

# Interactions in Higher-Spin Gravity: A Holographic Perspective

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Dissertation by Charlotte Sleight



# **Interactions in Higher-Spin Gravity: A Holographic Perspective**

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## Dissertation

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*To my dad, sister, and the wonderful memory of my mum.*

*“Die Probleme werden gelöst, nicht durch Beibringen neuer Erfahrungen,  
sondern durch Zusammenstellung des längst Bekannten.”*

— Ludwig Wittgenstein

# Zusammenfassung

Die vorliegende Dissertation widmet sich der Untersuchung der Höhere Spin-Gravitation auf einem Anti-de-Sitter-Raum (AdS). Es gibt Hinweise darauf, dass dem Hochenergiebereich der Quantengravitation eine große (Höhere Spin-) Symmetriegruppe zugrundeliegt und somit eine mögliche effektive Beschreibung dieses Bereichs durch eine Höhere Spin-Theorie gegeben ist. Die Realisierung einer Quantengravitation mit dieser Symmetrie ist insbesondere in der String Theorie bei hohen Energien gegeben, wenn die Stringspannung vernachlässigt werden kann.

Die Untersuchung der Höhere Spin-Theorien ist von physikalischer Relevanz, weil ihre unendlichdimensionale Symmetrie als Leitprinzip verwendet werden kann, um neue Erkenntnisse über die Quantengravitation bei Energien jenseits der heute experimentell erreichbaren Energieskalen zu gewinnen. Es muss dabei berücksichtigt werden, dass die verfügbaren nichtlinearen Formulierungen der Höhere-Spin-Theorien unendlich viele Hilfsfelder mit einschließen, so dass selbst grundlegende Eigenschaften wie Unitarität oder Kausalität noch nicht vollständig verstanden sind.

Zielsetzung dieser Arbeit ist es, die Wechselwirkungen zwischen höheren Spin-Feldern im Rahmen der Lagrangeformulierung zu untersuchen, da dies ohne Hilfsfelder möglich ist. Um dies zu ermöglichen, wird die vermutete Dualität zwischen der Höhere Spin-Gravitation im AdS-Raum und integrablen konformen Feldtheorien benutzt. Durch Anwendung von Techniken der konformen Feldtheorie werden in der Lagrangefunktion der Höheren Spin-Theorie alle Wechselwirkungsterme bis zur dritten Ordnung sowie die Selbstwechselwirkung des Skalarfeldes bis zur vierten Ordnung bestimmt. Zu diesem Zweck werden neue Methoden entwickelt: Diese umfassen eine neue Herangehensweise

zur Berechnung von Dreipunktamplituden mit externen Feldern beliebigen Spins und die explizite Form von Propagatoren für Eichfelder mit beliebigem ganzzahligen Spin im AdS-Raum. Weiterhin werden Techniken zur Berechnung konformer partieller Wellenzerlegungen von Vierpunktamplituden in Höhere Spin-Theorien entwickelt.

Die  $\alpha'$ -Korrekturen in der Stringtheorie spiegeln sich in der Nichtlokalität der Wechselwirkungen zwischen Höhere Spin-Feldern wider: Es treten beliebig hohe Ableitungen der Felder auf. In der Stringtheorie bei endlicher Stringsannung bedeuten diese höheren Ableitungsterme offensichtlich keine Einschränkung von fundamentalen Eigenschaften wie etwa der Kausalität. In der Höhere-Spin-Theorie jedoch ist dies Gegenstand aktueller Forschung. Im Kontext der Ergebnisse dieser Arbeit wird die Bedeutung der AdS/CFT Dualität für die Lokalität der Wechselwirkungen zwischen Höhere Spin-Feldern diskutiert. Insbesondere wird auf die holographische Interpretation der Korrelatoren in der konformen Feldtheorie als Streuamplituden der dualen Gravitationstheorie im AdS-Raum eingegangen.

# Abstract

In this thesis we investigate aspects of higher-spin gravity on an anti-de Sitter (AdS) background. Such theories are believed to give an effective description of quantum gravity in its high energy regime, where there have been hints that a large underlying (higher-spin) symmetry may become restored. In particular, through considerations of string theory in the high-energy (tensionless) limit. The study of higher-spin gauge theories is thus of physical relevance, for their infinite dimensional symmetry may serve as a guiding principle to help shed light on the nature of quantum gravity at energies which are beyond the reach of present day experiments.

To fulfil this potential, there is much still to be understood about higher-spin theories themselves. To date, the fully non-linear formulations of higher-spin theories include infinitely many additional auxiliary fields. With this work we take steps towards clarifying the nature of interactions amongst higher-spin fields in a metric-like Lagrangian framework, free from such auxiliary fields. To this end, we employ the conjectured holographic duality between higher-spin gravity theories on AdS and certain, solvable, conformal field theories (CFTs). In this way well-known methods in CFT may be applied, and with them we determine completely all cubic interactions and the quartic self-interaction of the scalar in the higher-spin Lagrangian on an AdS background. For this purpose a number of further original results were established, which include: Tools to evaluate three-point amplitudes in AdS space involving external fields of arbitrary integer spin, propagators for gauge fields of arbitrary integer spin on an AdS background, and techniques to determine conformal partial wave expansions of four-point amplitudes in higher-spin theories.



Reminiscent of the tail of  $\alpha'$  corrections in string theory, higher-spin interactions are non-local in the sense that they are unbounded in their number of derivatives. While in (finite tension) string theory this higher-derivative behaviour does not appear to pose a threat to fundamental properties like causality, in higher-spin theories this remains to be clarified. In the context of the results established in this work, we briefly discuss the implications of the AdS/CFT duality on the locality properties of interactions in higher-spin theory. In particular, by drawing on the holographic interpretation of CFT correlation functions as scattering amplitudes in the dual gravity theory on AdS.

# Thesis publications

This thesis is based on some of the authors work that was carried out during the period of May 2013 - 2016 at the Max-Planck-Institut für Physik (Werner-Heisenberg-Institut) München, under the supervision of Prof. Johanna Erdmenger. The corresponding publications are:

- [1] **Towards holographic higher-spin interactions: Four-point functions and higher-spin exchange.**

Xavier Bekaert, Johanna Erdmenger, Dmitry Ponomarev and Charlotte Sleight.

Published in JHEP: [JHEP 1503 \(2015\) 170](#)

- [2] **Quartic AdS Interactions in Higher-Spin Gravity from CFT**

Xavier Bekaert, Johanna Erdmenger, Dmitry Ponomarev and Charlotte Sleight.

Published in JHEP: [JHEP 1511 \(2015\) 149](#)

- [3] **Bulk quartic vertices from boundary four-point correlators**

Xavier Bekaert, Johanna Erdmenger, Dmitry Ponomarev and Charlotte Sleight.

[arXiv:1602.08570 \[hep-th\]](#), to be published in World Scientific.

Proceedings: “International Workshop on Higher Spin Gauge Theorie” (4-6 Nov. 2015, Singapore).

- [4] **Higher-spin Interactions from CFT: The Complete Cubic Couplings**

Charlotte Sleight and Massimo Taronna.

Published in Physical Review Letters: [Phys.Rev.Lett. 116 \(2016\) no.18, 181602](#)

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# Chapter 1

## Introduction

### 1.1 A tale of two theories

Existing physical theories provide a powerful framework to describe all phenomena observed so far in Nature. To date, the range of length scales accessed by physical experiment is colossal: From the deep subatomic distances of  $\sim 10^{-19}$  m probed by collisions of highly accelerated particles, to  $\sim 10^{25}$  m from deep sky surveys.

The two monumental pillars behind our theoretical understanding of Nature thus far are Quantum Field Theory (QFT) and General Relativity:

QFT describes the dynamics of elementary particles. All known elementary particles and their interactions are encoded in the Lagrangian of the Standard Model of particles physics [5–7], and consists of three fundamental ingredients: Matter fields, the Higgs boson responsible for particle masses and force carriers (gauge bosons). The gauge bosons account for three of the four known fundamental interactions amongst the constituents of visible matter: The electromagnetic, strong nuclear and weak nuclear forces.

The remaining fundamental force is gravity. Einstein’s theory of General Relativity [8] provides a beautiful theory of gravitation at the classical level, which elegantly geometrises dynamics. Gravity is a dominant force at large, cosmic, scales and indeed, General Relativity is the backbone of the cosmological standard model (or  $\Lambda$ CDM model). The latter provides a coherent framework for the history of the Universe from Big Bang nucleosynthesis up to the present day.

In spite of these successes, the theoretical tools of QFT — that have been so effective in describing interactions at the subatomic scale — are unfortunately not directly applicable to the gravitational force. The characteristic gravitational scale is the Planck scale  $\ell_{pl} \sim 10^{-35}$  m, with the gravitational coupling being  $\sim \ell_p^2/L^2$  in a process at length scales  $L$ . Unlike for the Standard Model, General Relativity therefore suffers a lack of predictive power at Planckian scales, where it becomes strongly coupled (in other words, it is non-renormalisable [9, 10]). An understanding of how to reconcile QFT and General Relativity is imperative, since gravity couples to matter and consequently we cannot forgo the need for a quantum description of it.

One approach, which has proven to be particularly successful in improving UV behaviour, is to consider coupling gravity to an enlarged spectrum of gauge and matter fields. A notable example is  $\mathcal{N} = 8$  supergravity, which contains one graviton, 8 gravitinos (which have spin  $3/2$ ), 28 vector bosons, 56 fermions and 70 scalar fields. Current results indicate its finiteness at seven-loop order [11–14], with speculations that it is all-loop finite. These promising results illustrate that the idea of increasing the particle content, and symmetry, is one worth pursuing.

## 1.2 Higher-spins and strings

From a Wilsonian perspective, both the Standard Model and General Relativity are effective field theories describing the relevant degrees of freedom up to a certain energy



scale. The ultimate physical theory is viewed as having an onion-like structure, with different layers corresponding to different energies, with different degrees of freedom being excited and governed by different Lagrangians. In this regard, the break down of General relativity at the Planck scale indicates that it should be substituted by a new, more fundamental theory.

The problem of formulating a consistent quantum theory that includes as its low-energy limit both the Standard Model and General Relativity is known as the quantum gravity problem. Because energies at which quantum-gravitational effects become significant are not directly accessible via existing experiments, the development of the subject has been largely driven by Gedankenexperiments and symmetry principles.

For example, considering General Relativity as an effective quantum theory with cut-off  $M_{pl}$ , counter-terms to the Einstein-Hilbert action appear in the form of higher curvature invariants (due to diffeomorphism invariance) and therefore additional derivatives. At one loop for example, the only correction that is generated is the Gauss-Bonnet term,

$$S_{1\text{-loop}} = S_{EH} + S_{GB} = \frac{1}{\ell_p^{d-2}} \int \sqrt{-g} [R + \alpha (R^2 - 4R^{\mu\nu} R_{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma})], \quad (1.1)$$

with  $\alpha \gg \ell_p^2$ . Such corrections ( $\mathcal{O}(R^2, R^3, \dots)$ ) lead to short distance violations of causality at the classical level [15], inducing (Shapiro) time *advances*. Causality cannot be restored by adding local higher curvature terms, but instead by introducing an infinite number of new massive particles of spins  $s > 2$  and mass  $\sim 1/\sqrt{\alpha}$ .

We see from above example that basic consistency requirements seem to imply that a theory of quantum gravity should admit a completion to a theory with an enlarged spectrum, which includes an infinite tower of higher-spin excitations.

A promising research program which addresses the problem of quantum gravity is string

theory, which indeed has a spectrum containing infinitely many massive excitations of increasing mass and spin. These arise from the different vibrational modes of the fundamental string, in an enticing unified manner. The string states dispose themselves on Regge trajectories in the (mass)<sup>2</sup> vs. spin plane, where for the first Regge trajectory we have, e.g. for the open bosonic spin in flat space,

$$\alpha' m_s^2 = s - 1, \tag{1.2}$$

with  $\alpha' = l_s^2$  the square of the string length. In fact, the higher-spin resonances are responsible for the improved UV behaviour [16–24] compared to General Relativity above, essentially because the string length  $l_s$  acts as a natural UV-cutoff.

An important observation from our experience with QFTs is that improved UV behaviour is typically the signal of an underlying symmetry. In addition to the example of supersymmetry mentioned previously, another demonstration of this phenomenon is given by the weak interaction: Fermi’s four-fermion theory is non-renormalisable, but describes the weak interaction remarkably well up to  $\sim 100$  GeV. Together with QED, it can be considered as the low-energy effective theory for the Standard Model of the electroweak interactions. In the latter, as opposed to the effective four-point interaction of Fermi’s theory, weak interactions are mediated by massive  $W^\pm$  and  $Z$  bosons. These arise from the spontaneous breaking of an  $SU(2) \times U(1)$  gauge symmetry, which emerges at very high energies.

One may then ask whether a similar phenomenon occurs in string theory. This question was indeed posed by Gross in 1988 [25], and subsequently explored in a number of works (for instance: [24, 26–32]). Just like the broken  $SU(2) \times U(1)$  symmetry of the electroweak interactions can be seen by examining weak scattering amplitudes at energies high enough so that the masses of  $W^\pm$  and  $Z$  can be neglected, by studying the high energy ( $\alpha' \rightarrow \infty$ ) behaviour of string scattering, Gross argued for the existence

of an infinite dimensional symmetry group that gets restored at high energies. Should this be the case, the symmetry would be generated by gauge symmetries associated to the infinite family of higher-spin gauge fields that emerge in this limit.<sup>1</sup>

With the presence of an infinite dimensional symmetry comes a lot of control. In this sense, uncovering such an underlying symmetry principle behind string theory could lead to a greater understanding of quantum gravity at high energies. For example, from the space-time viewpoint string theory is still formulated in a background-dependent manner, without an analogue of the powerful geometric background independent presentation of Einstein gravity. Further more, at present both the action principle and equations of motion for the second quantised form of string theory (string field theory) require the addition of infinitely many auxiliary fields.<sup>2</sup> In this sense much remains to be unravelled about the properties of high energy interactions in quantum gravity.

## 1.3 This thesis

In this thesis we aim to shed some light on the nature of interactions in the presence of higher-spin symmetry. To this end we employ a particular consequence of string theory known as *holography*, focusing on its most celebrated incarnation: The AdS/CFT correspondence [44–46].

In its most general form, the AdS/CFT correspondence is the statement that two systems are equivalent:

$$\text{AdS}_{d+1} \text{ QG} = \text{CFT}_d. \quad (1.3)$$

Quantum gravity in asymptotically anti-de Sitter spacetime  $\text{AdS}_{d+1}$  and non-gravitational conformal field theory<sup>3</sup> (CFT) in a flat  $d$ -dimensional spacetime. This relationship is

<sup>1</sup>Recall, for example, that on the first Regge trajectory in flat space we have  $m_s^2 \propto 1/\alpha' \rightarrow 0$ .

<sup>2</sup>At the free level, the second quantised formulation of all string models was completed in the '80s [33–40]. While at the interacting level, see for example: [41–43].

<sup>3</sup>A conformal field theory is a quantum field theory without any intrinsic scale.

called a duality. It is holographic since the gravitational theory lives in (at least) one extra dimension. Since AdS space is conformally flat, we can in fact think of the dual CFT as living on the boundary (solid black boundary of disc) of the gravity theory in asymptotically AdS, which is often represented as in figure 1.1 .

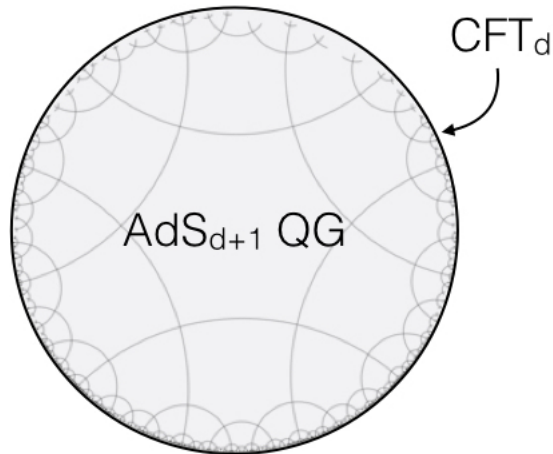


Figure 1.1: For a given dual pair (1.3), we can think of the  $\text{CFT}_d$  as living on the boundary (solid black boundary of disc) of the gravity theory in asymptotically  $\text{AdS}_{d+1}$  (entire disc).

The duality (1.3) is remarkable, as it opens up the possibility to study quantum gravity through its dual CFT, and vice versa. In particular, the dimensionless coupling  $\lambda$  in the  $\text{CFT}_d$  is typically related to the string length of the gravity theory in AdS via

$$\lambda \sim \left( \frac{R^2}{\alpha'} \right)^{d/2}, \quad (1.4)$$

where  $R$  is the AdS radius. This relationship between the CFT coupling and the string length illustrates the duality's strong-weak nature: For example, in taking the point-particle limit  $\alpha'/R^2 \rightarrow 0$ , the CFT coupling  $\lambda$  grows large. The latter limit therefore opens up the possibility to study strongly coupled systems from the perspective of the comparably well-understood General Relativity (at least when quantum corrections are suppressed), and has been extensively explored since the birth of the AdS/CFT corre-

spondence.<sup>4</sup>

Taking the opposite limit on the other hand –  $\alpha'/R^2 \rightarrow \infty$  – equation (1.4) tells us that the high-energy limit of string theory may be probed by taking the free theory limit  $\lambda \rightarrow 0$  in the dual CFT [49–61]. Free CFTs are constrained by higher-spin symmetry (as we shall demonstrate later), owing to the emergence of an infinite tower of conserved currents of increasing in the free limit. Since the dual quantum gravity theory should be governed by the same symmetry as its CFT counterpart, the above striking relationship makes the existence of a highly symmetric phase of string theory even more plausible, and more tractable to study.

The most famous manifestation of the holographic duality (1.3), crystallised by Maldacena in 1997 [44], is the conjecture that  $\mathcal{N} = 4$  Super Yang-Mills theory in  $d = 3 + 1$  dimensions is equivalent to type IIB superstring theory on  $\text{AdS}_5 \times S^5$ . In the high energy limit  $\alpha'/R^2 \rightarrow \infty$ , the string spectrum on the  $\text{AdS}_5 \times S^5$  background has been extrapolated, and could be identified with the operator spectrum of planar<sup>5</sup>  $\mathcal{N} = 4$  Super Yang-Mills theory at weak coupling ( $\lambda \rightarrow 0$ ) [62–68]. With control of the spectrum, studies towards understanding the spontaneous breaking of the local higher-spin symmetries in AdS were also undertaken from a holographic vantage point. From the CFT perspective, this corresponds to switching on the Yang-Mills coupling ( $\lambda \neq 0$ ), leading to the anomalous violation of the higher-spin ( $s > 2$ ) conserved currents. These studies were made possible by a remarkable feature of  $\mathcal{N} = 4$  Super Yang-Mills, which is that the spectrum of local operators is calculable for any coupling  $\lambda$  in the planar limit [69].

While the above example provides a concrete framework to study string theory in the

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<sup>4</sup>See, for example, [47, 48] for pedagogical introductions to applying the AdS/CFT correspondence to strongly coupled systems.

<sup>5</sup>I.e. when the number of colours goes to infinity. As shall be explained later, this limit corresponds to the weak coupling regime of the dual string theory.

$\alpha' \rightarrow \infty$  limit, in practice this is complicated by the exponentially growing spectrum, composed of infinitely many Regge trajectories. A streamlined version of this set-up is provided by instead considering interacting higher-spin gauge theories on AdS space, which are believed to provide an effective description of the first Regge trajectory of string theory in the tensionless limit. Such theories have been conjectured to be dual to the free fixed points of vector models [51, 70–76],<sup>6</sup> which indeed have linear spectra as opposed to the exponentially growing spectra of Matrix models, like that of  $\mathcal{N} = 4$  Super Yang-Mills theory. The duality between higher-spin theories and vector models therefore provides a toy model for investigating the nature of string interactions in its putative maximally symmetric phase. Moreover, there have been recent results on embedding such higher-spin gauge theories into string theory [82].

Interactions in higher-spin gauge theories have a long history, with some of their properties still remaining quite elusive. While no-go theorems [83–90] seem to indicate that Minkowski space breaks higher-spin symmetry (however see developments [91–94]), there are promising results for the existence of interacting theories in (A)dS.<sup>7</sup> Expanding about an (A)dS background, all possible structures for cubic interactions of bosonic gauge fields have been constructed for any triplet of spin [30, 96–114] and classified [90, 108, 112, 113], with consistent propagation at the non-linear level requiring the spectrum to be unbounded in spin [91, 92, 115].<sup>8</sup> Little is known about the properties of quartic and higher-order interactions in (A)dS within a standard Lagrangian framework, though background-independent fully non-linear equations of motion have been constructed by Vasiliev [73, 76, 118–127], which employ an infinite set of auxiliary fields.<sup>9</sup>

A complete understanding of higher-spin theories beyond cubic order in a standard La-

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<sup>6</sup>See also [77–81] for earlier closely related work.

<sup>7</sup>For a comprehensive review of no-go theorems for higher-spin interactions in flat spacetime, and how to evade them, see: [95].

<sup>8</sup>This comes from the structure of the higher-spin algebra, which is to large extent unique [116, 117].

<sup>9</sup>See also [128–132] for proposals of an action, using the machinery of auxiliary fields.

grangian framework is a problem which has remained open since the ball began rolling for studies of higher-spin interactions in the ‘70s. In particular, like the tail of  $\alpha'$  corrections in string theory, one expects the appearance of non-local interactions unbounded in their number of derivatives. The latter behaviour requires careful consideration for the consistency of interacting higher-spin theories.

### 1.3.1 Summary of main results

It is the goal of this work to employ the holographic duality (1.3) to shed some light on the above questions, regarding the nature of interactions in higher-spin gauge theories within a standard Lagrangian framework.

For simplicity, we consider the most streamlined set-up of the minimal bosonic higher-spin theory on  $\text{AdS}_{d+1}$ , which is a theory of even spin gauge fields represented by totally symmetric double-traceless<sup>10</sup> tensors  $\varphi_{\mu_1 \dots \mu_s}$ . These have linearised gauge transformations

$$\delta\varphi_{\mu_1 \dots \mu_s} = \nabla_{(\mu_1} \xi_{\mu_2 \dots \mu_s)}, \quad s = 2, 4, 6, \dots \quad (1.5)$$

In addition to the gauge fields above, the spectrum includes a parity even scalar which we denote by  $\varphi_0$ . This theory has been conjectured to be dual to the free scalar  $O(N)$  vector model in  $d$ -dimensions [70, 71], which is a CFT of an  $N$ -component real scalar field transforming in the fundamental representation of  $O(N)$ .

The leitmotiv of this thesis is that correlation functions of the free scalar  $O(N)$  vector model in the large  $N$  limit are considered equivalent to a perturbative Feynman diagram expansion in the minimal bosonic higher-spin theory on AdS. Assuming this holographic duality holds, with the knowledge of correlators in the free  $O(N)$  model (which, in a free theory, are straightforward to compute by Wick’s theorem) one can then in principle infer the on-shell form of the interactions in the higher-spin theory.

<sup>10</sup>This is the algebraic constraint:  $\varphi_{\mu_1 \dots \mu_{s-4} \rho \sigma}{}^{\rho \sigma} = 0$ , whose role will be explained later.

The basic idea (which is made more precise in the following chapter) is illustrated for the extraction of cubic interactions in AdS in figure 1.2.

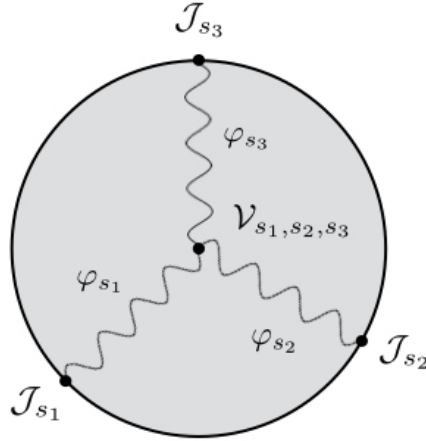


Figure 1.2: Holographic interpretation of the three-point function of operators  $\mathcal{J}_{s_i}$  in the free  $O(N)$  model at large  $N$ . The operators  $\mathcal{J}_{s_i}$  are inserted on the boundary of  $\text{AdS}_{d+1}$ , sourced by the boundary values of their dual fields  $\varphi_{s_i}$  in AdS. The wavy lines are the propagators of the  $\varphi_{s_i}$ . With the knowledge of  $\langle \mathcal{J}_{s_1} \mathcal{J}_{s_2} \mathcal{J}_{s_3} \rangle$  the on-shell form of the cubic interaction  $\mathcal{V}_{s_1, s_2, s_3}$  can then in principle be determined.

