the saxionic parts of the moduli to scale as $s \sim \frac{f_3}{h}$ and $t \sim \frac{f_3}{q}$. To trust the supergravity approximation we need to have small string coupling and large volume, therefore $|q|, |h| << |f_3|$ thus $|f_3| >> 1$ while $|h|, |q| \sim \mathcal{O}(1)$.

The uncompactified action of the fluxes is the one of the flux formulation of double field theory [50, 80, 51]

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \Big(\mathcal{L}^{HH} + \mathcal{L}_1^{QQ} + \mathcal{L}_2^{QQ} + \mathcal{L}^{HQ} + \mathcal{L}^{\text{R-R}} \Big) . \tag{3.3.15}$$

The various contributions are

$$\begin{aligned} \mathcal{L}^{HH} &= -\frac{e^{-\phi}}{12} H_{ijk}\, H_{i'j'k'}\, g^{ii'} g^{jj'} g^{kk'} , & \mathcal{L}^{HQ} &= \frac{1}{2} H_{mni}\, Q_{i'}{}^{mn}\, g^{ii'} , \\ \mathcal{L}_1^{QQ} &= -\frac{e^{\phi}}{4} Q_k{}^{ij}\, Q_{k'}{}^{i'j'}\, g_{ii'} g_{jj'} g^{kk'} , & \mathcal{L}_2^{QQ} &= -\frac{e^{\phi}}{2} Q_m{}^{ni}\, Q_n{}^{mi'}\, g_{ii'} , \\ \mathcal{L}^{\text{R}-\text{R}} &= -\frac{e^{\phi}}{12} F_{ijk}\, F_{i'j'k'}\, g^{ii'} g^{jj'} g^{kk'} . \end{aligned} \tag{3.3.16}$$

The reduction of this action on a threefold results in the one of $\mathcal{N} = 2$ gauged supergravity (3.3.6) [72]. The metric scales with the third root of the volume $g \sim \sqrt{f_3/q}$ such that all terms in the action have the common scaling

$$\mathcal{L}^{HH} \sim \mathcal{L}_1^{QQ} \sim \mathcal{L}_2^{QQ} \sim \mathcal{L}^{QH} \sim \mathcal{L}^{R-R} \sim \frac{hq^{\frac{3}{2}}}{f_3^{\frac{1}{2}}} . \tag{3.3.17}$$

In the perturbative regime $|q|, |h| << |f_3|$ the action becomes small. Since this action is the contraction of a generalized Riemann tensor and higher corrections are supposed to come in higher powers in this generalized Riemann tensor we conclude that α' corrections are suppressed. Notice that here the R-R flux appears but also when it is absent one can show for several examples that there is always a limit in which the generalized Ricci scalar becomes small [81].

Let us turn to the backreaction that is measured by the energy momentum tensor $T_{ij} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{ij}}$. We get an overall scaling that is now independent of the R-R flux

$$T_{ij}^{HH} \sim T_{1ij}^{QQ} \sim T_{2ij}^{QQ} \sim T_{ij}^{HQ} \sim T_{ij}^{\mathrm{R-R}} \sim hq\,. \qquad (3.3.18)$$

h and q being quantized the backreaction is of order one, therefore not severely large but still there is no controlled limit in which the backreaction is negligible. Also heuristically a backreaction is expected. Non-geometric fluxes as the T-duals of the usual fluxes are supposed to correspond to wound string backgrounds. While strings with momentum tend to expand a cycle, wound strings shrink a cycle [82, 83]. Therefore non-geometric fluxes on a cycle are expected to backreact on the cycle by shrinking it towards the self-dual radius.

When a cycle becomes small due to a flux, the winding modes of that cycle become lighter. A possible problem appears if they become even lighter than the scalars that have gotten their mass through the fluxes. Furthermore in the above model the Kaluza-Klein scale M_{KK} is parametrically of the same order as the mass of the moduli $M_{KK}/M_{\mathrm{mod}} \sim (hq)^{-1/2}$. By choosing $h > q$ we can at least tune the string scale up such that indeed higher spin modes are correctly integrated out. Nevertheless if we miss some modes, the gauged supergravity cannot be the effective action that describes the dynamics up to a certain energy scale.

Still, the gauged supergravity has the potential to describe the massless modes kinematically.[3] Since the different cycles of a Calabi-Yau are rather independent, the massless modes from the unaffected cycles should stay massless regardless of fluxes on other cycles. Furthermore, the modes that have gotten a mass from the fluxes, the winding modes as well as the KK modes might might get rather low but still stay massive along the backreaction. One can subsume this by saying that the kinematic spectrum is preserved along the adjustments. More technically, the topological data, thus the number of closed and (due to fluxes, see e.g. [71, 68, 69]) unclosed cycles is preserved. In this way the number of massless modes might be predicted correctly by the gauged supergravity while the dynamics of the breaking are not captured.

The massless spectrum is exactly the data that we can unambiguously extract from Gepner models as well. Gepner models are at a point in moduli space where all massive modes have a mass of order α' and there is no way to distinguish massive modes according to their origin. Therefore we restricted to the massless modes in the discussion in the last section.

To summarize, while a gauged supergravity seems not to be the correct effective action when turning on non-geometric fluxes it might still have predictive power in the sense that the massless modes are correctly foreseen. Since this is the quantity that can be compared to the CFT spectra, we have hope that we can find an agreement. The comparison might show discrepancies since additional mass terms of axions could be generated.

[3]With this we mean that the number of massless and massive degrees of freedom is predicted correctly.

3.4 Comparison of ACFTs and GSUGRA

We are ready to compare the asymmetric Gepner models of section 3.2 to the gauged supergravity predictions (3.3.10), (3.3.12) and (3.3.13). We summarized the results in table 3.5.

Gepner	$(N_V, N_{\text{ax}}, N_0)$	$(h^{2,1}, h^{1,1})$	constraints
(3 3 3 3 3)	$(80, 0, 74)$	$(83, 2)$	$N_V - N_{\text{ax}} \leq 81 - \Delta$ $N_V - 2N_{\text{ax}} \geq 79 - \Delta$ $81 - \Delta \leq N_V \leq 82$
$(5\,5\,5\,12_D)$	$(86, 2, 80)$	$(89, 3)$	$N_V - N_{\text{ax}} \leq 86 - \Delta$ $N_V - 2N_{\text{ax}} \geq 83 - \Delta$ $86 - \Delta \leq N_V \leq 88$
$(5\,5\,5\,12_A)$	$(86, 2, 80)$	$(88, 4)$	$N_V - N_{\text{ax}} \leq 84 - \Delta$ $N_V - 2N_{\text{ax}} \geq 80 - \Delta$ $84 - \Delta \leq N_V \leq 87$
(7 7 7 1 1)	$(74, 2, 70)$	$(75, 6)$	$N_V - N_{\text{ax}} \leq 69 - \Delta$ $N_V - 2N_{\text{ax}} \geq 63 - \Delta$ $69 - \Delta \leq N_V \leq 74$
$(6\,6\,6\,6_D)$	$(60, 4, 64)$	$(62, 6)$	$N_V - N_{\text{ax}} \leq 56 - \Delta$ $N_V - 2N_{\text{ax}} \geq 50 - \Delta$ $56 - \Delta \leq N_V \leq 61$
(10 10 4 4)	$(59, 5, 68)$	$(66, 8)$	$N_V - N_{\text{ax}} \leq 58 - \Delta$ $N_V - 2N_{\text{ax}} \geq 50 - \Delta$ $58 - \Delta \leq N_V \leq 65$

Table 3.5: Comparison of the ACFTs with the corresponding GSUGRA. The dashed conditions are satisfied if only one of the R-R-axions $\zeta_{1,2}$ remains massless after being fixed by (3.3.9). The underlined condition is a priori not satisfied but would be, if we allow for additional axion mass terms by other effects.

As one can see, the data from the asymmetric Gepner models and the gauged supergravity show a huge correlation. Notice that the bounds from the inequalities are in fact very narrow, for instance in the first example the bounds allow only six combinations of N_V and N_{Ax}. In the whole list only one inequality is not satisfied, but would be if the model had few more massless axions. Recalling the discussion of the last section that the generation of extra mass terms for axions is expected. It is therefore rather a surprise that only one relation is violated.

Also general expectations are fulfilled. The universal chiral multiplet from the vacuum sector of the CFT is the perfect candidate to host the dilaton and the dual of the B field, that both cannot generate a mass from the gauged supergravity viewpoint. Furthermore, conformal field theories that yield $\mathcal{N} = 1$ target space supersymmetry are necessarily left-right asymmetric and therefore non-geometric. This is reflected in the fact that the $\mathcal{N} = 1$ vacua appear only for simultaneous geometric and non-geometric fluxes.

These results are very encouraging and motivate us to the conjecture:

Certain Gepner models with asymmetric simple current extensions describe the fully backreacted solutions to partially broken $\mathcal{N} = 2$ gauged supergravities.

This is the main result of this thesis. Let us list the consequences of this conjecture

- Partial supersymmetry breaking is possible up to all orders in α'.
- Minima of gauged supergravity can lift to full string solutions.
- The strong constraint violating non-geometric fluxes are part of the string landscape and correspond to asymmetric conformal field theories.

Chapter 4

Asymmetric Gepner models with extended supersymmetry

In section 3 we found very encouraging hints that asymmetric Gepner models could actually be identified with $\mathcal{N} = 2$ gauged supergravities. Unfortunately, the identification of the underlying supergravity theory does not rely on an unambiguous procedure. Furthermore we could only compare the CFT data with inequalities derived from supergravity. The natural next step is to provide further evidence for the ACFT/GSUGRA conjecture. Instead of searching for more $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ examples we will change to a setting that is better under control and seek for more evidence there.

When having extended supersymmetry with eight or more supercharges the existence of a superpotential is forbidden. The consequence is that mass terms can only be generated via the super-Higgs effect [84, 85, 86, 87]. In the super-Higgs effect the number of degrees of freedom is a conserved quantity such that the uncertainties in the breaking mechanism become equalities. Notice that this statement holds presumably for all orders in α'. Asymmetric Gepner models with extended supersymmetry can therefore be compared to very clean expectations from the supergravity side. When furthermore increasing the dimensions to 6D or 8D there are no NS-NS gaugings at all. A heuristic reason for this is that the NS-NS fluxes have three legs that cannot be supported neither in 8D nor in the 6D supergravity of type II on K3. While supportable on the supergravity of type II on $\mathbb{T}^4$ an analysis of the spectrum shows that supersymmetry cannot be broken in this case either.

To collect convincing evidence for the ACFT/GSUGRA correspondence we need to check as many asymmetric Gepner models with extended supersymmetry in 4D, 6D and 8D as possible. With this intention we wrote a computer program to perform stochastic searches over the landscape of asymmetric Gepner models. It turned out that the computer program was actually so efficient that we are confident to say that we found a full classification of asymmetric Gepner models with extended supersymmetry. We were able to explain all classes of models by the following short list of mechanisms:

- Dimensional reduction of higher dimensional models.

- Special gauge enhancements as common for Gepner models.

- Orbifolds containing a $(-1)^{F_L}$ factor. Being a fermionic action these CFTs cannot have anything to do with supergravity plus NS-NS gaugings.

- The super-Higgs effect giving evidence for the ACFT/GSUGRA correspondence.

Examples for every type of model we found are stored under the URL [88]

This chapter is structured as follows. First we will review the super-Higgs effect to see what to expect from the supergravity side. Then we will state the CFT setup and the classification scheme. After that, in the third section, the classification and explanation of all asymmetric Gepner models will be described. We provide details about the computer program that performed the stochastic search in the appendix A.

4.1 Super-Higgs effect

When having at least 8 supercharges, supersymmetry forbids the existence of a superpotential. Therefore when turning on gaugings, the fields can only become massive through the super-Higgs effect [84, 85, 86, 87]. This implies tight constraints on the allowed breaking patterns since the degrees of freedom of both theories must match. Let us explain this mechanism in an example, all other cases can then be treated similarly using the list of multiplets in appendix B.

When taking $\mathcal{N}=8$ maximal supergravity in 4D the spectrum is

$$\begin{aligned} \text{massless} \qquad \mathcal{G}_{(8)} &= 1\cdot[2] + 8\cdot[\tfrac{3}{2}] + 28\cdot[1] + 56\cdot[\tfrac{1}{2}] + 70\cdot[0] \\ &= (2)_{\rm b} + (16)_{\rm f} + (56)_{\rm b} + (112)_{\rm f} + (70)_{\rm b}\,, \end{aligned} \tag{4.1.1}$$

where the first line counts the number of fields of each spin and the second line counts the degrees of freedom of each spin. Let us check whether a breaking to $\mathcal{N}=6$ is possible. The $\mathcal{N}=6$ supergravity multiplet is

$$\begin{aligned} \text{massless} \qquad \mathcal{G}_{(6)} &= 1\cdot[2] + 6\cdot[\tfrac{3}{2}] + 16\cdot[1] + 26\cdot[\tfrac{1}{2}] + 30\cdot[0] \\ &= (2)_{\rm b} + (12)_{\rm f} + (32)_{\rm b} + (52)_{\rm f} + (30)_{\rm b}\,, \end{aligned} \tag{4.1.2}$$

and the massive spin-3/2 supermultiplet is

$$\begin{aligned} \text{massive} \qquad \overline{\mathcal{S}}_{(6)} &= 2\cdot\Big([\tfrac{3}{2}] + 6\cdot[1] + 14\cdot[\tfrac{1}{2}] + 14\cdot[0]\Big) \\ &= (8)_{\rm f} + (36)_{\rm b} + (56)_{\rm f} + (28)_{\rm b}\,. \end{aligned} \tag{4.1.3}$$

Since this is a $\frac{1}{2}$-BPS short multiplet, it has to come in pairs to guarantee CPT invariance. As one can see, the $\mathcal{N}_8$ spectrum can be decomposed into $\mathcal{N}=6$ multiplets

$$\mathcal{G}_{(8)} \to \mathcal{G}_{(6)} + 2\times\overline{\mathcal{S}}_{(6)}\,. \tag{4.1.4}$$

Instead of looking at the concrete dynamics of $\mathcal{N}=8 \to \mathcal{N}=6$ supersymmetry breaking we only checked that the kinetic requirements are fulfilled. That is only a necessary but not a sufficient criteria for the possibility of a breaking from $\mathcal{N}=8$ to $\mathcal{N}=6$.

For instance, a similar breaking to $\mathcal{N}=5$ is kinematically forbidden as one can check by consulting the appendix B where all multiplets are collected.

4.2 Classification scheme and the CFT setup

For our analysis we add n additional simple currents to the partition function of the Gepner model. The partition function is

$$Z_{\mathrm{ACFT}}(\tau,\overline{\tau}) \sim \vec{\chi}^{T}(\tau)\, M(J_1)\ldots M(J_{\mathrm{n}})\, M(J_{\mathrm{GSO}}) \prod_{i=1}^{r} M(J_i)\, \vec{\chi}(\overline{\tau})\Big|_{\phi^{-1}_{\mathrm{bsm}}}, \tag{4.2.1}$$

where in the actual stochastic search the number of additional simple currents n was at most five. The simple currents we added were chosen stochastically and had non-trivial entries in all five minimal models.

Since we look for models with extended supersymmetry it is often more efficient to use the following searching scheme. We first added enhancement simple currents to get a certain amount of supersymmetry from the right-movers. While keeping the enhancement simple currents fixed, we added further simple currents to get an asymmetric model. The partition function is therefore

$$Z_{\mathrm{ACFT}}(\tau,\overline{\tau}) \sim \vec{\chi}^{T}(\tau)\, M(J_{\mathrm{break}})\, M(J_{\mathrm{enhance}})\, M(J_{\mathrm{GSO}}) \prod_{i=1}^{r} M(J_i)\vec{\chi}(\overline{\tau})\Big|_{\phi^{-1}_{\mathrm{bsm}}}. \tag{4.2.2}$$

Let us state some examples of this procedure. Imaging that an enhancement simple current gives the torus compactification in 4D with $\mathcal{N}_L, \mathcal{N}_R = 4$. Then the additional breaking simple currents break the left-moving supersymmetry down to $\mathcal{N}_L = 0, 1, 2$ while $\mathcal{N}_R$ is still four. Using this procedure we get models with $\mathcal{N} = 6, 5, 4$ target space supersymmetry in 4D.

Another example is a Gepner model with $k = 1, 3, 3, 4, 8$. When adding a D-invariant simple current (2.5.32), supersymmetry is enhanced to $\mathcal{N}_L, \mathcal{N}_R = 2$. This corresponds to the $K3 \times \mathbb{T}^2$ compactification of type II. Adding simple currents to break the left-moving supersymmetry to $\mathcal{N}_L = 1, 0$ leaves the right-moving supersymmetry untouched such that we get models with $\mathcal{N} = 3, 2$ target space supersymmetry in 4D.

We will sort the models according to the number of non-compact directions D and the amount of target space supersymmetry from the left and right $\mathcal{N}_L, \mathcal{N}_R$. We collect this information into the notation ${}^{D}\mathfrak{N}_{[\mathcal{N}_L,\mathcal{N}_R]}$. In 4D we will consider models that have at least $\mathcal{N}_R \geq 2$, in 6D and 8D models that have at least $\mathcal{N}_R \geq 1$. Since the number of non-compact dimensions D is always even, the number of supercharges is $Q = 2^{\frac{D}{2}}\,(\mathcal{N}_L + \mathcal{N}_R)$. We will state the spectrum in the form

$$
{}^{D}\mathfrak{N}_{[\mathcal{N}_L,\mathcal{N}_R]}:\begin{cases}(n_v^{(0)},n_s^{(0)},n_c^{(0)},n_o^{(0)})_L\otimes(n_v^{(0)},n_s^{(0)},n_c^{(0)},n_o^{(0)})_R & \text{supermultipl.}\\ (n_v^{(c)},n_s^{(c)},n_c^{(c)},n_o^{(c)})_L\otimes(n_v^{(c)},n_s^{(c)},n_c^{(c)},n_o^{(c)})_R & \text{supermultipl.}\\ \quad\cdots \qquad\qquad\qquad \cdots & \cdots\end{cases}
$$

The superscript (0) stands for the uncharged vacuum sector and (c) stands for the charged sector. The subscript stands for the $\widehat{\mathfrak{so}}(D-2)_1$ representation (v,s,c,o) which can be used to deduce the target space interpretation following the rules on page 19.

Let us state the different vacuum sectors of one side in different dimensions and how many gravitinos and additional fields they contribute. We use the notation from above (n_v,n_s,n_c,n_o).

D	Gravitinos	Vacuum sector	Additional fields
8	1	$(1,1,1,2)$	-
	0	$(1,0,0,6)$	6 vectors
6	2	$(1,2,2,4)$	-
	1^+	$(1,2,0,0)$	-
	1^-	$(1,0,2,0)$	-
	0	$(1,0,0,n)$	n vectors
4	4	$(1,4,4,6)$	-
	2	$(1,2,2,2)$	-
	1	$(1,1,1,0)$	-
	0	$(1,0,0,n)$	n vectors

Combining left- and right-moving vacuum sectors of the above that we see that we can have $\mathcal{N}=8,6,5,4,3,2,1$ but not $\mathcal{N}=7$ supersymmetry in 4D.

Supersymmetry also constraints the charged sector of the CFT. When one side contributes more than eight supercharges, the charged sector is always empty. The remaining non-trivial charged sectors are constrained are of the form

D	SUSY	Charged sector
6	$1^\pm$	$(0,0,1,2)^-$ or $(0,1,0,2)^+$
4	2	$(0,1,1,2)$
	1	$(0,0,1,1)$ and/or $(0,1,0,1)$

4.3 ACFTs in $D=8$

In case of 8 non-compact directions there are only three Gepner models with levels $k=(1,1,1)$, $k=(1,4)$, $k=(2,2)$. Most models we found have the spectrum of maximal

$\mathcal{N} = 2$ supergravity in 8 dimensions with only the gravity multiplet. Its bosonic massless spectrum is

$$\mathcal{G}_{(2)} = 1 \cdot [2] + 2 \cdot [\tfrac{3}{2}] + 6 \cdot [1] + 6 \cdot [\tfrac{1}{2}] + 7 \cdot [0] + 1 \cdot [\mathrm{t}_3]\,. \tag{4.3.1}$$

On the CFT side this originates from

$$^{8}\mathfrak{N}_{[1,1]}:\ \Big\{(1,1,1,2)_L \otimes (1,1,1,2)_R \qquad \mathcal{G}_{(2)}\,. \tag{4.3.2}$$

These models correspond to the torus compactification of type II on $\mathbb{T}^2$.

The class $^{8}\mathfrak{N}_{[0,1]}$

Only for the Gepner model with levels $k = (2, 2)$ we found another class of models. For later reference let us state one of the many simple currents that induces this model

$$J_{\mathrm{ACFT}} = (0, 2, 2)(0, -2, 2)(v). \tag{4.3.3}$$

The spectrum reveals an $\mathcal{N} = 1$ supersymmetry where all states come from the vacuum sector

$$^{8}\mathfrak{N}_{[0,1]}:\ \Big\{(1,0,0,6)_L \otimes (1,1,1,2)_R \qquad \mathcal{G}_{(1)} + 6 \cdot \mathcal{V}_{(1)}\,. \tag{4.3.4}$$

The multiplets have the field content

$$\mathcal{G}_{(1)} = 1 \cdot [2] + 1 \cdot [\tfrac{3}{2}] + 2 \cdot [1] + 1 \cdot [\tfrac{1}{2}] + 1 \cdot [0] + 1 \cdot [\mathrm{t}_2]\,, \tag{4.3.5}$$

and

$$\mathcal{V}_{(1)} = 1 \cdot [1] + 1 \cdot [\tfrac{1}{2}] + 2 \cdot [0]\,. \tag{4.3.6}$$

A closer inspection of the vector mutiplets shows that they transform in the gauge group $SU(2) \times SU(2)$.

Let us check whether this model can be interpreted as gauged $\mathcal{N} = 2$ supergravity. For this we decompose the $\mathcal{N} = 2$ supergravity multiplet in terms of $\mathcal{N} = 1$ multiplets as

$$\mathcal{G}_{(2)} \ \rightarrow\ \mathcal{G}_{(1)} + \overline{\mathcal{S}}_{(1)} + (2 - \alpha) \cdot \mathcal{V}_{(1)} + \alpha \cdot \overline{\mathcal{V}}_{(1)}\,, \tag{4.3.7}$$

where $\alpha = 0, 1, 2$ counts the number of massive vector multiplets. Clearly the model we found cannot be interpreted as broken $\mathcal{N} = 2$ supergravity. This was expected since the NS-NS fluxes in string theory have three legs and can not be supported by a two-dimensional compact space.

So either we have a counterexample to the ACFT/GSUGRA correspondence or we have to find an alternative explanation. For this we remember the fact that the $k = (2, 2)$ model corresponds to the rectangular torus with radii at the self-dual radius. As such it is natural to consider asymmetric torus orbifolds. Since this ACFT cannot have a GSUGRA dual with bosonic fluxes we focus on orbifolds with an fermionic action like $(-1)^{F_L}$. Recall that indeed simple currents are able to implement $(-1)^{F_L}$ orbifolds as can be seen in (2.8.10).

The partition function type IIB on the rectangular $\mathbb{T}^2$ with $r_i = \sqrt{\alpha'}$ is

$$Z_{\mathbb{T}^2} = (V_8 - S_8)(\tau)\,(\overline{V}_8 - \overline{S}_8)(\overline{\tau})\,\Lambda^{(2)}_{\vec{m},\vec{n}}(\tau,\overline{\tau})\,, \tag{4.3.8}$$

where we skipped the bosonic contribution $1/|\eta|^{16}$. V_8, O_8, C_8 and S_8 are the $\widehat{\mathfrak{so}}(8)_1$ characters from (2.2.19) and $\Lambda_{\vec{m},\vec{n}}$ denotes the momentum and winding modes

$$\Lambda^{(2)}_{\vec{m},\vec{n}}(\tau,\overline{\tau}) = \sum_{\vec{m},\vec{n}\in\mathbb{Z}^2} q^{\frac{1}{4}\sum_i (m_i-n_i)^2}\,\overline{q}^{\frac{1}{4}\sum_i (m_i+n_i)^2}\,. \tag{4.3.9}$$

Let us take a look at the asymmetric orbifold

$$\mathcal{A}_8 = \frac{\mathbb{T}^2}{(-1)^{F_L}\,S\,W}\,, \tag{4.3.10}$$

where S and W are defined by

$$S : (-1)^{\sum_i m_i} =: (-1)^{\vec{m}}\,, \qquad W : (-1)^{\sum_i n_i} =: (-1)^{\vec{n}}\,. \tag{4.3.11}$$

What the twisted sector looks like can be deduced by a modular S transformation which exchanges $O \leftrightarrow V$ and $C \leftrightarrow S$ and shifts $\vec{m}$ and $\vec{n}$ by $\frac{1}{2}$. The latter shift can also be deduced by writing S and W as shift operators. The partition function is therefore

$$\begin{aligned} Z_{\mathrm{ACFT}} = \frac{1}{2}\Big[&\quad (V_8 - S_8)(\overline{V}_8 - \overline{S}_8)\Lambda^{(2)}_{\vec{m},\vec{n}} \\ &+ (V_8 - S_8)(\overline{V}_8 + \overline{S}_8)(-1)^{\vec{m}+\vec{n}}\,\Lambda^{(2)}_{\vec{m},\vec{n}} \\ &+ (V_8 - S_8)(\overline{O}_8 - \overline{C}_8)\Lambda^{(2)}_{\vec{m}+\frac{\vec{1}}{2},\vec{n}+\frac{\vec{1}}{2}} \\ &+ (V_8 - S_8)(\overline{O}_8 + \overline{C}_8)(-1)^{\vec{m}+\vec{n}}\,\Lambda^{(2)}_{\vec{m}+\frac{\vec{1}}{2},\vec{n}+\frac{\vec{1}}{2}} \Big]\,. \end{aligned} \tag{4.3.12}$$

Since $(-1)^{F_L}$ eliminates all left-moving R-R states the $V_8 \otimes \overline{V}_8$ states are the only bosonic massless modes in the untwisted sector. These 64 bosonic modes combine into the $\mathcal{N}=1$ supergravity multiplet plus two vector multiplets[1]. In the twisted sector we have $V_8 \otimes \overline{O}_8$ such that we need an additional contribution from the rightmoving winding sector of the form $\overline{q}^{\frac{1}{2}}$ to get a level-matched massless state. This contribution can only come from

$$\Lambda^{(2)}_{\vec{m}+\frac{\vec{1}}{2},\vec{m}+\frac{\vec{1}}{2}} = q^0 \sum_{\vec{m}} \overline{q}^{\frac{1}{4}\sum_i (2m_i+1)^2}\,. \tag{4.3.13}$$

[1]Notice that this is the spectrum of a broken $\mathcal{N}=2$ supergravity

There are four combinations $m_1, m_2 = 0, 1$ giving four $\mathcal{N} = 1$ vector multiplets that are the $W^{\pm}$-bosons of an $SU(2) \times SU(2)$ gauge group whose Cartan generators are the two vectors from the untwisted sector. This is enough evidence to come to the conclusion that this orbifold is the ${}^8\mathfrak{N}_{[0,1]}$ model.