Taking the above approach, in this work we determine the complete set of cubic couplings and the quartic self interaction of the scalar  $\varphi_0$  in the action of the minimal bosonic higher-spin theory on AdS. In other words, using holography we fix the on-shell action

$$S[\varphi] = \sum_s \int_{\text{AdS}} \frac{1}{2} \varphi_s (\square - m_s^2 + \dots) \varphi_s + \sum_{s_i} \int_{\text{AdS}} \mathcal{V}_{s_1, s_2, s_3}(\varphi) + \int_{\text{AdS}} \mathcal{V}_{0,0,0,0}(\varphi_0) + \dots,$$

where the  $\dots$  in the free quadratic action denote terms which depend on the gauge fixing [133]. The cubic interaction for each triplet of spins takes the schematic form

$$\mathcal{V}_{s_1, s_2, s_3}(\varphi) = g_{s_1, s_2, s_3} \left[ \nabla^{\mu_1} \dots \nabla^{\mu_{s_3}} \varphi_{\nu_1 \dots \nu_{s_1}} \nabla^{\nu_1} \dots \nabla^{\nu_{s_2}} \varphi_{\rho_1 \dots \rho_{s_2}} \nabla^{\rho_1} \dots \nabla^{\rho_{s_3}} \varphi_{\mu_1 \dots \mu_{s_1}} + \mathcal{O}(\Lambda) \right], \quad (1.6)$$

where the terms of  $\mathcal{O}(\Lambda)$  and higher in the bracket are descending in the number of



derivatives, and are fixed uniquely. The overall coupling is given by

$$g_{s_1, s_2, s_3} = \frac{1}{\sqrt{N}} \frac{\pi^{\frac{d-3}{4}} 2^{\frac{3d-1}{2}} (-2\Lambda)^{(s_1+s_2+s_3)/2}}{\Gamma(d+s_1+s_2+s_3-3)} \sqrt{\frac{\Gamma(s_1+\frac{d-1}{2}) \Gamma(s_2+\frac{d-1}{2}) \Gamma(s_3+\frac{d-1}{2})}{\Gamma(s_1+1) \Gamma(s_2+1) \Gamma(s_3+1)}}. \quad (1.7)$$

Put in the context of existing results for cubic vertices of higher-spin gauge fields, the above result is new in the sense that the coefficients (1.7) together with the precise combination of structures in (1.6) were previously unknown in any higher-spin theory on AdS.

The quartic self interaction of the scalar takes the schematic form

$$\mathcal{V}_{0,0,0,0}(\varphi_0) = \sum_{r,n} a_{r,n} (\varphi_0 \nabla_{\mu_1} \dots \nabla_{\mu_r} \varphi_0 + \dots) \square^n (\varphi_0 \nabla^{\mu_r} \dots \nabla^{\mu_1} \varphi_0 + \dots), \quad r \in 2\mathbb{N}, \quad (1.8)$$

where the  $\dots$  denote a finite number of terms with no more than  $r$  derivatives. As opposed to the cubic vertices in  $\text{AdS}_{d+1}$  above, to extract the quartic interaction we work in  $\text{AdS}_4$ , in which the conformally coupled scalar affords certain simplifications. The coefficients in the derivative expansion can be expressed in terms of a generating function  $a_r(z) = \sum_n a_{r,n} z^n$ , whose explicit form is quite involved:

$$a_r(z) = \frac{\Lambda r}{N} \frac{\left( \pi e^{\gamma(2r+1)} \Gamma(r)^2 \left( (2r+1)^2 + \left( \Lambda z + r + \frac{9}{4} \right)^2 \right)^2 P\left(r + \frac{1}{2}; z\right)^2 - 4^2 \right)}{2^{r-4} \Gamma(r)^2 \left( \left( \Lambda z + r + \frac{9}{4} \right)^2 + \left( r - \frac{1}{2} \right)^2 \right)} \\ - \frac{i^r \pi^2}{2\sqrt{2}N} \frac{\Gamma\left(r + \frac{3}{2}\right)}{\Gamma\left(\frac{r}{2} + 1\right)^2} e^{\gamma(2r+\frac{5}{2})} P\left(r + \frac{1}{2}; z\right) P\left(\frac{3}{4}; z\right) \\ \times \left( (2r+1)^2 + \left( 2\Lambda z + 2r + \frac{9}{2} \right)^2 \right) \left( 9 + \left( 2\Lambda z + 2r + \frac{9}{2} \right)^2 \right),$$

where we introduced

$$P(a|z) = \prod_{k=0}^{\infty} \left[ \left(1 + \frac{a}{k}\right)^2 + \left(\frac{4\Lambda z + 4r + 9}{4k}\right)^2 \right] e^{-2a/k}, \quad (1.9)$$

and  $\gamma$  is the Euler-Mascheroni constant.

Consistent with standard expectations, the quartic interaction (1.8) is non-local in the sense that it has an unbounded number of derivatives. Drawing on the interpretation of CFT correlation functions as scattering amplitudes in AdS, the non-locality of a vertex can be quantified by studying the contributions in the conformal partial wave expansion of its four-point amplitude. Our findings suggest that the holographic duality rules out the appearance of certain pathological non-localities, however there are some subtleties which remain to be clarified. This is discussed in the final chapter of this work.

### Intermediate results

In obtaining the above results, we took a number of intermediate steps which may be of interest / applicable in a broader context. For example:

- Techniques to evaluate tree-level three-point amplitudes in AdS, involving external fields of arbitrary integer spin.
- Propagators in AdS for gauge fields of arbitrary integer spin, in a basis of harmonic functions.
- Methods to decompose a general tree-level four-point amplitude in AdS into conformal partial waves.

### Ambient space

We emphasise that obtaining many of the results in this work was facilitated by the application of the “ambient space formalism”. This is a powerful approach, with the

basic idea being that fields in Euclidean  $\text{AdS}_{d+1}$ , or their dual CFT operators, can be expressed in terms of fields in an ambient Minkowski space  $\mathbb{M}^{d+2}$ . The action of the  $\text{AdS}_{d+1}$  isometry group, or of the conformal group  $\text{SO}(d+1, 1)$ , can then be realised as the group of linear Lorentz transformations. For calculations in AdS, a key feature of this approach is the ability to represent expressions that are intrinsic to the AdS manifold (e.g. involving non-commuting covariant derivatives) in terms of simpler-flat space ones of the ambient space (e.g. commuting partial derivatives). Because of these simplifying features, the ambient framework has enjoyed a wide variety of applications in the AdS, higher-spin and conformal field theory literature. See for example: [134–152]

### 1.3.2 Outline

The outline of this thesis is as follows:

In chapter 2 we explain our approach to holographically reconstruct higher-spin interactions in detail. We review the precise dictionary between CFT correlation functions and amplitudes in the dual gravity theory on AdS, as well as the identification of CFT operators with fields in AdS. We further elaborate on the relevant aspects of the duality between higher-spin theories on AdS and free CFTs.

In chapter 3 we give a self contained review of the techniques in CFT which are relevant for this work. We recall how the conformal symmetry constrains two-, three- and four-point functions of operators, paying particular attention to operators of arbitrary integer spin. We further review the conformal partial wave expansion of four-point correlation functions, and introduce useful techniques for later application.

In chapter 4 we introduce the free scalar  $O(N)$  vector model. We apply the discussion on CFT correlators in chapter 3, together with useful tools to apply Wick's theorem, to establish the two- and three-point functions of all single-trace conserved currents

of arbitrary spin in the theory. We also compute the four-point function of the scalar single-trace operator, and determine all coefficients in its operator product expansion. With the latter result, we determine the conformal partial wave expansion and express it in an integral form for later application. This chapter is based on aspects of the original works: [2, 4].

Chapter 5 is dedicated to the computation of Witten diagrams. We review existing results for three-point Witten diagrams involving external scalars, and introduce an approach to extend them to external fields of arbitrary integer spin. We then turn to four-point Witten diagrams, deriving bulk-to-bulk propagators for gauge fields of arbitrary spin in a basis of AdS harmonic functions. Techniques are developed to decompose a general four-point Witten diagram into conformal partial waves. We apply the latter techniques to Witten diagrams with identical external scalars, including four-point exchanges of gauge fields with arbitrary spin and a general contact interaction dressed with derivatives. This chapter is based on aspects of the original works: [1, 2, 4].

Chapter 6 is a culmination of the preceding chapters, in which the intermediate results established throughout are put together to extract all cubic interactions in the minimal bosonic higher-spin theory on  $\text{AdS}_{d+1}$ , and the quartic self-interaction of the scalar for the theory on  $\text{AdS}_4$ . This chapter is based on aspects of the original works: [2, 4].

In chapter 7 we summarise the main results of this work, and discuss the non-locality of higher-spin interactions from a holographic perspective. This chapter is based on on-going work, and aspects of the original works: [2, 3].

Various technical details are relegated to the appendices (A-H). In particular, a detailed review of the ambient formalism is given in appendix A.

## Chapter 2

# The AdS/CFT correspondence

As stated in the introduction, the AdS/CFT correspondence is the conjecture that quantum gravity in asymptotically anti-de Sitter spacetime  $\text{AdS}_{d+1}$  is equivalent to a non-gravitational conformal field theory (CFT) in a flat  $d$ -dimensional spacetime,

$$\text{AdS}_{d+1} \text{ QG} = \text{CFT}_d. \quad (2.1)$$

The aim of this chapter is to review the most pertinent aspects of this duality, relevant for the applications this thesis.

### 2.1 Practicalities

For practical applications, the holographic correspondence is more conveniently formulated in terms of the generating functions of the two dual theories, in Euclidean signature.

In a CFT, the generating function  $F_{\text{CFT}}[\bar{\varphi}]$  of connected correlators admits the path-integral representation ,

$$\exp(-F_{\text{CFT}}[\bar{\varphi}]) = \int D\phi \exp\left(-S_{\text{CFT}}[\phi] + \int d^d y \bar{\varphi}(y) \mathcal{O}(y)\right). \quad (2.2)$$

We use  $\phi$  to denote the fundamental field(s) in the theory, governed by the conformally invariant action  $S_{\text{CFT}}[\phi]$ . The operator  $\mathcal{O}$  is built from the fields  $\phi$ , and is sourced by  $\bar{\varphi}(y)$ . The source is not dynamical, but a fixed function which is under our control.<sup>1</sup>

The basic idea behind the holographic duality (2.1) is to breathe life into the source  $\bar{\varphi}$ . We can think of the  $\text{CFT}_d$  as living on the boundary of the higher-dimensional  $\text{AdS}_{d+1}$ , in which the source is promoted to a dynamical field  $\varphi(y, z)$  governed by the bulk action  $S_{\text{AdS}}[\varphi]$  at the classical level. The only control we have over the field is the boundary value  $\bar{\varphi}(y)$  at  $z = 0$ . The duality then states that the physical quantity  $F_{\text{CFT}}[\bar{\varphi}]$  in the CFT coincides with the AdS one  $\Gamma_{\text{AdS}}[\bar{\varphi}]$ , [45, 46]. I.e.  $F_{\text{CFT}}[\bar{\varphi}] = \Gamma_{\text{AdS}}[\bar{\varphi}]$ , with

$$\exp(-\Gamma_{\text{AdS}}[\bar{\varphi}]) = \int_{\varphi|_{\partial\text{AdS}}=\bar{\varphi}} \mathcal{D}\varphi \exp\left(-\frac{1}{G}S_{\text{AdS}}[\varphi]\right), \quad (2.3)$$

and  $G$  the gravitational constant. We make the above identification more precise in the following section.

The crucial point, central to the work behind this thesis, is that connected correlators in the CFT can be computed holographically by instead functionally differentiating the bulk quantity  $\Gamma_{\text{AdS}}[\bar{\varphi}]$

$$\langle \mathcal{O}(y_1) \dots \mathcal{O}(y_n) \rangle_{\text{conn.}} = (-1)^n \frac{\delta}{\delta \bar{\varphi}(y_1)} \dots \frac{\delta}{\delta \bar{\varphi}(y_n)} \Gamma_{\text{AdS}}[\bar{\varphi}] \Big|_{\bar{\varphi}=0}, \quad (2.4)$$

with each differentiation sending a  $\varphi(y, z)$  particle into AdS.

Quantum effects in the gravity theory on AdS are measured in the dual CFT by the number of degrees of freedom

$$\left(\frac{R}{\ell_p}\right)^{d-1} \sim N_{\text{dof}}. \quad (2.5)$$

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<sup>1</sup>For example, it might be a background electric or magnetic field, or a background pressure density.

This can be understood by the fact that we can roughly measure the degrees of freedom in a CFT by the overall coefficient  $C_T$  of the energy momentum tensor two-point function.<sup>2</sup> From the preceding discussion, under the holographic duality we identify

$$\langle TT \rangle_{\text{CFT}} \sim C_T \sim N_{\text{dof}} \quad \leftrightarrow \quad \langle gg \rangle_{\text{AdS}} \sim \frac{R^{d-1}}{G} \sim \left( \frac{R}{\ell_p} \right)^{d-1}, \quad (2.6)$$

since the CFT energy momentum tensor is dual to the graviton in AdS (§2.1.1). This gives the relation (2.5).

The weak coupling expansion  $G \ll R$  in the bulk, with dominant contribution<sup>3</sup>

$$\exp(-\Gamma_{\text{AdS}}[\bar{\varphi}]) \approx \exp\left(-\frac{1}{G}S_{\text{AdS}}[\varphi]\right)\Big|_{\varphi|\partial\text{AdS}=\bar{\varphi}}, \quad (2.7)$$

therefore corresponds to a large  $N_{\text{dof}}$  expansion in the dual CFT, controlled by the dimensionless coupling

$$g := R^{1-d}G \sim 1/N_{\text{dof}}. \quad (2.8)$$

From the corresponding  $1/N_{\text{dof}}$  expansion admitted by  $F_{\text{CFT}}$ ,

$$F_{\text{CFT}} = N_{\text{dof}} F_{\text{CFT}}^{(0)} + F_{\text{CFT}}^{(1)} + \frac{1}{N_{\text{dof}}} F_{\text{CFT}}^{(2)} + \dots, \quad (2.9)$$

we can identify

$$\exp\left(-N_{\text{dof}} F_{\text{CFT}}^{(0)}[\bar{\varphi}]\right) = \exp\left(-\frac{1}{G}S_{\text{AdS}}[\varphi]\right)\Big|_{\varphi|\partial\text{AdS}=\bar{\varphi}}. \quad (2.10)$$

Correlators in the CFT may thus be computed in the large  $N_{\text{dof}}$  limit via a tree-level Feynman diagram expansion in AdS space.

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<sup>2</sup>Consider for example free theory: The stress tensors add, so increasing the number of fields (degrees of freedom) increases the two-point function coefficient.

<sup>3</sup>I.e. neglecting loops.

Equation (2.10) is essential for the results established in this work. It is from this equation that we extract interactions in the action of the minimal bosonic higher-spin theory from correlation functions in its dual CFT, as will be explained in more detail in the following sections. To this end, we first make more precise the identification between operators in CFT and fields on AdS in the large  $N_{\text{dof}}$  limit.

### 2.1.1 Field-operator map

What does it take for an operator  $\mathcal{O}$  of scaling dimension  $\Delta$  and spin- $s$  to be described by some field  $\varphi$  in an asymptotically  $\text{AdS}_{d+1}$  space? First of all, the CFT operator must be gauge invariant. This restricts the spectrum of local operators to be composed of traces. In the large  $N_{\text{dof}}$  limit, single-trace operators are identified with single particle states in the gravity theory, while multi-trace operators with multi-particle states.

An important feature of the AdS/CFT duality is that both theories are governed by the same symmetry group: The conformal group is isomorphic to  $SO(d, 2)$ , which is the isometry group of AdS. A second natural requirement for CFT operators and bulk fields to be identified, is then that they sit in the same representation of  $SO(d, 2)$ . This has the implication that  $\varphi$  is also spin- $s$ . Furthermore, it must transform in the same way as  $\mathcal{O}$  under dilatations, which correspond to boundary limit of the energy generators  $E$  in AdS:

The action of the dilatation generator  $D$  on  $\mathcal{O}$  is derived in §3.2, and is given by

$$[D, \mathcal{O}(y)] = -i(\Delta + y \cdot \partial) \mathcal{O}(y). \quad (2.11)$$

It acts on the bulk field  $\varphi$  through a Lie derivative with respect to the AdS Killing field



$E$  (which can be extracted from (A.7),  $E = J_{(d+1)d}$ ):<sup>4</sup>

$$\mathcal{L}_E \varphi_{\mu_1 \dots \mu_s} = -i(z \cdot \partial_z + y \cdot \partial_y + s) \varphi_{\mu_1 \dots \mu_s}. \quad (2.13)$$

Comparing the above with (2.11), to preserve conformal invariance as  $z \rightarrow 0$ , we require for boundary directions  $i$

$$(z \cdot \partial_z + s) \varphi_{i_1 \dots i_s} \sim \Delta \varphi_{i_1 \dots i_s} \implies \varphi_{i_1 \dots i_s} \sim z^{\Delta-s}. \quad (2.14)$$

The coefficient of  $z^{\Delta-s}$  is identified with the expectation value  $\langle \mathcal{O}(y) \rangle$ , since both have the same transformation properties under the conformal group.

The source  $\bar{\varphi}$  of  $\mathcal{O}$  has scaling dimension  $d - \Delta$ , which can be verified by demanding that (2.2) is invariant under dilatations. For  $\varphi$  to describe the source, we thus require<sup>5</sup>

$$\varphi^{i_1 \dots i_s} \sim z^{d+s-\Delta} \quad \text{as } z \rightarrow 0, \quad (2.15)$$

with the coefficient of  $z^{d+s-\Delta}$  given by the source  $\bar{\varphi}$ .

The above near-boundary behaviour places restrictions on the mass of the field  $\varphi_s$ , which can be seen as follows. At the linearised level, we have

$$(-\square + m_s^2) \varphi_{\mu_1 \dots \mu_s} = 0. \quad (2.16)$$

To extract the boundary behaviour, we need only consider the asymptotic form of the

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<sup>4</sup>For concreteness we work in Poincaré co-ordinates for  $\text{AdS}_{d+1}$

$$ds^2 = \frac{R^2}{z^2} (dz^2 + dy_i dy^i), \quad (2.12)$$

with  $R$  the AdS radius. The boundary of AdS is located at  $z = 0$ , with boundary directions  $y^i$ ,  $i = 1, \dots, d$ .

<sup>5</sup>Note that here we do not have  $-s$  (unlike in (2.14)) since the  $\mu$  indices are *raised* and receive a minus sign in the Lie derivative.

equation of motion as  $z \rightarrow 0$ , which in the boundary directions reads

$$\begin{aligned}
 - \left(\frac{z}{R}\right)^{d+1} \partial_z \left( \left(\frac{R}{z}\right)^{d-1} \partial^z \varphi_{i_1 \dots i_s} \right) - \frac{2sz}{R^2} \partial_z \varphi_{i_1 \dots i_s} \\
 + \frac{sd}{R^2} \varphi_{i_1 \dots i_s} + \left( m_s^2 - \frac{1}{R^2} s(s-1) \right) \varphi_{i_1 \dots i_s} = 0.
 \end{aligned}
 \tag{2.17}$$

The solution is given in terms of two integration constants  $A_{i_1 \dots i_s}(y)$  and  $B_{i_1 \dots i_s}(y)$ ,<sup>6</sup>

$$\varphi_{i_1 \dots i_s} \sim A_{i_1 \dots i_s}(y) z^{\Delta-s} + B_{i_1 \dots i_s}(y) z^{d-\Delta-s},
 \tag{2.18}$$

where  $\Delta$  satisfies

$$\Delta(\Delta - d) - s = m_s^2 R^2.
 \tag{2.19}$$

Should this hold, from the above discussion we can thus interpret  $A_{i_1 \dots i_s}(y)$  as the expectation value  $\langle \mathcal{O}_{i_1 \dots i_s}(y) \rangle$  in the presence of the source  $B_{i_1 \dots i_s}(y) = \bar{\varphi}_{i_1 \dots i_s}(y)$ .

To summarise, small excitations around a given CFT state are dual to perturbations of the gravitational background, where the latter satisfy equations of motion derived from the gravitational action.

### 2.1.2 Gauge fields and conserved currents

A special entry in the field-operator dictionary is reserved for operators/fields with spin- $s$  and dimension/energy  $\Delta = s + d - 2$ . Such representations of  $SO(d, 2)$  are *short* representations due to the appearance of zero norm states in the Fock space/ The latter form an invariant submodule, which can be factored out. This leaves us with a “shorter” unitary representation. For the bulk fields, this multiplet shortening can be

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<sup>6</sup>This can be obtained by Fourier transforming with respect to the boundary directions, and making an ansatz of the type  $\hat{\varphi}_s(p, z) \sim \hat{A}(p) z^{\Delta+s}$ .

seen as the emergence of a gauge symmetry

$$\delta\varphi_{\mu_1\dots\mu_s} = \nabla_{(\mu_1} \xi_{\mu_2\dots\mu_s)}. \quad (2.20)$$

In the dual CFT, this shortening manifests itself in the fact that the corresponding operators are *conserved*. This is explained further in §3.3.2, however for now let us note that invariance under the transformations induced by the bulk gauge symmetry (2.20) ensures conservation of the dual CFT operator:

$$0 = \delta \int_{\partial\text{AdS}} d^d y \bar{\varphi}^{i_1\dots i_s} \mathcal{O}_{i_1\dots i_s} = - \int_{\partial\text{AdS}} d^d y \xi^{i_2\dots i_s} \partial^{i_1} \mathcal{O}_{i_1\dots i_s} \implies \partial^{i_1} \mathcal{O}_{i_1\dots i_s} = 0.$$

Such representations saturate the unitarity bound for spin- $s > 0$  unitary irreducible representations of  $SO(d, 2)$ . States with lower dimension / energies will have negative norm, and should be excluded from the physical spectrum. This is justified from a CFT perspective in §3.3.2.

As a final comment, note that the above discussion applies only to representations of non-zero spin. The celebrated result for the unitarity bound of scalar fields in AdS was derived earlier by Breitenlohner and Freedman [153]. It reads

$$\Delta \geq \frac{1}{2}(d-2), \quad (2.21)$$

or more familiarly in terms of the mass of the bulk scalar field,

$$m^2 > - \left( \frac{d}{2R} \right)^2. \quad (2.22)$$

This is known as the Breitenlohner-Freedman (or BF) bound, and informed us that fields can be a little bit tachyonic in AdS and still be stable.

## 2.2 Higher-spin / vector model holography

With the dictionary between CFT operators and bulk fields in place, we are ready to explore the holographic duality in more detail. As outlined in the introduction, in this work we are interested in studying interactions in the high energy limit  $\alpha'/R^2 \rightarrow \infty$  through the holographic looking-glass. The logic is clear: In this regime the dual CFT with coupling  $\lambda$  is free (recall (1.4))

$$\lambda \sim \left(\frac{R^2}{\alpha'}\right)^{d/2} \rightarrow 0, \quad (2.23)$$

and one may then hope that its simplicity can be used to help us gain some insight in the bulk.

Let us first consider this limit in the context of well-known conjectured equivalence between  $\mathcal{N} = 4$  Super-Yang Mills in  $d = 3 + 1$  dimensions with  $SU(N)$  gauge group, and the type IIB super-string on  $\text{AdS}_5 \times S^5$ . Amongst the single-trace operators in the former, in the free limit there emerges an infinite tower of conserved currents of increasing spin. For example, bi-linears of the schematic form<sup>7 8</sup>

$$\mathcal{J}_{i_1 \dots i_s} = \frac{1}{N} \sum_{a=1}^6 \text{Tr} [\phi^a \partial_{(i_1} \dots \partial_{i_s)} \phi^a] + \dots, \quad s = 2, 4, 6, \dots, \quad (2.24)$$

where the  $\dots$  denote terms which make the operator a primary one (§3.2.1), and the  $\phi^a$  are the six scalar fields<sup>9</sup> in the adjoint representation of  $SU(N)$ . Recalling the field-operator map for conserved currents (§2.1.2), each conserved current (2.24) is dual to a gauge field of the same spin in the bulk. In this way the correspondence confirms

<sup>7</sup>Note that we use the partial derivative  $\partial$  as opposed to the gauge-covariant derivative  $\nabla$ , since we are working in the free theory.

<sup>8</sup>Since the scalar  $\phi^a$  has scaling dimension  $\frac{d}{2} - 1$ , the quantum numbers of (2.24) coincide with that for a spin- $s$  conserved current §3.3.2.

<sup>9</sup>Analogous gauge-invariant bi-linears also exist for the  $N^2 - 1$  free gauge fields and  $4(N^2 - 1)$  free complex fermions. At the free level different fundamental fields propagate independently so it is consistent to restrict attention to a subset of them.

that string theory in the regime  $\alpha'/R^2 \rightarrow \infty$  possesses an infinite tower of gauge fields: one for each spin  $s$ . As these states saturate the unitarity bound, they have the lowest mass for given spin and therefore lie on the first Regge trajectory.

Since the fields in  $\mathcal{N} = 4$  SYM are matrix valued, the single-trace operators in the theory are not restricted to bi-linears in the elementary fields. In general, any single-trace operator built from  $\phi^a$  can be expressed as a linear combination of the operators

$$\frac{1}{N^{n/2}} \text{Tr} [(\partial^{i_1} \dots \partial^{i_{m_1}} \phi^{a_1}) (\partial^{j_1} \dots \partial^{j_{m_2}} \phi^{a_2}) \dots (\partial^{k_1} \dots \partial^{k_{m_n}} \phi^{a_n})], \quad (2.25)$$

which are multi-linear in the elementary fields. These correspond to the exponentially growing number of single-particle states on sub-leading (for  $n > 2$ ) Regge trajectories in the bulk AdS theory, forming matter multiplets of the higher-spin algebra.

While the above set up gives a complete holographic description of the high-energy limit of string theory, the study of interactions is complicated by the sheer magnitude of the spectrum. A crucial simplifying observation is that the bi-linears (2.24) form a closed subsector in the free theory at  $\lambda \equiv 0$ .<sup>10</sup> This suggests that it is possible to consistently restrict attention to the first Regge trajectory in the tensionless limit  $\alpha'/R^2 \rightarrow \infty$ ,<sup>11</sup> which contains a single gauge field for each integer spin. The dynamics of the first Regge trajectory in this regime is thus captured by non-linear theories of higher-spin gauge fields on AdS,<sup>12</sup> and their study may shed some light on string theory in the tensionless limit (or equivalently, its maximally symmetric phase).

From a CFT perspective, the proliferation of single-trace operators (2.25) is a consequence of the elementary fields sitting in the adjoint of  $SU(N)$ . A toy model

<sup>10</sup>By this we mean that the OPEs of the bi-linears (2.24) contain no contributions from single-trace operators with  $k > 2$ . This can be confirmed by direct computation of the OPE coefficients via Wick's theorem, or see [81].

<sup>11</sup>In fact, this was verified recently [82] for the case of string theory on  $\text{AdS}_3 \times S^3 \times \mathbb{T}^4$ .

<sup>12</sup>See [95, 154–164] for a selection of reviews on various aspects of higher-spin theories.

of the above scenario was proposed by Szegin and Sundell [70], and Klebanov and Polyakov [71], who contemplated the holographic duals of CFTs in which the elementary fields are instead in the *fundamental* representation. In this case the only possible class of single-trace operators are bi-linears in the elementary fields, and one could therefore conceive that they give a holographic description of higher-spin gauge theories on AdS. More concretely, they proposed:

The free scalar  $O(N)$  vector model in  $d$ -dimensions

*is equivalent to*

The type A minimal bosonic higher-spin theory on  $\text{AdS}_{d+1}$ .

The theory on AdS is one of even spin gauge fields, represented by symmetric tensors  $\varphi_s$ , of spins  $s = 2, 4, 6, \dots$ . It also contains a parity even scalar, which we denote by  $\varphi_0$ .

The free scalar  $O(N)$  vector model is a theory of an  $N$  component free real scalar field  $\phi^a$ , transforming in the fundamental representation of  $O(N)$ . The single-trace spectrum consists of a scalar single-trace operator  $\mathcal{O} = \phi^a \phi^a$  of scaling dimension  $\Delta = d - 2$ , as well as an infinite tower of even-spin conserved currents

$$\mathcal{J}_{i_1 \dots i_s} = \frac{1}{\sqrt{N}} \phi^a \partial_{i_1} \dots \partial_{i_s} \phi^a + \dots, \quad \partial^{i_1} \mathcal{J}_{i_1 \dots i_s} = 0, \quad s = 2, 4, 6, \dots, \quad (2.26)$$

of scaling dimension  $\Delta_s = s + \Delta = s + d - 2$ .

While the crucial requirement that the CFT single-trace and bulk single-particle states are in one-to-one correspondence is satisfied, the bulk fields must be prescribed the near-boundary behaviour (§2.1.1):

$$\varphi_{i_1 \dots i_s} \sim z^{\Delta_s - s}, \quad (2.27)$$

in order to be able to identify

$$\begin{aligned}\varphi_s &\leftrightarrow \mathcal{J}_s, \\ \varphi_0 &\leftrightarrow \mathcal{O}.\end{aligned}$$

### Generalisations

The above duality is the simplest of its kind in the context of AdS higher-spin / vector model dualities. In particular, the spectrum of gauge fields in the bulk is truncated to those involving only even spin. The full non-minimal theory contains a higher-spin gauge field for *each* integer spin, and its dual CFT description is given by the free  $U(N)$  invariant theory with a *complex*  $N$  component scalar field  $\phi^a$ . The latter also contains conserved currents of odd spin, which accommodates for the extension of the bulk spectrum to include gauge fields of odd spin. Further generalisations of this set up have been considered, including for instance: Having a parity odd scalar in the bulk (the so-called type-B theory), Chan-Paton factors and supersymmetry [72–74, 76].<sup>13</sup> However in this work we focus on the simplest, most streamlined, case presented above, free from the latter extra decorations.

### Tests of the duality

Before we proceed to applications of the higher-spin / free vector model duality, let us note that although the duality is still a conjecture, its validity has withstood a number of non-trivial checks. Most well known is the heroic tree-level three-point function test by Giombi and Yin [178, 179], who verified using Vasiliev’s system that certain three-point amplitudes in the bulk higher-spin theory on  $\text{AdS}_4$  correctly reproduce the corresponding three-point functions in the free scalar  $O(N)$  vector model in  $3d$ . The duality has also been verified at the level of one-loop vacuum energy [180–183] (see also [184] in the context of adjoint-valued elementary fields).

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<sup>13</sup>Some approaches towards deriving the higher spin/vector model duality from first principles were investigated in [165–177].

### 2.2.1 AdS higher-spin interactions from CFT

As explained in the introduction, there is much to be learnt about interactions in higher-spin gauge theories within a standard Lagrangian framework. This makes the higher-spin / vector model duality even more profound, as it opens up the possibility to study higher-spin interactions on AdS space through the consideration of a drastically simpler (free) CFT. The role of this subsection is to put this idea on a more concrete footing.

Assuming the existence of a standard action principle  $S_{\text{HS AdS}}[\varphi]$  for the minimal bosonic higher-spin theory, we perform a weak field expansion around an AdS background in the power of the fields,<sup>14</sup>

$$S_{\text{HS AdS}}[\varphi] = G S_{\text{HS AdS}}^{(2)}[\varphi] + G^{3/2} S_{\text{HS AdS}}^{(3)}[\varphi] + G^2 S_{\text{HS AdS}}^{(4)}[\varphi] + \dots \quad (2.28)$$

Recalling the holographic equality (2.10), with the knowledge of the leading contribution to the generating function,  $F_{\text{free } O(N)}^{(0)}$ , of connected correlators in the large  $N$  free scalar  $O(N)$  vector model, one may in principle iteratively extract the on-shell interactions in the conjectured classical bulk higher-spin theory:

$$\exp\left(-\frac{1}{\sqrt{N}} F_{\text{free } O(N)}^{(0)}\right) = \prod_{n=2}^{\infty} \exp\left(-\sqrt{G}^{n-2} S_{\text{HS AdS}}^{(n)}[\varphi]\right) \Big|_{\varphi|_{\partial\text{AdS}}=\bar{\varphi}}. \quad (2.29)$$

With the quadratic kinetic term  $S_{\text{HS AdS}}^{(2)}[\varphi]$  known, we begin this procedure at cubic order ( $n = 3$ ). The cubic action of the minimal bosonic higher-spin theory takes the form

$$S_{\text{HS AdS}}^{(3)}[\varphi] = \sum_{s_i=0}^{\infty} \int_{\text{AdS}} \mathcal{V}_{s_1, s_2, s_3}[\varphi_{s_i}], \quad (2.30)$$

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<sup>14</sup>To do so, we re-define the fields  $\varphi \rightarrow \sqrt{G}\varphi$ . This also ensures a canonically normalised kinetic term.



where by applying the Noether procedure the possible structures that can enter the cubic vertex  $\mathcal{V}_{s_1, s_2, s_3}$  between gauge fields of spins  $s_1$ ,  $s_2$  and  $s_3$  have already been determined [108, 109, 113]. In this work, building on the latter results we completely fix the coupling constants between all vertex structures by solving

$$\langle \mathcal{J}_{s_1} \mathcal{J}_{s_2} \mathcal{J}_{s_3} \rangle = \sqrt{G} \frac{\delta}{\delta \bar{\varphi}_{s_1}} \frac{\delta}{\delta \bar{\varphi}_{s_2}} \frac{\delta}{\delta \bar{\varphi}_{s_3}} S_{\text{HS AdS}}^{(3)}[\varphi] \Big|_{\varphi_{s_i}|_{\partial \text{AdS}} = \bar{\varphi}_{s_i}}, \quad (2.31)$$

which equates the three-point correlator of the spin- $s_i$  conserved currents (2.26) in the free scalar  $O(N)$  vector model, to the tree-level three-point amplitude of the dual spin- $s_i$  gauge fields  $\varphi_{s_i}$ , generated by the bulk interaction  $\mathcal{V}_{s_1, s_2, s_3}$ . In particular, to this end we determine the three-point correlators of all single-trace conserved currents  $\mathcal{J}_{s_i}$ , and develop techniques to systematically compute three-point bulk amplitudes involving external fields of arbitrary integer spin.

With the above result for the cubic action, we may then proceed to quartic order. We restrict to the simplest case of solving for the quartic self-interaction of the scalar  $\varphi_0$ , which is dual to the scalar single-trace operator  $\mathcal{O}$ . Moreover, we focus on the set-up in  $\text{AdS}_4$ , where the conformally coupled scalar provides further simplifications (which will become clear in §5.). For clarity, the equation we solve is given diagrammatically below

$$\langle \mathcal{O}(y_1) \mathcal{O}(y_2) \mathcal{O}(y_3) \mathcal{O}(y_4) \rangle_{\text{conn.}} \quad (2.32)$$

$$= \text{Diagram 1} + \sum_s \text{Diagram 2} + \text{t- and u- channels}$$

The diagrammatic equation (2.32) shows the decomposition of a four-point correlator into contact and exchange diagrams. On the left, a circle represents the correlator with four external legs labeled  $\mathcal{O}(y_1)$ ,  $\mathcal{O}(y_2)$ ,  $\mathcal{O}(y_3)$ , and  $\mathcal{O}(y_4)$ . The contact diagram is a circle with four external legs meeting at a central vertex labeled  $\mathcal{V}_{0,0,0,0}$ . This is followed by a plus sign and a sum over  $s$ . The exchange diagram is a circle with four external legs and a wavy internal line connecting two vertices labeled  $\mathcal{V}_{s,0,0}$ . The sum is followed by a plus sign and the text "t- and u- channels".

The diagrams contributing to the right hand side of the equality consist of: the contact

amplitude generated by the quartic interaction  $\mathcal{V}_{0,0,0,0}$  we seek to establish, and the four-point exchanges of each spin- $s$  gauge field between two pairs of the bulk scalar. The iterative nature of this reconstruction procedure is thus apparent, for in order to compute the exchange diagrams we require all cubic interactions between two real scalars and a spin- $s$  gauge field in the theory.

There is quite a large gap in technical difficulty in applying this holographic approach to extracting the cubic vertices  $\mathcal{V}_{s_1, s_2, s_3}$ , and extending it to the quartic vertex above. In particular, in order to match the four-point correlator of the operator  $\mathcal{O}$  with the dual four-point bulk amplitudes as shown in equation (2.32), we put both sides of the equation on an equal footing by decomposing them into conformal partial waves. While it is well understood how to decompose CFT correlators into conformal partial waves (especially in free theories), as an intermediate step we tailor methods to establish conformal partial wave expansions of the bulk amplitudes.

We therefore spend the following chapters gathering the necessary ingredients and sharpening technical tools. This culminates in §6, in which we put together the latter intermediate results, to fix completely the cubic action and the scalar quartic self-interaction in the type A minimal bosonic higher-spin theory as described above.

## Quantum corrections

Let us note that a key feature of free CFTs is that the large  $N_{\text{dof}}$  expansion is *exact*. For example, in the expansion of the generating function of connected correlators,

$$F_{\text{CFT}} = N_{\text{dof}} F_{\text{CFT}}^{(0)} + F_{\text{CFT}}^{(1)} + \frac{1}{N_{\text{dof}}} F_{\text{CFT}}^{(2)} + \dots, \quad (2.33)$$

we have  $F_{\text{CFT}}^{(n \geq 1)} \equiv 0$ . An immediate implication of this property in the conjectured bulk higher-spin theory dual is the absence of any quantum corrections.<sup>15</sup> In other words, the interactions in the minimal bosonic higher-spin theory that we extract from equation (2.29) should (assuming validity of the holographic duality, in particular for finite  $N$ ) also hold in the quantum bulk theory.

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<sup>15</sup>This is a remarkable feature because it necessitates a precise cancellation of infinitely many loop diagrams in the bulk. This has been investigated in a series of papers [180, 182, 183, 185–188], where the simplest example of the vanishing of the one-loop vacuum energy  $F_{\text{CFT}}^{(1)}$  from the bulk was considered.



## Chapter 3

# Correlation functions in CFT

The work of this thesis is to a large part underpinned by techniques and concepts in conformal field theory (CFT). This chapter is intended to give a self-contained overview of the relevant ideas, in particular of the constraints imposed by conformal symmetry on two-, three- and four-point functions of local operators. The first few sections cover the basics of CFT, which can be safely skipped by the knowledgeable reader. Later sections move on to discuss conformal block expansions, and more recent developments relevant for applications in this work.

### 3.1 Conformal algebra and transformations

A *conformal transformation* is a change of co-ordinates that locally rescales the metric tensor,

$$g_{\mu\nu}(x) \rightarrow \Omega^2(x) g_{\mu\nu}(x). \quad (3.1)$$

A conformal field theory is a field theory that is invariant under these transformations, and whose physics looks the same at all length scales as a consequence. In this thesis we are concerned with CFTs defined on a fixed flat background, and for the remainder of this discussion we therefore restrict to the case  $g_{\mu\nu}(x) = \eta_{\mu\nu}$ .

The transformations (3.1) form a group, which for a fixed Minkowski background includes the Poincaré group as a subgroup, since the metric is invariant under Lorentz transformations and translations ( $\Omega(x) = 1$ ). In addition to the latter, the conformal group is generated by scale transformations

$$x^\mu \rightarrow \lambda x^\mu, \quad \Omega(x) = \lambda, \quad \lambda \in \mathbb{R}, \quad (3.2)$$

and special conformal transformations

$$x^\mu \rightarrow \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2}, \quad \Omega(x) = \frac{1}{x^2}. \quad (3.3)$$

The conformal algebra can be determined by considering the infinitesimal form of the above transformations. For example, the generator  $D$  associated to scale transformations can be obtained by considering its action on a test function

$$\delta f(x) = e^{i\lambda D} f(x) e^{-i\lambda D} = \lambda [iD, f(x)] + \mathcal{O}(\lambda^2) = \lambda (x \cdot \partial) f(x) + \mathcal{O}(\lambda^2), \quad (3.4)$$

which implies  $iD = (x \cdot \partial)$ . Similarly, the remaining generators can be determined

$$iP_\mu = \partial_\mu, \quad iM_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad iK_\mu = 2x_\mu (x \cdot \partial) - x^2 \partial_\mu, \quad (3.5)$$

which are associated to translations, special conformal transformations and Lorentz transformations, respectively. These annihilate the conformal vacuum. With the explicit form of the generators, the conformal algebra can be worked out to be

$$\begin{aligned} [D, K_\mu] &= iK_\mu, & [D, P_\mu] &= -iP_\mu, & [P_\mu, K_\nu] &= 2iM_{\mu\nu} - 2i\eta_{\mu\nu}D \\ [M_{\mu\nu}, P_\sigma] &= -i(\eta_{\mu\sigma}P_\nu - \eta_{\nu\sigma}P_\mu), & [M_{\mu\nu}, K_\sigma] &= -i(\eta_{\mu\sigma}K_\nu - \eta_{\nu\sigma}K_\mu), \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}), \end{aligned} \quad (3.6)$$

with all other commutators vanishing.

From an AdS/CFT point of view, perhaps the most important property of the conformal group on  $\mathbb{R}^{d-1,1}$  is that it is isomorphic to  $SO(d, 2)$ , which (as shown in §A) is the isometry group of  $\text{AdS}_{d+1}$ . This can be seen at the level of the generators by defining  $J_{AB}$ ,  $A, B = 0, \dots, d + 1$

$$J_{\mu\nu} = M_{\mu\nu}, \quad J_{\mu d} = \frac{1}{2}(K_\mu - P_\mu), \quad J_{\mu(d+1)} = \frac{1}{2}(K_\mu + P_\mu), \quad J_{(d+1)d} = D, \quad (3.7)$$

which satisfy the  $\mathfrak{so}(d, 2)$  commutation relations,

$$[J_{AB}, J_{CD}] = i(\eta_{BC}J_{AD} + \eta_{AD}J_{BC} - \eta_{AC}J_{BD} - \eta_{BD}J_{AC}), \quad (3.8)$$

where  $\eta_{AB} = (- + + \dots + -)$ .

## 3.2 Representations on operators

With the knowledge of the conformal algebra and its generators, we can classify operators according to their representations. These can be derived as induced representations of the subgroup that leave a point in space-time invariant, the *stability subgroup*.

The stability subgroup is generated by Lorentz generators  $M_{\mu\nu}$ , dilatations  $D$  and special conformal transformations  $K_\mu$ . This can be verified using their explicit forms (3.4) and (3.5). Operators at a fixed space-time point form a representation of this subgroup. On the other hand, the translation generator  $P_\mu$  acts exclusively on the co-ordinates  $x^\mu$ , so that

$$[P_\mu, \mathcal{O}^a(x)] = -i\partial_\mu \mathcal{O}^a(x). \quad (3.9)$$

This implies that an operator  $\mathcal{O}^a$  at a generic point in space-time can be obtained from

its value at, for example, the origin via

$$\mathcal{O}^a(x) = e^{ix \cdot P} \mathcal{O}^a(0) e^{-ix \cdot P}. \quad (3.10)$$

Therefore, the representation of an operator under the full conformal algebra can be determined (or *induced*) from its representation under the stability subalgebra through the action (3.10) of the translation generator. To wit, for a conformal generator  $G$

$$[G, \mathcal{O}^a(x)] = e^{ix \cdot P} [\tilde{G}, \mathcal{O}^a(0)] e^{-ix \cdot P}, \quad (3.11)$$

with

$$\tilde{G} = e^{-ix \cdot P} G e^{ix \cdot P} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \underbrace{[x \cdot P, [x \cdot P, \dots, [x \cdot P, G] \dots]]}_{n \text{ times}}, \quad (3.12)$$

using the Baker-Campbell-Hausdorff formula. In practice, for all generators of the conformal algebra this series truncates at either the first or second term.

### Warm up: Poincaré Representations

In a Poincaré invariant QFT, local operators at the origin transform in irreducible representations of the Lorentz group,

$$[M_{\mu\nu}, \mathcal{O}^a(0)] = (\Sigma_{\mu\nu})^a_b \mathcal{O}^b(0). \quad (3.13)$$

This defines the spin of the operator  $\mathcal{O}^a$ , given matrices  $\Sigma_{\mu\nu}$  forming a finite dimensional representation of the Lorentz algebra. Away from the origin, we then have

$$\begin{aligned} [M_{\mu\nu}, \mathcal{O}^a(x)] &= e^{ix \cdot P} [e^{-ix \cdot P} M_{\mu\nu} e^{ix \cdot P}, \mathcal{O}^a(0)] e^{-ix \cdot P} \\ &= e^{ix \cdot P} [M_{\mu\nu} + x_\mu P_\nu - x_\nu P_\mu, \mathcal{O}^a(0)] e^{-ix \cdot P} \\ &= (\Sigma_{\mu\nu}^a_b - ix_\mu \partial_\nu + ix_\nu \partial_\mu) \mathcal{O}^b(x), \end{aligned} \quad (3.14)$$



where in the second equality we employed (3.12).

### Dilatations and special conformal transformations

In a scale invariant theory, it is natural to diagonalise the dilatation operator acting on operators at the origin,

$$[D, \mathcal{O}(0)] = -i\Delta \mathcal{O}(0). \quad (3.15)$$

$\Delta$  is referred to as the *scaling dimension* of  $\mathcal{O}$ .

Applying again the Baker-Campbell-Hausdorff formula, we find

$$[D, \mathcal{O}(x)] = -i(\Delta + x \cdot \partial) \mathcal{O}(x). \quad (3.16)$$

Likewise, for the special conformal transformation,

$$[K_\mu, \mathcal{O}(x)] = (2x^\nu \Sigma_{\nu\mu} + 2i\Delta x_\mu - ix^2 \partial_\mu + ix_\mu x \cdot \partial + \mathcal{K}_\mu) \mathcal{O}(x), \quad (3.17)$$

where we define

$$[K_\mu, \mathcal{O}(0)] = \mathcal{K}_\mu \mathcal{O}(0). \quad (3.18)$$

#### 3.2.1 Primaries and descendants

In fact,  $K_\mu$  is a lowering operator for scaling dimension,

$$\begin{aligned} [D, [K_\mu, \mathcal{O}(0)]] &= [[D, K_\mu] + K_\mu D, \mathcal{O}(0)] \\ &= -i(\Delta - 1) [K_\mu, \mathcal{O}(0)]. \end{aligned} \quad (3.19)$$

As we shall see in §3.3.2, for unitary CFTs scaling dimensions are bounded from below. This implies that each representation of the conformal algebra must have an operator

of lowest dimension, which is then annihilated by  $K_\mu$  at  $x = 0$ ,

$$[K_\mu, \mathcal{O}(0)] = \mathcal{K}_\mu \mathcal{O}(0) = 0. \quad (3.20)$$

This is the definition of a *primary operator*.

Translation generators on the other hand are raising operators for scaling dimension,

$$[D, [P_\mu, \mathcal{O}(0)]] = -i(\Delta + 1)[P_\mu, \mathcal{O}(0)]. \quad (3.21)$$

Given a primary operator  $\mathcal{O}$  of dimension  $\Delta$ , we can construct operators of higher dimension by acting with translation generators,

$$\mathcal{O}(0) \rightarrow (-i)^n \partial_{\mu_1} \dots \partial_{\mu_n} \mathcal{O}(0), \quad \Delta \rightarrow \Delta + n. \quad (3.22)$$

These operators are called *descendants* of the operator  $\mathcal{O}$ . A representation of the conformal algebra is thus constructed from  $\mathcal{O}(0)$  and its descendants.

### Finite conformal transformations

In the preceding section, we derived the action of infinitesimal conformal transformations on operators, given by (3.9), (3.14), (3.16) and (3.17). For a primary operator of scaling dimension  $\Delta$  and rotation matrices  $\Sigma_{\mu\nu}$ , these can be summarised as

$$[Q_v, \mathcal{O}(x)] = -i \left( v \cdot \partial + \frac{\Delta}{d} (\partial \cdot v) - \frac{i}{2} \partial^\mu v^\nu \Sigma_{\mu\nu} \right) \mathcal{O}(x), \quad (3.23)$$

where  $v_\mu$  is a general conformal Killing vector

$$v_\mu = a_\mu + \omega_{\mu\sigma} x^\sigma + \lambda x_\mu + (b_\mu x^2 - 2x_\mu b \cdot x). \quad (3.24)$$

This corresponds to a combination of infinitesimal translations  $a_\mu$ , rotations  $\omega_{\mu\nu}$ , scalings  $\lambda$  and special conformal transformations parametrised by  $b_\mu$ .

The corresponding finite transformations are obtained by exponentiating the charge,

$$\mathcal{O}'(x') = U \mathcal{O}(x) U^{-1} = \Omega(x')^\Delta D(R(x')) \mathcal{O}(x'), \quad U = e^{Q_v}, \quad (3.25)$$

where

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x') R^\mu{}_\nu(x'), \quad R^\mu{}_\nu \in \text{SO}(d). \quad (3.26)$$

In (3.25),  $D(R)$  is a matrix implementing the action of  $R$  in the  $\text{SO}(d)$  representation of  $\mathcal{O}$ . For example,

$$\begin{aligned} \text{scalar representation:} & & D(R) &= 1 \\ \text{vector representation:} & & D(R)_\mu{}^\nu &= R_\mu{}^\nu \\ & & \vdots & \\ \text{spin-}s \text{ representation:} & & D(R)_{\mu_1 \dots \mu_s}{}^{\nu_1 \dots \nu_s} &= R_{(\mu_1}{}^{\nu_1} \dots R_{\mu_s)}{}^{\nu_s} - \text{traces.} \end{aligned}$$

### 3.3 Radial quantisation

In this section we introduce a particular way of quantising CFTs, which proves useful in establishing various properties of conformally invariant theories. Examples of which, that we rely upon later, include: The state-operator correspondence (§3.3.1), the subsequent convergence of the operator product expansion (§3.4) and unitarity bounds (§3.3.2).