Similar models with a $(-1)^{F_L}$ factor and shifts in the winding and/or momentum can also be constructed in other dimensions. For instance in 4D such models are classified in [89]. There they find models with $[0, 4]$ supersymmetry and maximal gauge group $SU(2)^6$. Similarly in 6D one gets $[0, 2]$ supersymmetry and gauge group $SU(2)^4$. Since these models will be important later let us give them a name, $\mathcal{A}_d$. Clearly due to the $(-1)^{F_L}$ factor all these models cannot have an interpretation as a supergravity with bosonic gaugings. They are therefore perfect candidates if we find models that cannot arise as supergravity with broken supersymmetry. All $\mathcal{A}_d$ models have in common that part of the symmetry is broken when going to the Coloumb branch.

Summary

We summarize the findings in 8D in the following table:

class	spectrum	realization
${}^8\mathfrak{N}_{[1,1]}$	$\mathcal{G}_{(2)}$	$\mathbb{T}^2$
${}^8\mathfrak{N}_{[0,1]}$	$\mathcal{G}_{(1)} + 6 \cdot \mathcal{V}_{(1)}^{SU(2)^2}$	$\mathcal{A}_8$

4.4 ACFTs in $D = 6$

The class ${}^6\mathfrak{N}_{[2,2]}$

To get maximal $\mathcal{N} = (2, 2)$ supersymmetry one can either use simple currents or one takes the pure model $k = (1, 1, 1, 2, 2)$ without further simple currents giving the spectrum

$$
{}^6\mathfrak{N}_{[2,2]} : \ \Big\{ (1, 2, 2, 4)_L \otimes (1, 2, 2, 4)_R \qquad \mathcal{G}_{(2,2)} \, . \tag{4.4.1}
$$

This vacuum sector correctly gives the $\mathcal{N} = (2, 2)$ supergravity multiplet, see appendix B. The target space interpretation of this model is type II on $\mathbb{T}^4$.

The classes ${}^6\mathfrak{N}_{[1,1]}$

The next model arises as pure $k = (2, 2, 2, 2)$ model. The field content is

$$
{}^6\mathfrak{N}_{[1,1]}(\mathrm{B}) : \ \begin{cases} (1, 0, 2, 0)_L \otimes (1, 0, 2, 0)_R & \mathcal{G}_{(0,2)} + \mathcal{T}_{(0,2)} \, , \\ 20 \times \big[(0, 0, 1, 2)_L \otimes (0, 0, 1, 2)_R \big] & 20 \cdot \mathcal{T}_{(0,2)} \, . \end{cases} \tag{4.4.2}
$$

In 6D, the existence of tensor multiplets is a signal for a chiral theory, in this case an $\mathcal{N} = (2, 0)$ theory. The target space interpretation is type IIB on a $K3$ surface. To obtain type IIA on a $K3$ surface we add the simple current (2.8.10) and obtain a model with spectrum

$$
{}^6\mathfrak{N}_{[1,1]}(\mathrm{A}): \begin{cases} (1,2,0,0)_L \otimes (1,0,2,0)_R & \mathcal{G}_{(1,1)}\,, \\ 20 \times \big[(0,1,0,2)_L \otimes (0,0,1,2)_R\big] & 20 \cdot \mathcal{V}_{(1,1)}\,. \end{cases} \tag{4.4.3}
$$

The class ${}^6\mathfrak{N}_{[0,2]}$

When compactifying the $\mathcal{A}_2$ model from the 8D classification on a $\mathbb{T}^2$ we get half-maximal supergravity in 6D. We can reproduce this model e.g. by taking $k = (1,1,1,2,2)$ and adding the simple current $J_{\mathrm{ACFT}} = (0,0,0)^3(0,2,2)(0,-2,2)(v)$ similar to (4.3.3). The spectrum is

$$
{}^6\mathfrak{N}_{[0,2]}: \Big\{ (1,0,0,8)_L \otimes (1,2,2,4)_R \qquad \mathcal{G}_{(1,1)} + 8 \cdot \mathcal{V}_{(1,1)}\,. \tag{4.4.4}
$$

The vectors transform in $SU(2) \times SU(2) \times U(1)^2$.

In the model $k = (2,2,2,2)$ we also found models with a different amount of left-moving scalars in the vacuum sector leading to $n_V = 4, 8, 12$ vector multiplets. We interpret the model with $n_V = 12$ as the model $\mathcal{A}_6$ with its gauge group $SU(2)^4$ and the other models of this series as lying on its Coloumb branch. The model with $n_V = 4$ is the model with the minimal gauge group $U(1)^4$. The models having $n_V = 8$ and $n_V = 12$ with the gauge groups $SU(2)^2 \times U(1)^2$ and $SU(2)^3 \times U(1)$ lie in the middle. This explanation excludes an interpretation as a gauged supergravity and indeed one can check that these models cannot arise as broken $\mathcal{N} = (2,2)$ supergravity.

The class ${}^6\mathfrak{N}_{[0,1]}$

Lastly let us look at models with $\mathcal{N} = (1,0)$ target space supersymmetry. There we found a single class of model. The spectrum is

$$
{}^6\mathfrak{N}_{[0,1]}: \begin{cases} (1,0,0,0)_L \otimes (1,0,2,0)_R & \mathcal{G}_{(0,1)} + \mathcal{T}_{(0,1)}\,, \\ 8 \times \big[(0,1,0,0)_L \otimes (0,1,0,2)_R\big] & 8 \cdot \mathcal{T}_{(0,1)}\,, \\ 8 \times \big[(0,0,1,0)_L \otimes (0,1,0,2)_R\big] & 8 \cdot \mathcal{V}_{(0,1)}\,, \\ 20 \times \big[(0,0,0,2)_L \otimes (0,1,0,2)_R\big] & 20 \cdot \mathcal{H}_{(0,1)}\,. \end{cases} \tag{4.4.5}
$$

The vacuum orbit reveals the $\mathcal{N} = (1,0)$ supersymmetry and the total matter spectrum is

$$
n_T = 1 + 8 = 9\,, \qquad n_V = 8\,, \qquad n_H = 20\,. \tag{4.4.6}
$$

This spectrum is anomaly free since it satisfies

$$
n_H - n_V + 29 n_T = 273\,. \tag{4.4.7}
$$

Actually we also find slight deviations from this model that had the modified vacuum sector

$$(1,0,0,n)_L \otimes (1,0,2,0)_R \qquad \mathcal{G}_{(0,1)} + \mathcal{T}_{(0,1)} + n \cdot \mathcal{V}_{(0,1)}. \tag{4.4.8}$$

where $n = 0, 1, 2, 3, 4$. Models with this spectrum have the same spectrum as above plus n additional vector/hyper pairs such that the anomaly is still canceled. The difference to the other vector and hypermultiplets is that these additional vector/hyper pairs arise in the NS-NS sector. Because of that they have a non-trivial minimal coupling and the correlator $\langle v\psi\overline{\psi}\rangle$ satisfies all selection rules. Therefore the hypermultiplet has a non-zero $U(1)$ charge under the additional vector field and these pairs can be made massive by the Higgs effect. These are signals for a gauge enhancement.

Possible ${}^6\mathfrak{N}_{[0,1]}$ models

Assume we found a IIB model with the spectrum

$$(n^B_{T,\mathrm{R-R}} + 1\,,\, n^B_{V,\mathrm{R-R}} + n^B_{V,\mathrm{NS-NS}}\,,\, n^B_{H,\mathrm{R-R}}) \tag{4.4.9}$$

Using charge conjugation we can deduce the spectrum of the associated IIA model. The charge conjugation acts only on the R-R states by exchanging vectors and tensors while the NS-NS states remain unaffected. The result is

$$\begin{aligned}(n^A_{T,\mathrm{R-R}} + 1, n^A_{V,\mathrm{R-R}} &+ n^A_{V,\mathrm{NS-NS}}, n^A_{H,\mathrm{R-R}}) \\ &= (n^B_{V,\mathrm{R-R}} + 1\,,\, n^B_{T,\mathrm{R-R}} + n^B_{V,\mathrm{NS-NS}}\,,\, n^B_{H,\mathrm{R-R}})\,.\end{aligned} \tag{4.4.10}$$

Clearly both, the IIB and the IIA model should be anomaly free and satisfy (4.4.7). From this we conclude

$$n^{B/A}_{T,\mathrm{R-R}} = n^{B/A}_{V,\mathrm{R-R}} \tag{4.4.11}$$

Actually (4.4.7) combines two terms in the gravitational anomaly

$$\mathcal{A}_G = \alpha\,\mathrm{Tr}(R^4) + \beta\left(\mathrm{Tr}\,R^2\right)^2, \tag{4.4.12}$$

where

$$\alpha \sim 244 - 29\,n^{B/A}_T - n^{B/A}_H + n^{B/A}_V\,, \qquad \beta \sim n^{B/A}_T - 8\,. \tag{4.4.13}$$

But in type II there are no Chern-Simons terms such that β should be zero right away. From this we conclude

$$n^{B/A}_{T,\mathrm{R-R}} = n^{B/A}_{V,\mathrm{R-R}} = 8\,. \tag{4.4.14}$$

The anomaly fixes the remaining numbers precisely to the model we found up to additional vector/hyper pairs in the NS-NS sector. It is therefore not at all a surprise that we discovered only one type of model, but rather dictated by anomaly cancellation.

Interpretation of ${}^{6}\mathfrak{N}_{[0,1]}$

Let us first check whether the model can be realized as a broken GSUGRA. It cannot be the fluxed supergravity of a $K3$ compactification since the NS-NS fluxes have three legs and the $K3$ does not have any three-cycles. Indeed when trying to decompose the $\mathcal{N}=(2,0)$ or $\mathcal{N}=(1,1)$ spectrum in $\mathcal{N}=(1,0)$ multiplets we find

$$\begin{aligned} \mathcal{G}_{(0,2)} + n_T \cdot \mathcal{T}_{(0,2)} &\rightarrow \mathcal{G}_{(0,1)} + 2\cdot\mathcal{S}^-_{(0,1)} + n_T\cdot\mathcal{T}_{(0,1)} + 2n_T\cdot\mathcal{H}_{(0,1)}\,,\\ \mathcal{G}_{(1,1)} + n_V\cdot\mathcal{V}_{(1,1)} &\rightarrow \mathcal{G}_{(0,1)} + 2\cdot\mathcal{S}^+_{(0,1)} + \mathcal{T}_{(0,1)} + n_V\cdot\mathcal{V}_{(0,1)} + 2n_V\cdot\mathcal{H}_{(0,1)}\,. \end{aligned} \tag{4.4.15}$$

The $(0,2)\rightarrow(0,1)$ breaking results in an anomalous $\mathcal{N}=(1,0)$ theory and is therefore forbidden. The $(1,1)\rightarrow(0,1)$ needs $n_V=244$ to be anomaly free, a number that is way beyond the $n_V=20$ of IIA on $K3$.

The only chance is a broken $\mathcal{N}=(2,2)$ supergravity since this comes from the $\mathbb{T}^4$ compactification which clearly has three-cycles. For this let us decompose the unique $\mathcal{N}=(2,2)$ spectrum into $\mathcal{N}=(1,0)$ multiplets

$$\mathcal{G}_{(2,2)} \rightarrow \mathcal{G}_{(0,1)} + 4\cdot\mathcal{S}^+_{(0,1)} + 2\cdot\mathcal{S}^-_{(0,1)} + 8\cdot\mathcal{V}_{(0,1)} + 5\cdot\mathcal{T}_{(0,1)} + 10\cdot\mathcal{H}_{(0,1)}\,. \tag{4.4.16}$$

Again the spectrum is anomalous excluding this breaking pattern to be visible in a CFT.

To find another realization let us first comment that when taking $k=(2,2,2,2)$ we could can realize the model using the simple current $J_{\mathrm{ACFT}}=(0,0,0)^2(0,1,1)^2(v)$ or alternatively $J_{\mathrm{ACFT}}=(0,0,0)^2(0,2,2)(0,-2,2)(c)$. They look very much like the simple currents (4.5.13) or (4.3.3) that both implemented an $(-1)^{F_L}$ action. Indeed in [56] they found an orbifold realization of this model as asymmetric orbifold

$$\frac{\mathbb{T}^4}{\mathbb{Z}\times\mathbb{Z}'}\,, \tag{4.4.17}$$

where

$$\Theta : z_i \rightarrow -z_i\,, \qquad i=1,2\,, \tag{4.4.18}$$

and $\mathbb{Z}'=\Theta S(-1)^{F_L}$ where S is a momentum shift along a single S_1. As expected, this orbifold contains an $(-1)^{F_L}$ action.

Summary

We summarize the classification of ACFTs in the following table:

class	spectrum beyond supergravity	realization
${}^6\mathfrak{N}_{[2,2]}$	–	${}^8\mathfrak{N}_{[1,1]}$ on $\mathbb{T}^2$
${}^6\mathfrak{N}_{[1,1]}$(B)	$21 \cdot \mathcal{T}_{(0,2)}$	IIB on $K3$
${}^6\mathfrak{N}_{[1,1]}$(A)	$20 \cdot \mathcal{V}_{(1,1)}$	IIB on $K3/(-1)^{F_L}$ = IIA on $K3$
${}^6\mathfrak{N}_{[0,2]}$	$(4, 8, 12) \cdot \mathcal{V}_{(1,1)}$	Coulomb-branch: $\mathcal{A}_6$
${}^6\mathfrak{N}_{[0,1]}$	$9 \cdot \mathcal{T}_{(0,1)} + (8+n) \cdot \mathcal{V}_{(0,1)}$ $+(20+n) \cdot \mathcal{H}_{(0,1)}$	gauge enhancement: $\mathbb{T}^4/\{\Theta, \Theta S(-1)^{F_L}\}$

As expected we did not find any model that can be explained by a gauged supergravity. All models have a natural explanation in terms of asymmetric orbifolds with a fermionic $(-1)^{F_L}$ action. Notice that, consistent with [90], we could not find an $\mathcal{N} = (1, 2)$ model.

4.5 ACFTs in $D = 4$

Let us turn to the most interesting case. In case of four non-compact directions the internal space can support fluxes so here we can really test the ACFT/GSUGRA conjecture. Our starting point is always a model with at least 2 gravitinos coming from the right such that the overall supersymmetry is $\mathcal{N} = 8, 6, 5, 4, 3, 2$. Notice that the models with $\mathcal{N} = 2$ are not genuine Calabi-Yau compactification since they are asymmetric with $(\mathcal{N}_L, \mathcal{N}_R) = (0, 2)$ supersymmetry.

The classes ${}^4\mathfrak{N}_{[m,4]}$

We begin in the $k = (1^3, 2^4)$ Gepner model and extend it with the simple current

$$J_{\text{ACFT}} = (0,0,0)^2(0,1,1); (0,4,0)^2(0,3,-1)^2(o)\,, \tag{4.5.1}$$

giving maximal supergravity that we interpret as type II on a $\mathbb{T}^6$. The vacuum orbit is

$${}^4\mathfrak{N}_{[4,4]}:\ \Big\{(1,4,4,6)_L \otimes (1,4,4,6)_R \qquad \mathcal{G}_{(8)}\,. \tag{4.5.2}$$

The $\mathcal{N} = 6$ model can be realized by the simple current

$$J_{\text{ACFT}} = (0,-2,0)(0,3,-1)(0,-2,2)(0,1,1)^2(0,2,2)^2(v)\,, \tag{4.5.3}$$

giving the spectrum

$${}^4\mathfrak{N}_{[2,4]}:\ \Big\{(1,2,2,2)_L \otimes (1,4,4,6)_R \qquad \mathcal{G}_{(6)}\,. \tag{4.5.4}$$

As discussed previously in section 4.1, this model can kinematically be realized as broken $\mathcal{N} = 8$ supergravity. Alternatively there is an orbifold description $\mathbb{T}^6/(\mathbb{Z}_2^L\, S)$ where Z_2^L is a leftmoving reflection on four of the internal coordinates while S is a $\mathbb{Z}_2$ shift along the orthogonal $\mathbb{T}^2$[91]. Notice that this orbifold is purely bosonic, in congruence with the other interpretation where the breaking happens due to bosonic NS-NS fluxes.

Going down in this pattern is achieved when adding two simple currents at the same time

$$\begin{aligned} J_{\text{ACFT},1} &= (0,-1,1)(0,3,-1)(0,2,0)(0,4,0)(0,2,2)(0,1,1)(0,3,-1)(v)\,, \\ J_{\text{ACFT},2} &= (0,-1,1)(0,2,2)(0,2,0)(0,-1,-1)(0,2,2)(0,1,1)(0,2,2)(o)\,, \end{aligned} \tag{4.5.5}$$

giving $\mathcal{N}=5$ supergravity with the spectrum

$$^4\mathfrak{N}_{[1,4]}: \ \Big\{ (1,1,1,0)_L \otimes (1,4,4,6)_R \qquad \mathcal{G}_{(5)}\,. \tag{4.5.6}$$

The $\mathcal{N}=5$ supergravity cannot be explained in terms of a broken $\mathcal{N}=8$ supergravity since the field content does not fit. But there is the orbifold realization $\mathbb{T}^6/(\mathbb{Z}_2^L\,S, \tilde{\mathbb{Z}}_2^L\,\tilde{S})$.

The class $^4\mathfrak{N}_{[0,4]}$

Although along the lines of the previous models, let us consider the models $^4\mathfrak{N}_{[0,4]}$ separately. Similar as above their spectrum is purely uncharged and is coming from the NS-NS sector

$$^4\mathfrak{N}_{[0,4]}: \ \Big\{ (1,0,0,n)_L \otimes (1,4,4,6)_R \qquad \mathcal{G}_{(4)} + n\cdot\mathcal{V}_{(4)}\,. \tag{4.5.7}$$

The models we found had $n_V = 0,2,4,6,8,10,14,18$. The model with $n_V = 18$ has the right amount of vectors to be the $\mathcal{A}_4$ model with gauge symmetry $SU(2)^6$. Analogous to the 8D case there are no bosons coming from the R-R sector since the $(-1)^{F_L}$ action eliminates the left-moving Ramond states. When going to the Coloumb branch of $\mathcal{A}_4$, the number of vectors decreases in steps of 2 until one reaches the minimal gauge group $U(1)^6$ with 6 vectors.

We still need to explain the models with $n_V \leq 6$. This is the first example where we have a non-trivial explanation in terms of a fluxes supergravity. Indeed decomposing the $\mathcal{N}=8$ spectrum in terms of $\mathcal{N}=4$ multiplets we find the massless spectrum

$$\mathcal{G}_{(8)} \to \mathcal{G}_{(4)} + (6-2k)\cdot\mathcal{V}_{(4)}\,, \tag{4.5.8}$$

where $k=0,1,2,3$. The $-2k$ factor comes from the fact that the massive vector multiplet contains two massive vectors. The models $n_V = 6,4,2,0$ can therefore be explained by flux compactifications on $\mathbb{T}^6$.

The class $^4\mathfrak{N}_{[2,2]}$

Let us proceed with $^4\mathfrak{N}_{[2,2]}$. Although symmetric we will discuss it as well since we have all techniques at hand to classify this class. The spectrum is

$$^4\mathfrak{N}_{[2,2]}: \ \begin{cases} (1,2,2,2)_L \otimes (1,2,2,2)_R & \mathcal{G}_{(4)} + 2\cdot\mathcal{V}_{(4)}\,, \\ n\times\big[(0,1,1,2)_L \otimes (0,1,1,2)_R\big] & n\cdot\mathcal{V}_{(4)}\,, \end{cases} \tag{4.5.9}$$

where $n_V = 22, 14, 10, 6, 4$. Clearly $n_V = 22$ is type II on $K3 \times \mathbb{T}^2$. To interpret the models with $n_V \in \{4, 6, 10, 14\}$ we looked at symmetric toroidal orbifolds with $\mathcal{N} = 4$ and found

$$\text{Orb}_{n,m} = \frac{\mathbb{T}^4 \times \mathbb{T}^2}{\mathbb{Z}_n\, S_m}\,, \tag{4.5.10}$$

where $\mathbb{Z}_n$ for $n \in \{2, 3, 4, 6\}$ is a crystallographic action on the $\mathbb{T}^4$ and S_m is a momentum shift of order m on $\mathbb{T}^2$. For consistency we need m to be a divisor of n. The result for the different values of m and n are

$\text{Orb}_{n,m}$	twisted sector vectors	massless spectrum
$(2,2)$	$(1,\theta) = (6,0)$	$\mathcal{G}_{(4)} + 6 \cdot \mathcal{V}_{(4)}$
$(3,3)$	$(1,\theta,\theta^2) = (4,0,0)$	$\mathcal{G}_{(4)} + 4 \cdot \mathcal{V}_{(4)}$
$(4,2)$	$(1,\theta,\theta^2,\theta^3) = (4,0,10,0)$	$\mathcal{G}_{(4)} + 14 \cdot \mathcal{V}_{(4)}$
$(6,3)$	$(1,\theta,\theta^2,\theta^3,\theta^4,\theta^5) = (4,0,0,6,0,0)$	$\mathcal{G}_{(4)} + 10 \cdot \mathcal{V}_{(4)}$

The spectra of the models $\text{Orb}_{4,4}$, $\text{Orb}_{6,6}$ and $\text{Orb}_{6,2}$ give $n_V = \{4, 4, 14\}$ which we therefore skip in the list. As it turns out data of the models of ${}^4\mathfrak{N}_{[2,2]}$ agree perfectly with the $\text{Orb}_{m,n}$ orbifold.

The class ${}^4\mathfrak{N}_{[1,2]}$

Let us come to models with $\mathcal{N} = 3$ supersymmetry in 4D. The massless spectrum is of the form

$$ {}^4\mathfrak{N}_{[1,2]} : \begin{cases} (1,1,1,0)_L \otimes (1,2,2,2)_R & \mathcal{G}_{(3)} + \mathcal{V}_{(3)}\,, \\ n \times \big[(0,2,0,2)_L \otimes (0,1,1,2)_R\big] & 6 \cdot \mathcal{V}_{(3)}\,, \\ n \times \big[(0,0,2,2)_L \otimes (0,1,1,2)_R\big] & 6 \cdot \mathcal{V}_{(3)}\,. \end{cases} \tag{4.5.11}$$

In the stochastic search we found models with

$$n_V = 2n + 1 \in \{3, 7, 11, 13, 19\}\,. \tag{4.5.12}$$

All these models can nicely be explained as a fluxed supergravity on $\mathbb{T}^4$ or $K3 \times \mathbb{T}^2$. The breaking patterns from $\mathcal{N} = 8$ and $\mathcal{N} = 4$ to $\mathcal{N} = 3$ are

$\mathcal{N}'$	$\mathcal{N}$	massless spectrum
8	3	$\mathcal{G}_3 + (3 - 2k) \cdot \mathcal{V}_3$
6	3	–
5	3	–
4	3	$\mathcal{G}_3 + (19 - 2k) \cdot \mathcal{V}_3$

and share characteristic properties with our models. First there may only be steps of two since the massive vector multiplet contains two vectors and the $K3 \times \mathbb{T}^2$ sets the upper bound to $n_V = 19$. For the models with $n_V = 3, 1$ there is an overlap between both interpretations.

The class ${}^4\mathfrak{N}_{[0,2]}$

Lastly let us look at models with $\mathcal{N} = 2$ supersymmetry. Since the symmetric ${}^4\mathfrak{N}_{[1,1]}$ models correspond to usual Calabi-Yau compactifications, we only consider the asymmetric ${}^4\mathfrak{N}_{[0,2]}$. We found three individual classes that we characterize by the difference of vector and hypermultiplets. The massless spectrum of all three classes is of the form

$$ {}^4\mathfrak{N}_{[0,2]}(\mathrm{A}) : \begin{cases} (1,0,0,m)_L \otimes (1,2,2,2)_R & \mathcal{G}_{(2)} + (m+1) \cdot \mathcal{V}_{(2)} \,, \\ (0,n,n,2k)_L \otimes (0,1,1,2)_R & 2n \cdot \mathcal{V}_{(2)} + k \cdot \mathcal{H}_{(2)} \,, \end{cases} \tag{4.5.13} $$

with $n \geq 1$. We count $n_V = m + 2n + 1$ vector multiplets and $n_H = k$ hypermultiplets.

In the first class the number of vector and hypermultiplets is connected by $n_H = n_V + 1$ with

$$ n_V \in \{1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 19, 20, 21, 22, 23\} \,. \tag{4.5.14} $$

We find it remarkable that we could close nearly all holes by running the program appropriately long. This makes us confident that our stochastic search was deep enough.
Let us try to interpret this list of models

- When compactiying the ${}^6\mathfrak{N}_{[0,1]}$ model on a further $\mathbb{T}^2$ we get the model $n_V = 19$. As for ${}^6\mathfrak{N}_{[0,1]}$, there are at most four additional vector/hyper pairs from a gauge enhancement. They are arising in the NS-NS sector and can therefore be Higgsed.

- When interpreting an $\mathcal{N} = 2$ model as broken higher supergravity one has to take into account that there are short and long massive multiplets in $\mathcal{N} = 2$ supergravity. When looking at the $\mathcal{N} = 4 \to \mathcal{N} = 2$ breaking we find the spectrum

$$ \begin{aligned} n_V &= 27 - 4(n^L_{3/2} + n^S_{3/2}) - (n^L_1 + 2n^S_1) \qquad \text{and} \\ n_H &= 20 + 4n^S_{3/2} - n^L_1 \,, \end{aligned} \tag{4.5.15} $$

  where the upper index signals short/long massive multiplets. To get $n_H - n_V = 1$ we need only long multiplets, so all models with $n_V \leq 19$ can be explained by the super-Higgs effect.

The second class, ${}^4\mathfrak{N}_{[0,2]}(\mathrm{B})$, obeys $n_V - n_H = 11$. The list of models we found is

$$ n_V \in \{13, 15, 17, 19, 21, 23\} \,. \tag{4.5.16} $$

Since we will interpret this class together with the third one, we will first of all state the third class, ${}^4\mathfrak{N}_{[0,2]}(\mathrm{C})$, where $n_H - n_V = 13$ with

$$n_V \in \{3, 4, 5, 7, 8, 9, 10, 11\}\,. \tag{4.5.17}$$

Models in this class had no left-moving Ramond states, thus $n = 0$ in (4.5.13). One can partly interpret the third class via the super-Higgs effect as a broken $\mathcal{N} = 4$ supergravity. The models with $n_V \leq 7$ appear if we have $n^S_{3/2} = 0$ and $n^S_1 = 6$. Alternatively the model with $n_V = 7$ results as the ${}^8\mathfrak{N}_{[0,1]} = \mathcal{A}_8$ model on $K3$. The $n_V > 7$ models with at most four additional vector/hyper pairs are interpreted as gauge enhancement.