The commutation relations considered in the previous section hold in any quantisation. The standard way to quantise in QFT is to choose a specific space-time foliation, with

each leaf endowed with its own Hilbert space. Hilbert spaces on different leaves are connected by a unitary evolution operator.

In practice, it is convenient to choose quantisations that respect the symmetries of a theory: if all surfaces are related by a symmetry transformation, the Hilbert space is the same on each surface. For example, in a theory with Poincaré symmetry typically we divide space-time into surfaces of equal time. The evolution operator is then the Hamiltonian, which moves us from one fixed time slice to another.

In a theory with conformal symmetry in  $\mathbb{R}^d$ , it is natural to foliate space-time with concentric spheres. States are then evolved from smaller spheres to larger spheres using the dilatation operator. This is naturally parameterised by co-ordinates  $(r, \mathbf{n})$ , where  $r \in \mathbb{R}$  is a radial co-ordinate and  $\mathbf{n}^\mu$  is a unit vector on  $S^{d-1}$ . The space-time metric is then

$$ds^2 = dr^2 + r^2 d\mathbf{n}^2. \quad (3.27)$$

To gain a more intuitive grasp of why the dilatation generator defines an evolution operator, we introduce a time co-ordinate  $t = \ln r$ . The metric then becomes

$$ds^2 = e^{2t} [dt^2 + d\mathbf{n}^2], \quad (3.28)$$

which is conformally equivalent to a cylinder. This maps  $\mathbb{R}^d$  to  $\mathbb{R} \times S^{d-1}$ , with the  $S^{d-1}$  spheres precisely those which define the foliation above. In the Schrödinger picture, states on the cylinder evolve as

$$\frac{\partial}{\partial t} |\psi(t)\rangle = -H |\psi(t)\rangle. \quad (3.29)$$

The crucial observation is that a time translation  $t \rightarrow t + \tau$  on the cylinder is generated by a rescaling  $r \rightarrow e^\tau r$ . In other words, the dilatation operator displaces points along

the time direction of the cylinder,

$$D = -ix \cdot \frac{\partial}{\partial x} \rightarrow -i \frac{\partial}{\partial t}. \quad (3.30)$$

We can hence identify the Hamiltonian  $H$  with the dilatation operator

$$H = iD. \quad (3.31)$$

We can use this observation to quantise a CFT on the cylinder, and in the next section we construct the Hilbert space. Quantising a CFT in this manner is known as *radial quantisation*, which dates back to Fubini, Hanson and Jackiw [189].

### 3.3.1 State-operator correspondence

Having laid down the basic framework of radial quantisation, we now construct a complete basis of states on a sphere of fixed radius. To begin, we assume the existence of a vacuum state  $|0\rangle$  that is invariant under all global conformal transformations

$$P_\mu |0\rangle = K_\mu |0\rangle = M_{\mu\nu} |0\rangle = D |0\rangle = 0. \quad (3.32)$$

The  $SO(d)$  angular momentum operators  $M_{\mu\nu}$  are the only generators that commute with  $D$ . States in the Hilbert space  $\mathcal{H}$  can therefore be classified according to two quantum numbers: Their scaling dimension  $\Delta$  and spin  $s$ ,

$$D|\Delta\rangle = -i\Delta|\Delta\rangle \quad \text{and} \quad M_{\mu\nu}|\Delta, s\rangle^a = (\Sigma)_{\mu\nu}^a{}_b|\Delta, s\rangle^b. \quad (3.33)$$

We can organise  $\mathcal{H}$  in this way by defining states on the sphere through insertions of local operators at its origin,<sup>1</sup>

$$\mathcal{O}_j(x) \rightarrow |\mathcal{O}_j\rangle \equiv \lim_{x \rightarrow 0} \mathcal{O}_j(x) |0\rangle. \quad (3.34)$$

---

<sup>1</sup>In this language, the vacuum is prepared by inserting the identity operator.

Since we diagonalise the dilatation generator acting on operators at the origin (3.15), such states will be eigenstates of  $D$ . We can determine their quantum numbers as follows. For a state created by a primary operator  $\mathcal{O}$  of scaling dimension  $\Delta$ , we have

$$D|\mathcal{O}\rangle = [D, \mathcal{O}(0)]|0\rangle + \mathcal{O}(0)D|0\rangle \quad (3.35)$$

$$= -i\Delta|\mathcal{O}\rangle. \quad (3.36)$$

Similarly, we can conclude

$$M_{\mu\nu}|\mathcal{O}\rangle = \Sigma_{\mu\nu}|\mathcal{O}\rangle \quad \text{and} \quad K_\mu|\mathcal{O}\rangle = 0. \quad (3.37)$$

When we act on  $|\mathcal{O}\rangle$  with the translation generator  $P_\mu$ , it raises the energy (scaling dimension) by one unit

$$DP_\mu|\mathcal{O}\rangle = [D, P]|\mathcal{O}\rangle + PD|\mathcal{O}\rangle \quad (3.38)$$

$$= -i(\Delta + 1)|\mathcal{O}\rangle.$$

This state is associated to the descendant operator  $-i\partial_\mu\mathcal{O}$ , since

$$P_\mu|\mathcal{O}\rangle = \lim_{x \rightarrow 0} [P_\mu, \mathcal{O}(x)]|0\rangle \quad (3.39)$$

$$\lim_{x \rightarrow 0} -i\partial_\mu\mathcal{O}(x)|0\rangle.$$

$$\partial_{\mu_1} \dots \partial_{\mu_n} \mathcal{O}(x) = [iP_{\mu_1}, [iP_{\mu_2}, \dots [iP_{\mu_n}, \mathcal{O}(x)] \dots ]]. \quad (3.40)$$

Owing to conformal symmetry, defining states in this manner accounts for the entire Hilbert space of the CFT. Any state created by an operator inserted at  $x \neq 0$  is a superposition of states (3.34),

$$\mathcal{O}(x)|0\rangle = e^{iPx}\mathcal{O}(0)e^{-iPx}|0\rangle = e^{iPx}|\mathcal{O}\rangle = \sum_n \frac{1}{n!} x^{\mu_1} \dots x^{\mu_n} |\partial_{\mu_1} \dots \partial_{\mu_n} \mathcal{O}\rangle. \quad (3.41)$$

This construction also works backwards: Given a state  $|\psi\rangle$  in radial quantisation, it is natural to decompose it into eigenstates  $|\mathcal{O}_j\rangle$  of the dilatation operator,

$$|\psi\rangle = \sum_j a_j |\mathcal{O}_j\rangle, \quad D|\psi\rangle = -i \sum_j \Delta_j a_j |\mathcal{O}_j\rangle. \quad (3.42)$$

From each eigenstate  $|\mathcal{O}_j\rangle$  we can construct a local operator with dimension  $\Delta_j$ . If the state is annihilated by the superconformal generator  $K_\mu$ , this operator will be a primary one. This feature of CFTs is known as the *state-operator correspondence*: States are in one-to-one correspondence with local operators.

This is a remarkable (and, as we shall see, useful) property of CFTs. In a typical quantum field theory, states and local operators are very different objects. While local operators live at a point in space-time, states live over an entire spatial slice.

### 3.3.2 Unitarity bounds, conserved currents and free scalars

For a theory to be unitarity, all states must have positive norm. This requirement on every state in a conformal multiplet places bounds on the dimensions of primary operators, in a way that we explain in the following. We continue in radial quantisation, which has the convenient feature that  $P_\mu^\dagger = K_\mu$ .<sup>2</sup>

For concreteness, consider the example of a scalar primary operator  $\mathcal{O}$  of dimension  $\Delta$ . The condition that the first level descendant  $P_\mu|\mathcal{O}\rangle$  has positive norm gives

$$(P_\mu|\mathcal{O}\rangle)^\dagger P_\nu|\mathcal{O}\rangle = \langle\mathcal{O}|K_\mu P_\nu|\mathcal{O}\rangle = \langle\mathcal{O}|[K_\mu, P_\nu]|\mathcal{O}\rangle = 2i\langle\mathcal{O}|(\delta_{\mu\nu}D + M_{\mu\nu})|\mathcal{O}\rangle = 2\Delta\delta_{\mu\nu}.$$

where we normalised  $\langle\mathcal{O}|\mathcal{O}\rangle = 1$ . For the above to be positive definite, we require

<sup>2</sup>This is most easily verified on the Minkowskian cylinder, where

$$P_\mu = ie^{-it} \left[ -i\mathbf{n}_\mu \partial_t + (\delta_{\mu\nu} - \mathbf{n}_\mu \mathbf{n}_\nu) \frac{\partial}{\partial \mathbf{n}_\nu} \right], \quad K_\mu = ie^{it} \left[ i\mathbf{n}_\mu \partial_t + (\delta_{\mu\nu} - \mathbf{n}_\mu \mathbf{n}_\nu) \frac{\partial}{\partial \mathbf{n}_\nu} \right].$$

$\Delta \geq 0$ . The operator saturating this bound is the identity operator, which has  $\Delta = 0$ .

Similarly, for spin- $s$  primaries  $\mathcal{O}_s$  one can verify (see e.g. [190]) that the same condition yields

$$(P_\nu|\mathcal{O}_s\rangle)^\dagger P_\mu|\mathcal{O}_s\rangle \text{ is positive definite} \implies \Delta \geq s + d - 2. \quad (3.43)$$

Moreover, the above bound implies that spin- $s$  states with dimension  $\Delta = s + d - 2$  are conserved currents:  $-iP^{\mu_1}|\mathcal{O}_{\mu_1\dots\mu_s}\rangle = 0 \implies \partial^{\mu_1}\mathcal{O}_{\mu_1\dots\mu_s}(x) = 0$ . Conserved currents of arbitrary spin will appear often throughout this work.<sup>3</sup>

For traceless symmetric tensors in generic CFTs, no further conditions arise by considering higher descendants. However, for scalar operators we can go one step further: Considering the second level descendant, it is possible to obtain

$$(P^2|\mathcal{O}\rangle)^\dagger P^2|\mathcal{O}\rangle = 2(2\Delta + 2 - d)\langle\mathcal{O}|K \cdot P|\mathcal{O}\rangle \geq 0 \implies \Delta \geq \frac{d}{2} - 1, \quad (3.44)$$

where it was used that  $\langle\mathcal{O}|K \cdot P|\mathcal{O}\rangle \geq 0$ , following from unitarity at the first level (3.43).

The scalar operator which saturates this bound satisfies the Klein-Gordon equation

$$P^2|\mathcal{O}\rangle = 0 \implies \square\mathcal{O} = 0. \quad (3.45)$$

I.e. it is a free scalar with scaling dimension  $\Delta = \frac{d}{2} - 1$ .

### 3.4 Operator product expansion

An invaluable tool in conformal field theory is the *operator product expansion* (OPE). This is a statement about what happens as local operators approach each other. Although it has a place in any quantum field theory, in a CFT the OPE gains additional

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<sup>3</sup>In fact for CFTs in  $d \geq 3$ , assuming the existence of exactly one stress tensor, the presence of currents with spin  $s \geq 3$  implies that the theory is free [117, 191, 192]. The CFT we consider in this work is precisely of this type.



and very powerful properties.

More concretely, the OPE asserts that the product of any two local operators inserted at nearby points can be closely approximated by a string of operators at one of these points. In a CFT, the OPE can thus be expressed in the form

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_{\mathcal{O}_k \text{ primary}} \mathcal{F}_{12k}(x_{12}, \partial_{x_2}) \mathcal{O}_k(x_2), \quad (3.46)$$

where  $\mathcal{F}_{12k}(x_{12}, \partial_{x_2})$  is a power series in  $\partial_{x_2}$ , and  $x_{ij} = x_i - x_j$ . From a primary operator  $\mathcal{O}_k$ , this generates the contributions of all of its descendants.<sup>4</sup>

The OPE is to be understood as a statement that holds inside correlation functions (within its radius of convergence),

$$\langle \mathcal{X} \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle = \sum_{\mathcal{O}_k \text{ primary}} \mathcal{F}_{12k}(x_{12}, \partial_{x_2}) \langle \mathcal{X} \mathcal{O}_k(x_2) \rangle, \quad (3.47)$$

where  $\mathcal{X}$  denotes any other operator insertions,  $\mathcal{X} = \mathcal{O}_{i_1}(y_1) \dots \mathcal{O}_{i_n}(y_n)$ . This relation can in fact be established using radial quantisation:

Let  $\mathcal{B}$  be a sphere that separates  $\{x_1, x_2\}$  from all other insertion points  $\{y_1, \dots, y_n\}$  in the correlation function (3.47). We can then quantise radially around the centre  $z$  of  $\mathcal{B}$ , inserting a complete basis of states  $\mathbb{1} = \sum_n |n\rangle\langle n|$

$$\langle 0 | \mathcal{X} \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) | 0 \rangle = \sum_n \langle 0 | \mathcal{X} | n \rangle \langle n | \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) | 0 \rangle. \quad (3.48)$$

By the state-operator correspondence, the states  $|n\rangle$  correspond to operators  $\mathcal{O}$  inserted

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<sup>4</sup>For simplicity, we have suppressed all  $SO(d)$  indices.

at  $z$ . We can thus write

$$\langle 0 | \mathcal{X} \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) | 0 \rangle = \sum_{\{\mathcal{O}\}} f_{\mathcal{O}}(x_{12}) \langle \mathcal{X} \mathcal{O}(z) \rangle, \quad f_{\mathcal{O}}(x_{12}) = \langle \mathcal{O} | \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) | 0 \rangle. \quad (3.49)$$

To bring the above in the desired form (3.47), we separate the contributions from the primaries and descendants, and re-sum the descendants. In this way, we obtain

$$\begin{aligned} \langle 0 | \mathcal{X} \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) | 0 \rangle &= \sum_{\mathcal{O}_k \text{ primary}} \tilde{\mathcal{F}}_{12k}(x_{12}, \partial_z) \langle \mathcal{X} \mathcal{O}_k(z) \rangle \\ &= \sum_{\mathcal{O}_k \text{ primary}} \mathcal{F}_{12k}(x_{12}, \partial_{x_2}) \langle \mathcal{X} \mathcal{O}_k(x_2) \rangle, \end{aligned} \quad (3.50)$$

where in the second equality we expanded  $\mathcal{O}_k(z) = \mathcal{O}_k(x_2) + (z - x_2) \cdot \partial_{x_2} \mathcal{O}_k(x_2) + \dots$ .

The above derivation indicates that for the OPE to hold, we require the existence of a sphere which separates  $x_1$  and  $x_2$  from all other insertion points  $y_i$ . In fact, in CFT the OPE is *convergent*, with radius of convergence equal to the distance to the closest insertion point  $y_i$  [193, 194]. This is by virtue of the state-operator correspondence, since convergence of the OPE is then guaranteed by the usual convergence of a complete set of states in quantum mechanics.

The form of the differential operators  $\mathcal{F}(x, \partial)$  appearing in the OPE (3.46) is fixed by conformal invariance. To see this, we simply apply the OPE to the three-point function<sup>5</sup>

$$\begin{aligned} \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_l(y) \rangle &= \sum_{\mathcal{O}_k \text{ primary}} \mathcal{F}_{12k}(x_{12}, \partial_{x_2}) \langle \mathcal{O}_k(x_2) \mathcal{O}_l(y) \rangle \\ &= \mathcal{F}_{12k}(x_{12}, \partial_{x_2}) \langle \mathcal{O}_l(x_2) \mathcal{O}_l(y) \rangle. \end{aligned} \quad (3.51)$$

The  $\mathcal{F}(x, \partial)$  are therefore determined by the two- and three-point functions of the cor-

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<sup>5</sup>This is valid in the domain  $|x_1 - x_2| < |y - x_2|$ . In the second line, we assumed an orthogonal basis of primaries.

responding primary operator, and the structures appearing in both of the latter are fixed by conformal symmetry. This is discussed in detail in the following sections.

With the above property of the OPE, higher-point functions are essentially determined with the knowledge of the two- and three-point functions in the theory. This is discussed in detail for four-point functions of scalar operators in §3.5.3, and plays a key role in this work.

## 3.5 Correlation functions of primary operators

The structure imposed by conformal symmetry on correlation functions is a central tool in obtaining some of the results in this work. In this section we therefore review these constraints in detail for the cases relevant for the applications in this thesis.

The absence of a length scale means the concept of the standard S-matrix ill-defined in a CFT. This makes correlation functions of gauge invariant operators key observables in conformally invariant theories. As we shall see below, conformal symmetry places powerful constraints on the form they can take. We focus on the correlation functions of primary operators, since those involving descendants can be immediately determined from the latter via (3.22).

In a conformally invariant theory, correlation functions are constrained to be invariant under conformal transformations (3.25)

$$\begin{aligned} \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle &= \langle U \mathcal{O}_1(x_1) U^{-1} \dots U \mathcal{O}_n(x_n) U^{-1} \rangle \\ &= \Omega(x'_1)^{\Delta_1} D_1(R(x'_1)) \dots \Omega(x'_n)^{\Delta_n} D_n(R(x'_n)) \langle \mathcal{O}_1(x'_1) \dots \mathcal{O}_n(x'_n) \rangle, \end{aligned} \quad (3.52)$$

where  $D_i(R)$  is the  $\text{SO}(d)$  representation of the primary operator  $\mathcal{O}_i$ , of scaling dimension  $\Delta_i$ . In the following, we investigate the consequences of the above constraints on

the structure of two-, three- and four-point functions.

When considering correlators of operators with spin, the framework of ambient space provides a powerful setting to explore the consequences of conformal symmetry. We give a brief review of the ambient formalism in this chapter, however the full details are given in §A.

### 3.5.1 Two-point functions

As a warm up, consider the two-point function of scalar primary operators  $\mathcal{O}_1$  and  $\mathcal{O}_2$  with scaling dimensions  $\Delta_1$  and  $\Delta_2$ . For rotation and translation invariance, we require

$$\langle \mathcal{O}_1(x) \mathcal{O}_2(y) \rangle = h(|x - y|), \quad (3.53)$$

for some function  $h$ . In a scale-invariant theory with scale-invariant boundary conditions, correlators must be invariant under the action of the dilatation operator  $D$ ,

$$0 = \langle [D, \mathcal{O}_1(x)] \mathcal{O}_2(y) \rangle + \langle \mathcal{O}_1(x) [D, \mathcal{O}_2(y)] \rangle, \quad (3.54)$$

which implies the following differential equation for  $h$

$$i(x \cdot \partial + \Delta_1 + y \cdot \partial + \Delta_2) h(|x - y|) = 0. \quad (3.55)$$

It is straightforward to see that the general solution is

$$h(|x - y|) = \frac{C}{|x - y|^{\Delta_1 + \Delta_2}}, \quad (3.56)$$

for some constant  $C$ . This gives the form of the two-point function in a scale invariant theory.

For theories exhibiting full conformal symmetry, there is a further constraint from

invariance under special conformal transformations:

$$\begin{aligned}
0 &= \langle [K_\mu, \mathcal{O}_1(x)] \mathcal{O}_2(0) \rangle + \langle \mathcal{O}_1(x) [K_\mu, \mathcal{O}_2(0)] \rangle \\
&= i \left( 2\Delta_1 x_\mu - x^2 \partial_\mu + x_\mu x \cdot \partial \right) \frac{C}{|x|^{\Delta_1 + \Delta_2}} \\
&= i (\Delta_1 - \Delta_2) \frac{C}{|x|^{\Delta_1 + \Delta_2}},
\end{aligned} \tag{3.57}$$

where in the second equality we used the fact that  $\mathcal{O}_2$  is primary. For a non-trivial two-point function, we therefore require  $\Delta_1 = \Delta_2$ .

To summarise, two-point functions of scalar operators with dimensions  $\Delta_1$  and  $\Delta_2$  in a CFT take the form

$$\langle \mathcal{O}_1(x) \mathcal{O}_2(y) \rangle = C \frac{\delta_{12}}{|x-y|^{\Delta_1 + \Delta_2}}, \tag{3.58}$$

for some constant  $C$ , and Kronecker delta  $\delta_{12}$ .

### Technical interlude

When considering correlators of operators with spin, a direct generalisation of the approach used for the scalar operators above becomes increasingly involved as the spin increases. Before considering the analogous constraints on the two-point functions of such operators, we make a brief pause to review some useful tools for dealing with spinning operators. The full details can be found in §A.

First, when considering operators with spin it is convenient to encode them in polynomials of auxiliary vectors. For example, a spin- $s$  primary operator  $\mathcal{O}_{\mu_1 \dots \mu_s}$  can be packaged in the polynomial

$$\mathcal{O}_{\mu_1 \dots \mu_s} \quad \rightarrow \quad \mathcal{O}_s(x|z) = \frac{1}{s!} \mathcal{O}_{\mu_1 \dots \mu_s}(x) z^{\mu_1} \dots z^{\mu_s}, \quad z^2 = 0, \tag{3.59}$$

with the null condition on the auxiliary vector  $z$  enforcing tracelessness. In this way,

tensorial operations (e.g. trace and index contractions) are mapped into relatively simple differential operations.

Second, investigating the consequences of conformal invariance is more transparent in the ambient space formalism, in which the  $SO(d, 2)$  symmetry is manifest. The basic idea behind this framework is due to Dirac [195], and is that the natural habitat for the conformal group  $SO(d, 2)$  is an ambient  $(d + 2)$ -dimensional ambient space  $\mathbb{R}^{d,2}$ . Here, the conformal group is realised as the group of linear isometries, and the constraints from conformal symmetry become as trivial as those from Lorentz symmetry. This is provided all CFT fields can be lifted to  $\mathbb{R}^{d,2}$ , which we quickly review below (see §A for full details).

In the ambient framework,  $d$ -dimensional flat space-time is realised as a hypercone in  $\mathbb{R}^{d,2}$ . Points  $x^\mu$  are represented by null rays in  $\mathbb{R}^{d,2}$ , which can be parameterised in light-cone ambient co-ordinates as

$$P^A(x) = (1, x^2, x^\mu), \quad P^A = (P^+, P^-, P^\mu). \quad (3.60)$$

Analogously, the auxiliary variables  $z^\mu$  in (3.59), are represented by

$$Z^A(x) = z^\mu \frac{\partial P^A}{\partial x^\mu} = (0, 2x \cdot z, z^\mu). \quad (3.61)$$

Therefore, the ambient counterpart of the generating function (3.59) for a spin- $s$  primary operator is given by

$$\mathcal{O}_s(P|Z) = \frac{1}{s!} \mathcal{O}_{A_1 \dots A_s}(P) Z^{A_1} \dots Z^{A_s}, \quad (3.62)$$

where to make the mapping one-to-one, we require

$$\mathcal{O}_s(\lambda P|Z) = \lambda^{-\Delta} \mathcal{O}_s(P|Z) \quad \text{and} \quad (P \cdot \partial_Z) \mathcal{O}_s(P|Z) = 0. \quad (3.63)$$

In other words,  $\mathcal{O}_s(P)$  must be homogeneous and tangent to the hypercone  $P^2 = 0$ . As usual,  $\Delta$  represents the scaling dimension of the operator.

### Two-point functions of operators with spin

We turn to applying the above technology to the study of conformal correlators involving operators with spin. We begin in this section with two-point functions.

Generalising the selection rule for scalar two-point functions considered previously, it can be shown that two-point functions of primaries with different spin or conformal dimension vanish. We therefore consider the two-point function of two identical spin- $s$  primary operators  $\mathcal{O}_{\mu_1 \dots \mu_s}$ , of scaling dimension  $\Delta$ . Employing the ambient framework that was reviewed in the previous section, this takes the form

$$\langle \mathcal{O}_s(P_1|Z_1) \mathcal{O}_s(P_2|Z_2) \rangle = H(P_1, P_2|Z_1, Z_2), \quad (3.64)$$

where  $H$  is a function of  $(P_1 \cdot P_2)$ ,  $(P_1 \cdot Z_2)$  and  $(P_2 \cdot Z_1)$ .<sup>6</sup> It must also be a  $SO(d, 2)$  singlet, satisfying homogeneity, tangentiality and spin constraints<sup>7</sup>

$$H(\{\lambda_i P_i\} | \{Z_i\}) = (\lambda_1 \lambda_2)^{-\Delta} H(\{P_i\} | \{Z_i\}), \quad \lambda_1, \lambda_2 > 0, \quad (3.65)$$

$$P_1 \cdot \frac{\partial}{\partial Z_1} H(\{P_i\} | \{Z_i\}) = P_2 \cdot \frac{\partial}{\partial Z_2} H(\{P_i\} | \{Z_i\}) = 0, \quad (3.66)$$

$$H(\{P_i\} | \{\alpha_i Z_i\}) = (\alpha_1 \alpha_2)^s H(\{P_i\} | \{Z_i\}). \quad (3.67)$$

The first condition arises from the fact that the ambient representative of  $\mathcal{O}_{\mu_1 \dots \mu_s}$  is homogeneous of degree  $-\Delta$ . The second condition ensures that  $H$  is tangent to the projective null cone in the  $(d+2)$ -dimensional ambient space. The third is just a fancy way of saying that it is a degree  $s$  polynomial in null auxiliary vectors  $Z_1$  and  $Z_2$ .

<sup>6</sup>Since the  $P_i$  and  $Z_i$  are null vectors, all other possible contractions are zero.

<sup>7</sup>For concision we often abbreviate functions  $\tilde{F}(P_1, \dots, P_n) = \tilde{F}(\{P_i\})$ ,  $i = 1, \dots, n$ .

To satisfy the transversality condition (3.66), it is straightforward to see that the auxiliary vectors  $Z_1$  and  $Z_2$  can only appear through the following building block

$$H_3 = \frac{1}{P_{12}} \left( (Z_1 \cdot Z_2) + \frac{2(Z_1 \cdot P_2)(Z_2 \cdot P_1)}{P_{12}} \right), \quad (3.68)$$

where the overall factor of  $P_{12} = -2P_1 \cdot P_2$  has been chosen for future convenience. The homogeneity and spin- $s$  conditions (3.65) and (3.67), can then be solved up to an overall coefficient as

$$\langle \mathcal{O}_s(P_1|Z_1) \mathcal{O}_s(P_2|Z_2) \rangle = C_{\mathcal{O}} \frac{(H_3)^s}{P_{12}^{\tau_s}}, \quad (3.69)$$

where we have introduced the twist  $\tau_s \equiv \Delta - s$  of a spin- $s$  operator with scaling dimension  $\Delta$ .

### 3.5.2 Three-point functions

Conformal invariance is also sufficiently powerful to fix three-point correlation functions. Like with the two-point functions of operators with spin above, constructing their form is most straightforward and systematic in the ambient formalism. As a warm up, we begin with three-point functions involving only scalar primary operators.

#### Scalar primary operators

Consider the three-point function of scalar primary operators  $\mathcal{O}_i$  with dimensions  $\Delta_i$ ,  $i = 1, 2, 3$ ,

$$\langle \mathcal{O}_1(P_1) \mathcal{O}_2(P_2) \mathcal{O}_3(P_3) \rangle = F(P_1, P_2, P_3). \quad (3.70)$$

Since the  $P_i$  are null vectors  $P_i^2 = 0$ , this is a function only of  $P_{12}$ ,  $P_{13}$  and  $P_{23}$ ,

$$F(P_1, P_2, P_3) \propto P_{23}^{k_1} P_{13}^{k_2} P_{12}^{k_3}, \quad (3.71)$$



for some constants  $k_i$  to be determined.

For scalar operators in the ambient formalism, there is only the homogeneity constraint

$$F(\{\lambda_i P_i\}) = \lambda_1^{-\Delta_1} \lambda_2^{-\Delta_2} \lambda_3^{-\Delta_3} F(\{P_i\}). \quad (3.72)$$

This is enough to fix the  $k_i$  in (3.71) completely,

$$k_1 = \frac{1}{2}(\Delta_1 - \Delta_2 - \Delta_3), \quad k_2 = \frac{1}{2}(\Delta_2 - \Delta_1 - \Delta_3), \quad k_3 = \frac{1}{2}(\Delta_3 - \Delta_2 - \Delta_1). \quad (3.73)$$

Conformal symmetry thus requires the three-point function takes the form

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = \frac{\mathbf{C}_{123}}{(P_{23})^{\frac{\Delta_1 - \Delta_2 - \Delta_3}{2}} (P_{13})^{\frac{\Delta_2 - \Delta_1 - \Delta_3}{2}} (P_{12})^{\frac{\Delta_3 - \Delta_1 - \Delta_2}{2}}}, \quad (3.74)$$

for some multiplicative constant  $\mathbf{C}_{123}$ .

As a final note, while the above form (3.74) is the most general one dictated by conformal symmetry, an additional constraint comes from Bose symmetry. This requires that it is invariant under the exchange of any two operator insertions, from which it follows that  $\mathbf{C}_{ijk}$  is symmetric in its indices  $i$ ,  $j$  and  $k$ .

### Two scalars and a spinning operator

We now turn things up a gear, and consider three-point functions involving primary operators with spin. The simplest case is given by those involving a single spinning operator and two scalars, as it involves just a single tensor structure.

For a three-point function involving scalar operators  $\mathcal{O}_1$  and  $\mathcal{O}_2$ , and a spin- $s$  operator  $\mathcal{O}_{\mu_1 \dots \mu_s}$  of dimension  $\Delta$ ,

$$\langle \mathcal{O}_1(P_1) \mathcal{O}_2(P_2) \mathcal{O}_s(P_3|Z) \rangle = F(P_1, P_2, P_3|Z), \quad (3.75)$$

we must satisfy the three constraints

$$F(\{\lambda_i P_i\} | Z) = \lambda_1^{-\Delta_1} \lambda_2^{-\Delta_2} \lambda_3^{-\Delta} F(\{P_i\} | Z), \quad (3.76)$$

$$P_3 \cdot \frac{\partial}{\partial Z} F(\{P_i\} | Z) = 0, \quad (3.77)$$

$$F(\{P_i\} | \alpha Z) = \alpha^s F(\{P_i\} | Z). \quad (3.78)$$

In this case, satisfying the transversality condition (3.77) is only possible if  $Z$  appears with  $P_1$  and  $P_2$  through the new building block

$$Y_3 = \frac{(Z \cdot P_2)}{P_{23}} - \frac{(Z \cdot P_1)}{P_{13}}, \quad (3.79)$$

where the overall factor of  $(P_{13})(P_{23})$  was chosen for later convenience. Combining this with the homogeneity (3.76) and spin conditions (3.78), we find that the most general form of the correlator (3.75) as fixed by conformal symmetry is

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_s(x_3|z) \rangle = C_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_s} \frac{Y_3^s}{(P_{12}^2)^{\frac{\Delta_1 + \Delta_2 - \tau_s}{2}} (P_{23}^2)^{\frac{\Delta_2 + \tau_s - \Delta_1}{2}} (P_{31}^2)^{\frac{\tau_s + \Delta_1 - \Delta_2}{2}}}. \quad (3.80)$$

Just like for the scalar three-point function in the previous section, we can constrain the overall coefficient using Bose symmetry. In this case, since  $Y_3 \leftrightarrow -Y_3$  under  $\mathcal{O}_1 \leftrightarrow \mathcal{O}_2$ , a similar argument shows that

$$C_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_s} = (-1)^s C_{\mathcal{O}_2 \mathcal{O}_1 \mathcal{O}_s}. \quad (3.81)$$

The immediate consequence of this condition is that the correlator (3.80) vanishes for odd spin  $s$ .

### The general case

We now consider the general case of a three-point function involving primary operators  $\mathcal{O}_{s_i}$  of integer spin  $s_i$  and dimension  $\Delta_i$ . This is more involved, owing to the increased number of possible tensorial structures compatible with conformal symmetry.

We write

$$\langle \mathcal{O}_{s_1}(P_1|Z_1) \mathcal{O}_{s_2}(P_2|Z_2) \mathcal{O}_{s_3}(P_3|Z_3) \rangle = \frac{F(P_1, P_2, P_3|Z_1, Z_2, Z_3)}{(P_{12})^{\frac{\tau_1+\tau_2-\tau_3}{2}} (P_{13})^{\frac{\tau_1+\tau_3-\tau_2}{2}} (P_{23})^{\frac{\tau_2+\tau_3-\tau_1}{2}}}, \quad (3.82)$$

where  $\tau_i = \Delta_i - s_i$ . Conformal symmetry requires<sup>8</sup>

$$F(\{\lambda_i P_i\} | \{\alpha_i Z_i\}) = F(\{P_i\} | \{Z_i\}) \prod_i (\alpha_i)^{s_i} \quad (3.83)$$

$$P_j \cdot \frac{\partial}{\partial Z_j} F(\{P_i\} | \{Z_i\}) = 0, \quad \text{for each } j = 1, 2, 3. \quad (3.84)$$

Restricting to parity-invariant correlators,<sup>9</sup> in this case there are six possible transverse building blocks from which the three-point function can be constructed [148],

$$Y_1 = \frac{(Z_1 \cdot P_3)}{P_{13}} - \frac{(Z_1 \cdot P_2)}{P_{12}}, \quad H_1 = \frac{1}{P_{23}} \left( (Z_2 \cdot Z_3) + \frac{2(Z_2 \cdot P_3)(Z_3 \cdot P_2)}{P_{23}} \right), \quad (3.85a)$$

$$Y_2 = \frac{(Z_2 \cdot P_1)}{P_{21}} - \frac{(Z_2 \cdot P_3)}{P_{23}}, \quad H_2 = \frac{1}{P_{13}} \left( (Z_1 \cdot Z_3) + \frac{2(Z_1 \cdot P_3)(Z_3 \cdot P_1)}{P_{13}} \right), \quad (3.85b)$$

$$Y_3 = \frac{(Z_3 \cdot P_2)}{P_{32}} - \frac{(Z_3 \cdot P_1)}{P_{31}}, \quad H_3 = \frac{1}{P_{23}} \left( (Z_2 \cdot Z_3) + \frac{2(Z_2 \cdot P_3)(Z_3 \cdot P_1)}{P_{13}} \right). \quad (3.85c)$$

Each of the above structures independently satisfy the transversality constraint (3.84) in more than three dimensions.

<sup>8</sup>We chose the denominator such that  $F(\{P_i\} | \{Z_i\})$  does not scale with the  $P_i$ .

<sup>9</sup>In three- and four-dimensions, we can construct parity odd correlators using the  $SO(d, 2)$ -invariant epsilon tensor.

Taking into account the spin and homogeneity conditions (3.83), the most general three-point function compatible with conformal symmetry is thus

$$\begin{aligned} & \langle \mathcal{O}_{s_1}(P_1|Z_1) \mathcal{O}_{s_2}(P_2|Z_2) \mathcal{O}_{s_3}(P_3|Z_3) \rangle \\ &= \sum_{n_i} \mathbf{C}_{n_1, n_2, n_3} \frac{\mathbf{Y}_1^{s_1-n_2-n_3} \mathbf{Y}_2^{s_2-n_3-n_1} \mathbf{Y}_3^{s_3-n_1-n_2} \mathbf{H}_1^{n_1} \mathbf{H}_2^{n_2} \mathbf{H}_3^{n_3}}{(P_{12})^{\frac{\tau_1+\tau_2-\tau_3}{2}} (P_{23})^{\frac{\tau_2+\tau_3-\tau_1}{2}} (P_{13})^{\frac{\tau_3+\tau_1-\tau_2}{2}}}, \end{aligned} \quad (3.86)$$

with theory-dependent coefficients  $\mathbf{C}_{n_1, n_2, n_3}$ .

### Back to Intrinsic

In the preceding sections, with the power of the ambient formalism we were able to write down the most general form of two- and three-point functions of primary operators of any scaling dimension and spin. In §4 we apply these results to a particular CFT, and for this purpose it is useful to give the dictionary which translates them back into the standard  $d$ -dimensional language.

For the scalar products of ambient vectors ubiquitous in the ambient expressions for the correlators:  $-2P_i \cdot P_j$ ,  $Z_i \cdot P_j$  and  $Z_i \cdot Z_j$ , this is straightforward to write down from the explicit forms of the ambient space-time and auxiliary vectors

$$P^A(x) = (1, x^2, x^\mu), \quad Z^A(x) = (0, 2x \cdot z, z^\mu). \quad (3.87)$$

Recalling that we work in light-cone ambient co-ordinates, this establishes the following dictionary between ambient and intrinsic scalar products

$$-2P_i \cdot P_j = x_{ij}^2, \quad Z_i \cdot P_j = -z_i \cdot x_{ij}, \quad Z_i \cdot Z_j = z_i \cdot z_j, \quad (3.88)$$

where by definition

$$P_i \equiv P(x_i), \quad Z_i \equiv Z(x_i). \quad (3.89)$$

Employing the above dictionary, one obtains the following intrinsic expressions of the building blocks found above:

$$Y_1 = \frac{z_1 \cdot x_{12}}{x_{12}^2} - \frac{z_1 \cdot x_{13}}{x_{13}^2}, \quad H_1 = \frac{1}{x_{23}^2} \left( z_2 \cdot z_3 + \frac{2z_2 \cdot x_{23} z_3 \cdot x_{32}}{x_{23}^2} \right), \quad (3.90a)$$

$$Y_2 = \frac{z_2 \cdot x_{23}}{x_{23}^2} - \frac{z_2 \cdot x_{21}}{x_{21}^2}, \quad H_2 = \frac{1}{x_{31}^2} \left( z_3 \cdot z_1 + \frac{2z_3 \cdot x_{31} z_1 \cdot x_{13}}{x_{31}^2} \right), \quad (3.90b)$$

$$Y_3 = \frac{z_3 \cdot x_{31}}{x_{31}^2} - \frac{z_3 \cdot x_{32}}{x_{32}^2}, \quad H_3 = \frac{1}{x_{12}^2} \left( z_1 \cdot z_2 + \frac{2z_1 \cdot x_{12} z_2 \cdot x_{21}}{x_{12}^2} \right). \quad (3.90c)$$

### 3.5.3 Four-point functions

Correlation functions with more than three operator insertions are no longer fixed by conformal kinematics alone. This is due to the existence of *cross ratios*, which are conformal invariants that can be built from four or more distinct space-time points. In this section, and for the remainder of this chapter, we focus our attention on the degree to which four-point functions of scalar primary operators can be constrained by conformal symmetry.

Out of four distinct points, one can build two independent cross ratios,

$$u = \frac{(x_1 - x_2)^2 (x_3 - x_4)^2}{(x_1 - x_3)^2 (x_2 - x_4)^2}, \quad v = u \Big|_{2 \leftrightarrow 4} = \frac{(x_1 - x_4)^2 (x_2 - x_3)^2}{(x_1 - x_3)^2 (x_2 - x_4)^2}. \quad (3.91)$$

Four-point functions may therefore depend non-trivially on  $u$  and  $v$ , in a way that cannot be fixed by conformal symmetry. For a four-point function of scalar operators  $\mathcal{O}_i$ , reverting briefly to the ambient formalism we have

$$\langle \mathcal{O}_1(P_1) \mathcal{O}_2(P_2) \mathcal{O}_3(P_3) \mathcal{O}_4(P_4) \rangle = \frac{g_{1234}(u, v)}{P_{12}^{k_1} P_{13}^{k_2} P_{14}^{k_3} P_{23}^{k_4} P_{24}^{k_5} P_{34}^{k_6}}, \quad (3.92)$$

with the  $k_i$  constrained by the homogeneity condition

$$\begin{aligned} \langle \mathcal{O}_1(\lambda_1 P_1) \mathcal{O}_2(\lambda_2 P_2) \mathcal{O}_3(\lambda_3 P_3) \mathcal{O}_4(\lambda_4 P_4) \rangle = & \quad (3.93) \\ \lambda_1^{-\Delta_1} \lambda_2^{-\Delta_2} \lambda_3^{-\Delta_3} \lambda_4^{-\Delta_4} \langle \mathcal{O}_1(P_1) \mathcal{O}_2(P_2) \mathcal{O}_3(P_3) \mathcal{O}_4(P_4) \rangle. & \end{aligned}$$

This implies the following relations amongst the  $k_i$ ,

$$\begin{aligned} k_1 = \Delta_1 + \Delta_2 - \Delta_3 - \Delta_4 + k_6, \quad k_2 = \Delta_1 - \Delta_2 + \Delta_3 - \Delta_4 + k_5 & \quad (3.94) \\ k_3 = 4\Delta_4 - k_5 - k_6, \quad k_4 = -\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 - k_5 - k_6. & \end{aligned}$$

For instance we can take the following convenient parametrisation

$$\langle \mathcal{O}_1(P_1) \mathcal{O}_2(P_2) \mathcal{O}_3(P_3) \mathcal{O}_4(P_4) \rangle = \left( \frac{P_{24}}{P_{14}} \right)^{\frac{\Delta_1 - \Delta_2}{2}} \left( \frac{P_{14}}{P_{13}} \right)^{\frac{\Delta_3 - \Delta_4}{2}} \frac{g_{1234}(u, v)}{P_{12}^{\frac{\Delta_1 + \Delta_2}{2}} P_{34}^{\frac{\Delta_3 + \Delta_4}{2}}}. \quad (3.95)$$

Like for the three-point functions, the function  $g_{1234}(u, v)$  is not completely unconstrained as it must be consistent with Bose-symmetry. I.e. invariance under permutations of the scalar operators. Since all permutations of  $\{x_1, x_2, x_3, x_4\}$  are generated by  $x_1 \leftrightarrow x_2$ ,  $x_3 \leftrightarrow x_4$  and  $x_1 \leftrightarrow x_3$ , it is sufficient to consider the constraints coming from  $\mathcal{O}_1(x_1) \leftrightarrow \mathcal{O}_2(x_2)$ ,  $\mathcal{O}_3(x_3) \leftrightarrow \mathcal{O}_4(x_4)$  and  $\mathcal{O}_1(x_1) \leftrightarrow \mathcal{O}_3(x_3)$ . These require<sup>10</sup>

$$g_{1234}(u, v) = v^{\frac{\Delta_4 - \Delta_3}{2}} g_{2134}\left(\frac{u}{v}, \frac{1}{v}\right), \quad (3.96)$$

$$g_{1234}(u, v) = v^{\frac{\Delta_1 - \Delta_2}{2}} g_{1243}\left(\frac{u}{v}, \frac{1}{v}\right), \quad (3.97)$$

$$g_{1234}(u, v) = u^{\frac{\Delta_1 + \Delta_2}{2}} v^{-\frac{\Delta_2 + \Delta_3}{2}} g_{3214}(v, u), \quad (3.98)$$

respectively. This gives the most general form of a scalar four-point function in a conformally invariant theory.<sup>11</sup>

<sup>10</sup>Note that under  $x_1 \leftrightarrow x_2$  and  $x_1 \leftrightarrow x_3$ , the cross ratios transform as  $u \rightarrow u/v$  and  $v \rightarrow 1/v$ . Under  $x_1 \leftrightarrow x_3$ , we have  $u \leftrightarrow v$ .

<sup>11</sup>We will see shortly that the functions  $g_{ijkl}(u, v)$  are not arbitrary, but for a given theory are related in a non-trivial way to its three-point functions.

The case relevant for applications in later chapters of this thesis is the four-point functions of *identical* scalar operators. In general, the Bose-symmetry constraints (3.96), (3.97) and (3.98) relate *different* functions  $g_{ijkl}(u, v)$ . However for four-point functions involving identical scalars  $\mathcal{O}$ , these constraints just apply to a single function  $g(u, v)$ ,

$$g(u, v) = g\left(\frac{u}{v}, \frac{1}{v}\right), \quad g(u, v) = \left(\frac{u}{v}\right)^\Delta g(v, u). \quad (3.99)$$

## 3.6 Conformal block decomposition

In the previous subsection, we observed that conformal kinematics fixes four-point functions up to a function of the cross ratios  $u$  and  $v$ . As we shall demonstrate below, this function can be completely determined from a combination of conformal invariance and the OPE (c.f. §4.38). The OPE coefficients encode the dynamics of the CFT.

For simplicity and relevance for applications in this thesis, we simply consider a four-point function of identical scalar operators  $\mathcal{O}$  with scaling dimension  $\Delta$ ,

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle = \frac{g(u, v)}{(x_{12}^2)^\Delta (x_{34}^2)^\Delta}, \quad (3.100)$$

where  $x_{ij}^2 = (x_i - x_j)^2$ . Applying the OPE twice, for example in the (12) and (34) channels, we can represent the four-point function as a double sum over the primary operators that appear in the OPE

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle = \sum_{\substack{\mathcal{O}_i, \mathcal{O}_j \\ \text{primary}}} \mathcal{F}_{\mathcal{O}\mathcal{O}_i}(x_{12}, \partial_{x_2}) \mathcal{F}_{\mathcal{O}\mathcal{O}_j}(x_{34}, \partial_{x_4}) \langle \mathcal{O}_i(x_2) \mathcal{O}_j(x_4) \rangle. \quad (3.101)$$

Choosing an orthonormal basis of operators, this collapses to a single sum

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle = \sum_{\mathcal{O}_i \text{ primary}} (c_{\mathcal{O}\mathcal{O}_i})^2 W_{\mathcal{O}_i}(x_1, x_2; x_3, x_4), \quad (3.102)$$

where we have extracted the OPE coefficients  $c_{\mathcal{O}\mathcal{O}_i}$  from the differential operators  $\mathcal{F}_{\mathcal{O}\mathcal{O}_i}$ . Owing to the relationship (3.51) between  $\mathcal{F}_{\mathcal{O}\mathcal{O}_i}$  and the three-point function  $\langle \mathcal{O}\mathcal{O}\mathcal{O}_i \rangle$ , this OPE coefficient is given by

$$c_{\mathcal{O}\mathcal{O}_i}^2 = \mathcal{C}_{\mathcal{O}\mathcal{O}\mathcal{O}_i}^2 / \mathcal{C}_{\mathcal{O}_i\mathcal{O}_i}, \quad (3.103)$$

where  $\mathcal{C}_{\mathcal{O}\mathcal{O}\mathcal{O}_i}$  is the overall three-point function coefficient and  $\mathcal{C}_{\mathcal{O}_i\mathcal{O}_i}$  the overall coefficient of the two-point function  $\langle \mathcal{O}_i\mathcal{O}_i \rangle$ . The conformal partial wave  $W_{\mathcal{O}_i}$  re-sums the contribution of the primary  $\mathcal{O}_i$  and all of its descendants to the correlator, and thus represents the contribution from the entire conformal multiplet.

It is often instructive to think of a conformal partial wave  $W_{\mathcal{O}_i}$  as the insertion of a projector onto the conformal multiplet of  $\mathcal{O}_i$ ,

$$\begin{aligned} W_{\mathcal{O}_i}(x_1, x_2; x_3, x_4) &= \langle \mathcal{O}(x_1) \mathcal{O}(x_2) | \mathcal{O}_i | \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle, \\ | \mathcal{O}_i | &= \sum_n | P_{\mu_n} \dots P_{\mu_1} \mathcal{O}_i \rangle \frac{1}{\mathcal{N}_{\mathcal{O}_i}} \langle P_{\mu_n} \dots P_{\mu_1} \mathcal{O}_i |, \end{aligned} \quad (3.104)$$

for some normalisation  $\mathcal{N}_{\mathcal{O}_i}$ . The identity is the sum of the projection operators over all primary operators

$$\mathbf{1} = \sum_{\mathcal{O}_i \text{ primary}} | \mathcal{O}_i |. \quad (3.105)$$

The projector  $| \mathcal{O}_i |$  commutes with all conformal generators, and thus the partial wave  $W_{\mathcal{O}_i}$  has the same transformation properties as the four point function (3.100). It may then be written as

$$W_{\mathcal{O}_i}(x_1, x_2; x_3, x_4) = \frac{G_{\mathcal{O}_i}(u, v)}{(x_{12}^2)^\Delta (x_{34}^2)^\Delta}. \quad (3.106)$$



The function  $G_{\mathcal{O}_i}(u, v)$  of cross ratios is known as a *conformal block*, and encodes the contribution of a primary  $\mathcal{O}_i$  and its descendants to the function  $g(u, v)$  in the correlator (3.100),

$$g(u, v) = \sum_{\mathcal{O}_i \text{ primary}} (c_{\mathcal{O}\mathcal{O}_i})^2 G_{\mathcal{O}_i}(u, v). \quad (3.107)$$

The space-time dependence of  $g(u, v)$  is completely encoded in the conformal blocks  $G_{\mathcal{O}_i}(u, v)$ , which are universal in the sense that they do not depend on the CFT under consideration but only on the conformal representation of  $\mathcal{O}_i$ . I.e. its spin and scaling dimension. The a priori arbitrary function  $g(u, v)$  is thus completely fixed by conformal invariance and the OPE.

While conformal blocks are central objects in CFT,<sup>12</sup> their explicit forms have been difficult to pin down in generality. Explicit formulas are known only for simple cases, like for external scalar operators in even spacetime dimensions [199, 200]. In other cases one has to resort to more indirect methods, like recursion relations [199, 201, 202] or efficient series expansions [203, 204]. There is also a general method to increase the spin of the external operators using differential operators [149, 205, 206].

Another indirect approach is given by so-called integral representations [151, 201, 207–212], which we introduce in the following section. This is most useful for our purposes, for it allows us to define conformal blocks as integral products of three-point structures. In this way it is more straightforward to identify conformal blocks arising from computations in theories which are not, a priori, CFTs. Such as gravity theories in anti-de Sitter space.

### 3.6.1 Integral representation

In this section we derive a useful representation of the projector (3.105), which leads to an integral expression for conformal blocks. This is motivated by the following ob-

<sup>12</sup>For example, they are basic ingredients in the conformal bootstrap program [196–198].

servation of Dolan and Osborn [201],<sup>13</sup> which is underpinned by the shadow formalism of Ferrara, Gatto, Grillo, and Parisi [207–209, 213].