Recall that the $n_V = 7$ model in ${}^4\mathfrak{N}_{[0,2]}(\mathrm{C})$ is $\mathcal{A}_8 \times K3$. Writing $\mathcal{A}_8 \times K3$ as orbifold reads

$$\frac{\mathbb{T}^4 \times \mathbb{T}^2}{\{\mathbb{Z}_2, (-1)^{F_L} SW\}}\,. \tag{4.5.18}$$

The $\mathbb{Z}_2$ is a reflection on the $\mathbb{T}^4$ and $(-1)^{F_L} SW$ is known from ${}^8\mathfrak{N}_{[0,1]} = \mathcal{A}_8$ in (4.3.10). Since there are two independent $\mathbb{Z}_2$ factors, we can introduce a discrete torsion [92]. Roughly speaking a discrete torsion is a sign $\epsilon = \pm 1$ between different twisted sectors that is not fixed by modular invariance and therefore a choice. The spectrum of the orbifold for both choices is

sector	$\epsilon = +1$	$\epsilon = -1$
untwisted	$\mathcal{G}_{(2)} + 3 \cdot \mathcal{V}_{(2)} + 4 \cdot \mathcal{H}_{(2)}$	$\mathcal{G}_{(2)} + 3 \cdot \mathcal{V}_{(2)} + 4 \cdot \mathcal{H}_{(2)}$
θ twisted	$16 \cdot \mathcal{H}_{(2)}$	$16 \cdot \mathcal{V}_{(2)}$
$(-1)^{F_L} SW$ twisted	$4 \cdot \mathcal{V}_{(2)}$	$4 \cdot \mathcal{H}_{(2)}$
total	$\mathcal{G}_{(2)} + 7 \cdot \mathcal{V}_{(2)} + 20 \cdot \mathcal{H}_{(2)}$	$\mathcal{G}_{(2)} + 19 \cdot \mathcal{V}_{(2)} + 8 \cdot \mathcal{H}_{(2)}$

While by construction it is clear that the $n_V = 7$ model of the ${}^4\mathfrak{N}_{[0,2]}(\mathrm{C})$ series should appear for $\epsilon = +1$ it is surprising that for $\epsilon = -1$ the $n_V = 19$ model of the ${}^4\mathfrak{N}_{[0,2]}(\mathrm{B})$ class appears.

For $\epsilon = +1$ the orbifold projects out every bosonic mode in the R-R sector similar to our observation in the Gepner realization. Having explained the models with the highest value of n_V the lower models can be explained as lying on the Higgs branch of these models. The models with up to four more n_V show the same structure as in the 6D case, where up to four Higgsable NS-NS vector/hyper pairs could become massless by a gauge enhancement.

Summary

We summarize the list of asymmetric Gepner models with $c = 9$ in the table

class	spectrum beyond supergravity	realization
${}^4\mathfrak{N}_{[4,4]}$	–	type IIB on $\mathbb{T}^6$
${}^4\mathfrak{N}_{[2,4]}$	–	sHiggs of ${}^4\mathfrak{N}_{[4,4]}$
${}^4\mathfrak{N}_{[1,4]}$	–	–
${}^4\mathfrak{N}_{[0,4]}$	$(0,2,4,6)\cdot\mathcal{V}_{(4)}$ $(6,8,10,14,18)\cdot\mathcal{V}_{(4)}$	sHiggs of ${}^4\mathfrak{N}_{[4,4]}$ Coulomb branch: $\mathcal{A}_4$
${}^4\mathfrak{N}_{[2,2]}$(A)	$22\cdot\mathcal{V}_{(4)}$	type IIB on $K3\times\mathbb{T}^2$
${}^4\mathfrak{N}_{[2,2]}$(B)	$(4,6,10,14)\cdot\mathcal{V}_{(4)}$	shift orbifolds $\mathrm{Orb}_{n,m}$
${}^4\mathfrak{N}_{[1,2]}$	$(3,7,11,13,19)\cdot\mathcal{V}_{(3)}$	sHiggs of ${}^4\mathfrak{N}_{[2,2]}$(A)
${}^4\mathfrak{N}_{[0,2]}$(A)	$(1,\ldots,19)\cdot\mathcal{V}_{(2)}+(2,\ldots,20)\cdot\mathcal{H}_{(2)}$ $(19,\ldots,23)\cdot\mathcal{V}_{(2)}+(20,\ldots,24)\cdot\mathcal{H}_{(2)}$	sHiggs of ${}^4\mathfrak{N}_{[2,2]}$(A) ${}^6\mathfrak{N}_{[0,1]}$ on $\mathbb{T}^2$
${}^4\mathfrak{N}_{[0,2]}$(B)	$(13,15,17,19)\cdot\mathcal{V}_{(2)}+(2,4,6,8)\cdot\mathcal{H}_{(2)}$ $(21,23)\cdot\mathcal{V}_{(2)}+(10,12)\cdot\mathcal{H}_{(2)}$	Higgs chain: ${}^8\mathfrak{N}_{[0,1]}$ on $K3_{\epsilon=-1}$ gauge enhancement
${}^4\mathfrak{N}_{[0,2]}$(C)	$(3,4,5,7)\cdot\mathcal{V}_{(2)}+(16,17,18,20)\cdot\mathcal{H}_{(2)}$ $(8,9,10,11)\cdot\mathcal{V}_{(2)}+(21,22,23,24)\cdot\mathcal{H}_{(2)}$	Higgs chain: ${}^8\mathfrak{N}_{[0,1]}$ on $K3_{\epsilon=+1}$ gauge enhancement

4.6 Conclusion

We performed an extensive stochastic search over asymmetric Gepner models and sorted the results into classes of models. The depth of our search makes us confident that we found all existing classes of models. The goal of the classification was to find more evidence or counterexamples to the conjecture that certain asymmetric Gepner models can be identified with fully backreacted gauged supergravities.

In 8D and 6D we did not expect to find any models that can be interpreted as gauged supergravity. Indeed all models we found could be identified as standard compactifications or orbifolds involving the fermionic $(-1)^{F_L}$ action. Partially we found gauge enhancement as common for Gepner models. In 4D there were again models that could be explained via fermionic orbifolds but also many models that we identified as gauged supergravity.

In total we needed very little ingredients for the classification and found no counterexample to the ACFT/GSUGRA conjecture. Instead we found a lot of models that support the ACFT/GSUGRA conjecture. This makes us confident that at least kinematically there is a close relationship between asymmetric CFTs and gauged supergravities. Going further this suggests that there are vacua of a gauged supergravity that can be uplifted to asymmetric conformal field theories in the sense that the solution correctly captures the kinematics of the fully backreacted uplifted string solution.

Chapter 5

Gravity in the non-associative R-flux space

In the previous chapters we outlined that asymmetric conformal field theories correspond to fully backreacted vacua of a gauged supergravity. It remains to answer what kind of target space interpretation the adjusted manifold has. Since the manifold is described by an asymmetric CFT, this cannot be an ordinary Riemannian manifold. Having in mind that the non-geometric fluxes are the T-duals of the geometric ones Hull took the idea of U-manifolds [55] to invent the notion of the T-fold [46] in case of non-geometric Q-flux backgrounds. A T-fold is a manifold where the transition functions between the patches of the manifold are not only diffeomorphism and gauge transformations but also T-dualities. Locally each patch can be described by ordinary geometry.

In case of the non-geometric R-flux one believes that even the local description breaks down [93, 94, 95]. There are hints that the R-flux corresponds to a non-associative manifold equipped with a non-associative star product. The star product was first derived in [96, 97] and later reproduced using CFT [98, 63, 64] or mode expansion methods [99]. A derivation from doubled sigma models [100, 101] or membrane sigma models [102, 103] was successful as well. A possible M-theory origin or uplift is discussed in [104, 105, 106] (see also the earlier works [107]). Consistency was checked in [108] and formal aspects of the deformation quantization of the non-associative brackets [109, 110, 111, 112, 113, 114] were looked at, too.

The following chapter will continue this line of research about the non-associative star product in R-flux backgrounds. In particular it will discuss whether it is possible to formulate a gravity theory on such a non-associative manifold and how the non-associative star product can be reconciled with the diffeomorphism symmetry of string theory. Notice here that the R-flux appears in the closed string sector that necessarily contains gravity. Let us outline the structure and logic of this chapter.

The first section 5.1 contains a review of the R-flux star product and its different versions. We will comments on its role in string theory and especially highlight that, in contrast to the open string star product, the R-flux star product cannot fully take care for the background. Then we review the main techniques to covariantize a symmetry with respect to a star product. The first, most traditional way is to introduce star symmetries whose generators act with the star product. For our purposes the star symmetries have the major drawback that there are only star $U(N)$ theories but no star diffeomorphism. Furthermore star symmetries can only exist if the star product is associative. In case the star product is non-associative, like the R-flux star product, we show that instead twisted symmetries appear naturally. Twisted symmetries keep the symmetry unchanged but change the way the symmetry acts on products by introducing a twisted Leibniz rule. We outline why twisted symmetries are fundamentally different than known gauge symmetries and why twisted symmetries might be completely unphysical. Nevertheless having a non-associative star product we cannot avoid twisted symmetries and since they are furthermore compatible with diffeomorphism we will try to construct a gravity invariant under non-associatively twisted diffeomorphism. In section 5.3 we start with a pedagogical introduction to the basic computational techniques and outlines the logic of the construction of twisted tensors in a simplified setting. Afterwards in 5.4 we describe how the most general situation can be treated technically. Then in 5.5 we follow the usual steps towards a gravity theory by introducing the notion of a covariant derivative, torsion and the Riemann tensor in section 5.6. To proceed in constructing an Einstein-Hilbert action we need to introduce a metric, which is done in 5.7. When introducing a metric we encounter major obstacles that we all can trace back to the non-associative nature of the R-flux star product. For instance we can show that in an non-associative algebra there cannot be an inverse. Furthermore since a scalar product combines three objects there is a bracketing ambiguity and every choice of bracketing contradicts one or the other known properties of a scalar product. Being agnostic about the appearing obstacles we try to get insights about a possible action in 5.8. We find that there is no obvious way how the R-flux corrections could trivialize to leave a gravity theory invariant under ordinary diffeomorphisms. Lastly we conclude in 5.9.

5.1 The R-flux star product

This section will review the non-associative R-flux star product that characterizes the non-geometric nature of the manifolds associated to certain asymmetric CFTs. Since the derivation of the R-flux star product was outlined in the master thesis of the author, we will skip these details here and simply state the star product.

Actually two different types of non-associative star products appear in the literature. The first type cannot involve a deformation of the multiplication of two functions (at least in first order in R) due to the structure of the performed conformal perturbation theory. Instead the tri-product

$$(f \vartriangle g \vartriangle h)(x) := f\, g\, h \,+\, \frac{l_s^4}{6} R^{\mu\nu\sigma}\; \partial_\mu f\, \partial_\nu g\, \partial_\sigma h + \mathcal{O}(R^2) \tag{5.1.1}$$

was found for permutations of tachyon vertex operators in their three-point function [98]. Here l_s denotes the string length. The tri-product leads to a non-trivial Jacobi identity

$Jac(x^i, x^j, x^k) = l_s^4 R^{ijk}$ that is similar to the one in the seminal work [96]. This is no coincidence since both, [96] and [98], essentially work with the WZW model [115]. Notice that a violated Jacobi identity is usually a signal for a non-associative algebra which can be seen by writing out the Jacobi identity with correct brackets.[1]

The second type of non-associativity appeared mainly when using string mode expansions and features a non-trivial commutator between the coordinates [97, 99, 63, 64]. In the convention of [108] the non-trivial commutation relations are

$$[x^i, x^j] = \frac{i}{3\hbar} \ell_s^4 R^{ijk} p_k \qquad \text{and} \qquad [x^i, p_j] = i\hbar \, \delta^i{}_j . \tag{5.1.2}$$

These brackets reproduce the Jacobi identity $Jac(x^i, x^j, x^k) = l_s^4 R^{ijk}$ from the tri-product. In deformation quantization such an algebra is realized by a star product. For (5.1.2) the star product was first constructed in [102] and further considered in [109, 110, 111, 112, 113, 114]. It reads

$$f \star g = f \cdot g + \tfrac{1}{2} i\hbar (\partial_i f \, \tilde{\partial}_p^i g - \tilde{\partial}_p^i f \, \partial_i g) + \tfrac{i l_s^4}{6\hbar} R^{ijk} p_k \, \partial_i f \, \partial_j g + \dots \ , \tag{5.1.3}$$

where $\tilde{\partial}_p$ denotes the derivative with respect to the momentum. Notice that this star product naturally lives on the full phase space (x, p). As one can check this star product is non-associative and violates the Jacobi identity. In first order in R we find for functions $f, g, h \in C^\infty(M)$

$$Ass(f, g, h) = Jac(f, g, h) = l_s^4 R^{ijk} \partial_i f \, \partial_j g \, \partial_k h \, , \tag{5.1.4}$$

where $Ass(a, b, c) = (a \star b) \star c - a \star (b \star c)$ measures the non-associativity.

The connection between the two- and tri-products (5.1.3) and (5.1.1) was explained in [114]. In string theory all vertex operators depend only on the coordinates while the momentum appearing in the exponent e^{ipX} is merely a number. It is therefore natural to multiply only functions $f \in C^\infty(M)$ that do not depend on the momentum. When multiplying those with (5.1.3) one recovers the tri-product (5.1.1) in zeroth order in the momentum p

$$f(x) \star (g(x)) \star h(x)) = f(x) \triangle g(x) \triangle h(x) + \mathcal{O}(p) \, . \tag{5.1.5}$$

Here we chose a bracketing where all brackets are nested to the right. Other types of bracketings are connected to the above one using the associator as explained later in (5.3.10). In light of (5.1.5) the authors of [117] suggest that string theory lies on the $p = 0$

[1] Strictly speaking, the violation of the Jacobi identity equals the associator $Ass(a, b, c) = (a \star b) \star c - a \star (b \star c)$ iff the associator is totally antisymmetric which by definition is the case for alternating algebras [116]. The star product we look at in this chapter is only alternative up to first order in the deformation parameter for functions $C^\infty(M)$ [1]. This is the order in which the algebra is derived. It therefore makes sense to say that the Jacobi identity and the non-associativity cannot be separated.

leaf in the phase space while higher orders in p come from membrane corrections. This would also clarifies why the star product lives on phase space.[2]

Regardless of the interpretation the two-product is more general than the tri-product so we will work exclusively with the two-product. More precisely we work with its full exponentiated version

$$\begin{aligned} f \star g &= \mu\Big[\exp\Big(\tfrac{1}{2}i\hbar(\partial_i \otimes \tilde{\partial}_p^i - \tilde{\partial}_p^i \otimes \partial_i) + \tfrac{il_s^4}{12\hbar}R^{ijk}\big(p_k\partial_i \otimes \partial_j - \partial_j \otimes p_k\partial_i\big)\Big)\ f \otimes g\Big] \\ &= \mu\left[\mathcal{F}^{-1}(f,g)\right] . \end{aligned} \tag{5.1.6}$$

Here and in the following μ turns all tensor products into multiplications $\mu(A\otimes B) = A\cdot B$, the operator $\mathcal{F}$ is called twist. Only at the very end we will set $p = 0$ to draw conclusions about string theory.

Let us comment on the closed string star product (5.1.6):

- The worldsheet of an open string is a disk. On the boundary of the disk there are two nonequivalent orderings in which vertex operators can be inserted. Therefore on the boundary, the D-brane, there is room for a non-commutativity between the coordinates. When placing a D-branes in a constant B-field background it turns out that the background B-field can fully be taken care of by introducing a non-commutative star product between the vertex operators. Let us postpone the discussion of the open string star product to the next section (5.2.1) but instead stress two crucial differences between the open string star product and the closed string star product (5.1.1):

  The first difference is that the R-flux star product does not fully capture the R-flux background. Since the R-flux star product always acts with derivatives, the star product cannot contribute mass terms. But in [98] they found that some parts of the internal metric acquire a mass due to the R-flux. That this would happen was expected since the R-flux induces a scalar potential in gauged supergravity (3.3.6). Therefore the closed string star product in the form of (5.1.1) can, at most, partly capture the R-flux background.

  The second difference to open strings is that for the closed strings one does not expect a star product since the worldsheet has the topology of a sphere where there is no ordering ambiguity. Furthermore in CFTs the non-associativity is forbidden by crossing symmetry. From this viewpoint it is no surprise that the tri-product (5.1.1) trivializes upon using momentum conservation [108]. For more general n-products derived from tachyon n-point scattering or the generalization of (5.1.5) it is shown that the $p = 0$ leaf yields only boundary terms [114] and therefore no physical contribution in scattering processes as well. The physical significance of the R-flux star product has therefore not been demonstrated so far.

[2] In [118] an interpretation of fields living in phase space was given in terms of Wigner probability functions.

- An important theoretical concern about a non-associative star product is that observables in quantum mechanics must associate by assumption. If they do not associate, they are proven to be non-observables [119]. A possible loophole in this argumentation is that the non-associativity between the coordinates might be allowed since the coordinates are not primary fields and therefore not observables. Still, a non-associativity between the observables, namely vertex operators, is forbidden. But then there cannot be any physical significance of star product at all and the star product correction must vanish in all amplitudes and not only as observed in the n-point functions of tachyon vertex operators.

- There is a nice explanation for the non-associative structure and the appearing non-vanishing Jacobi identity. Both could be a consequence of the quantization of the M2 branes. The quantization of their worldvolume theory requires the quantization of Nambu three-brackets which has not been achieved so far. Considering the tri-product (5.1.1) to be a remnant of this quantization is indeed very tempting. Understanding the string theory tri-product better would maybe shed light on the quantization of Nambu structures. First steps in this direction have already been taken. Using octonions the algebra (5.1.2) was uplifted to M-theory in [105] and the appropriate star product constructed in [106]. Intriguingly the uplifted star product does not fully trivialize in the $R \to 0$ limit.

Despite these caveats it is hard to negate that a structure is shining through and might be a signal for a lot more. Instead of interpreting the star product as part of the effective action, the non-commuting coordinates might rather be the manifestation of the non-geometric nature of the R-flux background. While physically probably not relevant in string theory it is an effect worth investigating on its own.

Most of the work so far has dealt with the interplay between the star product and the symmetry which is the diffeomorphism symmetry of closed strings. Since the above star product can be seen as easiest prototype example for a non-associative star product and a natural generalization to the Moyal-Weyl product where $[x, x] \sim const.$, the research was not only pushed by the string community but also by the non-commutative field theory community.[3] Without directly referring to string theory a series of papers [102, 110, 114, 112, 113] tried to formulate diffeomorphism in presence of the star product. As we will demonstrate soon, the Hopf-algebraic twisted diffeomorphism appear naturally in the R-flux background. The research started with the construction of the appropriate Hopf algebra which turns out to be a quasi-triangular quasi-Hopf algebra [102, 110, 111]. It went on with the formulation of the twisted diffeomorphism algebra in [114] and a comprehensive mathematical treatment of non-associative twisted symmetries in [112, 113]. They were able to define connections and proposed a Riemann tensor. Since [112, 113] is written in categorial language, the physical meaning and how this miraculously works remains far from obvious. Also a later review article [121] works only with a associative star product.

[3]Putting scalars fields in such an R-flux space was tried in [111] (see also the former work [120]).

Furthermore the physically most important steps like introducing vectors, a metric, the torsion and the Levi-Civita connection are missing.

In our group the research focused on the question how such a star product is theoretically consistent in string theory. In a paper of the author [108] different star products are investigated in double field theory. It is shown that upon imposing closure of the symmetry algebra one can derive relations that trivialize the closed string star product. The open string star product in contrast is not trivial but the non-associative defect vanishes upon using the equation of motion (see also the former work about the open string product [122, 123, 124, 125, 126]). These two results are very interesting and [108] suggests to turn the logic around: By demanding associativity one can learn about crucial consistency relations.

Project goals

Let us now formulate the project goals for the publication [1] that this chapter is based on. The next step after [108] is to investigate the mechanism of how the proposed twisted diffeomorphism in the R-flux space go together with the associativity of observables and how they reduce to the diffeomorphisms of closed string theory. Since the answer has already been known for scalars (boundary terms or closure conditions), we started to focus on the gravity sector. But the construction of a graviton vertex operator failed [98]. As such it is not at all clear what the star product looks like for the physically more relevant massless fields like the graviton. Nevertheless it is tempting to speculate that the star product also somehow appears in the Einstein-Hilbert action. It would surely be interesting to see how these two highly non-trivial structures, the R-flux star product and the Einstein-Hilbert action with its underlying differential geometry, can evade all obstacles and be harmonized. Due to the complexity of gravity and the experience of [108] we could imagine that a rather involved mechanism ensures the consistency.

To analyze this question one needs to translate the existing mathematical work about twisted differential geometry [112, 113] into a physically accessible language, fill the gaps and construct a star Einstein-Hilbert action that, by some mechanism, reduces to the usual Einstein-Hilbert action or appears to be diffeomorphism invariant. Having such an action would in principle show not only that a star product can also emerge in the gravity sector but also how this would look like.

5.2 Symmetries in presence of a star product

Being our main goal, let us review the existing techniques to reconcile symmetries with a star product. Although being highly problematic, twisted symmetries do arise naturally for non-associative star products.

Instead of using the closed string star product we will mainly look at the Moyal-Weyl plane with its star product

$$[x^i, x^j] = i\theta^{ij} \qquad \leftrightarrow \qquad f \star g = \mu\left(e^{\frac{i}{2}\theta^{ij}\partial_i \otimes \partial_j} f \otimes g\right) , \tag{5.2.1}$$

where θ is a constant non-commutativity parameter. In contrast to (5.1.6) this star product is associative and furthermore cyclic under an integral $\int f \star g = \int g \star f$. This star product appears in the effective action of a D-brane in a constant background $B \sim \theta^{-1}$ field [127, 128, 129]. When deriving the effective action one faces a choice of regularization. Pauli-Villar's regularization results in a theory of fluctuations around the B-field background with a standard $U(1)$ gauge symmetry. Point-splitting regularization on the other hand gives a theory where the background can be fully taken care of by introducing the star product (5.2.1) between the vertex operator.[4] When choosing the latter option the effective Lagrangian of a single D-brane in flat space for slowly varying fields is the DBI-action with all multiplications replaced by a $\star$

$$S \sim \int \sqrt{\det \eta + F}_{\star} \quad \sim \int F_{ij} \star F^{ij} + \mathcal{O}(F^4)\,, \tag{5.2.2}$$

where we neglected a constant piece and

$$F_{ij} = \partial_i A_j - \partial_j A_i - i[A_i, A_j]_\star = \partial_i A_j - \partial_j A_i + \theta^{kl}\partial_k A_i \partial_l A_j + \mathcal{O}(\theta^2) \tag{5.2.3}$$

with the star commutator $[A, B]_\star = A \star B - B \star A$. The usual $U(1)$ symmetry is lost because of the non-commutative nature of the star product. This leads to the non-trivial contribution from the commutator, which is definitely a signal for a non-abelian symmetry. The above action can be straightforwardly generalized to $U(N)$ by adding a trace.

5.2.1 Star symmetries

Let us present the first and probably most intuitive way to covariantize the symmetry with respect to the star product by introducing so called star $U(N)$ transformations

$$A_i \quad \rightarrow \quad U_\star(x) \star A_i \star U_\star(x)^{-1} - i\, \partial_i U_\star(x) \star U_\star(x)^{-1}\,, \tag{5.2.4}$$

where the inverse is defined with respect to the star product

$$U_\star(x) \star U_\star(x)^{-1} = U_\star(x)^{-1} \star U_\star(x) = 1\,. \tag{5.2.5}$$

The star $U(1)$ symmetry is non-abelian and therefore fundamentally different from its $U(1)$ cousin. Nevertheless there must be a connection between these theories when remembering that the star $U(1)$ theory and the $U(1)$ theory with a background field describe the same physical system. Indeed one can identify their gauge equivalence classes with the famous Seiberg-Witten map [130]. Showing the invariance of the action (5.2.2) goes along the lines of showing the invariance of the action of ordinary non-abelian Yang-Mills. In a first step we find that the field strength transforms in the adjoint representation

$$F_{ij} \quad \rightarrow \quad U_\star(x) \star F_{ij} \star U_\star(x)^{-1}\,. \tag{5.2.6}$$

[4] The particularly simple form arises only in the Seiberg-Witten limit which decouples the gravitational bulk degrees of freedom [130].

Then, using cyclicity $\int f \star g = \int g \star f$, and

$$\begin{aligned} F_{ij} \star F_{kl} \quad \rightarrow \quad & (U_\star(x) \star F_{ij} \star U_\star(x)^{-1}) \star (U_\star(x) \star F_{kl} \star U_\star(x)^{-1}) \\ & = U_\star(x) \star (F_{ij} \star F_{kl}) \star U_\star(x)^{-1} \end{aligned} \tag{5.2.7}$$

invariance of the action is readily checked. The last relation shows manifestly that the star product behaves covariantly under the star $U(1)$ transformations. In words, the star product of two fields in the adjoint representation is again a field in the adjoint representation. For infinitesimal transformations $U(x) = 1 + i\lambda(x) + \mathcal{O}(\lambda^2)$ the transformation law for a field ϕ in the adjoint representation reads

$$\delta^\star_\lambda \phi = i[\lambda, \phi]_\star \,. \tag{5.2.8}$$

Inserting this transformation law into the covariance relation (5.2.7) results in the usual product rule

$$\delta^\star_\lambda(\phi \star \psi) = \delta^\star_\lambda \phi \star \psi + \phi \star \delta^\star_\lambda \psi \,. \tag{5.2.9}$$

For a detailed discussion of the star Yang-Mills theory with its intriguing features like Morita-equivalence, UV/IR mixing or asymptotic freedom even for $U(1)$ we want to refer to the excellent reviews [131, 132, 133].

Important is that the star gauge group is restricted to star $U(N)$ since all other choices for the gauge group do not close under the star commutator [134, 135]. The absence of a well defined star $SO(n)$ and star $Sp(N)$ group is reflected in string theory: Both gauge groups appear in orientifold constructions where the D-branes lie on top of orientifold planes. Demanding the B-field term in the Polyakov action to be invariant under the orientifold action $X \rightarrow -X$ and $\sigma \rightarrow \pi - \sigma$ one deduces

$$B_{\parallel\,\parallel} \rightarrow -B_{\parallel\,\parallel}, \qquad B_{\perp\,\parallel} \rightarrow B_{\perp\parallel}, \qquad B_{\perp\,\perp} \rightarrow -B_{\perp\perp} \,, \tag{5.2.10}$$

where we used the notation of [135] to highlight directions parallel and perpendicular to the branes. Steadiness demands $B_{\parallel\,\parallel} = 0$ which eliminates the source of non-commutativity. Therefore no non-commutative version of $SO(N)$ or $Sp(N)$ theories appears in string theory.

As a direct consequence it is not possible to naively write down a gravity theory in the vielbein formalism with its local $SO(d,1)$ invariance. Furthermore the constant θ^{ij} in (5.2.1) directly violates general coordinate invariance. There are three attempts to overcome these obstacles and to formulate a non-commutative theory of gravity: The first consists of modifying gravity [136], the second in restricting to a subclass of coordinate transformations preserving θ^{ij} [137, 138]. This subclass turns out to be volume preserving such that these attempts work in unimodular gravity. Both ways are not fully successful and especially unattractive from the string theory viewpoint that predicts standard Einstein-Hilbert gravity. We will therefore stick to the third approach that we will present in the next section.