We first define for a given primary operator  $O$  with quantum numbers  $[\Delta, s]$ , an associated dual (or shadow) operator<sup>14</sup>

$$\tilde{\mathcal{O}}(x_1|z_1) = \kappa_{\Delta,s} \int d^d x_2 \frac{1}{(x_{12}^2)^{d-\Delta}} \left( z_1 \cdot I(x_1 - x_2) \cdot \hat{\partial}_{z_2} \right)^s \mathcal{O}(x_2|z_2), \quad (3.109)$$

which has quantum numbers  $[d - \Delta, s]$ . The normalisation  $\kappa_{\Delta,s}$  is chosen such that  $\tilde{\tilde{\mathcal{O}}} = \tilde{\mathcal{O}}$ , which requires

$$\kappa_{\Delta,s} = \left( \frac{2}{\pi} \right)^{\frac{d}{4}} \frac{\Gamma(d - \Delta + s)}{\Gamma(\Delta - \frac{d}{2})} \frac{1}{(\Delta - 1)_s}. \quad (3.110)$$

Dolan and Osborn observed that inserting the invariant projection operator

$$\mathcal{P}_{\mathcal{O}_i} = \kappa_{d-\Delta,s} \int d^d x \mathcal{O}_i(x) |0\rangle \langle 0| \tilde{\mathcal{O}}_i(x), \quad (3.111)$$

in a four-point function yields the contributions from the  $[\Delta_i, s]$  and  $[d - \Delta_i, s]$  conformal multiplets,

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{P}_{\mathcal{O}_i} \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle = \mathbf{c}_{\mathcal{O}\mathcal{O}\mathcal{O}_i}^2 W_{\mathcal{O}_i}(x_1, x_2; x_3, x_4) + \mathbf{c}_{\mathcal{O}\mathcal{O}\tilde{\mathcal{O}}_i}^2 W_{\tilde{\mathcal{O}}_i}(x_1, x_2; x_3, x_4).$$

This entails an integral representation of their total contribution,

$$\begin{aligned} \mathbf{c}_{\mathcal{O}\mathcal{O}\mathcal{O}_i}^2 W_{\mathcal{O}_i}(x_1, x_2; x_3, x_4) + \mathbf{c}_{\mathcal{O}\mathcal{O}\tilde{\mathcal{O}}_i}^2 W_{\tilde{\mathcal{O}}_i}(x_1, x_2; x_3, x_4) \\ = \kappa_{d-\Delta,s} \int d^d x \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}_i(x) \rangle \langle \tilde{\mathcal{O}}_i(x) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle, \end{aligned} \quad (3.112)$$

<sup>13</sup>Later formulated in the ambient framework in [151].

<sup>14</sup> $\hat{\partial}_z$  is a differential operator which accommodates tracelessness (§A.2.3), and  $I_{\mu\nu}(x)$  is the inversion tensor

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}; \quad z_1 \cdot I(x) \cdot z_2 = z_1 \cdot z_2 - 2 \frac{z_1 \cdot x z_2 \cdot x}{x^2}. \quad (3.108)$$

as a product of two three-point functions. The factor of  $\kappa_{d-\Delta,s}$  in the above ensures that the projector (3.111) acts trivially when inserted into a correlator involving  $\mathcal{O}_i$ .

An integral expression for the non-shadow contribution can be obtained by integrating over complex scaling dimensions with poles at dimensions  $\Delta_i$  and  $d - \Delta_i$ ,

$$\int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + \left(\Delta_i - \frac{d}{2}\right)^2} \left( W_{\mathcal{O}_{\frac{d}{2}+i\nu}}(x_1, x_2; x_3, x_4) + W_{\mathcal{O}_{\frac{d}{2}-i\nu}}(x_1, x_2; x_3, x_4) \right). \quad (3.113)$$

This projects out the shadow contribution, since the  $\nu$ -contour may only be closed around the poles with  $\frac{d}{2} \pm i\nu = \Delta_i$ ,<sup>15</sup>

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + \left(\Delta_i - \frac{d}{2}\right)^2} \left( W_{\mathcal{O}_{\frac{d}{2}+i\nu}}(x_1, x_2; x_3, x_4) + W_{\mathcal{O}_{\frac{d}{2}-i\nu}}(x_1, x_2; x_3, x_4) \right) \\ = \frac{2\pi}{\left(\Delta_i - \frac{d}{2}\right)} W_{\mathcal{O}_i}(x_1, x_2; x_3, x_4). \end{aligned}$$

We then arrive to the following integral representation of a single conformal block

$$\begin{aligned} W_{\mathcal{O}_i}(x_1, x_2; x_3, x_4) &= \frac{\left(\Delta_i - \frac{d}{2}\right)}{2\pi} \int_{-\infty}^{\infty} d\nu \frac{1}{\nu^2 + \left(\Delta_i - \frac{d}{2}\right)^2} \frac{\kappa_{\frac{d}{2}-i\nu,s}}{(2\pi)^{\frac{d}{4}}} \frac{\Gamma\left(\frac{\frac{d}{2}-i\nu+s}{2}\right)^2}{\Gamma\left(\frac{\frac{d}{2}+i\nu+s}{2}\right)^2} \\ &\times \int d^d x \langle\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}_{\frac{d}{2}+i\nu}(x) \rangle\rangle \langle\langle \tilde{\mathcal{O}}_{\frac{d}{2}-i\nu}(x) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle\rangle, \end{aligned} \quad (3.114)$$

where we introduced the notation

$$\langle\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}_{\frac{d}{2}\pm i\nu}(x) \rangle\rangle = C_{\mathcal{O}\mathcal{O}\mathcal{O}_{\frac{d}{2}\pm i\nu}} \langle\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}_{\frac{d}{2}\pm i\nu}(x) \rangle\rangle, \quad (3.115)$$

<sup>15</sup>The partial wave  $W_{\mathcal{O}_{\frac{d}{2}\pm i\nu}}(x_1, x_2; x_3, x_4)$  decays fast enough (exponentially) only for  $\text{Im}(\nu) \rightarrow \mp\infty$ . Applying the residue theorem, we must therefore close the contour in the lower half plane for  $W_{\mathcal{O}_{\frac{d}{2}+i\nu}}$  thus picking up a contribution from the pole at  $i\nu = \delta_i - \frac{d}{2}$ , while for  $W_{\mathcal{O}_{\frac{d}{2}-i\nu}}$  we close in the upper-half plane around the pole at  $i\nu = \frac{d}{2} - \delta_i$ .

I.e.  $\langle\langle\bullet\rangle\rangle$  denotes the removal the overall coefficient from the three-point function.<sup>16</sup>

Similarly we can express the conformal block expansion of a give four-point function in the following form<sup>17</sup>

$$\langle\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle\rangle = \sum_s \int_{-\infty}^{\infty} d\nu g_s(\nu) W_{\mathcal{O}_{\frac{d}{2}+i\nu,s}}(x_1, x_2; x_3, x_4), \quad (3.117)$$

which we refer to as the *contour integral representation*. For each spin- $s$  primary operator in the  $\mathcal{O}\mathcal{O}$  OPE of scaling dimension  $\Delta_i$ , the meromorphic function contains a pole at  $i\nu = \Delta_i - \frac{d}{2}$ . The residue of  $g_s(\nu)$  at this pole gives the square of the corresponding OPE coefficient, such that the usual conformal block decomposition (3.102) is recovered upon application to the residue theorem.

This way of representing the conformal block expansion and conformal blocks will prove instrumental in later sections of this thesis, in which we apply the conformal block expansion in the context of the AdS/CFT duality.

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<sup>16</sup>In establishing (3.114), we used that

$$C_{\mathcal{O}\mathcal{O}\mathcal{O}_{\frac{d}{2}-i\nu}} = \frac{1}{(2\pi)^{\frac{d}{4}}} \frac{\Gamma\left(\frac{\frac{d}{2}-i\nu+s}{2}\right)^2}{\Gamma\left(\frac{\frac{d}{2}+i\nu+s}{2}\right)^2} C_{\mathcal{O}\mathcal{O}\mathcal{O}_{\frac{d}{2}+i\nu}}. \quad (3.116)$$

<sup>17</sup>Which has origins in the early literature, for example [214–216].

## Chapter 4

# Correlators in the free $O(N)$ model

Having discussed the general structure and properties of two-, three- and four-point correlation functions in CFTs, in this chapter we move on to consider them in the context of the simplest examples of such theories: free conformal field theories. In this case, exact results for the correlators can be obtained by simple application of Wick's theorem.

Our consideration of these simple CFTs is motivated by the conjectured equivalence between higher-spin gauge theories on AdS and free vector model CFTs. In particular, we compute two-, three- and four-point functions of single-trace operators in the free scalar  $O(N)$  vector model, which are dual to single-particle states in the higher-spin theory. Moreover, we apply the conformal partial wave expansion techniques introduced in the previous chapter to the four-point function of the scalar single-trace operator. This is in the view to compare with the corresponding processes in the dual higher-spin theory on AdS.

### 4.1 The free scalar $O(N)$ vector model

Some of the simplest examples of CFTs are given by free theories without a mass scale. These theories are *solvable* in the sense that the path integral is Gaussian, which can

therefore be straightforwardly be evaluated to determine all correlators in the theory.

In this chapter we focus on the example of the free scalar  $O(N)$  vector model, which is conjectured to be dual to the type A minimal bosonic higher-spin theory on AdS (§2.2). This is a theory of an  $N$  component real scalar field  $\phi^a$ , transforming in the fundamental representation of  $O(N)$ . The action is simply

$$S[\phi] = \sum_{a=1}^N \int d^d x \frac{1}{2} \partial_i \phi^a \partial^i \phi^a, \quad (4.1)$$

with equation of motion  $\partial^2 \phi^a = 0$ . As we saw in §3.3.2, free massless scalar fields are primary of scaling dimension  $\Delta_\phi = \frac{d}{2} - 1$ .

Correlation functions of operators in the theory can be defined by the path integral

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \int d[\phi] \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) e^{-S[\phi]}, \quad (4.2)$$

where the measure is normalised such that  $\langle \mathbb{1} \rangle = 1$ . In particular, by evaluating the Gaussian integral the two-point function of the fundamental scalar is given by

$$\langle \phi^a(x_1) \phi^b(x_2) \rangle = \frac{1}{(d-2) \text{Vol}(S^d)} \frac{\delta^{ab}}{|x_{12}|^{d-2}}, \quad \text{Vol}(S^d) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}. \quad (4.3)$$

It is conventional to re-define  $\phi \rightarrow \sqrt{(d-2) \text{Vol}(S^d)}$ , to obtain a canonical unit normalisation

$$\langle \phi^a(x_1) \phi^b(x_2) \rangle = \frac{\delta^{ab}}{|x_{12}|^{d-2}}. \quad (4.4)$$

Since the theory is free, we may determine all correlation functions of operators in the theory by expressing them in terms of the two-point function (4.3) through Wick's theorem.

As explained in §2.2.1, from the correlation functions of operators in the singlet sector,

we can study the possible interactions in the dual minimal bosonic higher-spin theory on AdS. For this purpose, in the following sections we determine the explicit forms of the appropriate 2-, 3- and 4-pt functions. We work in Euclidean signature throughout.

### 4.1.1 Singlet sector

By definition, operators in the singlet sector are invariants under the  $O(N)$  symmetry group. Operators in this sector have no free  $O(N)$  indices, and are thus composite in the elementary scalar  $\phi^a$ . We distinguish between *single-trace* operators, consisting of a single contraction of the  $O(N)$  indices, and *multi-trace* operators, consisting of two or more contractions. As discussed in §2.1.1, these are dual to single- and multi- particle states in anti-de Sitter space, respectively.

#### Single-trace operators

The spectrum of single-trace operators in the vector model are bi-linears in the elementary scalar  $\phi^a$ . There is just a single scalar primary operator

$$\mathcal{O} = \frac{1}{\sqrt{N}} : \phi^a \phi^a : , \quad \Delta = 2\Delta_\phi = d - 2, \quad (4.5)$$

whose scaling dimension is simply twice the scaling dimension of the elementary scalar.<sup>1</sup> The rest of the single-trace spectrum comprises of an infinite tower of even spin conserved currents of the schematic form

$$\mathcal{J}_{i_1 \dots i_s} \sim \frac{1}{\sqrt{N}} : \phi^a \partial_{i_1} \dots \partial_{i_s} \phi^a : + \dots, \quad s \in 2\mathbb{N}. \quad (4.6)$$

The  $\dots$  denote further symmetric singlet bi-linear structures, which ensure that the operator is primary and whose form we determine explicitly below. Recall (§3.3.2) that spin- $s$  conserved operators are primary with scaling dimension  $\Delta_{J_s} = s + d - 2 = s + \Delta$ ,

<sup>1</sup>Recall that in a free theory, scaling dimensions are additive.

which can be verified in the above by counting derivatives.

To compute correlation functions of the conserved currents (4.6), it is convenient to employ the index-free notation

$$\mathcal{J}_s(x|z) = \mathcal{J}_{i_1 \dots i_s}(x) z^{i_1} \dots z^{i_s}, \quad z^2 = 0. \quad (4.7)$$

In this way, they can be packaged in the compact form

$$\mathcal{J}_s(x|z) = \frac{1}{\sqrt{N}} f^{(s)}(z \cdot \partial_{y_1}, z \cdot \partial_{y_2}) : \phi^a(y_1) \phi^a(y_2) : \Big|_{y_1, y_2 \rightarrow x}, \quad (4.8)$$

where the function  $f^{(s)}(x, y)$  is given in terms of a Gegenbauer polynomial [217],

$$f^{(s)}(x, y) = (x + y)^s C_s^{(\Delta_\phi - \frac{1}{2})} \left( \frac{x - y}{x + y} \right). \quad (4.9)$$

Recall that  $\Delta_\phi$  is the scaling dimension of  $\phi^a$ .

This way of representing the currents can be derived by demanding that the expression (4.8) is annihilated by the conformal boost generator  $K_i$ , which gives rise to the differential equation,<sup>2</sup>

$$[(\Delta_\phi + x \partial_x) \partial_x + (\Delta_\phi + y \partial_y) \partial_y] f^{(s)}(x, y) = 0, \quad (4.10)$$

whose solution is expressed in terms of the Gegenbauer polynomials above.

---

<sup>2</sup>This equation can be generalised to deal with scalar single-trace operators built out of constituents with different dimensions  $\Delta_1/2$  and  $\Delta_2/2$ . The corresponding primary is in this case a Jacobi polynomial  $f(x, y) = (x + y)^s P_s^{(\Delta_2/2-1, \Delta_1/2-1)} \left( \frac{x-y}{x+y} \right)$ .



### Double-trace operators

There are also operators in the singlet sector composed of higher traces, which are dual to multi-particle states in the gravity theory on AdS. At the level of four-point functions studied in this work, we need only consider double-trace operators. In particular, in studying the conformal block expansion of the four-point function  $\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle$ , we will encounter contributions from double-trace operators of the schematic form

$$[\mathcal{O}\mathcal{O}]_{n,s} \sim \frac{1}{N} : \phi^a \phi^a : \overleftrightarrow{\partial}_{i_1} \dots \overleftrightarrow{\partial}_{i_s} \left( \overleftrightarrow{\partial} \cdot \overleftrightarrow{\partial} \right)^n : \phi^b \phi^b :, \quad (4.11)$$

where  $s$  is the spin, and  $\Delta_{n,s} = 2\Delta + 2n + s$  the scaling dimension. The primary condition (3.20) fixes their precise form, which we determine for all  $s$  and  $n$  in §D.

## 4.2 Singlet sector correlation functions

We now consider correlation functions of operators in the singlet sector introduced above. Since the theory is free, we simply apply Wick's theorem and express them in terms of two-point functions of the fundamental scalar (4.3).

Our computations are drastically simplified by using the Schwinger-parametrised form of the fundamental scalar two-point function

$$\langle \phi^a(x_1) \phi^b(x_2) \rangle = \frac{\delta^{ab}}{\Gamma\left(\frac{\Delta}{2}\right)} \int_0^\infty \frac{dt}{t} t^{\frac{\Delta}{2}} e^{-tx_{12}^2}, \quad (4.12)$$

which allows for a seamless application of Wick's theorem.

Conformal symmetry fixes the form of 2- and 3-pt functions up a set of theory dependent coefficients (§3.5). The goal of the following sections is to determine them for the free scalar  $\mathcal{O}(N)$  vector model in  $d$ -dimensions.

### 4.2.1 2-point functions

As we reviewed in §3.5.1, conformal invariance determines two-point functions up a single overall coefficient, with those of spin- $s$  conserved currents (4.8) taking the form

$$\langle \mathcal{J}_s(x_1|z_1) \mathcal{J}_s(x_2|z_2) \rangle = \mathbf{C}_{\mathcal{J}_s} \frac{(\mathbf{H}_3)^s}{(x_{12}^2)^\Delta}. \quad (4.13)$$

The purpose of this section is to determine the coefficient  $\mathbf{C}_{\mathcal{J}_s}$ , which essentially depends on the choice of normalisation of the currents (4.8).

Employing the generating function (4.9) for the conserved current, applying Wick's theorem we have

$$\begin{aligned} \langle \mathcal{J}_s(x_1|z_1) \mathcal{J}_s(x_2|z_2) \rangle &= \frac{1}{N} f^{(s)}(z_1 \cdot \partial_{y_1}, z_1 \cdot \partial_{y_2}) f^{(s)}(z_2 \cdot \partial_{\bar{y}_1}, z_2 \cdot \partial_{\bar{y}_2}) \\ &\times \left[ \langle \phi^a(y_1) \phi^a(y_2) \rangle \langle \phi^b(\bar{y}_1) \phi^b(\bar{y}_2) \rangle + \phi^a(y_2) \leftrightarrow \phi^b(\bar{y}_1) + \phi^a(y_2) \leftrightarrow \phi^b(\bar{y}_2) \right] \Big|_{y_1, y_2 \rightarrow x_1; \bar{y}_1, \bar{y}_2 \rightarrow x_2}. \end{aligned} \quad (4.14)$$

To extract the overall two-point coefficient, due to conformal invariance it is sufficient to restrict attention to terms with zero contractions between the null auxiliary vectors. We thus set to zero  $z_1 \cdot z_2$ , and match with the corresponding term in (4.13). This is straightforward using the Schwinger-parameterised form of the fundamental scalar two-point function (4.12), through which we obtain

$$\begin{aligned} \mathbf{C}_{\mathcal{J}_s} &= \left[ \frac{1 + (-1)^s}{2} \right] \frac{2^{s+1}}{\Gamma(\frac{\Delta}{2})^2} \int_0^\infty \frac{dt_1}{t_1} \frac{dt_2}{t_2} t_1^{\frac{\Delta}{2}} t_2^{\frac{\Delta}{2}} (t_1 + t_2)^{2s} C_s^{(\Delta_\phi - \frac{1}{2})} \left( \frac{t_1 - t_2}{t_1 + t_2} \right) C_s^{(\Delta_\phi - \frac{1}{2})} \left( \frac{t_1 - t_2}{t_1 + t_2} \right) e^{-(t_1 + t_2)} \\ &= \left[ \frac{1 + (-1)^s}{2} \right] \frac{2^{s+2-\Delta}}{\Gamma(\frac{\Delta}{2})^2} \int_0^\infty \frac{dq}{q} q^{2s+\Delta} e^{-q} \int_{-1}^1 dp (1-p^2)^{\frac{\Delta}{2}-1} C_s^{(\Delta_\phi - \frac{1}{2})}(p) C_s^{(\Delta_\phi - \frac{1}{2})}(p) \\ &= \left[ \frac{1 + (-1)^s}{2} \right] \frac{2^{s+2-\Delta}}{\Gamma(\frac{\Delta}{2})^2} \Gamma(\Delta + 2s) \int_{-1}^1 dp (1-p^2)^{\frac{\Delta}{2}-1} C_s^{(\Delta_\phi - \frac{1}{2})}(p) C_s^{(\Delta_\phi - \frac{1}{2})}(p) \\ &= \left[ \frac{1 + (-1)^s}{2} \right] 2^{s+1} \frac{(\Delta - 1)_s (\Delta - 1)_{2s}}{\Gamma(s + 1)} \end{aligned} \quad (4.15)$$

where in the second equality we used the change of variables

$$q = t_1 + t_2, \quad p = \frac{t_1 - t_2}{t_1 + t_2}, \quad (4.16)$$

and in the third the orthogonality relation for Gegenbauer polynomials.

In the same way, we can determine the coefficient of the two-point functions of the double-trace operators (4.11)

$$\langle [\mathcal{OO}]_{n,s}(x_1|z_1) [\mathcal{OO}]_{n,s}(x_2|z_2) \rangle = C_{[\mathcal{OO}]_{n,s}} \frac{(H_3)^s}{(x_{12}^2)^{2\Delta+2n}}. \quad (4.17)$$

Using the explicit form (D.2) of the operators  $[\mathcal{OO}]_{n,s}$ , together with the Schwinger parameterisation (4.12), we find

$$C_{[\mathcal{OO}]_{n,s}} = \left( 1 + (-1)^n \frac{4}{N} \frac{\Gamma(s)}{2^s \Gamma(\frac{s}{2})} \frac{(\frac{d}{2} - 1)_{n+\frac{s}{2}}}{(\frac{d-1}{2})_{\frac{s}{2}} (d-2)_{n+\frac{s}{2}}} \right) \quad (4.18)$$

$$\times \frac{(-1)^n 4^{n+s} \Gamma(n+1) (-d-2n+4)_n (d-2)_{n+s}^2 (\frac{d}{2} + s)_n (\frac{3d}{2} + n + s - 4)_{\frac{d}{2} + n + s - 1}}{\Gamma(s+1) (\frac{d}{2} - 1)_n^2 (\frac{3d}{2} + 2n + s - 4)_{\frac{d}{2} - 1}},$$

where we inserted  $\Delta = d - 2$ .

### 4.2.2 3-point functions of single-trace operators

#### Warm-up: s-0-0

As a warm up, we consider the simplest case of two scalar single trace operators  $\mathcal{O}$  and a spin- $s$  conserved current. In §3.5.2, we saw that conformal symmetry fixes its form up to an overall factor

$$\langle \mathcal{J}_s(x_1|z_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \rangle = C_{\mathcal{J}_s \mathcal{O} \mathcal{O}} \frac{1}{(x_{12}^2)^{\frac{\Delta}{2}} (x_{13}^2)^{\frac{\Delta}{2}} (x_{23}^2)^{\frac{\Delta}{2}}} Y_1^s, \quad (4.19)$$

which we determine in the following.

Applying the same tools we used for the two-point functions in the previous section, we have

$$\begin{aligned}
& \langle \mathcal{J}_s(x_1|z_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \rangle \tag{4.20} \\
&= \frac{4}{(\sqrt{N})^3} f(z_1 \cdot \partial_{y_1}, z_1 \cdot \partial_{y_2}) [\langle \phi^a(y_1) \phi^b(x_2) \rangle \langle \phi^a(y_2) \phi^c(x_3) \rangle \langle \phi^b(x_2) \phi^c(x_3) \rangle + y_1 \leftrightarrow y_2] \Big|_{y_1, y_2 \rightarrow x_1} \\
&= \frac{1}{\sqrt{N}} \frac{8}{\Gamma(\frac{\Delta}{2})^3} \left[ \frac{1 + (-1)^s}{2} \right] (z_1 \cdot \partial_{y_1} + z_1 \cdot \partial_{y_2})^s C_s^{(\Delta_\phi - \frac{d}{2})} \left( \frac{z_1 \cdot \partial_{y_1} - z_1 \cdot \partial_{y_2}}{z_1 \cdot \partial_{y_1} + z_1 \cdot \partial_{y_2}} \right) \\
&\quad \times \int_0^\infty \left( \prod_{i=1}^3 \frac{dt_i}{t_i} t_i^{\frac{\Delta}{2}} \right) e^{-t_1(y_1-x_2)^2 - t_2(y_2-x_3)^2 - t_3(x_2-x_3)^2} \Big|_{y_1, y_2 \rightarrow x_1; \bar{y}_1, \bar{y}_2 \rightarrow x_2}.
\end{aligned}$$

To extract  $C_{\mathcal{J}_s \mathcal{O} \mathcal{O}}$ , as with the two-point function in the previous section, by conformal symmetry we need only identify the coefficients of  $(x_{12} \cdot z_1)^s$  in (4.20) and (4.19). This leads to

$$\begin{aligned}
C_{\mathcal{J}_s \mathcal{O} \mathcal{O}} &= \frac{1}{\sqrt{N}} \frac{2^{s+2}}{\Gamma(\frac{\Delta}{2})^3} \left[ \frac{1 + (-1)^s}{2} \right] \int_0^\infty \left( \prod_{i=1}^3 \frac{dt_i}{t_i} t_i^{\frac{\Delta}{2}} \right) t_1^s e^{-t_1 - t_2 - t_3} \\
&= \frac{8}{\sqrt{N}} \left[ \frac{1 + (-1)^s}{2} \right] \frac{2^s \left(\frac{\Delta}{2}\right)_s (\Delta - 1)_s}{\Gamma(s + 1)}. \tag{4.21}
\end{aligned}$$

Likewise, we can also determine the analogous three-point function for a spin- $s$  double-trace operator for general  $n$ ,

$$\langle [\mathcal{O}\mathcal{O}]_{n,s}(x_1|z_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \rangle = C_{[\mathcal{O}\mathcal{O}]_{n,s} \mathcal{O}\mathcal{O}} \frac{(x_{23}^2)^n}{(x_{12}^2)^{\Delta+n} (x_{13}^2)^{\Delta+n}} \Upsilon_1^s, \tag{4.22}$$

with

$$C_{[\mathcal{O}\mathcal{O}]_{n,s} \mathcal{O}\mathcal{O}} = (-1)^{n+s} 2^{n+s} \frac{(d-2)_{n+s}^2}{\Gamma(s+1)} \left( 1 + (-1)^n \frac{4}{N} \frac{\Gamma(s)}{2^s \Gamma(\frac{s}{2})} \frac{\left(\frac{d}{2} - 1\right)_{n+\frac{s}{2}}}{\left(\frac{d-1}{2}\right)_{\frac{s}{2}} (d-2)_{n+\frac{s}{2}}} \right). \tag{4.23}$$

**General case: s1-s2-s3**

With the approach clear, we now tackle the more involved general case of a three-point function involving currents (4.8) for a generic triplet of spin  $\{s_1, s_2, s_3\}$ . As we saw in §3.5.2, the most general three-point function takes the form

$$\begin{aligned} & \langle \mathcal{J}_{s_1}(x_1|z_1) \mathcal{J}_{s_2}(x_2|z_2) \mathcal{J}_{s_3}(x_3|z_3) \rangle \\ &= \sum_{n_i} C_{s_1, s_2, s_3}^{n_1, n_2, n_3} \frac{Y_1^{s_1-n_2-n_3} Y_2^{s_2-n_3-n_1} Y_3^{s_3-n_1-n_2} H_1^{n_1} H_2^{n_2} H_3^{n_3}}{(x_{12}^2)^{\frac{\tau_1+\tau_2-\tau_3}{2}} (x_{23}^2)^{\frac{\tau_2+\tau_3-\tau_1}{2}} (x_{31}^2)^{\frac{\tau_3+\tau_1-\tau_2}{2}}}, \end{aligned} \quad (4.24)$$

built from the six basic conformal structures (3.90). Each individual term is independently invariant under conformal transformations, and thus conformal symmetry does not determine the coefficients  $C_{s_1, s_2, s_3}^{n_1, n_2, n_3}$ .

The requirement that the currents are conserved relates the coefficients  $C_{s_1, s_2, s_3}^{n_1, n_2, n_3}$  amongst each other, reducing the number of independent forms. Conservation constraints on the structure of three-point functions involving conserved vector operators and the stress-energy tensor were first studied by Osborn and Petkou in [218]. This was extended to a generic triplet of spins  $\{s_1, s_2, s_3\}$  in [148, 219, 220]. In [220], the general structure required by conservation in  $d > 3$  was found to be

$$\begin{aligned} \langle \mathcal{J}_{s_1}(x_1|z_1) \mathcal{J}_{s_2}(x_2|z_2) \mathcal{J}_{s_3}(x_3|z_3) \rangle &= \sum_{l=0}^{\frac{1+\min(s_1, s_2, s_3)}{2}} c_l {}_2F_1 \left( \frac{1}{2} - l, -l, 3 - \frac{d}{2} - 2l, -\frac{1}{2} \frac{\Lambda}{H_1^2 H_2^2 H_3^2} \right) \\ &\times \frac{e^{Y_1+Y_2+Y_3} {}_0F_1(\Delta, -\frac{1}{2}H_1) {}_0F_1(\Delta, -\frac{1}{2}H_2) {}_0F_1(\Delta, -\frac{1}{2}H_3)}{(x_{12}^2)^{\frac{\tau_1+\tau_2-\tau_3}{2}} (x_{23}^2)^{\frac{\tau_2+\tau_3-\tau_1}{2}} (x_{31}^2)^{\frac{\tau_3+\tau_1-\tau_2}{2}}} \Lambda^{2l}, \end{aligned} \quad (4.25)$$

with

$$\Lambda = Y_1 Y_2 Y_3 + \frac{1}{2} [Y_1 H_1 + Y_2 H_2 + Y_3 H_3], \quad (4.26)$$

and where  $l$  takes both integer and half integer values. The coefficients  $c_l$  are left unfixed by current conservation, and depend on the theory. In particular, for a generic triplet of spins there are  $1 + \min(s_1, s_2, s_3)$  independent structures, which was first counted in [218, 221] (see also [29, 222, 223]). In the following we determine the coefficients  $c_l$  for the  $d$ -dimensional free scalar  $O(N)$  vector model.<sup>3</sup> As we shall see, our results confirm a prediction made in [220], that the free scalar correlator corresponds to the  $l = 0$  structure in the above.

As with the two-point functions in §4.2.1, we can focus on terms with no contractions amongst the auxiliary vectors. In the calculations we may then set  $z_i \cdot z_j = 0$ , giving

$$\begin{aligned}
& \langle \mathcal{J}_{s_1}(x_1|z_1) \mathcal{J}_{s_2}(x_2|z_2) \mathcal{J}_{s_3}(x_3|z_3) \rangle \tag{4.27} \\
&= \frac{8N}{\Gamma(\frac{\Delta}{2})^3} \int_0^\infty \left( \prod_{i=1}^3 \frac{dt_i}{t_i} t_i^{\frac{\Delta}{2}} \right) f^{(s_1)}(-2t_3 z_1 \cdot x_{12}, -2t_2 z_1 \cdot x_{13}) f^{(s_2)}(-2t_3 z_2 \cdot x_{21}, -2t_1 z_2 \cdot x_{23}) \\
&\quad \times f^{(s_3)}(-2t_1 z_3 \cdot x_{32}, -2t_2 z_3 \cdot x_{31}) e^{-t_1 x_{23}^2 - t_2 x_{31}^2 - t_3 x_{12}^2} \\
&= \sum_{n_i=0}^{s_i} \mathbf{D}_{s_1, s_2, s_3}^{n_1, n_2, n_3} \frac{(z_1 \cdot x_{12})^{s_1 - n_1} (z_2 \cdot x_{21})^{s_2 - n_2} (z_1 \cdot x_{13})^{n_1} (z_3 \cdot x_{31})^{n_3} (z_2 \cdot x_{23})^{n_2} (z_3 \cdot x_{32})^{s_3 - n_3}}{(x_{12}^2)^{\frac{\Delta}{2} + s_1 + s_2 - n_1 - n_2} (x_{31}^2)^{\frac{\Delta}{2} + n_1 + n_3} (x_{23}^2)^{\frac{\Delta}{2} + s_3 + n_2 - n_3}},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{D}_{s_1, s_2, s_3}^{n_1, n_2, n_3} &= \frac{N(-1)^{n_1 + n_2 + n_3} 2^{3 + s_1 + s_2 + s_3}}{s_1! s_2! s_3!} \binom{s_1}{n_1} \binom{s_2}{n_2} \binom{s_3}{n_3} (\Delta - 1)_{s_1} (\Delta - 1)_{s_2} (\Delta - 1)_{s_3} \tag{4.28} \\
&\times \frac{\Gamma(s_1 + \frac{\Delta}{2}) \Gamma(s_2 + \frac{\Delta}{2}) \Gamma(s_3 + \frac{\Delta}{2}) \Gamma(n_1 + n_3 + \frac{\Delta}{2}) \Gamma(n_2 - n_3 + s_3 + \frac{\Delta}{2}) \Gamma(-n_1 - n_2 + s_1 + s_2 + \frac{\Delta}{2})}{\Gamma(n_1 + \frac{\Delta}{2}) \Gamma(n_2 + \frac{\Delta}{2}) \Gamma(n_3 + \frac{\Delta}{2}) \Gamma(-n_1 + s_1 + \frac{\Delta}{2}) \Gamma(-n_2 + s_2 + \frac{\Delta}{2}) \Gamma(-n_3 + s_3 + \frac{\Delta}{2})}.
\end{aligned}$$

By matching with the corresponding expansion of the general form for the correlator

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<sup>3</sup>These results extend to general dimensions the results [224–226] for the same correlators in three-dimensions. See also [227] for an earlier result for the stress-tensor three-point function in a free scalar theory in general dimensions.

(4.24), we can write down the following recursion relation for the coefficients  $\mathbb{C}_{s_1, s_2, s_3}^{n_1, n_2, n_3}$

$$\begin{aligned} \mathbb{C}_{s_1, s_2, s_3}^{n_1, n_2, n_3} &= 2^{-n_1 - n_2 - n_3} \left[ \mathbb{D}_{s_1, s_2, s_3}^{n_2, s_2 - n_3, s_3 - n_1} \right. \\ &\quad - \sum_{k_1 + k_2 + k_3 = 1}^{n_1 + n_2 + n_3} (-1)^{k_1 + k_2 + k_3} \binom{k_1 + k_2 - n_1 - n_2 + s_3}{k_1} \binom{k_1 + k_3 - n_1 - n_3 + s_2}{k_3} \\ &\quad \left. \times \binom{k_2 + k_3 - n_2 - n_3 + s_1}{k_2} 2^{-k_1 - k_2 - k_3 + n_1 + n_2 + n_3} \mathbb{C}_{s_1, s_2, s_3}^{(n_1 - k_1, n_2 - k_2, n_3 - k_3)} \right], \end{aligned} \quad (4.29)$$

where the summation assumes  $k_i \leq n_i$ . Solving for the explicit form of the coefficients, we find:

$$\begin{aligned} \mathbb{C}_{s_1, s_2, s_3}^{n_1, n_2, n_3} &= - \frac{N (-1)^{n_1 + n_2 + n_3} 2^{s_1 + s_2 + s_3 - (n_1 + n_2 + n_3) + 3}}{n_1! n_2! n_3! (s_1 - n_2 - n_3)! (s_2 - n_3 - n_1)! (s_3 - n_1 - n_2)!} \\ &\quad \times \frac{\Gamma(s_1 + \frac{\Delta}{2}) \Gamma(s_2 + \frac{\Delta}{2}) \Gamma(s_3 + \frac{\Delta}{2})}{\Gamma(n_1 + \frac{\Delta}{2}) \Gamma(n_2 + \frac{\Delta}{2}) \Gamma(n_3 + \frac{\Delta}{2})} (\Delta - 1)_{s_1} (\Delta - 1)_{s_2} (\Delta - 1)_{s_3}. \end{aligned} \quad (4.30)$$

A nice observation is that the  $n_i$  dependence can be re-summed in terms of a Bessel function, giving the following compact form for the correlation function

$$\begin{aligned} &\langle \mathcal{J}_{s_1}(x_1|z_1) \mathcal{J}_{s_2}(x_2|z_2) \mathcal{J}_{s_3}(x_3|z_3) \rangle \\ &= N \left( \prod_{i=1}^3 \mathbf{c}_{s_i} q_i^{\frac{1}{2} - \frac{\Delta}{4}} \Gamma(\frac{\Delta}{2}) J_{\frac{\Delta-2}{2}}(\sqrt{q_i}) \right) \frac{\mathbf{Y}_1^{s_1} \mathbf{Y}_2^{s_2} \mathbf{Y}_3^{s_3}}{(x_{12}^2)^{\Delta/2} (x_{23}^2)^{\Delta/2} (x_{31}^2)^{\Delta/2}}, \end{aligned} \quad (4.31)$$

where we introduced

$$q_1 = 2\mathbf{H}_1 \partial_{\mathbf{Y}_2} \partial_{\mathbf{Y}_3}, \quad q_2 = 2\mathbf{H}_2 \partial_{\mathbf{Y}_3} \partial_{\mathbf{Y}_1}, \quad q_3 = 2\mathbf{H}_3 \partial_{\mathbf{Y}_1} \partial_{\mathbf{Y}_2}. \quad (4.32)$$

Normalising the two-point functions (4.13) canonically,<sup>4</sup> the  $\mathbf{c}_{s_i}$  are given by

$$\mathbf{c}_{s_i}^2 = \frac{\sqrt{\pi} 2^{-\Delta - s_i + 3} \Gamma(s_i + \frac{\Delta}{2}) \Gamma(s_i + \Delta - 1)}{N s_i! \Gamma(s_i + \frac{\Delta-1}{2}) \Gamma(\frac{\Delta}{2})^2}. \quad (4.33)$$

---

<sup>4</sup>I.e. by redefining  $\mathcal{J}_{s_i} \rightarrow \frac{1}{\sqrt{\mathbf{c}_{s_i}}} \mathcal{J}_{s_i}$ .

A nice check of this result is that it is consistent with that established for the  $s=0$  correlator in the previous section, which was already available in the literature [228]. Furthermore, using the identity

$$\Gamma(\alpha + 1) x^{-\alpha} J_\alpha(2x) = 2^{-\alpha} {}_0F_1\left(\alpha + 1; -\frac{x^2}{4}\right), \quad (4.34)$$

our result confirms the conjecture made in [220], that the free scalar correlator corresponds to the  $l = 0$  term in (4.25).

With the results (4.13) and (4.31) for the two- and three-point functions of higher-spin conserved currents, in §6 we are able to fix the cubic action of the dual type A minimal bosonic higher-spin theory on  $\text{AdS}_{d+1}$ , using the conjectured holographic higher-spin / vector model duality. Furthermore, together with the double-trace operator 3- and 2-point functions (4.22) and (4.17), as discussed in §3.6 these expressions also determine the conformal block expansion of the scalar single-trace operator four-point function, to which we now turn our attention. The latter is a key result for later applications (§6), as it will allow us to determine the on-shell quartic self interaction of the scalar in the dual higher-spin theory.

### 4.2.3 Scalar 4-point function

For the remainder of this chapter we consider the four-point function of the scalar single-trace operator [199],

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle = \frac{g_{\mathcal{O}}(u, v)}{(x_{12}^2 x_{34}^2)^\Delta}, \quad (4.35)$$

which has disconnected and connected parts,

$$g_{\mathcal{O}}(u, v) = g_{\text{disc.}}(u, v) + g_{\text{conn.}}(u, v), \quad (4.36)$$



$$g_{\text{disc.}}(u, v) = 1 + u^\Delta + \left(\frac{u}{v}\right)^\Delta, \quad g_{\text{conn.}}(u, v) = \frac{4}{N} \left( u^{\frac{\Delta}{2}} + \left(\frac{u}{v}\right)^{\frac{\Delta}{2}} + u^{\frac{\Delta}{2}} \left(\frac{u}{v}\right)^{\frac{\Delta}{2}} \right), \quad (4.37)$$

where we used canonical normalisation for the  $\langle \mathcal{O}\mathcal{O} \rangle$  two-point function. As usual, we can obtain the above expressions by simply Wick contracting. The individual contributions can be understood pictorially in figure 4.1.

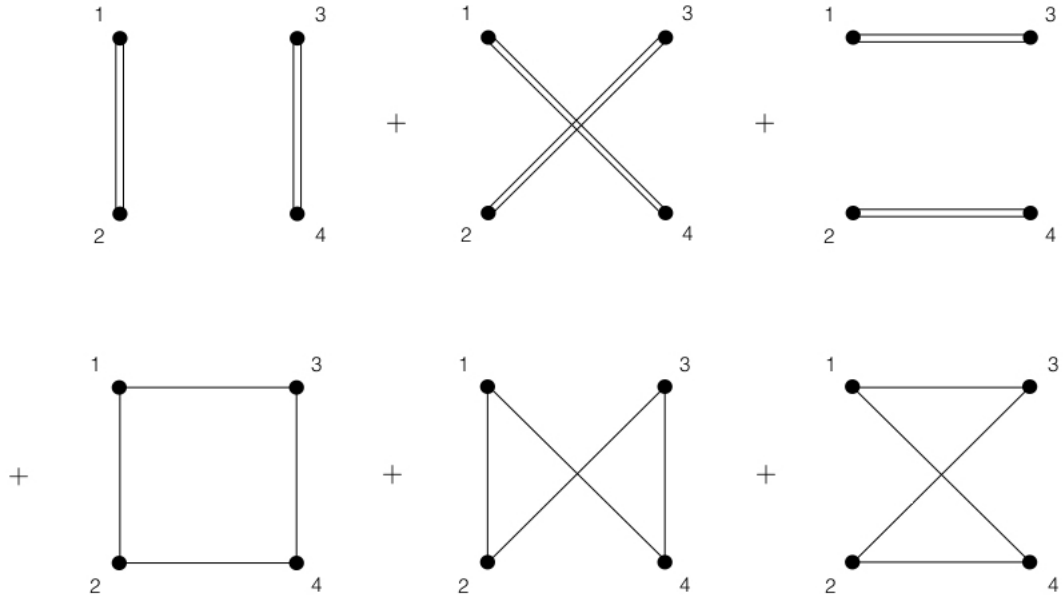


Figure 4.1: Contributions to the full four-point function (4.35) of the scalar single-trace operator  $\mathcal{O}$ . The first line constitutes the three disconnected terms, while the second line comprises the connected part of the correlator.

### OPE and conformal block expansion

For later application, we require the conformal block expansion of the four-point function (4.35). As explained in §3.5.3, although functions (4.37) of the cross ratios are invariant under conformal transformations, those in four-point functions they are com-

pletely determined by the  $\mathcal{O}\mathcal{O}$  OPE. Schematically, the latter takes the form<sup>5</sup>

$$\mathcal{O}\mathcal{O} \sim \mathbb{1} + \sum_s c_{\mathcal{O}\mathcal{O}\mathcal{J}_s} \mathcal{J}_s + \sum_{s,n} c_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{s,n}} [\mathcal{O}\mathcal{O}]_{s,n} + \text{descendants}, \quad (4.38)$$

with OPE coefficients  $c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}$  and  $c_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{s,n}}$  of the conserved currents and double-trace operators, respectively. Using the results for the two- and three-point functions established in the previous sections, together with their relationship (3.103) to the corresponding OPE coefficients, we find

$$\begin{aligned} c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}^2 &= \left[ \frac{(-1)^s + 1}{2} \right] \frac{1}{N} \frac{2^{s+3} \left(\frac{\Delta}{2}\right)_s}{s! (\Delta + s - 1)_s}, \\ c_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{s,n}}^2 &= \frac{\left[ \frac{(-1)^s + 1}{2} \right] 2^{s+1} \left(\frac{\Delta}{2}\right)_n^2 (\Delta)_{s+n}^2}{s! n! \left(s + \frac{\Delta}{2} + 1\right)_n (\Delta - 1 + n)_n (2\Delta + 2n + s - 1)_s \left(\frac{3\Delta}{2} - 1 + n + s\right)_n} \\ &\quad \times \left( 1 + (-1)^n \frac{4}{N} \frac{\Gamma(s)}{2^s \Gamma\left(\frac{s}{2}\right)} \frac{\left(\frac{\Delta}{2}\right)_{n+\frac{s}{2}}}{\left(\frac{\Delta+1}{2}\right)_{\frac{s}{2}} (\Delta)_{n+\frac{s}{2}}} \right). \end{aligned} \quad (4.40)$$

With the above results, we can write down the conformal block decomposition of the four-point function (4.35) (recall the discussion §3.6 on the conformal block decomposition of four-point correlators). For example, in the (12)(34) channel

$$\begin{aligned} \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle & \\ &= \frac{1}{(x_{12}^2 x_{34}^2)^\Delta} \left( 1 + \sum_s c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}^2 G_{\Delta+s,s}(u,v) + \sum_{s,n} c_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{s,n}}^2 G_{2\Delta+2n+s,s}(u,v) \right), \end{aligned} \quad (4.41)$$

where the first term on the RHS is the contribution from the identity operator, the second from the conformal multiplet of each conserved currents (4.8) and the third from the multiplets of each of the double-trace operators (4.11).

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<sup>5</sup>That there are no contributions from higher-trace operators is clear from the triviality of their three-point functions with two insertions of  $\mathcal{O}$ . This can be seen immediately by applying Wick's theorem.

To compare with the corresponding calculations of Witten diagrams in the dual higher-spin theory in §5, it is useful to express the expansion (4.41) of the correlator in a crossing-symmetric form. To this end, it is instrumental to understand the microscopic interpretation of each term in the connected part of the correlator (4.35): In free CFTs, four-point correlators can be straightforwardly separated into terms which independently satisfy crossing [229]. Recall that for four-point correlator of identical scalar operators, one requirement is (3.99)

$$v^\Delta g_{\mathcal{O}}(u, v) = u^\Delta g_{\mathcal{O}}(v, u). \quad (4.42)$$

In a free theory, the function  $g_{\mathcal{O}}(u, v)$  is a Laurent polynomial in  $\sqrt{u}$  and  $\sqrt{v}$ , meaning that the independent solutions of (4.42) can be expressed in a basis of the form

$$v^\Delta p_{ij}(u, v) = u^{\frac{i-j}{2}} v^{\frac{j}{2}} + v^{\frac{i-j}{2}} u^{\frac{j}{2}} = u^\Delta p_{ij}(v, u). \quad (4.43)$$

In the  $u \rightarrow 0$  limit, a given conformal block behaves as [201]

$$G_{\Delta_k, s}(u, v) \sim u^{\frac{\tau}{2}} \hat{g}_{\tau, s}(v) = u^{\frac{\tau}{2}} (1-v)^s {}_2F_1\left(\frac{\tau}{2} + s, \frac{\tau}{2} + s; \tau + 2s; 1-v\right), \quad (4.44)$$

where  $\tau = \Delta_k - s$  is the twist of an operator with quantum numbers  $[\Delta_k, s]$ , and  $\hat{g}_{\tau, s}(v)$  is sometimes referred to in the literature as a collinear conformal block. In this way we can identify the independent crossing-symmetric contributions

$$\frac{4}{N} \left( u^{\frac{\Delta}{2}} + \left( \frac{u}{v} \right)^{\frac{\Delta}{2}} \right) = \frac{4}{N} p_{3\Delta, 2\Delta}(u, v) = \sum_s c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}^2 G_{\Delta+s, s}(u, v), \quad (4.45)$$

and

$$\frac{4}{N} u^{\frac{\Delta}{2}} \left( \frac{u}{v} \right)^{\frac{\Delta}{2}} = \frac{2}{N} p_{3\Delta, \Delta}(u, v) = \sum_{s, n} c_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{s, n}}^2 G_{2\Delta+2n+s, s}(u, v). \quad (4.46)$$

With the above understanding, we can express the four-point correlator in a manifestly

crossing-symmetric form by writing,

$$\begin{aligned} \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle_{\text{conn.}} & \quad (4.47) \\ & = \mathcal{G}(y_1, y_2; y_3, y_4) + \mathcal{G}(y_1, y_3; y_2, y_4) + \mathcal{G}(y_1, y_4; y_3, y_2) \end{aligned}$$

with

$$\begin{aligned} \mathcal{G}(y_1, y_2; y_3, y_4) & \quad (4.48) \\ & = \frac{4}{N} \frac{1}{(x_{12}^2 x_{34}^2)^\Delta} \left( a \cdot p_{3\Delta, 2\Delta}(u, v) + \frac{b}{2} \cdot p_{3\Delta, \Delta}(u, v) \right) \\ & = \frac{1}{(x_{12}^2 x_{34}^2)^\Delta} \left( a \cdot \sum_s c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}^2 G_{\Delta+s, s}(u, v) + b \cdot \sum_{s, n} c_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{s, n}}^2 G_{2\Delta+2n+s, s}(u, v) \right). \end{aligned}$$

We are at liberty to choose any  $a$  and  $b$  such that  $2a + b = 1$ . This simply parametrises the freedom to add to terms which vanish under symmetrisation. In other words  $\mathcal{G}$  with different solutions for  $a$ ,  $b$  and  $c$  are related by the addition of  $\tilde{\mathcal{G}}$  with the property that

$$\tilde{\mathcal{G}}(y_1, y_2; y_3, y_4) + \tilde{\mathcal{G}}(y_1, y_3; y_2, y_4) + \tilde{\mathcal{G}}(y_1, y_4; y_3, y_2) = 0 \quad (4.49)$$

### Contour Integral form

In §6.2.2 we employ the conformal block expansion of the scalar singlet four-point function (4.41) in  $3d$  to extract the quartic self-interaction of the bulk scalar in the type A minimal bosonic higher-spin theory on  $\text{AdS}_4$ . This is achieved by matching with the conformal block expansions of the dual Witten diagrams in AdS. To do so, it will be convenient to employ the contour integral form (3.117)

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle = \int_{-\infty}^{\infty} d\nu \sum_s g_s(\nu) W_{\frac{d}{2}+i\nu, s}(x_1, x_2; x_3, x_4), \quad (4.50)$$

which we determine in the following for  $d = 3$ .