5.2.2 Twisted symmetries and Hopf algebras

Let us start with a heuristic introduction to Hopf algebras neglecting both the notion of the antipode and the distinction between algebra and modules. For details see e.g. [117]. A Hopf algebra H has two operations which are important in the following: A multiplication μ, which takes two objects multiplying them into one, and its opposite operation, the coproduct Δ, which takes a single object and converts it into two

$$\begin{aligned} \mu &: H \otimes H \to H \,, \\ \Delta &: H \to H \otimes H \,. \end{aligned} \tag{5.2.11}$$

Using the infinitesimal transformation δ of any symmetry group one can define a Hopf algebra[5]. The multiplication μ is the usual algebra multiplication $\mu(\delta_1 \otimes \delta_2) = \delta_1\delta_2$ and the coproduct is defined by

$$\Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta \,. \tag{5.2.12}$$

We can translate the variation of a product of fields[6] ϕ and ψ into the Hopf algebra language as follows

$$\delta(\phi\psi) = (\delta\phi)\,\psi + \phi\,(\delta\psi) \quad \Leftrightarrow \quad \delta(\mu(\phi \otimes \psi)) = \mu \circ \Delta(\delta)(\phi \otimes \psi) \,. \tag{5.2.13}$$

Notice that the coproduct (5.2.12) encodes the usual Leibniz rule. Let us introduce the star product (5.2.1) between all fields as

$$f \star g = \mu \left(e^{\frac{i}{2}\theta^{ij}\partial_i \otimes \partial_j} f \otimes g \right) = \mu \circ \mathcal{F}^{-1}(f \otimes g) := \mu_\star(f \otimes g) \,, \tag{5.2.14}$$

where we defined the inverse of a twist $\mathcal{F}$. A short computation shows that Hopf algebras provide an alternative way to covariantize the symmetry with respect to the star product.

$$\begin{aligned} \delta(\phi \star \psi) &= \delta\big(\mu \circ \mathcal{F}^{-1}(\phi \otimes \psi)\big) = \mu \circ \Delta(\delta) \circ \mathcal{F}^{-1}(\phi \otimes \psi) \\ &= \mu \circ \mathcal{F}^{-1} \circ \mathcal{F} \circ \Delta(\delta) \circ \mathcal{F}^{-1}(\phi \otimes \psi) \\ :&= \mu_\star \circ \Delta_\star(\delta)(\phi \otimes \psi) \,. \end{aligned} \tag{5.2.15}$$

Here we introduced a twisted coproduct $\Delta_\star := \mathcal{F} \circ \Delta(\delta) \circ \mathcal{F}^{-1}$ that together with the star multiplication $\mu^\star$ still defines a Hopf algebra. This means that instead of changing how the symmetry acts on fields as in the case of the star $U(1)$, see (5.2.4), we keep the symmetry action unchanged but twist the Leibniz rule to covariantize the star product with respect to the symmetry. The result is called twisted symmetry.

[5]More strictly speaking, the δ of a symmetry form the universal enveloping algebra of the symmetry and we construct the canonical Hopf algebra of the universal enveloping algebra.

[6]For this to work the fields have to be in the same representation of the universal enveloping algebra as the δ.

We expand the concrete form of the adjusted Leibniz rule

$$\Delta_\star := \mathcal{F} \circ \Delta(\delta) \circ \mathcal{F}^{-1} = \delta \otimes 1 + 1 \otimes \delta + \frac{i}{2}\theta^{ij} \left([\delta, \partial_i] \otimes \partial_j + \partial_i \otimes [\delta, \partial_j] \right) + \mathcal{O}(\theta^2)\,. \tag{5.2.16}$$

The additional terms are nothing else than the variation of the star product [139]

$$\frac{i}{2}\theta^{ij} \left([\delta, \partial_i] \otimes \partial_j + \partial_i \otimes [\delta, \partial_j] \right) + \mathcal{O}(\theta^2) = \delta(\star)\,. \tag{5.2.17}$$

Therefore the twisted Leibniz rule is actually

$$\delta(\phi \star \psi) = (\delta\phi) \star \psi + \phi \star (\delta\psi) + \phi\, \delta(\star)\, \psi\,. \tag{5.2.18}$$

Taking the δ as generators of $U(N)$ one can show that the action (5.2.2) is invariant under the twisted $U(N)$ transformations [140].

In case the δ generate the diffeomorphism the last term containing the variation of the star product $\delta(\star)$ is only zero for the volume preserving linear affine transformations $\delta x^\mu = L^\mu{}_\nu x^\nu + a^\nu$ if also the θ is transformed [141]. When looking at a non-commutative version of ordinary Einstein-gravity, as predicted by string theory, this term is therefore not zero. Apart from that the introduction of twisted diffeomporphism does not cause problems and the construction of a non-commutative gravity theory was successful [142, 143].

5.2.3 Comparison of star and twisted symmetries

The action (5.2.2) is invariant both under star $U(N)$ and twisted $U(N)$. It is necessary to compare both to decide which one is the physical symmetry.

The first part of this section will mainly follow [139] while the second part about L_∞ algebras is based on unpublished results obtained together with Ralph Blumenhagen and Matthias Traube.

Twisted symmetries in string theory

In [139] they observe that twisted symmetries are fundamentally different from ordinary symmetries since they act not only on the physical fields but also on the product, see e.g. (5.2.18). This is strange by itself and prohibits the use of the Noether theorem. Therefore there are no Ward identities making it impossible to quantize twisted theories in the usual way. Lacking the Noether theorem the usual conserved current can instead be deduced from the equation of motion as an integrability condition.[7]

[7]Another concern with twisted theories going back to [144] is the following: When adding matter to the $U(N)$ theory it is necessary to add new degrees of freedom since the gauge transformations have to be in the same representation of the universal enveloping algebra as the matter fields. But this point will not be important in the following.

For star $U(N)$ the situation is somewhat different. Having (5.2.9) the standard Noether procedure works and equipped with Ward identities a quantization is possible. That star $U(N)$ theories are rather ordinary is underpinned by the existence of the Seiberg-Witten-map [130] which relates the star $U(N)$ to an ordinary $U(N)$ transformation order by order in θ.

Since twisted symmetries are in case of $U(N)$ accompanied by the star symmetry one can argue that the existence of Ward identities for the twisted symmetry is guaranteed by the parallel existence of the star symmetry. But this argumentation does not hold for the case of gravity since no star diffeomorphisms exist.

All in all [139] comes to the conclusion that the twisted diffeomorphisms cannot appear as symmetry of string theory. In general they doubt that twisted symmetries can be physical. They support their claims with a calculation that the action of the induced gravity on the D-brane is different from the proposed action of twisted gravity[142]. But of course this computation does not tell anything about the truly closed string star product.

Star and twisted symmetries in light of L_∞ algebras

For a rather long time it is known that any consistent gauge algebra is equivalent to an L_∞ algebra [145, 146, 147, 148, 149, 150]. A gauge transformation δ_ϵ onto a field Φ should obey the two consistency conditions

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\Phi = \delta_{-C(\varepsilon_1,\varepsilon_2,\Phi)}\Phi\,, \tag{5.2.19}$$

and

$$\sum_{\text{cycl}} \left[\delta_{\varepsilon_1}, [\delta_{\varepsilon_2}, \delta_{\varepsilon_3}]\right] = 0\,, \tag{5.2.20}$$

to define a Lie-algebra. Notice that we allowed for field dependent gauge parameters in the closure relation. In any classical perturbative theory we are able to unambiguously identify[8]

$$\delta_\varepsilon\Phi = \sum_{n\geq 0} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\,\ell_{n+1}(\varepsilon, \underbrace{\Phi,\dots,\Phi}_{n\text{ times}}) \tag{5.2.21}$$

and

$$C(\varepsilon_1,\varepsilon_2,\Phi) = \sum_{n\geq 0} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\,\ell_{n+2}(\varepsilon_1,\varepsilon_2, \underbrace{\Phi,\dots,\Phi}_{n\text{ times}})\,. \tag{5.2.22}$$

where the ℓ_n products are graded commutative

$$\ell_n(\dots,x_1,x_2,\dots) = (-1)^{1+\deg(x_1)\deg(x_2)}\ell_2(\dots,x_2,x_1,\dots)\,. \tag{5.2.23}$$

[8]Notice that in the quantum case the ordering according to powers of fields and parameters is more involved, since contractions between fields can lower the power of a field, see e.g. the upcoming publication [151] also of the author.

The gauge parameters have degree 0 while all fields have degree 1 and

$$\deg\big(\ell_n(x_1,\dots,x_n)\big) = n-2+\sum_{i=1}^{n}\deg(x_i)\,. \tag{5.2.24}$$

Since so far we have nothing else than fields and gauge parameters we set objects to be zero if their degree differs from 0 or 1. Therefore the vector space is $V = V_0 \oplus V_{-1}$.

As one can show [146] (since this source is not available anymore consult the more recent [150]) the consistency relations (5.2.19) and (5.2.20) are satisfied if and only if the ℓ_n satisfy

$$\begin{aligned}\mathcal{J}_n(x_1,\dots,x_n) := \sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} \gamma(\sigma;x) \\ \ell_j\big(\,\ell_i(x_{\sigma(1)}\,,\dots,x_{\sigma(i)})\,,x_{\sigma(i+1)},\dots,x_{\sigma(n)}\big) = 0\,,\end{aligned} \tag{5.2.25}$$

where the permutations are restricted to the ones with

$$\sigma(1)<\dots<\sigma(i), \qquad \sigma(i+1)<\dots<\sigma(n)\,. \tag{5.2.26}$$

The sign $\gamma(x,\sigma)=\pm1$ can be read off using (5.2.23). Inserting exactly two gauge parameters into the $\mathcal{J}_n=0$ conditions produces relations that are equivalent to the closure (5.2.19) while inserting exactly three gauge parameters gives relations equivalent to (5.2.20). The ℓ_n products together with the relations $\mathcal{J}_n=0$ define an L_∞ algebra.[9]

The $\mathcal{J}_n$ have the schematic form

$$\mathcal{J}_2 = \ell_1\ell_2 - \ell_2\ell_1\,, \qquad \mathcal{J}_3 = \ell_1\ell_3+\ell_2\ell_2+\ell_3\ell_1\,, \tag{5.2.27}$$

and $\mathcal{J}_n=0$ reads explicitly

$$\ell_1\big(\ell_2(x_1,x_2)\big) = \ell_2\big(\ell_1(x_1),x_2\big)+\ell_2\big(x_1,\ell_1(x_2)\big)\,, \tag{5.2.28}$$

showing that ℓ_1 is a derivative with respect to ℓ_2 satisfying the Leibniz rule, and

$$\begin{aligned}0 = \;& \ell_1\big(\ell_3(x_1,x_2,x_3)\big) \\ &+\ell_2\big(\ell_2(x_1,x_2),x_3\big)+\ell_2\big(\ell_2(x_2,x_3),x_1\big)+\ell_2\big(\ell_2(x_3,x_1),x_2\big) \\ &+\ell_3\big(\ell_1(x_1),x_2,x_3\big)+\ell_3\big(x_1,\ell_1(x_2),x_3\big)+\ell_3\big(x_1,x_2,\ell_1(x_3)\big)\,,\end{aligned} \tag{5.2.29}$$

revealing that the L_∞ relations allow for a mild violation of the Jacobi identity, in the sense that the violation is a total derivative term. This is the reason why mathematicians call L_∞ algebras strong homotopy (sh) Lie-algebras.

Usually a perturbative action is constructed by demanding invariance under a certain gauge symmetry. Using that the gauge algebra is an L_∞ algebra Hohm and Zwiebach

[9] For more details and a class of highly non-trivial examples where the gauge algebra closes with field dependent gauge parameters consult the recent publication [6] also of the author.

could show in [150] that any perturbative and gauge invariant action is controlled by an L_∞ algebra as well. For this they need to add a degree -2 space V_{-2} containing equations of motion and defined a scalar product $\langle \cdot, \cdot \rangle$ that is non-zero only for the contraction $\langle A, E \rangle$ of a field $A \in V_{-1}$ and an equations of motion $E \in V_{-2}$. The action is

$$S = \tfrac{1}{2}\langle A, \ell_1(A)\rangle - \tfrac{1}{3!}\langle A,\ \ell_2(A, A)\rangle - \tfrac{1}{4!}\langle A, \ell_3(A, A, A)\rangle + \dots\,, \tag{5.2.30}$$

and the equations of motion are

$$\mathcal{F} = \ell_1(A) - \tfrac{1}{2}\ell_2(A, A) - \tfrac{1}{3}\ell_3(A, A, A) + \dots\,. \tag{5.2.31}$$

E.g. for Yang-Mills the scalar product is

$$\langle A, E\rangle = \int \eta^{\mu\nu} \mathrm{tr}(A_\mu E_\nu)\,, \tag{5.2.32}$$

and the non-vanishing L_∞ products are

$$\begin{aligned}
\ell_1(\lambda) &= \partial\lambda \in X_{-1} \\
\ell_1(A) &= \Box A - \partial(\partial\cdot A) \in X_{-2} \\
\ell_2(\lambda_1, \lambda_2) &= -[\lambda_1, \lambda_2] \in X_0 \\
\ell_2(A, \lambda) &= -[A, \lambda] \in X_{-1} \\
\ell_2(E, \lambda) &= -[E, \lambda] \in X_{-2} \\
\ell_2(A_1, A_2)_\star &= -\partial[A_1, A_{2\star}] - [\partial_\star A_1 - \partial A_{1\star}, A_2] + (1 \leftrightarrow 2) \in X_{-2} \\
\ell_3(A_1, A_2, A_3) &= -\big[A_1, [A_2, A_{3\star}]\big] + \text{sym.} \in X_{-2}\,,
\end{aligned} \tag{5.2.33}$$

where the lower star $_\star$ denotes a free spacetime index while the other two free indices are assumed to be contracted. These ℓ_n products define an L_∞ algebra and satisfy all defining relations (5.2.25). To show this the Jacobi identity of the bracket is needed at several stages of the computation.

Now let us replace every multiplication in ℓ products (5.2.33) and the scalar product (5.2.32) with a Moyal-Weyl star product (5.2.1). The resulting action and gauge transformations are exactly the ones of the star $U(N)$ Yang-Mills theory. Since the Moyal-Weyl star product obeys the Jacobi identity the star ℓ_n still obey all L_∞ relations (5.2.25). Without obstructions the star $U(N)$ Yang-Mills theory is a ordinary perturbative field theory from the L_∞ viewpoint.

In contrast twisted symmetries do not define an L_∞ algebra since the twisted Leibniz rule (5.2.18)

$$\delta(\phi \star \psi) = (\delta\phi) \star \psi + \phi \star (\delta\psi) + \phi\, \delta(\star)\, \psi \tag{5.2.34}$$

manifestly contradicts the L_∞ relation (5.2.28) demanding that ℓ_1 is a derivation of ℓ_2. This is a further argument why twisted and star transformations are fundamentally different in nature.

Let us introduce another setting where no L_∞ algebra can be defined. Instead of looking at a D-brane with a constant B field background we allow B to be slightly non-constant. In first order in $\partial\theta \sim \partial B^{-1}$ the product that accounts for the background is [123]

$$\begin{aligned} f \circ g &= f \star g + \frac{(2\pi i\,\alpha')^2}{12}\big(\theta^{im}\partial_m\theta^{jk}\big)\big(\partial_i\partial_j f \,\star\, \partial_k g + \partial_k f \,\star\, \partial_i\partial_j g\big) \,+\, \mathcal{O}\big((\partial\theta)^2, \partial^2\theta\big) \\ &\approx f\cdot g \,+\, \frac{(2\pi i\,\alpha')}{2}\theta^{ij}\,\partial_i f\,\partial_j g + \frac{(2\pi i\,\alpha')^2}{8}\theta^{ij}\theta^{kl}\,\partial_i\partial_k f\,\partial_j\partial_l g \\ &\quad + \frac{(2\pi i\,\alpha')^2}{12}\big(\theta^{im}\partial_m\theta^{jk}\big)\big(\partial_i\partial_j f\,\partial_k g - \partial_i\partial_j g\,\partial_k f\big) + \mathcal{O}\big((\partial\theta)^2, \partial^2\theta, \theta^2\big)\,, \end{aligned} \tag{5.2.35}$$

where $\star$ is the Moyal-Weyl star product (5.2.1) that appears for constant $\theta = B^{-1}$ field backgrounds. Due to a defect proportional to $\theta^{[\underline{im}}\partial_m\theta^{\underline{jk}]} \sim dB$ the $\circ$ product is not associative and does not obey the Jacobi identity. The non-vanishing Jacobi identity is one of several sources that causes violations of the L_∞ relations. Another problem with the L_∞ structure appears when trying to write down the corresponding L_∞ action. Following [123] the integral should have the DBI measure that spoils a nice behavior under partial integration needed to derive the L_∞ equations of motion.

Let us go into more detail about another place where the $\circ$ contradicts the L_∞ relations. For this take the $\mathcal{L}_2 = 0$ relation in (5.2.28) that we repeat for convenience

$$\ell_1\big(\ell_2(x_1,x_2)\big) = \ell_2\big(\ell_1(x_1),x_2\big) + \ell_2\big(x_1,\ell_1(x_2)\big)\,. \tag{5.2.36}$$

Inserting two gauge parameters λ_1 and λ_2 the left side becomes

$$\partial(-[\lambda_1,\lambda_2]_\star) = \underbrace{-[\partial\lambda_1,\lambda_2]_\star}_{\ell_2\left(\ell_1(\lambda_1),\lambda_2\right)}\ \underbrace{-[\lambda_1,\partial\lambda_2]_\star}_{\ell_2\left(\lambda_1,\ell_1(\lambda_2)\right)} - i\,\partial\theta^{ij}\,\partial_i\lambda_1\,\partial_j\lambda_2 + \mathcal{O}(\theta^2,\partial\theta)\,. \tag{5.2.37}$$

The extra term violates the L_∞ relation in an unrepairable way.[10] The extra term comes from the fact that the derivative acts on the star product as well which can nicely be summarized in the notation

$$i\,\partial\theta^{ij}\,\partial_i\lambda_1\,\partial_j\lambda_2 = [\lambda_1,\lambda_2]_{\partial(\star)}\,. \tag{5.2.38}$$

Writing the extra term in this form shows that we actually encountered the twisted Leibniz rule of a Hopf-algebra (see (5.2.18) or (5.2.34))[11]. The extra term reflect for finite transformations that, being non-associative, there is a bracketing ambiguity in (5.2.4) such that the covariance of star products cannot be realized in the usual way anymore. We see that Hopf algebras and therefore twisted symmetries appear naturally when considering non-associative star products. In particular a non-associative star symmetry seems impossible. This is the reason why there is also no literature about a star Yang-Mills with the

[10] We checked that also the introduction of a $\ell_0 \in X_{-2}$ does not help.

[11] This suggests to introduce a Hopf L_∞ algebra.

non-associative $\circ$ product even though it is a natural generalization to the well understood star $U(N)$ Yang-Mills theory.

Let us now discuss a conjectural consequence of the lacking L_∞ algebra of twisted symmetries. Recall that the perturbative string theory in form of bosonic closed string field theory is governed by an L_∞ algebra [147] while in superstring field theory an A_∞ algebras appears. In the recent work [152] Ashoke Sen introduces the notion of consistent truncations of the algebra of superstring field theory. In this light one can argue that a given perturbative action can only be a consistent truncation of string theory if it has the structure of an L_∞ or an A_∞ algebra. This line of arguments led Hohm and Zwiebach to construct the L_∞ algebra of any perturbative action governed by a gauge symmetry in [150].

Let us check this conjecture in the above example. As shown there is no L_∞ algebra for any twisted symmetry and in particular not for the star product of D-branes in non-constant B field backgrounds (5.2.35). The $\circ$ product should therefore not appear in string theory although it was derived from bosonic string theory. Indeed the derivation [123] does not respect that by the Freed-Witten anomaly [153] there may not be any gauge flux $H = dB$ on a D-brane. Therefore the $\circ$ is indeed forbidden. Notice that both the defect of the Jacobi identity and the non-associativity are proportional precisely to $H = dB$ that leaded to violated L_∞ relations. Without having a string theory in behind it might be no surprise that there is no consistent way to quantize twisted symmetries.

Following these arguments one can argue that twisted symmetries may not appear in string theory. Clearly the arguments rely on a conjecture that is far away from settled but still the conjecture holds surprisingly precise in certain circumstances. Therefore it strengthens the doubts of [139] that twisted symmetries can have something to do with string theory.

5.2.4 Summary

We saw that the only way to reconcile diffeomorphism with a non-associative star product is through twisted diffeomorphism that naturally appears in the context of non-associative star products. In the following we will therefore try to construct a general relativity that is invariant under twisted diffeomorphisms. We discussed many concerns with twisted symmetries that stress the necessity to reconcile the star product with the twisted symmetry by an appropriate mechanism.

5.3 Basic properties of the R-flux Hopf algebra

This section starts with the introduction of operators that are used throughout the rest of the chapter. Then we will exemplify the logic of the twisted Leibniz rule and the construction of twisted tensors in a simplified setting. Afterwards we will clarify how stringy vertex operators can be tensors.

5.3.1 Operators

Let us introduce important operators that control the non-commutativity and the non-associativity. We begin by noting that the partial derivatives in the star product (5.1.6) are not appropriate when star multiplying higher tensors. By demanding compatibility with the exterior derivative the authors of [117] suggest to use Lie derivatives instead

$$\begin{aligned} f \star g = \mu\Big[\exp\Big(&(\tfrac{1}{2} i\hbar (\mathcal{L}_{\partial_i} f\, \mathcal{L}_{\tilde{\partial}_p^i} g - \mathcal{L}_{\tilde{\partial}_p^i} f\, \mathcal{L}_{\partial_i} g) && (5.3.1)\\ &+ \big(\tfrac{i l_s^4}{6\hbar} R^{ijk}\, \mathcal{L}_{p_k \partial_i} f\, \mathcal{L}_{\partial_j} g - \tfrac{i l_s^4}{6\hbar} R^{ijk}\, \mathcal{L}_{\partial_j} f\, \mathcal{L}_{p_k \partial_i} g \big) \Big) f \otimes g \Big] \\ = \mu \left[\mathcal{F}^{-1}(f,g) \right] . && (5.3.2) \end{aligned}$$

Since the twist $\mathcal{F}^{-1}$ is merely a phase factor, inversion is simply done by changing the sign in the exponent or, due to antisymmetry of R^{ijk}, by changing the arguments

$$\mathcal{F}^{-1}(f,g) = \mathcal{F}(g,f) \,. \tag{5.3.3}$$

Using this, the non-commutativity

$$\begin{aligned} f \star g =& \mu\big[\mathcal{F}^{-1}(f,g)\big] = \mu\big[\mathcal{F}(g,f)\big] = \mu\big[\mathcal{F}^{-1} \underbrace{\mathcal{F}\mathcal{F}}_{:=\overline{\mathcal{R}}}(g,f)\big] \\ :=& \overline{\mathcal{R}}(g) \star \overline{\mathcal{R}}(f) \end{aligned} \tag{5.3.4}$$

can be seen to be controlled by the operator $\overline{\mathcal{R}} = \mathcal{F}^2$ which is the inverse of what is called the universal $\mathcal{R}$-matrix $\mathcal{R} = \mathcal{F}^{-2}$. To get an intuition about the $\mathcal{R}$-matrix let us choose $f, g \in C^\infty(M)$ and work up to linear order in the R-flux. We compare

$$f \star g = fg + \tfrac{i l_s^4}{6\hbar} R^{ijk} p_k\, \partial_i f\, \partial_j g + \ldots \tag{5.3.5}$$

with

$$\begin{aligned} \overline{\mathcal{R}}(g) \star \overline{\mathcal{R}}(f) &= g \star f - 2 \tfrac{i l_s^4}{6\hbar} R^{ijk} p_k\, \partial_i g \star \partial_j f + \ldots \\ &= gf + \tfrac{i l_s^4}{6\hbar} R^{ijk} p_k\, \partial_i g\, \partial_j f - 2 \tfrac{i l_s^4}{6\hbar} R^{ijk} p_k\, \partial_i g\, \partial_j f + \ldots \\ &= fg + \tfrac{i l_s^4}{6\hbar} R^{ijk} p_k\, \partial_i f\, \partial_j g + \ldots \end{aligned} \tag{5.3.6}$$

to see that the $\mathcal{R}$-matrix controls the non-commutativity due to the factor -2 from the square in $\overline{\mathcal{R}} = \mathcal{F}^2$. Actually also the non-associativity is controlled by an exponential operator. To see what is happening take $f, g, h \in C^\infty(M)$ and expand

$$\begin{aligned} (f \star g) \star h =& f \star (g \star h) \\ &+ \tfrac{l_s^4}{6} R^{ijk}\, \partial_i f \;\; \partial_j g \;\; \partial_k h \\ &+ \tfrac{l_s^4}{6} \tfrac{i l_s^4}{6\hbar} R^{ijk} R^{abc}\, p_c \, (\partial_i f \;\; \partial_a \partial_j g \;\; \partial_b \partial_k h + \text{cycl.}) \\ &+ \mathcal{O}(R^3) \,. \end{aligned} \tag{5.3.7}$$

The terms in the third line can be taken care of by star products of the second line such that

$$(f \star g) \star h = f \star (g \star h) + \sum_{n=1}^{\infty} \frac{1}{n!} (\tfrac{l_s^4}{6})^n \, R^{i_1 j_1 k_1} \ldots R^{i_n j_n k_n} (\partial_{i_1} \ldots \partial_{i_n} f) \star \Big((\partial_{j_1} \ldots \partial_{j_n} g) \star (\partial_{k_1} \ldots \partial_{k_n} h) \Big) . \quad (5.3.8)$$

This formula holds even for f, g and h with arbitrary dependence of x and p. Let us collect the result in to an operator ϕ called associator

$$\phi(f,g,h) = \exp \Big(\tfrac{l_s^4}{6} R^{ijk} \mathcal{L}_{\partial_i} \otimes \mathcal{L}_{\partial_j} \otimes \mathcal{L}_{\partial_k} \Big) (f \otimes g \otimes h) \quad (5.3.9)$$

that reorders the brackets in the following way:

$$(f \star g) \star h = f^{\phi} \star (g^{\phi} \star h^{\phi}) := f \star (g \star h)|_{\phi} . \quad (5.3.10)$$

Similarly the inverse associator $\overline{\phi}$ shifts the brackets to the left. Again, being a phase factor the inverse can be obtained by switching the sign in the exponent or, due to antisymmetry of R^{ijk}, by switching the arguments. This is important to keep in mind when interchanging the arguments of the associator with the $\mathcal{R}$ matrix

$$f^{\phi} \star (g^{\phi} \star h^{\phi}) = f^{\overline{\phi}} \star (\overline{\mathcal{R}}(h^{\overline{\phi}}) \star \overline{\mathcal{R}}(g^{\overline{\phi}})) . \quad (5.3.11)$$

5.3.2 Warmup with configuration space functions

Following [142, 143, 114] we will now demonstrate the first steps towards a twisted tensor formalism. To warm up we only work with twisted diffeomorphisms $\xi = \xi^i(x) \star \partial_i$ that solely depend on the configuration space and consider the full phase space $\xi = \xi^i \star \partial_i + \tilde{\xi}_i \star \tilde{\partial}_p^i$ later. With this simplification the crucial steps in the construction become clearer. A twisted scalar is defined to be an object transforming with the twisted Lie derivative

$$\delta_\xi f = \mathcal{L}_\xi^\star f := \xi^i \star \partial_i f . \quad (5.3.12)$$

We demand that the star product of two twisted scalars $f \star g$ should be a twisted scalar again

$$\delta_\xi (f \star g) = \mathcal{L}_\xi^\star (f \star g) = \xi^i \star \partial_i (f \star g) = \xi^i \star (\partial_i f \star g) + \xi^i \star (f \star \partial_i g) . \quad (5.3.13)$$

Note that the spacetime derivative commutes with the star product since the star product depends only on the momentum. Let us use an associator in the first term and reformulate the second term with the $\mathcal{R}$-matrix and an inverse associator

$$\begin{aligned} \xi^i \star (f \star \partial_i g) &= (\xi^i \star f) \star \partial_i g|_{\overline{\phi}} \\ &= (\overline{\mathcal{R}}(f) \star \overline{\mathcal{R}}(\xi^i)) \star \partial_i g|_{\phi} = \overline{\mathcal{R}}(f) \star (\overline{\mathcal{R}}(\xi^i) \star \partial_i g)|_{\phi^2} . \end{aligned} \quad (5.3.14)$$

Plugging in (5.3.13) we find the twisted Leibniz rule

$$\mathcal{L}^\star_\xi(f\star g) = \big(\mathcal{L}^\star_{\xi^{\overline{\phi}}} f^{\overline{\phi}}\big)\star g^{\overline{\phi}} + \overline{\mathcal{R}}(f^{\phi^2})\star\big(\mathcal{L}^\star_{\overline{\mathcal{R}}(\xi^{\phi^2})} g^{\phi^2}\big). \tag{5.3.15}$$

Notice that the twisted Leibniz rule was obtained using the $\mathcal{R}$-matrix and the associator ϕ while the concrete form of the star product did not matter.

Next, $w_i = \partial_i f$ should be a twisted covector and therefore

$$\delta_\xi(\partial_i f) = \partial_i(\delta_\xi f) = \xi^j\star(\partial_j\partial_i f) + (\partial_i\xi^j)\star(\partial_j f)\,. \tag{5.3.16}$$

From that formula we can deduce the transformation law for a general twisted covector w_i to be

$$\delta_\xi\omega_i = \mathcal{L}^\star_\xi\omega_i = \xi^j\star(\partial_j\omega_i) + (\partial_i\xi^j)\star\omega_j\,. \tag{5.3.17}$$

The transformation of a twisted vector v^i follows from $v^i\star w_i$ which should transform as a twisted scalar. We find

$$\delta_\xi v^i = \mathcal{L}^\star_\xi v^i = \xi^j\star(\partial_j v^i) - \overline{\mathcal{R}}(v^j)\star\overline{\mathcal{R}}(\partial_j\xi^i)\,. \tag{5.3.18}$$

Since the Lie derivative of a vector field usually matches the commutator we define

$$[v,w]_\star = v^j\star(\partial_j w^i) - \overline{\mathcal{R}}(w^j)\star\overline{\mathcal{R}}(\partial_j v^i)\,. \tag{5.3.19}$$

The same expression can also be obtained by the most intuitive twisted generalization of the commutator

$$[,]_\star = [,]\circ\mathcal{F}^{-1}\,. \tag{5.3.20}$$

The similarity to the definition of the star product $\mu_\star = \mu\circ\mathcal{F}^{-1}$ guarantees covariance under the twisted diffeomorphism and shows that the star commutator is $\mathcal{R}$-antisymmetric. The twisted Lie derivative is defined and closes

$$[\mathcal{L}^\star_\xi,\mathcal{L}^\star_\eta]_\star\, v = \mathcal{L}^\star_{[\xi,\eta]_\star}\, v\,. \tag{5.3.21}$$

A necessary requirement for this formula to hold is that the content of the commutator $[,]_\star$ is enclosed by bracks. Therefore when evaluating the left side we first have to use an associator

$$\big[\mathcal{L}^\star_\xi,\mathcal{L}^\star_\eta\big]_\star\, v := \mathcal{L}^\star_{\xi^\phi}\big(\mathcal{L}^\star_{\eta^\phi}\, v^\phi\big) - \mathcal{L}^\star_{\overline{\mathcal{R}}(\eta^\phi)}\big(\mathcal{L}^\star_{\overline{\mathcal{R}}(\xi^\phi)}\, v^\phi\big)\,. \tag{5.3.22}$$

Consequently also operators do not associate, either, and composite operators act with appropriate associators. We reflect this in the definition of a composition $\bullet$

$$(O\bullet O')(z) := O^\phi\big(O'^\phi(z^\phi)\big) \tag{5.3.23}$$

in presence of a bracketing ambiguity. When looking at the literature this composition is standard in the Hopf-algebra framework[154].[12]

Constructing higher twisted tensors is straightforward.

5.3.3 Vertex operators as twisted tensors

Having a definition of twisted tensors allows us to turn to the question how e.g. a twisted scalar actually looks like. For sure the basic coordinate functions x^i are twisted scalars. But then, by construction, $x^i \star x^j$ is again a twisted scalar again whereas $x^i \cdot x^j$ is not. Only for $i = j$ the antisymmetric star product does not yield any contribution and $x^i \star x^i = x^i \cdot x^i$. This means that functions $f(x)$ are only twisted scalars if they are defined with the star product. Therefore in general also the physical fields receive inner correction through the star product.

From the string theory viewpoint this means that the star product should not only appear between the vertex operators but within the vertex operator. This is not expected from the computation of the three-point function of tachyon vertex operators, where uncorrected tachyon vertex operators e^{ipX} were used since they are conformal primaries [108].

At least for the tachyon vertex operator $e^{iq_i x^i}$ with constant q_i the issues can be resolved by the observation that $(q_i x^i)^{\star n} = (q_i x^i)^n$, which can be proven e.g. by induction. The corrections from the star product in $e^{iq_i x^i}$ vanish and the uncorrected tachyon vertex operator indeed indeed turns out to be a twisted scalar. By the same arguments all functions that have a representation via a Fourier transform

$$f(x) = \int dk \; e^{ik_i x^i} \tilde{f}(k) \tag{5.3.24}$$

are twisted scalars without any adjustments. This is the reason why in the recent publication [106] they restrict themselves to Schwarz functions.

While the tachyon vertex operator behaves as it should, the higher vertex operators like the graviton vertex operators $V_G = h_{ij} \;\; \partial X^i \overline{\partial} X^j \cdot e^{ikX}$ are trickier. To see the problem let us first recall the open string star product. There the propagator of the coordinates in the Seiberg-Witten limit is

$$\langle x^i(\tau) x^j(\tau') \rangle = \tfrac{i}{2} \theta^{ij} \epsilon(\tau - \tau') \,, \tag{5.3.25}$$

leading to the non-commutative star product (5.2.1) when computing the correlators of vertex operators. Due to the $\epsilon(\tau - \tau')$ the polynomials $P[\partial X, \partial^2 X, \ldots]$ in a vertex operator

[12]The recent attempt to formulate a consistent non-associative quantum mechanics [117] uses the composition $(A \tilde{\bullet} B)(C) = A \star (B \star C)$. As one can show this is the only way to avoid negative probabilities. This composition $\tilde{\bullet}$ actually amounts to a brute force rebracketing to the right. Neglecting the associators and treating the star product as associative garantuees consistency but a consistent embedding into the Hopf-algebra framework is of course not possible using $\tilde{\bullet}$. These problems suggest that the usual quantization fails and the only way to make sense of non-associative quantum mechanics is in a membrane quantization where such effects might be rather natural.

$V = P[\partial X, \partial^2 X, \dots]e^{ikX}$ are not affected by the non-commutativity and the star product acts essentially between the e^{ikx} factors.

For the graviton vertex operator we cannot apply this line of arguments. Furthermore how to interpret e.g. $(\partial X^i \star \overline{\partial} X^j) \star e^{ikX}$ is pure speculation, especially taking into consideration that the true graviton vertex operator is not available [98].[13] We will therefore not pursue this any further and keep the formalism completely general.

The most promising way to resolve the problems is the setting of freely acting asymmetric torodial orbifolds since they are still free CFTs in contrast to the perturbed CFT of [98].

5.4 Working with the full phase space

The following section presents the technical tools to handle the full phase space. To shorten the formulas we introduce a doubled notation where the configuration space and the momentum space are packed together in the phase space $X_I = (\frac{p_i}{i\hbar}, x^i)$ such that a vector becomes

$$V = V^I(X) \star \partial_I = V^i(x,p) \star \partial_i + \tilde{V}_i(x,p) \star i\hbar\, \tilde{\partial}_p^i \,. \tag{5.4.1}$$

This notation is borrowed from double field theory [155, 156, 157] in the sense that capital indices run from $I = 0, \dots, 2(D-1)$ and that the index splits into covariant and covariant pieces $X_I = (\frac{p_i}{i\hbar}, x^i)$.[14] Let us furthermore adopt the notation $P, \tilde{P}, M$ of [114] defined by

$$\begin{aligned} P^I &= (\tilde{P}^i, P_i) = (i\hbar\, \tilde{\mathcal{L}}_{\partial_p^i} \;,\; \mathcal{L}_{\partial_i})\,, \\ M^I &= (M^i, 0) = (\tfrac{il_s^4}{6\hbar}\mathcal{L}_{R^{ijk}p_j\partial_k}, 0) = -F^{IJK}X_J P_K\,. \end{aligned} \tag{5.4.2}$$

$\tilde{M}_i$ is zero here since the only non-vanishing component of F^{IJK} is $F^{ijk} = \frac{l_s^4}{6}R^{ijk}$. Therefore a contraction like $F^{IJK}{F_I}^{MN}$ always results in zero. Using the above operators the star product becomes

$$f \star g = \mu\left[\exp\left(\tfrac{1}{2}(P_\mu \otimes \tilde{P}^\mu - \tilde{P}^\mu \otimes P_\mu) + \tfrac{1}{2}\big(M^\mu \otimes P_\mu - P_\mu \otimes M^\mu\big)\right) f \otimes g\right]. \tag{5.4.3}$$

5.4.1 Star products and derivatives

The star product also acts non-trivially onto the basis forms $dX^I = (dx^i, \frac{dp_i}{i\hbar})$. Using $[\mathcal{L}, d] = 0$, one obtains

$$\begin{aligned} &P^I(dX^J) = 0\,, \\ &M^I(dX^J) = F^{IJK}dX_K \qquad \text{or} \qquad M^i(dx^j) = \tfrac{l_s^4}{6}R^{ijk}\,\tfrac{dp_k}{i\hbar}\,. \end{aligned} \tag{5.4.4}$$

[13] As one can check, a simple $\mathcal{O}(R)$ correction to the uncorrected vertex operator does not help.

[14] Let us stress that the phase space $X_I = (\frac{p_i}{i\hbar}, x^i)$ we are considering here is substantially different from double field theory with its space $X_I = (\tilde{x}_i, x^i)$ where the $\tilde{x}$ are the T-dual winding coordinates. In contrast to double field theory our framework is not based on an $O(D,D)$ invariance but rather on double-dimensional (star)diffeomorphism and the $O(D,D)$ scalar product of double field theory is not (star)diffeomorphism invariant.

By enforcing the duality between TM and T^*M in the form $\delta_J{}^I = \partial_J \star dx^I = \partial_j \star dx^i + i\hbar\,\partial_p^j \star \frac{dp_i}{i\hbar}$ one finds

$$\begin{aligned} &P^I(\partial^J) = 0\,, \\ &M^I(\partial^J) = F^{IJK}\partial_K \qquad \text{or} \qquad M^i(i\hbar\,\tilde{\partial}_p^j) = \tfrac{l_s^4}{6}R^{ijk}\partial_k\,. \end{aligned} \tag{5.4.5}$$

Notice that the partial derivatives are meant as basis vectors of the tangent space $e_I = \partial_I$. This notation is introduced since we will see soon that the star product acts in the same way onto basis vectors and the derivatives.

Let us share two observations that make the following computations simpler. The first is that P_i and therefore the associator ϕ acts trivially onto any basis vector. When a basis vector is involved we therefore refrain from writing brackets and the associator. Secondly the only non-vanishing action onto a basis vector is given by M^i when acting on a basis vector with an upper index thus dx^i or $\tilde{\partial}^i$. In both cases the result has a lower index. Therefore $M(M(\partial^I)) = 0$ such that the star product terminates already after the first order when acting on basis vectors. More explicitly we get

$$f \star \partial_i = f \cdot \partial_i \quad \Rightarrow \quad f \star \partial_i - \partial_i \star f = 0\,, \tag{5.4.6}$$

but a purely linear correction in

$$f \star i\hbar\,\tilde{\partial}_p^i = f \cdot i\hbar\,\tilde{\partial}_p^i - \tfrac{1}{2}P_j f \cdot M^j(i\hbar\,\tilde{\partial}_p^i) = f \cdot i\hbar\,\tilde{\partial}_p^i + \tfrac{l_s^4}{12}R^{ijk}\partial_j f \star \partial_k\,, \tag{5.4.7}$$

and

$$f \star i\hbar\,\tilde{\partial}_p^i - i\hbar\,\tilde{\partial}_p^i \star f = \tfrac{l_s^4}{6}R^{ijk}\partial_j f \star \partial_k\,. \tag{5.4.8}$$

Both relations can be combined into

$$f \star \partial^I - \partial^I \star f = F^{IJK}\partial_J f \star \partial_K\,. \tag{5.4.9}$$

Since dx^I behaves like ∂^I we get

$$f \star dx^I - dx^I \star f = F^{IJK}\partial_J f \star dx_K\,. \tag{5.4.10}$$

These commutation relations lie at the heart of all computations, since they not only tell us how to switch a derivative with the star product, but also how to commute derivatives with star product, e.g. in

$$\partial^I(f \star g) = \partial^I f \star g + f \star \partial^I g - F^{IJK}\partial_J f \star \partial_K g\,. \tag{5.4.11}$$

All relations so far still hold when replacing ∂^I with $P^I = \mathcal{L}_{\partial^I}$. At this point it becomes clear that we indeed work with a Hopf algebra and the twisted coproduct from the discussion around (5.2.15) can be read off as

$$\Delta_\star(P^I) = P^I \otimes 1 + 1 \otimes P^I - F^{IJK}P_J \otimes P_K\,. \tag{5.4.12}$$

The twisted coproduct for M^I is

$$\Delta_\star(M^I) = M^I \otimes 1 + 1 \otimes M^I + F^{IJK}\partial_J \otimes \partial_K \,. \tag{5.4.13}$$

The last relation with the switched sign compared to (5.4.12) can be anticipated from $[P^I, M^J] = F^{IJK}\, P_K$.

5.4.2 Formulas for vectors

As consequence of the non-commutativity between ∂^I and the star product the two conventions $V^I \star \partial_I$ and $\partial_I \star V^I$ are not equivalent. We will stick to the first convention since it will make the definition of a twisted scalar easier.

When multiplying a scalar onto a vector it makes a crucial difference whether the scalar is multiplied from the left

$$f \star (V^I \star \partial_I) = (f \star V^I) \star \partial_I \,, \tag{5.4.14}$$

or the right

$$(V^I \star \partial_I) \star f = (V^I \star f) \star \partial_I - F^{IJK}\ (V_I \star \partial_J f) \star \partial_K \,. \tag{5.4.15}$$

Here we used the commutator of the star product and the ∂_I in (5.4.9) and the fact that the associator trivializes when acting onto basis vectors.

Looking back one sees that the form of the additional terms can essentially be deduced from the index structure. This becomes visible in the $\mathcal{R}$-matrix as well

$$\overline{\mathcal{R}}(f) \otimes \overline{\mathcal{R}}(V^I \star \partial_I) = \overline{\mathcal{R}}(f) \otimes \overline{\mathcal{R}}(V^I) \star \partial_I + F^{IJK}\ \overline{\mathcal{R}}(\partial_I f) \otimes \overline{\mathcal{R}}(V_J) \star \partial_K \,. \tag{5.4.16}$$

Iterating this expression yields

$$\begin{aligned}
\overline{\mathcal{R}}(V^I \star \partial_I) \otimes \overline{\mathcal{R}}(U^J \star \partial_J) = {} & \overline{\mathcal{R}}(V^I) \star \partial_I \otimes \overline{\mathcal{R}}(U^J) \star \partial_J \\
& - F^{MNI}\ \overline{\mathcal{R}}(V_N) \star \partial_I \otimes \overline{\mathcal{R}}(\partial_M U^J) \star \partial_J \\
& + F^{MNJ}\ \overline{\mathcal{R}}(\partial_M V^I) \star \partial_I \otimes \overline{\mathcal{R}}(U_N) \star \partial_J \\
& - F^{ABI} F^{MNJ}\ \mathcal{R}(\partial_M V_B) \star \partial_I \otimes \mathcal{R}(\partial_A U_N) \star \partial_J \,.
\end{aligned} \tag{5.4.17}$$

The last two relations will be put in use later. The multiple additional terms seem to complicate the computations but in the end they will organize themselves in such a way that the shape of the twisted Leibniz rule (5.3.15) is preserved even when considering the full phase space. Remarkably the same holds true for arbitrary quasi-triangular quasi-Hopf algebras [112, 113].

5.4.3 The contraction

Now let us formalize the pairing which we already used

$$\partial_I \star dx^J = dx^J \star \partial_I = \delta_I^J \,. \tag{5.4.18}$$

The symmetric definition is well defined since a small computation shows that the star acts trivial in this formula. As a nice byproduct two basis vectors commute. Recall that we have already fixed the convention of vectors to be $V = V^I \star \partial_I$. When contracting a vector V with a form ω, the basis vector and the basis form are only then directly next to each other if the vector is on the left and the form is on the right $\langle V, \omega\rangle$ and if the following convention is chosen

$$V = V^I \star \partial_I \,, \qquad \omega = dx^I \star \omega_I \,. \tag{5.4.19}$$

Using this convention we can contract without additional terms

$$V \star \omega = (V^I \star \partial_I) \star (dx^J \star \omega_J) = V^I \star \delta_I{}^J \star \omega_J = V^I \star \omega_I \,. \tag{5.4.20}$$

When having an expression $\omega \star V$ one either has to use the $\mathcal{R}$-matrix to interchange ω with V or use (5.4.9) to bring the basis vectors and forms together resulting in a δ.

5.4.4 The commuting derivative ∂

Next we introduce a derivative operator which commutes with the star product making computations a lot easier. Recall that the usual exterior derivative commutes with the Lie derivative $[\mathcal{L}, d] = 0$ and consequently also with the star product. Furthermore due to $dx^I \star \partial_I = dx^I \cdot \partial_I$ this property holds for the star exterior derivative as well. Actually this argumentation does not rely on the antisymmetrization. Let us therefore introduce the derivative one-form

$$\partial := dx^I \star \partial_I \tag{5.4.21}$$

where dx^I is meant as basis form and ∂_I acts as derivative. This operator ∂ does not assume antisymmetrization while it becomes the exterior derivative when additionally assuming antisymmetrization $d = \partial\wedge$. It is instructive to see ∂ at work:

$$\begin{aligned} dx^I \star \partial_I (f \star g) &= dx^I \star \partial_I f \star g + dx^I \star f \star \partial_I g - F^{IJK} dx_I \star \partial_J f \star \partial_K g \\ &= dx^I \star \partial_I f \star g + f \star dx^I \star \partial_I g - F^{IJK} dx_K \star \partial_J f \star \partial_I g \\ &\qquad - F^{IJK} dx_I \star \partial_J f \star \partial_K g \\ &= dx^I \star \partial_I f \star g + f \star dx^I \star \partial_I g \,. \end{aligned} \tag{5.4.22}$$

In the first line we used the product rule (5.4.11) while in the second line we switched the basis vector dx^I to the right using (5.4.10). As one can see the additional terms from both operations cancel. To summarize ∂ commutes with the star product and therefore the $\mathcal{R}$-matrix.

5.5 Tensor calculus

In the following we will repeat the analysis of section 5.3.2 and develop the notion of tensors for twisted diffeomorphisms generated by a full phase space vector

$$\xi = \xi^I(X) \star \partial_I = \xi^i(x,p) \star \partial_i + \tilde{\xi}_i(x,p) \star i\hbar\, \tilde{\partial}_p^i\,. \tag{5.5.1}$$

5.5.1 Scalars

A twisted scalar $f = f(x,p) = f(X)$ is a quantity transforming as

$$\delta_\xi f = \mathcal{L}_\xi^\star f = \xi^I \star \partial_I f = \xi^i \star \partial_i f + \tilde{\xi}_i \star i\hbar\, \tilde{\partial}_p^i f\,. \tag{5.5.2}$$

If we had instead chosen the convention $V = \partial_I \star V^I$, additional terms would have appeared from commuting the derivative. Using (5.4.11) we see that the Leibniz rule is now more involved

$$\begin{aligned}\mathcal{L}_\xi^\star(f \star g) &= \xi^I \star \partial_I(f \star g)\\ &= \xi^I \star (\partial_I f \star g) + \xi^I \star (f \star \partial_I g) - F^{IJK}\xi_I \star (\partial_J f \star \partial_K g)\,.\end{aligned} \tag{5.5.3}$$

Nevertheless the additional term can neatly be taken care of by the $\mathcal{R}$-matrix

$$\mathcal{L}_\xi^\star(f \star g) = \left(\mathcal{L}_\xi^\star f\right) \star g\big|_{\overline{\phi}} + \overline{\mathcal{R}}(f) \star \left(\mathcal{L}^\star_{\overline{\mathcal{R}}(\xi)} g\right)\big|_{\phi^2}\,. \tag{5.5.4}$$

This formula can most easily be derived using the star commuting derivative operator ∂ (5.4.21) by writing $\mathcal{L}_\xi^\star = \xi \star \partial$. As we can see here the $\mathcal{R}$-matrix is indeed capable of organizing all additional terms such that we get the same form for the Leibniz rule that we have encountered before (5.3.15).

5.5.2 Vectors

Following the discussion around (5.3.20) we define the transformation of a vector V as

$$\delta_\xi V = \mathcal{L}_\xi^\star V = [\xi, V]_\star := [\,,] \circ \mathcal{F}^{-1}\,. \tag{5.5.5}$$

Writing the commutator out explicitly results in

$$\begin{aligned}\mathcal{L}_\xi^\star V &= \xi(V) - \overline{\mathcal{R}}(V)\big(\overline{\mathcal{R}}(\xi)\big)\\ &= \xi^I \star \partial_I V^J \star \partial_J - \overline{\mathcal{R}}(V)^I \star \partial_I(\overline{\mathcal{R}}(\xi)^J) \star \partial_J\,.\end{aligned} \tag{5.5.6}$$

The I-th component of $(\mathcal{L}_\xi^\star V)^I$ can be computed using (5.4.17)

$$\begin{aligned}\mathcal{L}_\xi^\star V^I =& \xi^J \star \partial_J V^I - \overline{\mathcal{R}}(V^J) \star \partial_J\big(\overline{\mathcal{R}}(\xi^I)\big)\\ &- \overline{\mathcal{R}}(\partial_M V^J) \star \partial_J\big(\overline{\mathcal{R}}(\xi_N)\big) F^{MNI}\,.\end{aligned} \tag{5.5.7}$$

In a rather long computation with nice cancellations we checked the consistency of this algebra when applying consecutive twisted diffeomorphisms

$$[\mathcal{L}^\star_\xi, \mathcal{L}^\star_\eta]_\star f = \xi(\eta(f))|_\phi - \overline{\mathcal{R}}(\eta)\big(\overline{\mathcal{R}}(\xi)(f)\big)|_\phi = \mathcal{L}^\star_{[\xi,\eta]_\star} f \,. \tag{5.5.8}$$

Most easily shown with Hopf-algebra techniques [114] the closure of the algebra is guaranteed by the Jacobi identity

$$[U,[V,W]_\star]_\star = [[U,V]_\star, W]_\star|_{\overline{\phi}} + [\overline{\mathcal{R}}(V), [\overline{\mathcal{R}}(U), W]_\star]_\star|_{\phi^2} \,. \tag{5.5.9}$$

5.5.3 One-forms

The variation of a twisted one-form can be deduced from the variation of the one-form $\partial f = dx^I \star \partial_I f$ where we used the star commuting derivative operator ∂. This makes the computation very easy as we can switch δ and ∂ without extra terms. One gets

$$\begin{aligned} \delta^\star_\xi \partial f &= \partial \delta^\star_\xi f \\ &= \partial(\xi^I \star \partial_I f) = \partial \xi^I \star \partial_I f + \xi^I \star \partial_I \partial f \,. \end{aligned} \tag{5.5.10}$$

From this we deduce for a general one-form $\omega = dx^I \star \omega_I$

$$\mathcal{L}^*_\xi \omega = \xi^I \star \partial_I \omega + \partial \xi^J \star \omega_J \,, \tag{5.5.11}$$

or in components

$$\mathcal{L}^*_\xi \omega_I = \xi^J \star \partial_J \omega_I + \partial_I \xi^J \star \omega_J \,. \tag{5.5.12}$$

Again the consistency of this definition must be checked and indeed a contraction between a form and a vector transforms as a scalar as it should

$$\mathcal{L}^*_\xi (V^I \star \omega_I) = \mathcal{L}^*_\xi V^I \star \omega_I|_{\overline{\phi}} + \overline{\mathcal{R}}(V^I) \star \mathcal{L}^*_{\overline{\mathcal{R}}(\xi)} \omega_I|_{\phi^2} \,. \tag{5.5.13}$$

5.5.4 Tensor products and twisted p-forms

Having forms and vectors we can also build higher tensors using the star covariant tensor product $\otimes_\star := \otimes \circ \mathcal{F}^{-1}$. Being dressed with the appropriate twist the Leibniz rule takes the meanwhile familiar twisted form

$$\mathcal{L}^*_\xi (V \otimes_\star W) = \mathcal{L}^*_\xi V \otimes_\star W|_{\overline{\phi}} + \overline{\mathcal{R}}(V) \otimes_\star \mathcal{L}^*_{\overline{\mathcal{R}}(\xi)} W|_{\phi^2} \,. \tag{5.5.14}$$

Using this rule we can deduce the transformation law for an arbitrary element say in $T \in TM \otimes_\star TM$ only if we have a split

$$T = A_\alpha \otimes_\star B^\alpha \in TM \otimes_\star TM \tag{5.5.15}$$

at hand. To be more general we introduced a possible internal summation over α. Both $A_\alpha = A^I_\alpha \star \partial_I \in TM$ as well as $B^\alpha = B^{\alpha I} \star \partial_I \in TM$ are vectors. For later reference we commute the basis vectors to the right using (5.4.9)

$$\begin{aligned} T &= A^I_\alpha \star \partial_I \otimes_\star B^{\alpha J} \star \partial_J \\ &= (A^I_\alpha \star B^{\alpha J} - F^{MNJ} A_{\alpha M} \star \partial_N B^{\alpha I}) \star (\partial_I \otimes_\star \partial_J)\,. \end{aligned} \tag{5.5.16}$$

To define symmetric and antisymmetric tensors we need a definition of the transpose. The most natural definition of a transpose of the above T is $T^T = \overline{\mathcal{R}}(B^\alpha) \otimes_\star \overline{\mathcal{R}}(A_\alpha)$. The computation using (5.4.17) simplifies due to nice cancellations and a similar form as in (5.5.16) appears

$$\begin{aligned} T^T &= \overline{\mathcal{R}}(B^{\alpha I} \star \partial_I) \otimes_\star \overline{\mathcal{R}}(A^J_\alpha \star \partial_J) \\ &= (A^J_\alpha \star B^{\alpha I} - F^{MNI} A_{\alpha M} \star \partial_N B^{\alpha J}) \star (\partial_I \otimes_\star \partial_J)\,. \end{aligned} \tag{5.5.17}$$

The $\mathcal{R}$-matrix does nothing but interchanging the indices whenever a tensor is written in this form.

Scalar products between higher tensors are computed using $dx^I \star \partial_J = \delta^I_J$ after commuting the basis vectors and forms together. For a two-vector $A \otimes_\star B \in TM^2$ and a two-form $\omega \otimes_\star \alpha \in T^*M^2$ we get

$$\begin{aligned} \langle A \otimes_\star B\ ,\ \omega \otimes_\star \alpha \rangle_\star = (A^I \star B^J) \star (\omega_J \star \alpha_I) - F^{IJK}(A_I \star \partial_J B^M) \star (\omega_M \star \alpha_K) \\ -F^{IJK}(A_I \star B^M) \star (\partial_J \omega_M \star \alpha_K)\,. \end{aligned} \tag{5.5.18}$$

Using (5.5.16) and the observation that the transposition interchanges only indices (5.5.17) one can infer the transposition symmetry

$$\langle A \otimes_\star B\ ,\ \omega \otimes_\star \alpha \rangle_\star = \langle \overline{\mathcal{R}}(B) \otimes_\star \overline{\mathcal{R}}(A)\ ,\ \overline{\mathcal{R}}(\alpha) \otimes_\star \overline{\mathcal{R}}(\omega) \rangle_\star\,. \tag{5.5.19}$$

With this we can straightforwardly define higher p-forms $\omega \in \wedge^p_\star T^*M$ where the star wedge is $\wedge_\star = \wedge \circ \mathcal{F}^{-1}$. Star wedging two one-forms then amounts to an $\mathcal{R}$-antisymmetrization

$$\omega \wedge_\star \alpha = \omega \otimes_\star \alpha - \overline{\mathcal{R}}(\alpha) \otimes_\star \overline{\mathcal{R}}(\omega)\,. \tag{5.5.20}$$

Alternatively one can define the star wedge by the $\mathcal{R}$-antisymmetrization of their action onto vectors due to

$$\langle A \otimes_\star B, \omega \wedge_\star \alpha \rangle_\star = \langle A \otimes_\star B, \omega \otimes_\star \alpha \rangle_\star - \langle \overline{\mathcal{R}}(B) \otimes_\star \overline{\mathcal{R}}(A), \omega \otimes_\star \alpha \rangle_\star\,. \tag{5.5.21}$$

Having a star wedge and star p-forms let us take an aside and clarify how to translate ordinary tensors into their star version. Let us look at a two-form ω when acting on two vector fields X and Y

$$\omega(X, Y) = -\omega(\overline{\mathcal{R}}(Y), \overline{\mathcal{R}}(X))\,. \tag{5.5.22}$$

We can always shift the basis forms to the left using (5.4.9) to get

$$\omega(X,Y) := \langle X \otimes_\star Y \ , \ (dx^I \otimes_\star dx^J) \star \omega_{IJ} \rangle_\star \,. \tag{5.5.23}$$

The defining property (5.5.22) then becomes

$$\omega(X,Y) + \omega(\overline{\mathcal{R}}(Y), \overline{\mathcal{R}}(X)) = (X^I \star Y^J + F^{IMN} X_M \star \partial_N Y^J) \star (\omega_{IJ} + \omega_{JI}) \,. \tag{5.5.24}$$

Therefore usual non-star antisymmetric tensors give rise to $\mathcal{R}$ antisymmetric twisted p-forms. By the same computation also usual non-star symmetric tensors can be translated into $\mathcal{R}$-symmetric twisted tensors. But this does not mean that a transition from non-star tensors to twisted tensors is trivial. Looking at (5.5.16) one sees that writing ω with the basis forms to the left as in (5.5.23) yields internal corrections in ω. Of course one can see (5.5.23) as fundamental without corrections but then the transformation law governed by (5.5.14) becomes very complicated. Furthermore as discussed in 5.3.3 the star product has to be used in the explicit expression of the tensor. As such a direct translation of ordinary higher tensors to their star version is not as non-trivial as one might think. Here we do not see a way to fix the ambiguities and therefore we will proceed with fully general tensors.

Lastly we note that the exterior derivative $d = \partial \wedge_\star$ commutes with the star product

$$d(\omega \otimes_\star \alpha) = d\omega \otimes_\star \alpha + \omega \otimes_\star d\alpha \,. \tag{5.5.25}$$

5.6 Differential geometry

In the following section we will develop consistent definitions for the most important objects of general relativity, the covariant derivative, the torsion and the Riemann tensor.

5.6.1 Covariant derivative

Having a consistent notion of twisted tensors let us go on with connections and covariant derivatives. We start with computing the anomalous variation $\Delta^\star_\xi := \delta^\star_\xi - \mathcal{L}^\star_\xi$ of the derivative of a covector $\partial\omega$. For the definition of the one-form ∂ recall (5.4.21). Using the star commutativity of ∂ one finds

$$\Delta^\star_\xi \, \partial\omega = \partial\partial\xi^J \star \omega_J \,. \tag{5.6.1}$$

We introduce a connection Γ to compensate for the anomalous transformation of $\Delta^\star_\xi \, \partial\omega$. By commuting the basis vectors to the left and right on cost of correction terms in Γ we can always write Γ in the form

$$\Gamma = dx^I \star dx^J \star \Gamma_{IJ}{}^K \star \partial_K \,. \tag{5.6.2}$$

The covariant derivative is $\nabla = \partial - \Gamma$ where the Γ acts with a contraction. Notice that we have a choice whether ∇ acts from the left or right onto the form ω. Since ∂ commutes with the star product this equals the choice from which side Γ is multiplied on. The two choices are denoted as

$$\begin{aligned} &\overrightarrow{\nabla}\omega = \nabla(\omega) = \partial\omega - \Gamma \star \omega \\ \text{and} \qquad &\overleftarrow{\nabla}\omega = (\omega)\nabla = \partial\omega - \omega \star \Gamma\,. \end{aligned} \tag{5.6.3}$$

Recalling the convention that all basis forms are on the left one realizes that the first definition $\overrightarrow{\nabla}$ contains the especially simple term $\Gamma^K \star \omega_K$ while the second form $\overleftarrow{\nabla}$ gives additional terms from the $\mathcal{R}$-matrix. Nevertheless both definitions are fine and cancel out the anomalous variation (5.6.1) if $\Delta^\star_\xi \Gamma = \partial\partial\xi$

$$\begin{aligned} \Delta^\star_\xi(\Gamma \star \omega) &= \Delta^\star_\xi \Gamma \star \omega = \partial\partial\xi \star \omega\,, \\ \Delta_\xi(\omega \star \Gamma) &= \overline{\mathcal{R}}(\omega) \star \Delta_{\overline{\mathcal{R}}(\xi)}\Gamma = \overline{\mathcal{R}}(\omega) \star \partial\partial\overline{\mathcal{R}}(\xi) \\ &= \overline{\mathcal{R}}(\omega) \star \overline{\mathcal{R}}(\partial\partial\xi) = \partial\partial\xi \star \omega\,. \end{aligned} \tag{5.6.4}$$

Both definitions can also be seen as consequence of the usual axiomatic definition. One defines the covariant derivative as a map $T^*M \to T^*M \otimes_\star T^*M$ obeying $\nabla f = \partial f$ for scalars and the Leibniz rule. We deduce

$$\begin{aligned} \overrightarrow{\nabla}\omega &= \nabla(dx^I \star \omega_I) = \nabla(dx^I) \star \omega_I|_{\overline{\phi}} + \overline{\mathcal{R}}(dx^I) \star \overline{\mathcal{R}}(\partial)\omega_I \\ &= \partial\omega - \Gamma \star \omega\,, \end{aligned} \tag{5.6.5}$$

and analogously for $\overleftarrow{\nabla}$

$$\begin{aligned} \overleftarrow{\nabla}\omega &= (dx^I \star \omega_I)\nabla = (dx^I)\overline{\mathcal{R}}(\nabla) \star \overline{\mathcal{R}}(\omega_I) + dx^I \star \partial\omega_I \\ &= \partial\omega - \omega \star \Gamma\,. \end{aligned} \tag{5.6.6}$$

The same procedure can be repeated for vectors. We find

$$\Delta^\star_\xi\, \partial V = -\overline{\mathcal{R}}(V) \star \partial\partial\overline{\mathcal{R}}(\xi) \; = -\partial\partial\xi \star V. \tag{5.6.7}$$

Again this anomalous term can be compensated by a covariant derivative. Being non-commutative we again encounter two consistent covariant derivatives

$$\begin{aligned} &\overrightarrow{\nabla}V = \nabla(V) = \partial V + \Gamma \star V \\ \text{and} \qquad &\overleftarrow{\nabla}V = (V)\nabla = \partial V + V \star \Gamma\,. \end{aligned} \tag{5.6.8}$$

For vectors $\overleftarrow{\nabla}$ is especially simple giving $V^I \star \Gamma_I$. As such using $\overrightarrow{\nabla}$ for covectors, and $\overleftarrow{\nabla}$ for vectors seems to be the most convenient choice. This convention is furthermore justified by

$$\begin{aligned} \overleftarrow{\nabla}V \star \omega|_{\overline{\phi}} + V \star \overrightarrow{\nabla}\omega|_{\phi^2} &= \partial(V \star \omega) + (V \star \Gamma) \star \omega|_\phi - V \star (\Gamma \star \omega)|_{\phi^2} \\ &= \partial(V \star \omega)\,, \end{aligned} \tag{5.6.9}$$

showing that here the covariant derivative, similar to ∂, acts without $\mathcal{R}$-matrices. Using, for instance, only $\overrightarrow{\nabla}$ gives a covariant derivative that is also compatible with ∂, but now $\mathcal{R}$-matrices appear explicitly

$$\begin{aligned}\overrightarrow{\nabla}(V \star \omega) &= \overrightarrow{\nabla} V \star \omega|_{\overline{\phi}} + \overline{\mathcal{R}}(V) \star (\overline{\mathcal{R}}(\overrightarrow{\nabla})\omega)|_{\phi^2} = \partial(V \star \omega)\,,\\ \overleftarrow{\nabla}(V \star \omega) &= V \star \omega \overleftarrow{\nabla}|_{\phi} + V\overline{\mathcal{R}}(\overleftarrow{\nabla}) \star \overline{\mathcal{R}}(\omega)|_{\overline{\phi}^2} = \partial(V \star \omega)\,.\end{aligned} \tag{5.6.10}$$

When defining the directional covariant derivative we face another choice of convention. Taking the covariant derivative of a vector $Y = Y^I \star \partial_I$ into the direction of $X = X^I \star \partial_I$ we can multiply X from the left or the right onto ∇Y. Together with the choices $\overleftarrow{\nabla}$ and $\overrightarrow{\nabla}$ we have in total four possible conventions. As we will see when defining the torsion tensor we should rather use a convention where X and Y are placed together and, additionally, have a bracket around them. This leaves two choices of which we take

$$\begin{aligned}\nabla_X Y := \overleftarrow{\nabla}_{\overrightarrow{X}} Y :&= X \star \overleftarrow{\nabla} Y|_{\phi} = X \star \partial Y + (X \otimes_\star Y) \star \Gamma\\ &= X \star \partial Y + \langle X \otimes_\star Y\ ,\ \Gamma\rangle_\star\,.\end{aligned} \tag{5.6.11}$$

While not saying it explicitly, this convention was also used in non-commutative but associative gravity [142]. In components we find

$$\begin{aligned}\nabla_X Y = X^I \star \partial_I Y^J \star \partial_J &+ (X^I \star Y^J) \star \Gamma_{IJ}{}^K \star \partial_K\\ &- F^{MNJ}(X_M \star \partial_N Y^I) \star \Gamma_{IJ}{}^K \star \partial_K\,.\end{aligned} \tag{5.6.12}$$

For later reference let us note

$$\begin{aligned}\nabla_{\overline{\mathcal{R}}(Y)} \overline{\mathcal{R}}(X) = \overline{\mathcal{R}}(Y)^I \star \partial_I \overline{\mathcal{R}}(X)^J \star \partial_J &+ (X^J \star Y^I) \star \Gamma_{IJ}{}^K \star \partial_K\\ &- F^{MNI}(X_M \star \partial_N Y^J) \star \Gamma_{IJ}{}^K \star \partial_K\,.\end{aligned} \tag{5.6.13}$$

5.6.2 Torsion

By defining a torsion let us fill a crucial gap in the literature. The torsion can be defined as the antisymmetrized covariant derivative $\nabla\wedge$ of the frame dx^I. Being in a holonomic frame we have

$$T^K = \overrightarrow{\nabla}^{\wedge_\star} dx^K = \Gamma^K = (dx^I \wedge_\star dx^J) \star \Gamma_{IJ}{}^K\,, \tag{5.6.14}$$

leading to the familiar constraint

$$\Gamma_{[IJ]}{}^K = 0\,. \tag{5.6.15}$$

Probably the definition $T = \nabla_X Y - \nabla_Y X - [X, Y]$, which uses only vectors is more familiar. To guarantee that all definitions for the torsion match we use the T^K of (5.6.14) and turn the wedge into an antisymmetrization of the arguments according to (5.5.19)

$$\begin{aligned}T(X, Y) :&= \langle X \otimes_\star Y\ ,\ T^K \star \partial_K\rangle_\star\\ &= \langle X \otimes_\star Y\ ,\ dx^I \wedge_\star dx^J \star \Gamma_{IJ}{}^K \star \partial_K\rangle_\star\\ &= \langle X \otimes_\star Y - \overline{\mathcal{R}}(Y) \otimes_\star \overline{\mathcal{R}}(X)\ ,\ dx^I \otimes_\star dx^J \star \Gamma_{IJ}{}^K \star \partial_K\rangle_\star\,.\end{aligned} \tag{5.6.16}$$

Adding and subtracting $X\star\partial Y = X(Y)$ we recognize the covariant derivative (5.6.11) and an intuitive star version of the torsion

$$T(X,Y) = \nabla_X Y - \nabla_{\overline{\mathcal{R}}(Y)}\overline{\mathcal{R}}(X) - [X,Y]_\star\,. \tag{5.6.17}$$

Notice that we needed a convention where X and Y are placed together. If they were separated like $X\star\Gamma\star Y$ the appearance of $\mathcal{R}(\Gamma)$ would have spoiled the result from (5.6.16). In components we find

$$T(X,Y) = (X^I\star Y^J + F^{IMN}X_M\star\partial_N Y^J)\star(\Gamma_{IJ}-\Gamma_{JI})\,. \tag{5.6.18}$$

For basis vectors we reproduce the usual constraint

$$\langle T(\partial_I,\partial_J), dx^K\rangle_\star = \Gamma_{IJ}{}^K - \Gamma_{JI}{}^K\,. \tag{5.6.19}$$

5.6.3 Riemann and Ricci tensor

When interpreting the connection $\Gamma = dx^K\otimes_\star dx^I\star\Gamma_{KI}{}^L\star\partial_L$ as a matrix-valued one-form $\Gamma_K{}^L := dx^I\star\Gamma_{KI}{}^L$ the Riemann tensor can be defined as its exterior covariant derivative

$$R_K{}^L = \nabla^{\wedge_\star}\Gamma_K{}^L = d\Gamma_K{}^L - \Gamma_K{}^P\wedge_\star\Gamma_P{}^L\,. \tag{5.6.20}$$

Tensoriality is most easily verified in the form

$$R = dx^K\star R_K{}^L\star\partial_L = d\Gamma - \Gamma\wedge_\star\Gamma\,, \tag{5.6.21}$$

when using the anomalous transformation $\Delta^\star_\xi\Gamma = \partial\partial\xi$, nilpotency of d in $d\partial\xi = dd\xi = 0$ and the $\mathcal{R}$-antisymmetry of $\wedge_\star$. This definition can also be found in [113] but a connection to a vector formalism was not established. To achieve this we proceed as in the case of the torsion and contract R with vectors

$$\begin{aligned} R(X,Y,Z) :&= \langle (X\otimes_\star Y)\otimes_\star Z\ ,\ R\rangle_\star \\ &= \langle (X\otimes_\star Y)^\phi\ ,\ (Z^{K\phi}\star R_K{}^{L\phi}\star\partial_L)\rangle_\star\,. \end{aligned} \tag{5.6.22}$$

Similar to the torsion, one expects this to equal the common definition

$$R(X,Y,Z) = -\Big(\big(\nabla_X\bullet\nabla_Y\big)Z - \big(\nabla_{\overline{\mathcal{R}}(Y)}\bullet\nabla_{\overline{\mathcal{R}}(X)}\big)Z - \nabla_{[X,Y]_\star}Z\Big)\,, \tag{5.6.23}$$

where a minus was introduced to match both definitions and where we used the bullet composition from (5.3.23). The evaluation of (5.6.23) necessitates a careful treatment of the order of operations. Indeed already in $\nabla_X Y$ two consecutive operations have appeared. First, we have to act with ∇ from the right and afterwards we contract the result with X from the left. We will call the contraction in the following i_X. Furthermore the brackets enclose X and Y. In total

$$\begin{aligned} \nabla_X Y :&= \big(i_X\,(Y)\big)\overleftarrow{\nabla} = i^\phi_X(Y^\phi\overleftarrow{\nabla}^\phi) \\ &= i_X\partial Y + i^\phi_X(Y^\phi\star\Gamma^\phi) = X\star\partial Y + (X\otimes_\star Y)\star\Gamma\,. \end{aligned} \tag{5.6.24}$$

The composition in (5.6.23) should actually read

$$(\nabla_X \bullet \nabla_Y)Z = \big[(i_X \bullet i_Y)(Z)\big]\ (\overleftarrow{\nabla} \bullet \overleftarrow{\nabla})\,. \tag{5.6.25}$$

Using this, we showed in appendix C that the definitions (5.6.23) and (5.6.22) indeed match. This relies heavily on the definition $\nabla_X Y := \overleftarrow{\nabla}_{\overrightarrow{X}}$ for vectors and the other way round for forms. Other conventions will produce different results which only for vectors $X, Y, Z \sim \partial_I$ give the same result.

How the Riemann acts on components can be read off when putting

$$\begin{aligned} R = {} & dx^K \otimes_\star dx^I \wedge_\star dx^J \\ & \star \left[\partial_I \Gamma_{KJ}{}^L - \Gamma_{KI}{}^P \star \Gamma_{PJ}{}^L - F_I{}^{AB} \partial_A \Gamma_{KJ}{}^P \star \Gamma_{PB}{}^L\right] \star \partial_L \end{aligned} \tag{5.6.26}$$

and

$$\begin{aligned} (X \otimes_\star Y) \otimes_\star Z = \Big[& (X^J \star Y^I) \star Z^K \\ & - F^{ABJ} X^\phi_A \star \partial_B (Y^{I\phi} \star Z^{K\phi}) - F^{ABI} \star (X^J \star Y_A) \star \partial_B Z^K \\ & - F^{ABJ} F^{CDI} X^\phi_A \star \partial_B (Y^\phi_C \star \partial_D Z^{K\phi}) \Big] \star \partial_J \otimes_\star \partial_I \otimes_\star \partial_K\,. \end{aligned} \tag{5.6.27}$$

together in (5.6.23). As one can see from

$$\begin{aligned} R_{IJK}{}^L := & \langle R(\partial_I, \partial_J, \partial_K), dx^L \rangle_\star \\ = & 2\, \partial_{[\underline{I}} \Gamma_{K\underline{J}]}{}^L - 2\, \Gamma_{K[\underline{I}}{}^M \star \Gamma_{M\underline{J}]}{}^L - 2\, F_{[\underline{I}}{}^{AB}\, \partial_A \Gamma_{K\underline{J}]}{}^M \star \Gamma_{MB}{}^L \end{aligned} \tag{5.6.28}$$

the components of the star curvature contain a correction term even if internal corrections in the metric and the connection are absent. Further corrections appear in the Bianchi identities. While for torsion free connections the first Bianchi identity holds

$$R_{IJK}{}^L + R_{KIJ}{}^L + R_{JKI}{}^L = 0 \tag{5.6.29}$$

the second Bianchi identity has a correction proportional to the associator

$$\begin{aligned} & \frac{1}{2} \nabla_{[\underline{I}} R_{\underline{JK}]M}{}^N = \\ & \quad \Big[\big(\Gamma_{[\underline{I}M}{}^A \star \Gamma_{\underline{J}A}{}^B\big) \star \Gamma_{\underline{K}]B}{}^N - \Gamma_{[\underline{I}M}{}^A \star \big(\Gamma_{\underline{J}A}{}^B \star \Gamma_{\underline{K}]B}{}^N\big) \Big] + O(F)\,. \end{aligned} \tag{5.6.30}$$

A further evidence for the convention $\overrightarrow{\nabla}$ for covectors and $\overleftarrow{\nabla}$ for vectors appears in the computation of the second Bianchi identity. Only for the particular choice, the terms $\sim \partial\Gamma \star \Gamma$ and $\sim \Gamma \star \partial\Gamma$ cancel each other out. Having a Riemann tensor the Ricci tensor is

$$\mathrm{Ric}(Y, Z) := \langle R(\partial_I, Y, Z), dx^I \rangle_\star\,. \tag{5.6.31}$$

5.7 Scalar products and the metric

Since both the definition of a Levi-Civita connection and a Ricci scalar need a metric let us define a star metric as an $\mathcal{R}$-symmetric element in $T^*M \otimes_\star T^*M$

$$g(X,Y) = g(\overline{\mathcal{R}}(Y), \overline{\mathcal{R}}(X))\,, \tag{5.7.1}$$

which we write as

$$g(X,Y) := \langle X \otimes_\star Y\ ,\ (dx^I \otimes_\star dx^J) \star g_{IJ}\rangle_\star\,. \tag{5.7.2}$$

The symmetry translates in components in

$$\begin{aligned} 0 &= g(X,Y) - g(\overline{\mathcal{R}}(Y), \overline{\mathcal{R}}(X)) \\ &= (X^I \star Y^J + F^{IMN} X_M \star \partial_N Y^J) \star (g_{IJ} - g_{JI}) \end{aligned} \tag{5.7.3}$$

such that g_{IJ} must indeed be symmetric.

5.7.1 Scalar product

Having a metric we can define a scalar product. Since we have three objects, the metric and two vectors, there are several inequivalent choices how to define the star scalar product. Let us explain the two most convenient choices where the vectors are placed together. However, we find that both have a major disadvantage.

$\mathcal{R}$-symmetric scalar product

Let us define an $\mathcal{R}$-symmetric scalar product by

$$(V,W)_\star := \langle (V \otimes_\star W), g\rangle_\star\,. \tag{5.7.4}$$

Using (5.5.19) the $\mathcal{R}$-symmetry of g translates into the $\mathcal{R}$-symmetry of the vectors

$$\langle (V \otimes_\star W), g\rangle_\star = \langle \overline{\mathcal{R}}(W) \otimes_\star \overline{\mathcal{R}}(V), g\rangle_\star\,. \tag{5.7.5}$$

For $v = v \star \partial_i$ and $w = w \star \partial_i$ the scalar product reads

$$(v,w)^g_\star := (v^i \star w^j) \star g_{ij}\,. \tag{5.7.6}$$

The metric as a star duality map

Recall that the metric can also be regarded as the duality map $TM \to T^*M$ acting through $g(W) = \langle W, g\rangle_\star \in T^*M$. This element of T^*M can then be contracted with another vector defining a star scalar product $\langle V, g(W)\rangle_\star$. For $v = v \star \partial_i$ and $w = w \star \partial_i$ the scalar product reads

$$\langle v, g(w)\rangle_\star := v^i \star (w^j \star g_{ij})\,. \tag{5.7.7}$$

Clearly this scalar product corresponds to the second choice of bracketing and is related to (5.7.6) by an associator. The bracketing (5.7.7) has the disadvantage that it is not $\mathcal{R}$-symmetric in contrast to (5.7.6). In turn, the $\mathcal{R}$-symmetric (5.7.6) has the disadvantage that there is no way to find a dual vector which induces the scalar product.

5.7.2 The inverse metric and the Levi-Civita connection

To truly speak of a star duality map between dual vector spaces TM and T^*M makes only sense if there is an inverse such that $g^{-1}(g(V)) = V$. Only then TM and T^*M are connected bijectively. If there is no g^{-1}, there is, for instance, no appropriate scalar product on T^*M but only on TM. But even if we renounce on a duality we need an inverse if we want to find a Levi-Civita connection $\nabla g = 0$. Assuming $\Gamma_{[IJ]}{}^K = 0$ we arrive at

$$\begin{aligned}(dx^I \otimes_\star & dx^J \otimes_\star dx^K) \\ &\star \left[\partial_I g_{JK} + \partial_J g_{IK} - \partial_K g_{IJ}\right] = (dx^I \otimes_\star dx^J) \star 2\,\Gamma_{IJ}{}^L \star dx^K \star g_{LK} \\ &\qquad\qquad\qquad\qquad\qquad\quad = 2\,\Gamma \star g\,,\end{aligned} \tag{5.7.8}$$

To express Γ in terms of the metric one needs to solve an equation $G^{-1}(G(V)) = V$. This is the reason for defining the inverse by $G^{-1}(G(V)) = V$.

Let us start with an easy example $v = v^i \star \partial_i$. The inverse must fulfill

$$\left(v^k \star g_{kj}\right) \star (g^{-1\star})^{ji} = v^i\,. \tag{5.7.9}$$

Let us stress that this definition of an inverse deviates from the usual definition

$$g_{ij} \star (g^{-1\star})^{jk} = \delta_i{}^k \tag{5.7.10}$$

and it is very important to realize that due to the appearing associator an inverse satisfying (5.7.10) does not satisfy (5.7.9). This is only true in the associative case where an inverse was constructed in [142].

Now take the definition (5.7.9) and expand it in derivatives of v^k

$$\begin{aligned}\left(v^k \star g_{kj}\right) \star (g^{-1\star})^{ji} &= \left(v^k \cdot g_{kj}\right) \star (g^{-1\star})^{ji} + \partial_I v^k \ldots \\ &= v^k \cdot \left(g_{kj} \star (g^{-1\star})^{ji}\right) + \partial_I v^k \ldots\end{aligned} \tag{5.7.11}$$

Since this has to equal v^i, we conclude that an inverse satisfying (5.7.9) also has to satisfy (5.7.10). This is a contradiction unless the associator vanishes $\phi(\cdot, g_{kj}, (g^{-1})^{ji}) = 1$. Star products which satisfy this property are called alternative. But as one can see in (5.3.7) the R-flux star product is in general not alternative. For arbitrary functions of x and p

this is spoiled at linear order in the R-flux while for functions of only x this is spoiled at quadratic order in the R-flux. We conclude that in general there is no inverse for the R-flux star product. However looking for instance at scalars we have already identified a class of scalars for which $\phi(f, f^{\star-1}, \cdot) = 1$ namely the exponentials $f(x) = \exp(ikx)$ and therefore all functions with a Fourier transform. Their $\star$-inverse is $f^{\star-1} = \exp(-ikx)$.

The above arguments can be made more concrete when working in linear order in the R-flux and assuming $g = g(x)$. Due to (5.7.11) we want to find solutions to

$$g_{ij} \star g_R^{\star-1jk} = \delta_i^k + \mathcal{O}(R^2)\,, \qquad g_L^{\star-1ij} \star g_{jk} = \delta_i^k + \mathcal{O}(R^2) \tag{5.7.12}$$

Due to the non-commutativity it is a priory not guaranteed that the left and right inverse coincide and indeed they do not. Up to linear order the solutions are

$$\begin{aligned} g_R^{\star-1ij} &= g^{ij} - \tfrac{il_s^4}{6\hbar} R^{abc}\, p_c\, g^{im}\, \partial_a g_{mn}\, \partial_b g^{nj} + \tfrac{l_s^4}{12} R^{abc}\, \partial_a g^{im}\, \partial_b g_{mn}\, \partial_c g^{nj}\,, \\ g_L^{\star-1ij} &= g^{ij} - \tfrac{il_s^4}{6\hbar} R^{abc}\, p_c\, \partial_a g^{im}\, \partial_b g_{mn}\, g^{nj} - \tfrac{l_s^4}{12} R^{abc}\, \partial_a g^{im}\, \partial_b g_{mn}\, \partial_c g^{nj}\,. \end{aligned} \tag{5.7.13}$$

It is instructive to write them in the form

$$\begin{aligned} g_R^{\star-1ij} &= 2g^{ij} - g^{im} \star (g_{mn} \star g^{nj})\,, \\ g_L^{\star-1ij} &= 2g^{ij} - (g^{im} \star g_{mn}) \star g^{nj}\,. \end{aligned} \tag{5.7.14}$$

One can observe that they differ by an associator such that in the associative case both coincide. When neglecting the brackets the expressions coincide with the inverse in presence of an associative star product [143]. Just like there the inverse is not symmetric anymore while the symmetric part is given by the usual inverse. Furthermore note that one can start with a momentum independent g_{ij} but ends up with a momentum dependent inverse. This shows that we have to work with the whole phase space and a restriction to fields that depend only on the configuration space is not justified from the beginning. Also the two inverses we found do not satisfy the defining equation (5.7.9) of an inverse, e.g.

$$\big(v^k \star g_{kj}\big) \star (g_R^{-1\star})^{ji} = v^i + \tfrac{l_s^4}{6} R^{abc}\, \partial_a v^k\, \partial_b g_{kj}\, \partial_c g^{ji} + \mathcal{O}(R^2)\,, \tag{5.7.15}$$

Without inverse we see no convincing way to define a Levi-Civita connection and it is not clear how to contract the Ricci tensor into a Ricci scalar to write down an action. Furthermore we do not see a way how the duality between TM and T^*M can be made consistent.[15]

[15] In a private communication Richard Szabo and Paolo Aschieri told us that they found a way to write down an operator inverse. While this allows them to obtain Levi-Civita connection this solution has several drawbacks: The no-go theorems we proved here still hold and the inverse as individual object does not exist, only an inverse of the operators $f\star$ or $\star f$. Therefore one cannot define a Ricci scalar and a action and nor does there exist a scalar product on T^*M, therefore the $TM \leftrightarrow T^*M$ duality is manifestly broken.

5.8 A glimpse at a star Einstein-Hilbert action

Let us be agnostic about the problems we cannot give a definite answer to, and still try to gain insights into how the star product might be reconciled with the diffeomorphism symmetry of string theory. In the following we collect possible R-flux corrections in the action and see whether there are mechanisms that make them vanish. We will work in linear order in R to have at least something like an inverse.

A natural candidate for an action is

$$S = \int d^d x\, d^d p\, \sqrt{g} \star \mathrm{Ric}\,. \tag{5.8.1}$$

Some comments are in place:

- We embed the usual spacetime metric into the full metric via

$$g = dx^i \otimes_\star dx^j \star g_{ij}(x) + \tfrac{dp_i}{i\hbar} \otimes_\star \tfrac{dp_i}{i\hbar} \star \eta^{ij}\,. \tag{5.8.2}$$

  We see the metric and $\sqrt{g}$ are elementary objects which depend only on the coordinates. Therefore we neglect the possibility of internal corrections. This is reasonable: If we find problems for the easiest case there will be problems in complicated cases, too. For $g = g(x)$ the correction term in (5.6.28) vanishes in the Ricci scalar.

- The action is twisted diffeomorphism invariant if the measure $\mu = \sqrt{g}$ transforms as twisted scalar density $\delta_\xi \mu = \xi^I \star \partial_I \mu + \partial_I \xi^I \star \mu$

- The momentum enters only through the star product and therefore only linearly. So we naturally restricted to $p = 0$ due to $\int dp\, p = 0$.

Let us analyze the corrections coming from the star product. The first class of terms in the action has the form

$$S_1 = \int d^d x\, \sqrt{g} \star g^{-1} \star \partial\Gamma = \int d^d x\, \sqrt{g} \star g^{-1} \star \partial(\partial g \star g^{-1})\,. \tag{5.8.3}$$

Right now we do neither want to fix a bracketing, nor the order of the terms and whether we use the left or right inverse. Instead we want to examine whether there is a choice such that the non-associativity indeed only gives boundary terms as suggested in [108]. For sure, terms coming from the star product will produce only boundary terms but we have spotted other sources of R-flux corrections like the linear correction in the inverse metric (5.7.13). Since the corrections in the inverse metric are purely antisymmetric, the corrections can be trivially absent if they are coupled to symmetric indices. A small computation shows that there is another possibility

$$\begin{aligned}\int d^d x \left(g_L^{*-1}\right)^{[ij]} \psi_{ij} &= \int d^d x\, R^{abc}\, \partial_a g^{im}\, \partial_b g_{mn}\, \partial_c (g^{nj} \psi_{ij}) \\ &= \int d^d x\, \partial_c \Big(R^{abc}\, \partial_a g^{im}\, \partial_b g_{mn}\, g^{nj}\, \psi_{ij} \Big)\,.\end{aligned} \tag{5.8.4}$$

Therefore the corrections from the left-inverse of the metric give boundary terms if they the left-inverse is located completely to the left. Similarly the right-inverse gives boundary terms when placed to the right. As such for the first class of terms we can find an order where the corrections linear in R give only boundary terms. The second class of terms in the action has the schematic form

$$S_2 = \int d^d x \sqrt{g} \star g^{-1} \star \Gamma \star \Gamma = \int d^d x \sqrt{g} \star g^{-1} \star \partial g \star g^{-1} \star \partial g \star g^{-1} \,. \tag{5.8.5}$$

Having three inverse metrics, not all of them can be placed completely to the right or the left. Furthermore the index structure is such, that the inverse metric is not always contracted with symmetric indices. As such the correction from (5.7.13) does not vanish. For the second class of terms there is therefore no obvious reason why the corrections linear in R should vanish or give boundary terms.

5.9 Conclusion

Let us compare the results with the project goals. We have first succeeded in translating the mathematical work of [112, 113] into an accessible language and secondly we filled the existing gaps of the literature. We have come to a point where we do not see a way to overcome the problems in the construction of a non-associative star gravity. All problems are connected to the introduction of a metric. When proceeding we could not find a way to reconcile the star product and its twisted diffeomorphisms with string theory, for instance by producing only boundary terms.

One possible conclusion one can draw is that it is not possible to construct a gravity with non-associative twisted diffeomorphisms as symmetry. Since gravity is present in all string theories, this would mean that non-associative twisted diffeomorphism cannot appear in string theory. Of course it could be the case that we miss a solution to our problems.

This conclusion would complete the critique of [139] where they came to a similar conclusion in case of the open string star product. We want to stress that this does not rule out the existence of the star product in the gravity sector but instead it reveals something about the twisted symmetry we used. It supports the arguments of section 5.2.3 that twisted symmetries are not physical and cannot play a role in string theory. Of course there might be a completely different solution to reconcile gravity with a star product. What this looks like is completely unclear, especially since the twisted Leibniz rule appears rather naturally for a non-associative star product, see (5.2.37). But in light of the problems that the action (5.8.1) had it is likely that the star product does not simply appear between the vertex operators.

The most promising way to tackle this problem is, in our opinion, a conformal field theory computation in which one constructs the graviton vertex operator to derive its correlators. Only then one can answer if and, if yes, how the star product enters the gravity sector of string theory. Being a free theory the most promising CFT to look at is a freely acting asymmetric torodial orbifold.

Appendix A

Details about the computer program

In this appendix we will discuss details about the computer program that performed main parts of the computations that lead to the classification of asymmetric Gepner models in four, six and eight dimensions with at least eight supercharges in chapter 4. Since the stochastic search demands a high performance program we did not implement the program in mathematica but rather in Java. Instead of stating the code we put an emphasize on the structure and other major issues that had to be dealt with.

A.1 Object oriented structure of the program

Let us outline the structure of the Java program by describing the different classes we used. As Java is an object oriented programming language, this section requires basic knowledge of object oriented programming and, in particular, we will use its specific terminology of classes, objects, constructors and inheritance. To make this section accessible to those who are not familiar with these terms, we start out with a short introduction into object oriented programming before we outline the structure of the program.

A.1.1 A lightning review of object oriented programming

Object oriented programming is characterized by defining *objects* that are abstract models of some actor in the program. An object stores the variables that define the state of the actor and provides functions to deal with the variables.

Typically the variables are hidden from the rest of the program and only accessible through functions that serve as interface to communicate with the object. In this way the object manages the information it carries independently from the rest of the program while the remaining program is fully blind to the actual implementation. This justifies the notion that an object is an independent actor. Having independent actors and well defined interfaces is in particular useful for bigger projects since it structures the program into many independent modules that can be taken care of by different groups.

The programming code to define an object is written into a document called *class*. One can create as many objects of one class as needed. When a new object is initialized there are certain functions in the class that must be called. They are called *constructors* and usually ask for the variables that are needed to characterize the object.

Still there are often functions that are independent of the objects, like functions that implement certain mathematical operations. These functions are collected in *static* functions. They can be called even without initializing an object and can be used to program in the usual procedural programming style if desired.

A frequently used concept in object oriented programming is *inheritance*. Let us explain this using an example from the program. A state in a Gepner will form an object of the class *GepnerState*. This object carries the quantum numbers of the state and has functions to compute the conformal dimension, charge and so on. While a simple current is a state in a Gepner model as well, the simple current object should also provide us with additional functions, for instance, to compute orbits. As such the class *SimpleCurrent* inherits from the class *GepnerState*. This means that *SimpleCurrent* inherits all the functionality including variables and functions from the *GepnerState* class, but, on top of that, we can add functions specific to simple currents. Since an object of the *SimpleCurrent* class is an enhanced *GepnerState* object we can treat an object of the *SimpleCurrent* also as an *GepnerState* object and hand it over to functions that ask for a *GepnerState* object. This is called *polymorphism*.

A.1.2 General structure

Considering the size of the program a project oriented programming style might seem overpowered. A closer look, however, reveals that Gepner models have a structure that naturally translates into an object oriented program structure. States in a minimal model, a state in a Gepner model, simple currents and a full Gepner model are exactly the independent actors that are supposed to be implemented using objects. Therefore an object oriented programming style was chosen for the program. It turned out that, although being a small program, the modularized structure was very useful when it came to testing, modifying and extending the program.

Let us shortly summarize the structure by stating the purpose of each class. The way the names of classes and functions are written follows the usual nomenclature of Java programs.

- *MinimalPrimary* represents a primary of a minimal model.
- *GepnerState* represents a state of a Gepner model by collecting the tensored minimal models and the $\widehat{\mathfrak{so}}(D-2)_1$ label into a single object.
- *SimpleCurrent* inherits from *GepnerState* and provides the additional functionality of a simple current.
- *ExceptionalInvariant* inherits from *SimpleCurrent* to realize exceptional invariants in a unified way.

- *GepnerModel* collects the data of a full Gepner model, thus it stores the levels and the simple currents. Furthermore it provides functions to compute the spectrum.

After this overview we will now turn to the internal structure of each class by stating their most important variables and functions.

A.1.3 MinimalPrimary

The smallest independent actor is a primary of a single minimal model. Such a primary is represented by objects of the class *MinimalPrimary*. It stores the quantum numbers of a primary $(l\ m\ s)$ and the level k into integer variables. Due to performance issues the conformal dimension is stored in a variable instead of being computing on demand.

The constructors ask for $(l\ m\ s)$ and the level k. To compute the conformal dimension the triplet $(l\ m\ s)$ is first brought into the standard range (2.5.28) using the modulo relations (2.5.22) and the $\mathbb{Z}_2$ identifications (2.5.23). Then the conformal dimension is computed using (2.5.26) and all variables are stored in the standard range. The main functions in the class *MinimalPrimary* return the conformal dimension, the charge, the level and the triplet $(l\ m\ s)$ as array of integers.

Worth mentioning are the additional static functions. One of them returns a list *MinimalPrimary* objects that contains all chiral and/or antichiral primaries according to (2.5.29). Another function returns all simple currents for a given level following (2.5.30). Yet another function implements the fusion rules (2.5.30) by taking two minimal primaries and returning the fused minimal primary. This fusion is unambiguous since the program only fuses a state and a simple current such that the result is a single state.

A.1.4 GepnerState

The next step in the hierarchy is a full character of a highest weight in a Gepner model (2.8.4) represented by an object of the class *GepnerState*. *GepnerState* collects several minimal model objects in an array and has an integer variable $s = -1, 0, 1, 2$ for the representation of $\widehat{\mathfrak{so}}(D-2)_1$. There are functions that return the conformal dimension, charge and the quantum numbers of the characters. These functions are doubled to have separate functions that return the appropriate values before and after applying the bosonic string map (2.2.3).

There is a static function that fuses two states into a third one. It uses the adding routine from the minimal primary class and adds the $\widehat{\mathfrak{so}}(D-2)_1$ state according to the dimension dependent fusion rules in (2.2.21).

There is also a function *getMonodromyCharge(SimpleCurrent)* that computes the monodromy charge of the Gepner states with respect to a simple current that must be handed over to the function. The function uses the definition (2.3.5) since the ingredients of the formula are all easy to compute for the computer.

A.1.5 StateList

The class *StateList* collects several objects *GepnerState* into one object. *StateList* inherits from the standard class *ArrayList* that provides the functions to store and manage the list of *GepnerState* objects. On top of this we added several handy functions to the *StateList* class. One of them erases all states with $h \neq 1/2$ to leave a list with only massless states. Other functions count the number of states in the list with a particular $\widehat{\mathfrak{so}}(D-2)_1$ label. These numbers are needed to count the number of multiplets.

A.1.6 SimpleCurrent

The class *SimpleCurrent* represents a simple current. Since a simple current is also a state in a Gepner model, it inherits from the class *GepnerState*. As such all functions from *GepnerState* like computing conformal dimensions etc. are inherited from *GepnerState* and also simple current objects can be collected into a *StateList*. On top of that there are variables that store the specific data of a simple current and functions that compute the partition function that is associated to a simple current.

When a simple current object is initialized, first a function is called to compute the length $\mathcal{N}_a$ of the simple current by repeated self fusion until the vacuum appears. After storing the length in a variable, another function is called to compute the monodromy parameter r^a of the simple current using (2.3.9). In case r is odd and $\mathcal{N}_a$ is even where the simple current does not lead to a modular invariant partition function, the fake values $r = -100$ and length $N = -100$ are stored to signal that this simple current has no associated modular invariant partition function.

In case the simple current is viable we implemented a function *getOrbit(GepnerState)* to compute the associated modular invariant partition function. More precisely *getOrbit(GepnerState)* computes all states the a state is coupling according to (2.3.14) and (2.3.16) and returns these states in a *StateList* object.

A.1.7 ExceptionalInvariant

Recall that the modular invariant partition functions of the minimal models follow an ADE classification (2.2.42) where the A represents the diagonal partition function and the D series can be implemented by a simple current (2.5.32). Lacking a simple current for the E invariant, the E invariant has to be implemented in another way. Let us explain how to solve this problem elegantly.

The only purpose of simple currents is to generate a modular invariant partition function. As such *getOrbit(GepnerState)* is the only function of the *SimpleCurrent* class that the remaining program calls. *getOrbit(GepnerState)* returns only the information to which states the a given states is coupling to, while it is completely unimportant how these states were computed. This consideration suggests that we write a class *ExceptionalInvariant* that inherits from the *SimpleCurrent* class and overrides only the *getOrbit(GepnerState)* function according to (2.2.42). Due to polymorphism the program can treat an object of the class *ExceptionalInvariant* on the same level as *SimpleCurrent* objects fully blind to the actual nature as exceptional invariant.

When inheriting, the compiler demands all constructors to be overridden as well. The constructors of the *ExceptionalInvariant* class only check, whether the level is appropriate to support an exceptional invariant.

Notice that this treatment of exceptional invariants would produce errors if the program called a function like *getConformalDimension()* in the *ExceptionalInvariant* class. Therefore when writing the rest of the program one has to make sure that these functions are only used indirectly through the *getOrbit()* function. Although it worked out well in this small program this is bad programming style and very error-prone in bigger projects. Actually Java provides an alternative concept called *interfaces* that could eliminate this potential source of errors. An *interface* is a list of (not yet implemented) functions. If a class inherits from an *interface* it must implement every function of this list. Here, an interface, say *PartitionFunction*, could represent a matrix in a partition function (2.2.11) and the list of functions in the interface would only consist of the *getOrbit()* function. Then, both *SimpleCurrent* and *ExceptionalInvariant* would inherit from the *PartitionFunction* interface. When computing the modular invariant partition function the program would use a list of *PartitionFunction* objects. But having a list of *PartitionFunction* objects the program can only call the *getOrbit()* and no other function even if the *SimpleCurrent* class might as well inherit from the *GepnerState* class. The advantage is that the programm can treat every object only in exactly the way it is used for and the purpose of each object is clearly separated. A disadvantage of this solution is that *PartitionFunction* objects cannot be collected into a *StateList* object. Therefore we did not use *interfaces* in our implementation.

A.1.8 GepnerModel

Finally a whole Gepner model is represented by an object of the class *GepnerModel*. The information carried by a GepnerModel object consists of the levels of the minimal models and the chosen invariants. It collects all simple currents in a *StateList* object and provides functions to get the orbits of any state with respect to all the simple currents. There are also functions that return the whole spectrum of the Gepner model.

When the constructor is called, the program first checks whether the central charge is correct. Afterwards the program creates *SimpleCurrents* objects for J_{GSO}, J_i (see (2.8.5) and (2.8.7)) and possibly for the D and E invariants and adds them to the simple current list.

Since we construct asymmetric CFTs with additional simple currents there are functions that add further simple currents to the Gepner model. After adding the asymmetric simple currents, it is usually the spectrum that is asked for in the next step. Let us explain the procedure to compute the spectrum step by step. Every step of the computation is implemented as individual function. If an intermediate step is called, the function calls all prior steps if necessary. Roughly speaking, the procedure goes from the right to the left in the partition function (3.2.1).

In the first step the function *getInstates()* is called to get a list of all states that can come from the right in (3.2.1). Notice that it is enough to create all (anti)chiral primaries in the minimal sector with $s = 0$ and total conformal dimension 1/2. The reason for this is that all other states appear in the orbits of J_i and J_{GSO}. Then the simple currents for the ADE classification and the additional asymmetric simple currents follow. The resulting usually very large *StateList* object is then reduced to the massless states with conformal dimension 1/2. Notice that the orbit of the vacuum is treated separately but of course it undergoes the same procedure.

Having acquired all states that appear on the left we still have to find out to which states they couple to on the right. Due to the placement of the simple currents we have $\mathcal{N}_R \geq 1$ such that there is always a fixed structure for the right-moving states that we collected in section 4.2. The rightmoving supersymmetry is deduced by computing the orbit of the left-moving vacuum state by following (3.2.1) from the left to the right.

Having knowledge about all massless states there are functions which implement the interpretation and output of the result. For instance of them returns the field content while another one returns the multiplet structure. There are different output functions that write the results into the console in different levels of precision. A typical output for a Gepner model with only A invariants is

```
Gepner Model (c = 9.0):
k = [1, 1, 2, 2, 2, 10]
Invariants:  [ 1, 1, 1, 1, 1, 1]

Nonstandard Simple Currents:
{{0,0,2},{0,1,1},{0,1,1},{0,−2,2},{0,2,2},{0,−11,1},S}
{{0,0,2},{0,1,1},{0,1,1},{0,−3,1},{0,2,2},{0,2,2},V}

Charged sector:  (+1,−1,0) = (2,2,4) plus 1 vacuum orbit
Vacuum sector:  (+1,−1,0) = (1,1,0) coupling to (2,2,2)
Susy:  N=3, nV=1+2
```

A more elaborate output then prints every orbit separately. The program correctly subtracts these vacuum orbits in the charged sector. Another output function writes all orbits into the console

```
Multiplicity:  1

Vacuum orbit:
{{1,2,1},{0,1,1},{2,3,1},{2,3,1},{0,1,1},{0,1,1},S} q=2.5
{{0,1,1},{1,2,1},{0,1,1},{0,1,1},{2,3,1},{10,11,1},C} q=1.5
{{0,0,0},{0,0,0},{0,0,0},{0,0,0},{0,0,0},{0,0,0},V} q=0.0

Charged orbits (q=1 & q=-1!):
```

7. `Orbit of` $\{\{0,0,0\},\{0,0,0\},\{0,0,0\},\{2,-2,0\},\{2,-2,0\},\{0,0,0\},O\}$ `q=1.0:`
$\{\{0,1,1\},\{1,2,1\},\{0,1,1\},\{0,1,1\},\{2,3,1\},\{10,11,1\},C\}$ `q=1.5`
$\{\{0,0,0\},\{0,0,0\},\{0,0,0\},\{0,0,0\},\{0,0,0\},\{0,0,0\},V\}$ `q=0.0`
$\{\{1,2,1\},\{0,1,1\},\{2,3,1\},\{2,3,1\},\{0,1,1\},\{0,1,1\},S\}$ `q=2.5`

22. `Orbit of` $\{\{0,0,0\},\{1,-1,0\},\{0,0,0\},\{0,0,0\},\{0,0,0\},\{8,-8,0\},O\}$ `q=1.0:`
$\{\{1,2,1\},\{1,2,1\},\{0,1,1\},\{0,1,1\},\{0,1,1\},\{8,9,1\},S\}$ `q=2.5`
$\{\{1,1,0\},\{0,0,0\},\{0,0,0\},\{0,0,0\},\{2,-2,0\},\{2,-2,0\},O\}$ `q=1.0`

and so on.

A.1.9 Main

The *main* class is easy to describe. It creates a *GepnerModel* object with chosen levels and invariants and hands over additional simple currents. Then it calls the choosen output functions in the *GepnerModel* class.

A.1.10 SearchForSimpleCurrent

This class is a routine to search for new simple currents that generate models with the desired properties. It requires the specification of a Gepner model, thus the levels and invariants, how many simple currents should be added simultaneously and over which minimal factors the search should run to. According to this choice the program creates the simple currents by fetching the simple currents of the selected minimal models and equipping them with any possible value for the $\widehat{\mathfrak{so}}(2)_1$ label. The result is a usually very long list of simple currents that will be used to create new models. If one chooses to add more than one simple current simultaneously, the program combines the simple currents in every possible way and computes the models one by one.

Without further modification such a search would not be too productive since the amount of models that are to be tested grows exponentially when increasing complexity. As such we decided to perform a stochastic search e.g. by using only a certain percentage of all simple currents of a single minimal model for the search. Which ones are taken is chosen randomly. Similar thresholds can be chosen at many other stages, e.g. to test just 10% of all models. To nevertheless make the stochastic search as complete as possible, we made the search more efficient by optimizing the overall performance of the program, see next section. When running the stochastic search the program prints detailed information only about so far unknown model. For already known models statistics about the frequency are collected and printed in certain intervals.

A.1.11 Additional classes

Recall that we had a series of models (3.2.11) and (3.2.12) in case we added a specific simple current in a minimal model with odd level. The corresponding Calabi-Yau manifolds had a certain structure (3.2.11) and (3.2.12). To find more examples we implemented a small routine that scans the list [38]. This search resulted in the list 3.2.

A.2 Performance and other issues

In this section we will clarify several other issues. The first one is that computer programs only have a finite precision for non-integer numbers causing rounding errors. Numbers affected by such an error change their value giving false results when testing equality with another number. To solve this problem we implemented a function that tests equality between two variables by examining how close to zero their difference is.

Let us now turn to the performance issues unavoidable in a program scanning over billions of models. Our efforts to increase the performance were very successful and eventually increased the speed by factors of thousands.

Most of the time is spent for computing the spectrum of a model using the *getOrbit()* function of the *SimpleCurrent* class. Indeed, actual tests showed that by far the most computational time is consumed by this function (in a single run approximately 95%). We optimized the *getOrbit()* routine by treating the not uncommon case $r = 0$ differently. Having $r = 0$ the argument delta function (2.3.16) does not change over the orbit of a simple current and must be computed only once. This spares us from million modulo operations giving an enormous performance boost when computing the orbits of simple currents with $r = 0$.

Next step is going through all the methods that are called by the *getOrbit()* function. We checked for situations where computations were performed redundantly or without need. Take e.g. a *MinimalModel* object. We found that the program will always ask for the conformal dimension, often more than once. In contrast, the charge is needed in only very few cases. As such the conformal dimension is computed in the constructor of the *MinimalModel* class and stored in a variable while the charge is only computed on demand. Since *getOrbit()* calls for the conformal dimensions endlessly, this had massive effects on the speed.

Another improvement was gained by optimizing the routines for frequently used mathematical operations like modulo. The standard implementations in the Java-API are optimized for wide applicability and not for performance. This is fine since there are few programs whose bottleneck is the modulo function. But in our case they are. As such we implemented functions optimized for each special situation. Let us list some of them

- Probably the most commonly used function is the modulo 1 function. According to our tests, fastest way to implement this function uses the *Math.rint(double)* function of the Java API, a function which relies on the native C interface to be especially quick. Given a double d this function yields the nearest integer a that is close to the double. If $a > m$ we subtract a by one to ensure $m > a$. The result of the modulo operation is then $d - a$.

- We often check whether an integer i is even or not. This can be implemented on bit level by the logical & operator via $((i \,\&\, 1) == 0)$. Since this operation is very easy to do for the CPU it is extremely fast.

- Actually the above operation to test whether an integer is even computes i modulo 2 using &. Similarly we can implement the modulo 4 by $i \,\&\, 3$, too. This is needed for the $\widehat{\mathfrak{so}}(D-2)_1$ labels and the s of the minimal models which are both defined modulo four. Therefore every time a Gepner state is created, e.g. by fusion, this function is called several times.

Of course were many other places not listed here where we optimized the program a tiny bit.

When performing the stochastic search we saved lot of computational time by not computing redundantly. A glance at (3.2.1) shows that for any additional simple current the orbits coming from the right through J_{GSO} and J_i are the same. To not compute them time and time again we saved these "preorbits" in a *StateList* object. Then, when an asymmetric simple current is added we can skip the first steps in (3.2.1) and directly compute the orbits with respect to the asymmetric simple current.

We implemented the possibility to add standard simple currents that are treated like J_{GSO} and J_i and whose orbit is included in the above preorbits. This was used to perform stochastic searches with enhancing simple currents as described after (4.2.2).

We observed that some classes of models appear very often, while others are rare. To close the holes in the rare series we performed stochastic searches that only sought for a specific series. In a first step we computed the vacuum sector and read off the number of supersymmetry. Only if the result agreed with the series we are looking for the model is further analyzed. Furthermore, the charged orbit is only computed if supersymmetry allowed for non-trivial charged orbits. This procedure saves us from computing the charged orbits for every model that is to be tested.

Appendix B

Supermultiplets

To deduce possible breaking patterns we list the multiplets of supergravity in several dimensions. The list is based on [158] and [87].

B.1 Supergravity in $D = 8$

Let us start with the on-shell degrees of freedom of the different particles in eight dimensions.

name		symbol	on-shell d.o.f.
massless	spin 2	$[2]$	$20_{\rm b}$
massless	spin 3/2	$[\frac{3}{2}]$	$40_{\rm f}$
massless	spin 1	$[1]$	$6_{\rm b}$
massless	spin 1/2	$[\frac{1}{2}]$	$8_{\rm f}$
massless	spin 0	$[0]$	$1_{\rm b}$
massless	p-form	$[{\rm t}_p]$	$\binom{6}{p}_{\rm b}$
massive	spin 3/2	$\overline{[\frac{3}{2}]}$	$48_{\rm f}$
massive	spin 1	$\overline{[1]}$	$7_{\rm b}$
massive	spin 1/2	$\overline{[\frac{1}{2}]}$	$8_{\rm f}$
massive	spin 0	$\overline{[0]}$	$1_{\rm b}$
massive	p-form	$\overline{[{\rm t}_p]}$	$\binom{7}{p}_{\rm b}$

(B.1.1)

The relevant multiplets are

$$\begin{aligned} \mathcal{G}_{(2)} &= \mathcal{G}_{(1)} + \mathcal{S}_{(1)} + 2\cdot\mathcal{V}_{(1)}\,, \\ \overline{\mathcal{S}}_{(1)} &= \mathcal{S}_{(1)}\,, \\ \overline{\mathcal{V}}_{(1)} &= \mathcal{V}_{(1)}\,. \end{aligned} \tag{B.1.2}$$

and listed in table B.1.

$\mathcal{N}$	spin	mass	content
2	2		$\mathcal{G}_{(2)} = 1 \cdot [2] + 2 \cdot [\frac{3}{2}] + 6 \cdot [1] + 6 \cdot [\frac{1}{2}] + 7 \cdot [0] + 1 \cdot [\mathrm{t}_3]$
			$+ 3 \cdot [\mathrm{t}_2]$
1	2		$\mathcal{G}_{(1)} = 1 \cdot [2] + 1 \cdot [\frac{3}{2}] + 2 \cdot [1] + 1 \cdot [\frac{1}{2}] + 1 \cdot [0] + 1 \cdot [\mathrm{t}_2]$
1	3/2		$\mathcal{S}_{(1)} = 1 \cdot [\frac{3}{2}] + 2 \cdot [1] + 3 \cdot [\frac{1}{2}] + 2 \cdot [0] + 2 \cdot [\mathrm{t}_2] + 1 \cdot [\mathrm{t}_3]$
1	3/2	long	$\overline{\mathcal{S}}_{(1)} = 1 \cdot \overline{[\frac{3}{2}]} + 1 \cdot \overline{[1]} + 2 \cdot \overline{[\frac{1}{2}]} + 1 \cdot \overline{[0]} + 1 \cdot \overline{[\mathrm{t}_2]} + 1 \cdot \overline{[\mathrm{t}_3]}$
1	1		$\mathcal{V}_{(1)} = 1 \cdot [1] + 1 \cdot [\frac{1}{2}] + 2 \cdot [0]$
1	1	long	$\overline{\mathcal{V}}_{(1)} = 1 \cdot \overline{[1]} + 1 \cdot \overline{[\frac{1}{2}]} + 1 \cdot \overline{[0]}$

Table B.1: Supergravity multiplets in $D = 8$. The first column shows the amount of supersymmetry, the second column indicates the maximal spin of the multiplet, the third column specifies whether the fields are massless (no indication) or massive (long).

B.2 Supergravity in $D = 6$

The on-shell degrees of freedom of the different particles in six dimensions are

name	symbol	on-shell d.o.f.
massless spin 2	$[2]$	9_b
massless spin 3/2	$[\frac{3}{2}]^\pm$	6_f
massless spin 1	$[1]$	4_b
massless spin 1/2	$[\frac{1}{2}]^\pm$	2_f
massless spin 0	$[0]$	1_b
massless two-form	$[\mathrm{t}_2]^\pm$	3_b
massive spin 3/2	$\overline{[\frac{3}{2}]}$	16_f
massive spin 1	$\overline{[1]}$	5_b
massive spin 1/2	$\overline{[\frac{1}{2}]}$	4_f
massive spin 0	$\overline{[0]}$	1_b
massive two-form	$\overline{[\mathrm{t}_2]}$	10_b

. (B.2.1)

The $\pm$ indicates the chirality of the fermionic fields, and whether the two-tensor is self- or anti-self-dual. The multiplets relevant for our discussion are summarized in table B.2. At the level of the field content, the following relations can be obtained

$$\begin{aligned}
\mathcal{G}_{(2,2)} &= \mathcal{G}_{(0,2)} + 4\cdot\mathcal{S}_{(0,2)} + 5\cdot\mathcal{T}_{(0,2)}\,,\\
\mathcal{G}_{(2,2)} &= \mathcal{G}_{(1,1)} + 2\cdot\mathcal{S}^+_{(1,1)} + 2\cdot\mathcal{S}^-_{(1,1)} + 4\cdot\mathcal{V}_{(1,1)}\,,\\
\mathcal{G}_{(0,2)} &= \mathcal{G}_{(0,1)} + 2\cdot\mathcal{S}^-_{(0,1)}\,, & \mathcal{G}_{(1,1)} &= \mathcal{G}_{(0,1)} + 2\cdot\mathcal{S}^+_{(0,1)} + \mathcal{T}_{(0,1)}\,,\\
\mathcal{S}_{(0,2)} &= \mathcal{S}^+_{(0,1)} + 2\cdot\mathcal{V}_{(0,1)}\,, & \mathcal{S}^+_{(1,1)} &= \mathcal{S}^+_{(0,1)} + 2\cdot\mathcal{T}_{(0,1)}\,,\\
\mathcal{T}_{(0,2)} &= \mathcal{T}_{(0,1)} + 2\cdot\mathcal{H}_{(0,1)}\,, & \mathcal{S}^-_{(1,1)} &= \mathcal{S}^-_{(0,1)} + 2\cdot\mathcal{V}_{(0,1)} + \mathcal{H}_{(0,1)}\,,\\
& & \mathcal{V}_{(1,1)} &= \mathcal{V}_{(0,1)} + 2\cdot\mathcal{H}_{(0,1)}\,,\\
\overline{\mathcal{S}}_{(2)} &= \mathcal{S}^+_{(1,1)} + \mathcal{S}^-_{(1,1)}\,,\\
\overline{\mathcal{V}}_{(2)} &= \mathcal{V}_{(1,1)}\,,\\
\overline{\mathcal{S}}_{(1)} &= \mathcal{S}^+_{(0,1)} + \mathcal{S}^-_{(0,1)} + 2\cdot\mathcal{V}_{(0,1)} + 2\cdot\mathcal{T}_{(0,1)} + \mathcal{H}_{(0,1)}\,,\\
\overline{\mathcal{V}}_{(1)} &= \mathcal{V}_{(0,1)} + 2\cdot\mathcal{H}_{(0,1)}\,.
\end{aligned} \tag{B.2.2}$$

$\mathcal{N}$	spin	mass	content
$(2,2)$	2		$\mathcal{G}_{(2,2)} = 1\cdot[2] + 4\cdot[\frac{3}{2}]^+ + 4\cdot[\frac{3}{2}]^- + 16\cdot[1]$
			$+ 20\cdot[\frac{1}{2}]^+ + 20\cdot[\frac{1}{2}]^- + 25\cdot[0]$
			$+ 5\cdot[\mathrm{t}_2]^+ + 5\cdot[\mathrm{t}_2]^-$
$(0,2)$	2		$\mathcal{G}_{(0,2)} = 1\cdot[2] + 4\cdot[\frac{3}{2}]^- + 5\cdot[\mathrm{t}_2]^-$
$(0,2)$	3/2		$\mathcal{S}_{(0,2)} = 1\cdot[\frac{3}{2}]^+ + 4\cdot[1] + 5\cdot[\frac{1}{2}]^-$
$(0,2)$	1		$\mathcal{T}_{(0,2)} = 1\cdot[\mathrm{t}_2]^+ + 4\cdot[\frac{1}{2}]^+ + 5\cdot[0]$
$(1,1)$	2		$\mathcal{G}_{(1,1)} = 1\cdot[2] + 2\cdot[\frac{3}{2}]^+ + 2\cdot[\frac{3}{2}]^- + 4\cdot[1]$
			$+ 2\cdot[\frac{1}{2}]^+ + 2\cdot[\frac{1}{2}]^- + 1\cdot[0]$
			$+ 1\cdot[\mathrm{t}_2]^+ + 1\cdot[\mathrm{t}_2]^-$
$(1,1)$	3/2		$\mathcal{S}^{\pm}_{(1,1)} = 1\cdot[\frac{3}{2}]^{\pm} + 2\cdot[1] + 4\cdot[\frac{1}{2}]^{\pm} + 1\cdot[\frac{1}{2}]^{\mp}$
			$+ 2\cdot[0] + 2\cdot[\mathrm{t}_2]^{\pm}$
$(1,1)$	1		$\mathcal{V}_{(1,1)} = 1\cdot[1] + 2\cdot[\frac{1}{2}]^+ + 2\cdot[\frac{1}{2}]^- + 4\cdot[0]$
2	3/2	short	$\overline{\mathcal{S}}_{(2)} = 1\cdot\overline{[\frac{3}{2}]} + 2\cdot\overline{[1]} + 4\cdot\overline{[\frac{1}{2}]} + 2\cdot\overline{[0]} + 2\cdot\overline{[\mathrm{t}_2]}$
2	1	short	$\overline{\mathcal{V}}_{(2)} = 1\cdot\overline{[1]} + 2\cdot\overline{[\frac{1}{2}]} + 3\cdot\overline{[0]}$
$(0,1)$	2		$\mathcal{G}_{(0,1)} = 1\cdot[2] + 2\cdot[\frac{3}{2}]^- + 1\cdot[\mathrm{t}_2]^-$
$(0,1)$	3/2		$\mathcal{S}^+_{(0,1)} = 1\cdot[\frac{3}{2}]^+ + 2\cdot[1] + 1\cdot[\frac{1}{2}]^-$
$(0,1)$	3/2		$\mathcal{S}^-_{(0,1)} = 1\cdot[\frac{3}{2}]^- + 2\cdot[\mathrm{t}_2]^-$
$(0,1)$	1		$\mathcal{V}_{(0,1)} = 1\cdot[1] + 2\cdot[\frac{1}{2}]^-$
$(0,1)$	0		$\mathcal{H}_{(0,1)} = 1\cdot[\frac{1}{2}]^+ + 2\cdot[0]$
$(0,1)$	1		$\mathcal{T}_{(0,1)} = 1\cdot[\mathrm{t}_2]^+ + 2\cdot[\frac{1}{2}]^+ + 1\cdot[0]$
1	3/2	long	$\overline{\mathcal{S}}_{(1)} = 1\cdot\overline{[\frac{3}{2}]} + 2\cdot\overline{[1]} + 4\cdot\overline{[\frac{1}{2}]} + 2\cdot\overline{[0]} + 2\cdot\overline{[\mathrm{t}_2]}$
1	1	long	$\overline{\mathcal{V}}_{(1)} = 1\cdot\overline{[1]} + 2\cdot\overline{[\frac{1}{2}]} + 3\cdot\overline{[0]}$

Table B.2: Supergravity multiplets in $D = 6$. The first column shows the amount of supersymmetry, the second column indicates the maximal spin of the multiplet, the third column specifies whether the fields are massless (no indication) or massive (long or short).

B.3 Supergravity in $D = 4$

In this section we collect some information about the multiplet structure in four dimensions. The on-shell degrees of freedom of the various fields are summarized as follows

name	symbol	on-shell d.o.f.
massless spin 2	$[2]$	2_{b}
massless spin 3/2	$[\frac{3}{2}]$	2_{f}
massless spin 1	$[1]$	2_{b}
massless spin 1/2	$[\frac{1}{2}]$	2_{f}
massless spin 0	$[0]$	1_{b}
massive spin 3/2	$\overline{[\frac{3}{2}]}$	4_{f}
massive spin 1	$\overline{[1]}$	3_{b}
massive spin 1/2	$\overline{[\frac{1}{2}]}$	2_{f}
massive spin 0	$\overline{[0]}$	1_{b}

(B.3.1)

In table B.3 on pages 143 and 144 the massless and massive multiplets in four dimensions are summarized. This data has been taken from [87] and has been included here for completeness.

$\mathcal{N}$	spin	mass	content
8	2		$\mathcal{G}_{(8)} = 1 \cdot [2] + 8 \cdot [\frac{3}{2}] + 28 \cdot [1] + 56 \cdot [\frac{1}{2}] + 70 \cdot [0]$
6	2		$\mathcal{G}_{(6)} = 1 \cdot [2] + 6 \cdot [\frac{3}{2}] + 16 \cdot [1] + 26 \cdot [\frac{1}{2}] + 30 \cdot [0]$
6	3/2		$\mathcal{S}_{(6)} = 1 \cdot [\frac{3}{2}] + 6 \cdot [1] + 15 \cdot [\frac{1}{2}] + 20 \cdot [0]$
6	3/2	$\frac{1}{2}$ BPS	$\overline{\mathcal{S}}_{(6)} = 2 \cdot \overline{[\frac{3}{2}]} + 12 \cdot \overline{[1]} + 28 \cdot \overline{[\frac{1}{2}]} + 28 \cdot \overline{[0]}$
5	2		$\mathcal{G}_{(5)} = 1 \cdot [2] + 5 \cdot [\frac{3}{2}] + 10 \cdot [1] + 11 \cdot [\frac{1}{2}] + 10 \cdot [0]$
5	3/2		$\mathcal{S}_{(5)} = 1 \cdot [\frac{3}{2}] + 6 \cdot [1] + 15 \cdot [\frac{1}{2}] + 20 \cdot [0]$
5	3/2	$\frac{2}{5}$ BPS	$\overline{\mathcal{S}}_{(5)} = 2 \cdot \overline{[\frac{3}{2}]} + 12 \cdot \overline{[1]} + 28 \cdot \overline{[\frac{1}{2}]} + 28 \cdot \overline{[0]}$
4	2		$\mathcal{G}_{(4)} = 1 \cdot [2] + 4 \cdot [\frac{3}{2}] + 6 \cdot [1] + 4 \cdot [\frac{1}{2}] + 2 \cdot [0]$
4	3/2		$\mathcal{S}_{(4)} = 1 \cdot [\frac{3}{2}] + 4 \cdot [1] + 7 \cdot [\frac{1}{2}] + 8 \cdot [0]$
4	3/2	$\frac{1}{4}$ BPS	$\overline{\mathcal{S}}^{1/4}_{(4)} = 2 \cdot \overline{[\frac{3}{2}]} + 12 \cdot \overline{[1]} + 28 \cdot \overline{[\frac{1}{2}]} + 28 \cdot \overline{[0]}$
4	3/2	$\frac{1}{2}$ BPS	$\overline{\mathcal{S}}^{1/2}_{(4)} = 2 \cdot \overline{[\frac{3}{2}]} + 8 \cdot \overline{[1]} + 12 \cdot \overline{[\frac{1}{2}]} + 8 \cdot \overline{[0]}$
4	1		$\mathcal{V}_{(4)} = 1 \cdot [1] + 4 \cdot [\frac{1}{2}] + 6 \cdot [0]$
4	1	$\frac{1}{2}$ BPS	$\overline{\mathcal{V}}_{(4)} = 2 \cdot \overline{[1]} + 8 \cdot \overline{[\frac{1}{2}]} + 10 \cdot \overline{[0]}$
3	2		$\mathcal{G}_{(3)} = 1 \cdot [2] + 3 \cdot [\frac{3}{2}] + 3 \cdot [1] + 1 \cdot [\frac{1}{2}]$
3	3/2		$\mathcal{S}_{(3)} = 1 \cdot [\frac{3}{2}] + 3 \cdot [1] + 3 \cdot [\frac{1}{2}] + 2 \cdot [0]$
3	3/2	long	$\overline{\mathcal{S}}^{l}_{(3)} = 1 \cdot \overline{[\frac{3}{2}]} + 6 \cdot \overline{[1]} + 14 \cdot \overline{[\frac{1}{2}]} + 14 \cdot \overline{[0]}$
3	3/2	$\frac{1}{3}$ BPS	$\overline{\mathcal{S}}^{1/3}_{(3)} = 2 \cdot \overline{[\frac{3}{2}]} + 8 \cdot \overline{[1]} + 12 \cdot \overline{[\frac{1}{2}]} + 8 \cdot \overline{[0]}$
3	1		$\mathcal{V}_{(3)} = 1 \cdot [1] + 4 \cdot [\frac{1}{2}] + 6 \cdot [0]$
3	1	$\frac{1}{3}$ BPS	$\overline{\mathcal{V}}_{(3)} = 2 \cdot \overline{[1]} + 8 \cdot \overline{[\frac{1}{2}]} + 10 \cdot \overline{[0]}$
2	2		$\mathcal{G}_{(2)} = 1 \cdot [2] + 2 \cdot [\frac{3}{2}] + 1 \cdot [1]$
2	3/2		$\mathcal{S}_{(2)} = 1 \cdot [\frac{3}{2}] + 2 \cdot [1] + 1 \cdot [\frac{1}{2}]$
2	3/2	long	$\overline{\mathcal{S}}^{l}_{(2)} = 1 \cdot \overline{[\frac{3}{2}]} + 4 \cdot \overline{[1]} + 6 \cdot \overline{[\frac{1}{2}]} + 4 \cdot \overline{[0]}$
2	3/2	$\frac{1}{2}$ BPS	$\overline{\mathcal{S}}^{1/2}_{(3)} = 2 \cdot \overline{[\frac{3}{2}]} + 4 \cdot \overline{[1]} + 2 \cdot \overline{[\frac{1}{2}]}$

2	1		$\mathcal{V}_{(2)} = 1 \cdot [1] + 2 \cdot [\frac{1}{2}] + 2 \cdot [0]$
2	1	long	$\overline{\mathcal{V}}^{l}_{(2)} = 1 \cdot \overline{[1]} + 4 \cdot \overline{[\frac{1}{2}]} + 5 \cdot \overline{[0]}$
2	1	$\frac{1}{2}$ BPS	$\overline{\mathcal{V}}^{1/2}_{(2)} = 2 \cdot \overline{[1]} + 4 \cdot \overline{[\frac{1}{2}]} + 2 \cdot \overline{[0]}$
2	1/2		$\mathcal{H}_{(2)} = 2 \cdot [\frac{1}{2}] + 4 \cdot [0]$
2	1/2	$\frac{1}{2}$ BPS	$\overline{\mathcal{H}}_{(2)} = 2 \cdot \overline{[\frac{1}{2}]} + 4 \cdot \overline{[0]}$
1	2		$\mathcal{G}_{(1)} = 1 \cdot [2] + 1 \cdot [\frac{3}{2}]$
1	3/2		$\mathcal{S}_{(1)} = 1 \cdot [\frac{3}{2}] + 1 \cdot [1]$
1	3/2	long	$\overline{\mathcal{S}}^{l}_{(1)} = 1 \cdot \overline{[\frac{3}{2}]} + 2 \cdot \overline{[1]} + 1 \cdot \overline{[\frac{1}{2}]}$
1	1		$\mathcal{V}_{(1)} = 1 \cdot [1] + 1 \cdot [\frac{1}{2}]$
1	1	long	$\overline{\mathcal{V}}^{l}_{(1)} = 1 \cdot \overline{[1]} + 2 \cdot \overline{[\frac{1}{2}]} + 1 \cdot \overline{[0]}$
1	1/2		$\mathcal{C}_{(1)} = 1 \cdot [\frac{1}{2}] + 2 \cdot [0]$
1	1/2	long	$\overline{\mathcal{C}}_{(1)} = 1 \cdot \overline{[\frac{1}{2}]} + 2 \cdot \overline{[0]}$

Table B.3: Supergravity multiplets in $D = 4$. The first column shows the amount of supersymmetry, the second column indicates the maximal spin of the multiplet, the third column specifies whether the fields are massless (no indication) or massive (long or shortened). For more details see [87].

Appendix C

Computing the star Riemann tensor

In this appendix we provide the details on the evaluation of

$$-R(X,Y,Z) = (\nabla_X \bullet \nabla_Y)Z - (\nabla_{\overline{\mathcal{R}}(Y)} \bullet \nabla_{\overline{\mathcal{R}}(X)})Z - \nabla_{[X,Y]_\star} Z\,. \tag{C.0.1}$$

As discussed in the main text after (5.6.23), we need to interpret the $\bullet$ as a composition of left and right actions of the directional covariant derivative

$$(\nabla_X \bullet \nabla_Y)Z = \big[(i_X \bullet i_Y)(Z)\big]\ (\overleftarrow{\nabla} \bullet \overleftarrow{\nabla})\,. \tag{C.0.2}$$

Recall that in $\nabla_X Y$ first $\overleftarrow{\nabla}$ is carried out and afterwards X acts as a contraction denoted by i_X. In addition, we have to respect the order of ∇_X and ∇_Y. Indicating the order by a subscript, we have altogether

$$(\nabla_X \bullet \nabla_Y)Z = \big[(i_{X(4)} \bullet i_{Y(2)})(Z)\big]\ (\overleftarrow{\nabla}_{(1)} \bullet \overleftarrow{\nabla}_{(3)})\,. \tag{C.0.3}$$

We apply the first covariant derivative by bringing Z and $\overleftarrow{\nabla}_{(1)}$ together. The scalar product between Z and the first matrix index of Γ is carried out directly $Z \star \Gamma = Z^M \star \Gamma_M$ followed by bringing i_Y together with Z. Thus, the computation proceeds as

$$\begin{aligned}
&\big[(i_{X(4)} \bullet i_{Y(2)})(Z)\big]\ (\overleftarrow{\nabla}_{(1)} \bullet \overleftarrow{\nabla}_{(3)})\\
&\quad = (i_{X(4)} \bullet i_{Y(2)})^\phi\ \big[Z^\phi\ (\overleftarrow{\nabla}_{(1)} \bullet \overleftarrow{\nabla}_{(3)})^\phi\big]\\
&\quad = (i_{X(4)} \bullet i_{Y(2)})^\phi\ \big[\partial Z^\phi\ \overleftarrow{\nabla}^\phi_{(3)}\big] + (i_{X(4)} \bullet i_{Y(2)})^\phi\ \big[Z^{M\phi} \star (\Gamma_M\ \ \overleftarrow{\nabla}_{(3)})^\phi\big]\\
&\quad = \big[i^\phi_{X(4)}(Y^\phi \star \partial Z^\phi)\big]\ \overleftarrow{\nabla}_{(3)} + \big(\big[i^\phi_{X(4)}(Y^\phi \star Z^{M\phi})\big]^{\overline{\phi}} \star \Gamma^{\overline{\phi}}_M\big)\ \overleftarrow{\nabla}^{\overline{\phi}}_{(3)}\,.
\end{aligned} \tag{C.0.4}$$

Next, the second covariant derivative $\overleftarrow{\nabla}_{(3)}$ and afterwards i_X are applied

$$\begin{aligned}
=& i^{\phi\phi'}_{X(4)}\ \big[(Y^\phi \star \partial Z^\phi)^{\phi'} \overleftarrow{\nabla}^{\phi'}_{(3)}\big] + i^{\phi\phi'}_{X(4)}\big[(Y^\phi \star Z^{K\phi})^{\phi'} \star (\Gamma_K\ \ \overleftarrow{\nabla}_{(3)})^{\phi'}\big]\\
=& X^\phi \star \partial(Y^\phi \star \partial Z^\phi) + [X^\phi \star (Y^\phi \star \partial Z^{K\phi})] \star \Gamma_K\\
&+ X^{\phi\phi'} \star \partial\big[(Y^\phi \star Z^{K\phi})^{\phi'} \star \Gamma^{\phi'}_K\big] + \big[(X \star Y) \star Z^K\big] \star (\Gamma_K{}^P \star \Gamma_P)\,.
\end{aligned} \tag{C.0.5}$$

In this formula we placed the brackets and the derivative ∂ in such a way that they reflect, which objects have to be contracted with each other. For instance in the first term of (C.0.5), the derivative is contracted with X. After applying the Leibniz rule for $\partial(Y^\phi \star \partial Z^\phi)$, this contraction must be kept in mind.

When computing the other terms in (C.0.1), one realizes that the first term in (C.0.5) is canceled partly by $(\nabla_{\overline{\mathcal{R}}(Y)} \bullet \nabla_{\overline{\mathcal{R}}(X)})Z$ and partly by $\nabla_{[X,Y]_\star}Z$. The other term from $\nabla_{[X,Y]_\star}Z$ cancels the $X \star \partial Y \star Z \star \Gamma$ part in the third term of (C.0.5). The remaining two terms which have to cancel in (C.0.5) arise from the second and third term and are both of the form $X \star Y \star \partial Z \star \Gamma$. In one term ∂ is contracted with Y and in the other ∂ is contracted with X. These terms cancel crosswise against similar terms appearing in $(\nabla_{\overline{\mathcal{R}}(Y)} \bullet \nabla_{\overline{\mathcal{R}}(X)})Z$.

After all these cancellations, the Riemann tensor (C.0.1) simplifies to

$$\begin{aligned}-R(X,Y,Z) =& \big((X \star Y) \star Z^M\big) \star \partial\Gamma_M + \big((X \star Y) \star Z^M\big) \star ({\Gamma_M}^P \star \Gamma_P) \\ &- X \leftrightarrow^{\overline{\mathcal{R}}} Y\,.\end{aligned} \tag{C.0.6}$$

Recalling the discussion after (C.0.5), in the first term of (C.0.6), X is contracted with ∂. This is in contrast to the rule that always a vector is contracted with the nearest neighboring form[1]. To bring this into the usual notation, we switch the first term with its $\mathcal{R}$-permuted term and find with (5.5.16) and (5.5.17)

$$\begin{aligned}-R(X,Y,Z) =& -\big((X \star Y) \star Z^M\big) \star \partial\Gamma_M + \big((X \star Y) \star Z^M\big) \star ({\Gamma_M}^P \star \Gamma_P) \\ &- X \leftrightarrow^{\overline{\mathcal{R}}} Y\,.\end{aligned} \tag{C.0.7}$$

Now the notation matches the one in (5.6.22), where the vector Y is contracted with the form ∂ (see also (5.6.26) and (5.6.27)). By utilizing (5.5.19) to transfer the antisymmetrization on the vector side towards the form side, we indeed find

$$-R(X,Y,Z) = \big((X \star Y) \star Z^K\big) \star \big(-d\Gamma_K + {\Gamma_K}^P \wedge_\star \Gamma_P\big)\,. \tag{C.0.8}$$

This matches the definition of the star Riemann curvature as the exterior covariant derivative of the connection Γ in (5.6.22).

[1]Notice that the contraction in the second term $\sim \Gamma \star \Gamma$ comes out correctly according to this rule.

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