As explained in §3.6.1, the poles of the meromorphic function  $g_s(\nu)$  encode the contributions of each spin- $s$  conformal multiplet. In the case of the scalar singlet four-point function (4.41), it thus takes the form

$$g_s(\nu) = g_{\mathcal{J}_s}(\nu) + g_{[\mathcal{O}\mathcal{O}]_s}(\nu), \quad (4.51)$$

with  $g_{\mathcal{J}_s}(\nu)$  generating the contribution from the spin- $s$  conserved current multiplets, and  $g_{[\mathcal{O}\mathcal{O}]_s}(\nu)$  that of each of the spin- $s$  double trace operators (4.11).

More precisely, when closing the  $\nu$ -contour in the lower half plane as prescribed in §3.6.1,  $g_{\mathcal{J}_s}(\nu)$  contains a single simple pole corresponding to the dimension of a spin- $s$  conserved current. I.e. at  $\frac{d}{2} + i\nu = \Delta + s$

$$g_{\mathcal{J}_s}(\nu) = \frac{c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}(\nu)}{\nu^2 + \left(\Delta + s - \frac{d}{2}\right)^2}, \quad (4.52)$$

$$c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}(\nu) = \left[ \frac{(-1)^s + 1}{2} \right] \frac{2^{-2i\nu+s+5} \Gamma(i\nu + \frac{1}{2}) \Gamma\left(\frac{2s+2i\nu+3}{4}\right)^2}{\pi^{3/2} N(2i\nu + 2s + 1) \Gamma(i\nu) \Gamma\left(\frac{2s+2i\nu+1}{4}\right)^2}. \quad (4.53)$$

The function  $c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}(\nu)$  is chosen so that

$$c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}(\nu) \Big|_{\frac{d}{2} + i\nu = \Delta + s} = \frac{i}{2\pi} (2\Delta + 2s - d) c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}^2, \quad (4.54)$$

i.e. such that the single-trace contribution in (4.39) is recovered upon application of Cauchy's residue theorem.

Similarly, for the spin- $s$  double-trace operators,  $g_{[\mathcal{O}\mathcal{O}]_s}(\nu)$  has a string of simple poles at double-trace dimensions:  $\frac{d}{2} + i\nu = 2\Delta + 2n + s$ ,  $n = 0, 1, 2, \dots, \infty$

$$g_{[\mathcal{O}\mathcal{O}]_s}(\nu) = c_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_s}(\nu) \Gamma\left(\frac{2\Delta + s - \frac{d}{2} - i\nu}{2}\right), \quad (4.55)$$

which are accounted for by the poles of the Gamma function factor, and  $c_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_s}(\nu)$

generates the corresponding OPE coefficients at each pole:

$$\begin{aligned}
& c_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_s}(\nu) \tag{4.56} \\
&= \left[ \frac{(-1)^s + 1}{2} \right] \left( \frac{\Gamma(i\nu + \frac{1}{2}) 2^{-2i\nu+3s+2} \Gamma(s + \frac{3}{2}) \Gamma(\frac{2s+2i\nu+3}{4})^2}{\pi \Gamma(i\nu)(2i\nu + 2s + 1) \Gamma(s + 1) \Gamma(s - i\nu + \frac{1}{2}) \Gamma(s + i\nu + \frac{1}{2})} \right. \\
&\quad \left. + \frac{1}{N} \frac{i^s \Gamma(i\nu + \frac{1}{2}) 2^{-2i\nu+3s+\frac{3}{2}} \Gamma(\frac{s+1}{2}) \Gamma(s + \frac{3}{2}) \Gamma(\frac{2s+2i\nu+3}{4})^2}{\pi s!(s/2)! \Gamma(\frac{3}{2} - \frac{i\nu}{2}) \Gamma(\frac{3}{2} + \frac{i\nu}{2}) \Gamma(i\nu)(2i\nu + 2s + 1) \Gamma(s - i\nu + \frac{1}{2}) \Gamma(s + i\nu + \frac{1}{2})} \right).
\end{aligned}$$

with

$$c_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_s}(\nu) \Big|_{\frac{d}{2}+i\nu=2\Delta+2n+s} = \frac{i}{2\pi} n! (-1)^n c_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{s,n}}^2. \tag{4.57}$$

## Chapter 5

# Witten diagrams in higher-spin theory

With this chapter we turn to the bulk side of the story. In the preceding chapter we computed correlation functions of single-trace operators in the singlet sector of the free scalar  $O(N)$  vector model. In the context of holography, correlation functions of single-trace operators correspond to so-called Witten diagrams in the weak coupling regime of the dual gravity theory on AdS. In the present context, the dual theory is one of higher-spin gauge fields on AdS. With this setting in mind, in this chapter we introduce methods for computing Witten diagrams in theories containing fields of arbitrary rank. These methods are underpinned by the ambient space framework, which is reviewed in detail in §A.

In particular, we demonstrate how to compute three-point Witten diagrams involving external fields of arbitrary spin and mass (and also a parity even scalar), as well as methods to establish conformal partial wave expansions of four-point exchange and contact diagrams. This is all in the view to bring the Witten diagrams in the type A minimal bosonic higher-spin theory into the appropriate form, to match their counterparts in the free scalar  $O(N)$  vector model. All computations are at tree-level.

Before we proceed, let us first settle the notation. We denote points on  $\text{AdS}_{d+1}$  by  $x^\mu$ , and often work in Poincaré co-ordinates  $x^\mu = (z, y^i)$ ,

$$ds^2 = \frac{R^2}{z^2} (dz^2 + dy_i dy^i), \quad (5.1)$$

with  $y^i$  representing points on the boundary of AdS,  $i = 1, 2, \dots, d$ . We set the AdS radius  $R$  is set to one for all computations in this chapter. When working in ambient space, we use  $X$  to represent points on the AdS manifold and  $P$  for those on the conformal boundary.

## 5.1 Witten diagrams

Back in §2.1, we saw that the AdS/CFT duality can be phrased as the equivalence between the generating functional of correlators in a CFT and the full AdS partition function of its putative dual gravity theory, which in Euclidean signature reads

$$\langle \exp \left( \int d^d y \bar{\varphi}(y) \mathcal{O}(y) \right) \rangle_{\text{CFT}} = \int_{\varphi|_{\partial\text{AdS}} = \bar{\varphi}} \mathcal{D}\varphi \exp \left( -\frac{1}{G} S_{\text{AdS}}[\varphi] \right). \quad (5.2)$$

The bulk field  $\varphi$ , which is dual to the single-trace CFT operator  $\mathcal{O}$  of scaling dimension  $\Delta$  and spin- $s$ , is subject to the boundary condition (2.15)

$$\lim_{z \rightarrow 0} \varphi(z, y) z^{\Delta-d-s} = \bar{\varphi}(y), \quad (5.3)$$

in order to define the path integral. In particular, this implies that correlators of single-trace operators  $\mathcal{O}_i$  can be computed by functionally differentiating the gravity partition function with respect to  $\bar{\varphi}_i$

$$\langle \mathcal{O}_1(y_1) \dots \mathcal{O}_n(y_n) \rangle = \frac{\delta}{\delta \bar{\varphi}_1(y_1)} \dots \frac{\delta}{\delta \bar{\varphi}_n(y_n)} \int_{\varphi_i|_{\partial\text{AdS}} = \bar{\varphi}_i} \mathcal{D}\varphi \exp \left( -\frac{1}{G} S_{\text{AdS}}[\varphi_i] \right) \Big|_{\bar{\varphi}_i=0}. \quad (5.4)$$



When the gravity theory in AdS is weakly coupled,<sup>1</sup> we can perform a semi-classical expansion of the correlators generated by the gravity partition functional,

$$\begin{aligned} \langle \mathcal{O}_1(y_1) \dots \mathcal{O}_n(y_n) \rangle & \quad (5.5) \\ &= \frac{\delta}{\delta \bar{\varphi}_1(y_1)} \dots \frac{\delta}{\delta \bar{\varphi}_n(y_n)} \exp \left( -\frac{1}{G} S_{\text{AdS}} [\varphi_i |_{\partial \text{AdS}} = \bar{\varphi}_i] \right) \Big|_{\bar{\varphi}_i=0} + \text{loops}, \end{aligned}$$

in which quantum loop corrections are subdominant. This corresponds to a large  $N_{\text{dof}}$  expansion in the dual CFT, and the corresponding Feynman diagrams in AdS space are known as *Witten diagrams*.

As explained in §2.2.1, by studying the tree-level contribution to the relation (5.5) for non-linear higher-spin theories on AdS, we can extract the nature of their on-shell interactions by employing their conjectured equivalence with free conformal field theories. To do so, we therefore need to develop effective techniques for the systematic evaluation of tree-level Witten diagrams in theories containing fields of arbitrary rank. We first warm up by considering the simplest index-free example of a scalar field in AdS, the amplitudes of which we later show can be related to those with external higher-spin fields.

### 5.1.1 Warm-up example: Scalar field in AdS

Consider the example of a scalar field  $\phi$  in  $\text{AdS}_{d+1}$ , with classical action

$$S = \frac{1}{G} \int_{\text{AdS}} \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi + \frac{1}{2!} m^2 \phi^2 + \frac{1}{3!} g \phi^3 + \frac{1}{4!} \lambda \phi^4 + \dots, \quad (5.6)$$

for some overall coupling  $G$ . According to the field-operator map §2.1.1, the bulk scalar  $\phi$  will be dual to some single-trace scalar operator  $\mathcal{O}$  with scaling dimension  $\Delta$ , which

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<sup>1</sup>In the sense of  $\hbar \rightarrow 0$ , not necessarily small curvature/non-stringy.

is related to the (mass)<sup>2</sup> of  $\phi$

$$(mR)^2 = \Delta(\Delta - d). \quad (5.7)$$

In the weak coupling regime  $G \ll 1$  we can make the saddle point approximation

$$\exp(-F_{\text{CFT}}[\bar{\phi}]) \approx \exp(-S[\bar{\phi}]), \quad (5.8)$$

where we neglect loop corrections, which are suppressed in  $G$ . Here,  $F_{\text{CFT}}[\bar{\phi}]$  is the generating function of connected correlators of  $\mathcal{O}$  with source  $\bar{\phi}$ , and  $S[\bar{\phi}]$  is the on-shell action (5.6) subject to the boundary condition

$$\lim_{z \rightarrow 0} \phi(z, y) z^{\Delta-d} = \bar{\phi}(y). \quad (5.9)$$

In the following, we demonstrate how to evaluate correlation functions of  $\mathcal{O}$  at leading order in  $N_{\text{dof}}$  using the holographic equality (5.8).

To do so, we must evaluate the on-shell action  $S[\bar{\phi}]$ . The first step is to solve the non-linear classical equation of motion,

$$\frac{\delta S}{\delta \phi} = (-\square + m^2)\phi + \frac{g}{2!}\phi^2 + \frac{\lambda}{3!}\phi^3 + \dots = 0, \quad (5.10)$$

subject to the boundary condition (5.9). To express the solution in terms of the boundary value  $\bar{\phi}$ , we solve perturbatively in  $\bar{\phi}$  by using integral kernels.<sup>2</sup> We write

$$\phi(x) = \phi_0(x) + \phi_1(x) + \phi_2(x) + \dots, \quad (5.11)$$

where  $\phi_n$  is the solution at order  $n+1$  in  $\bar{\phi}$ . To begin, the solution of the linear equation

$$(-\square + m^2)\phi_0 = 0, \quad (5.12)$$

---

<sup>2</sup>This is justified either for small source  $\bar{\phi}$  or small couplings  $\{g, \lambda, \dots\}$ .

can be constructed from the boundary data via the corresponding *bulk-to-boundary propagator*. This is an integral kernel

$$\phi_0(z, y) = \int_{\partial\text{AdS}} d^d y' K_\Delta(z, y; y') \bar{\phi}(y'), \quad (5.13)$$

where

$$(-\square + m^2) K_\Delta(z, y; y') = 0, \quad \lim_{z \rightarrow 0} (z^{\Delta-d} K_\Delta(z, y; y')) = \frac{1}{2\Delta - d} \delta^d(y - y'). \quad (5.14)$$

Increasing the order in  $\bar{\phi}$ , we must solve

$$\begin{aligned} (-\square + m^2) \phi_0 &= 0, \\ (-\square + m^2) \phi_1 + \frac{g}{2!} \phi_0^2 &= 0, \\ (-\square + m^2) \phi_2 + g \phi_0 \phi_1 + \frac{\lambda}{3!} \phi_0^3 &= 0, \\ &\vdots \end{aligned} \quad (5.15)$$

Through use of the *bulk-to-bulk propagator*

$$(-\square + m^2) \Pi_\Delta(x; x') = \frac{1}{\sqrt{|g|}} \delta^{d+1}(x - x'), \quad (5.16)$$

the solution can be determined order-by-order in terms of the linear solution  $\phi_0$  and thus the boundary value  $\bar{\phi}$ ,

$$\begin{aligned} \phi_0(x) &= \int_{\partial\text{AdS}} d^d y' K_\Delta(z, y; y') \bar{\phi}(y'), \\ \phi_1(x) &= -\frac{g}{2!} \int_{\text{AdS}} d^{d+1} x' \Pi_\Delta(x; x') \phi_0^2(x'), \\ \phi_2(x) &= -\frac{\lambda}{3!} \int_{\text{AdS}} d^{d+1} x' \Pi_\Delta(x; x') \phi_0^3(x') - g \int_{\text{AdS}} d^{d+1} x' \Pi_\Delta(x; x') \phi_0(x') \phi_1(x'), \\ &\vdots \end{aligned} \quad (5.17)$$

The value of the action for this classical solution is given by the diagrammatic expansion

$$S[\bar{\phi}] = \frac{1}{2} \text{circle} + \frac{1}{6} \text{circle with 3 lines} + \frac{1}{8} \text{circle with 4 lines} + \frac{1}{24} \text{circle with 6 lines} + \dots \quad (5.18)$$

Correlation functions of  $\mathcal{O}$  can be computed at leading order in  $N_{\text{dof}}$  by taking functional derivatives of (5.18). In the following subsections we give explicit examples for the two- and three-point functions, for which we only require the bulk-to-boundary propagator (5.14). We therefore first discuss the latter in more detail.

### Propagators

To compute two- and three-point Witten diagrams at tree-level, as can be seen from the diagrammatic expansion (5.18) we require the form of the boundary-to-bulk propagator (5.14).<sup>3</sup> Its most familiar expression is given in the Poincaré patch

$$K_{\Delta}(z, y; y') = C_{\Delta,0} \left( \frac{z}{z^2 + (y - y')^2} \right)^{\Delta}, \quad (5.19)$$

where the overall coefficient is fixed by the near-boundary behaviour (5.14)

$$C_{\Delta,0} = \frac{\Gamma(\Delta)}{2\pi^{d/2}\Gamma(\Delta + 1 - \frac{d}{2})}. \quad (5.20)$$

However, we consider the evaluation of Witten diagrams in ambient space §A, in which the propagator takes the form

$$K_{\Delta}(X; P) = \frac{C_{\Delta,0}}{(-2X \cdot P)^{\Delta}}. \quad (5.21)$$

As we shall see, the ambient framework proves effective in extending the results for the scalar to those with higher-spin external legs. In particular, the ambient expression for

<sup>3</sup>We postpone a derivation until §5.2.2, where it is considered together with the higher-spin propagators.

the propagator admits a Schwinger-Parameterised form

$$K_{\Delta}(X; P) = \frac{C_{\Delta,0}}{\Gamma(\Delta)} \int_0^{\infty} \frac{dt}{t} t^{\Delta} e^{2tP \cdot X}, \quad (5.22)$$

which turns out to dramatically simplify the evaluation of contact Witten diagrams.

### Two-point Witten diagram

We begin with the two-point Witten diagrams. Unlike their higher-point counterparts, their computation via the above prescription requires careful treatment of IR divergences which arise from integrating over the infinite volume of AdS space [230]. However these issues can be circumvented by noting that, as in flat space, at tree-level they can simply be given by the corresponding propagator (5.16)/(5.14)

$$\langle \mathcal{O}(y_1) \mathcal{O}(y_2) \rangle = K_{\Delta}(y_1; y_2) = \Pi_{\Delta}(y_1; y_2), \quad (5.23)$$

which can be seen pictorially in figure 5.1.

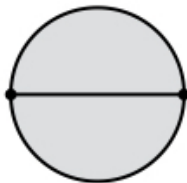
$$\langle \mathcal{O}(y_1) \mathcal{O}(y_2) \rangle = \frac{\delta}{\delta \bar{\phi}(y_1)} \frac{\delta}{\delta \bar{\phi}(y_2)} S[\bar{\phi}] = \mathcal{O}(y_1) \text{---} \text{---} \text{---} \mathcal{O}(y_2)$$


Figure 5.1: Bulk interpretation of the  $\langle \mathcal{O}\mathcal{O} \rangle$  CFT two-point function, in the large  $N_{\text{dof}}$  limit. The propagator departs from one boundary insertion point to the other.

### Three-point Witten diagram

For Witten diagrams of three or more external points, there are no issues of IR divergences.<sup>4</sup> We thus proceed using the prescription for computing Witten diagrams as described at the beginning of this subsection. The three-point correlator of  $\mathcal{O}$  at

<sup>4</sup>This can be verified straightforwardly by power counting.

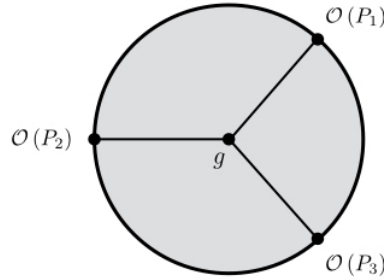


Figure 5.2: Tree-level three-point Witten diagram generated by the bulk vertex  $g\phi^3$ . This gives the holographic computation of the dual scalar operator three-point function at leading order in  $1/N_{\text{dof}}$ .

leading order in  $1/N_{\text{dof}}$  can thus be computed holographically via

$$\begin{aligned} \langle \mathcal{O}(y_1) \mathcal{O}(y_2) \mathcal{O}(y_3) \rangle &= \frac{\delta}{\delta \bar{\phi}(y_1)} \frac{\delta}{\delta \bar{\phi}(y_2)} \frac{\delta}{\delta \bar{\phi}(y_3)} S[\bar{\phi}] \\ &= g \int_{\text{AdS}} d^{d+1}x K_{\Delta}(x; y_1) K_{\Delta}(x; y_2) K_{\Delta}(x; y_3), \end{aligned} \quad (5.24)$$

and is depicted in figure 5.2. An effective way to evaluate the bulk integral (5.24), is to employ the Schwinger-parameterised form of the propagator in the ambient formalism.

This gives

$$\begin{aligned} \langle \mathcal{O}(P_1) \mathcal{O}(P_2) \mathcal{O}(P_3) \rangle &= g \int_{\text{AdS}} dX K_{\Delta}(X; P_1) K_{\Delta}(X; P_2) K_{\Delta}(X; P_3) \\ &= g \left( \frac{C_{\Delta,0}}{\Gamma(\Delta)} \right)^3 \int_0^{\infty} \prod_{i=1}^3 \left( \frac{dt_i}{t_i} t_i^{\Delta} \right) \int_{\text{AdS}} dX e^{2(t_1 P_1 + t_2 P_2 + t_3 P_3) \cdot X}. \end{aligned} \quad (5.25)$$

The integral over AdS is straightforward to evaluate using (B.1), and yields

$$\begin{aligned} \langle \mathcal{O}(P_1) \mathcal{O}(P_2) \mathcal{O}(P_3) \rangle & \\ &= g \pi^{\frac{d}{2}} \Gamma\left(\frac{3\Delta - d}{2}\right) \left( \frac{C_{\Delta,0}}{\Gamma(\Delta)} \right)^3 \int_0^{\infty} \prod_{i=1}^3 \left( \frac{dt_i}{t_i} t_i^{\Delta} \right) e^{(-t_1 t_2 P_{12} - t_1 t_3 P_{13} - t_2 t_3 P_{23})}. \end{aligned} \quad (5.26)$$

The remaining integrals can be dealt with through the change of variables

$$t_1 = \sqrt{\frac{m_2 m_3}{m_1}}, \quad t_2 = \sqrt{\frac{m_1 m_3}{m_2}}, \quad t_3 = \sqrt{\frac{m_1 m_2}{m_3}}, \quad (5.27)$$

which gives the final form for the three-point amplitude

$$\begin{aligned} \langle \mathcal{O}(P_1) \mathcal{O}(P_2) \mathcal{O}(P_3) \rangle &= g \frac{1}{2} \pi^{\frac{d}{2}} \Gamma\left(\frac{3\Delta - d}{2}\right) \left(\frac{C_{\Delta,0}}{\Gamma(\Delta)}\right)^3 \int_0^\infty \prod_{i=1}^3 \left(\frac{dm_i}{m_i} m_i^{\frac{\Delta}{2}}\right) \exp(-m_i P_{jk}) \\ &= g \frac{1}{2} \pi^{\frac{d}{2}} \Gamma\left(\frac{3\Delta - d}{2}\right) \left(C_{\Delta,0} \frac{\Gamma(\frac{\Delta}{2})}{\Gamma(\Delta)}\right)^3 \frac{1}{P_{13}^{\Delta/2} P_{23}^{\Delta/2} P_{12}^{\Delta/2}}. \end{aligned} \quad (5.28)$$

This result is straightforward to generalise to different interacting scalars  $\phi_i$  of mass  $m_i^2 R^2 = \Delta_i (\Delta_i - d)$ , connected via the cubic vertex

$$\hat{\mathcal{V}}_{0,0,0} = g \phi_1 \phi_2 \phi_3, \quad (5.29)$$

In precisely the same way as above, the corresponding amplitude at tree-level can be determined to be

$$\begin{aligned} &\langle \mathcal{O}_1(P_1) \mathcal{O}_2(P_2) \mathcal{O}_3(P_3) \rangle \\ &= g \pi^{\frac{d}{2}} \Gamma\left(\frac{-d + \sum_{i=1}^3 \Delta_i}{2}\right) \frac{C_{\Delta_1,0} C_{\Delta_2,0} C_{\Delta_3,0}}{\Gamma(\Delta_1) \Gamma(\Delta_2) \Gamma(\Delta_3)} \int_0^\infty \prod_{i=1}^3 \left(\frac{dt_i}{t_i} t_i^{\Delta_i}\right) e^{(-t_1 t_2 P_{12} - t_1 t_3 P_{13} - t_2 t_3 P_{23})} \\ &= g \mathbf{C}(\Delta_1, \Delta_2, \Delta_3; 0) \frac{1}{P_{13}^{\frac{\Delta_1 + \Delta_3 - \Delta_2}{2}} P_{23}^{\frac{\Delta_2 + \Delta_3 - \Delta_1}{2}} P_{12}^{\frac{\Delta_1 + \Delta_2 - \Delta_3}{2}}}, \end{aligned} \quad (5.30)$$

where in the second equality we use the same change of variables (5.27), and we introduced

$$\begin{aligned} &\mathbf{C}(\Delta_1, \Delta_2, \Delta_3; 0) \\ &= \frac{1}{2} \pi^{\frac{d}{2}} \Gamma\left(\frac{-d + \sum_{i=1}^3 \Delta_i}{2}\right) C_{\Delta_1,0} C_{\Delta_2,0} C_{\Delta_3,0} \frac{\Gamma\left(\frac{\Delta_1 + \Delta_2 - \Delta_3}{2}\right) \Gamma\left(\frac{\Delta_1 + \Delta_3 - \Delta_2}{2}\right) \Gamma\left(\frac{\Delta_2 + \Delta_3 - \Delta_1}{2}\right)}{\Gamma(\Delta_1) \Gamma(\Delta_2) \Gamma(\Delta_3)}. \end{aligned} \quad (5.31)$$

Note that (5.30) coincides with the structure (3.74) for the three-point correlator for the dual scalar operators, as required for the holographic duality to hold. In a sense this matching of space-time dependence (unlike matching the multiplicative constants) is a bit trivial at the three-point level, since everything is fixed by  $SO(d, 2)$  kinematics.

## 5.2 Propagators of arbitrary spin

With a firm understanding of tree-level three-point Witten diagrams for external scalars, for the remainder of this chapter we extend these results to four-point Witten diagrams in theories involving fields with arbitrary integer spin, and also to three-point Witten diagrams with spinning external legs. In this section we begin with the corresponding boundary-to-bulk and bulk-to-bulk propagators.

### 5.2.1 Bulk-to-bulk propagators

In this subsection we derive the explicit form of the bulk-to-bulk propagators for spin- $s$  fields propagating on an AdS background.<sup>5</sup> In particular we employ the ambient space formalism, which is reviewed in detail in §A. Let us note that this discussion only applies to gauge fields, for otherwise the only available quadratic actions require non-local terms or auxiliary fields [243–245].<sup>6</sup>

The linearised dynamics of an off shell spin- $s$  gauge field  $\varphi_s$  in AdS coupled to a conserved source  $J_s$  is governed by the action

$$S[\varphi_s] = \frac{s!}{2} \int_{\text{AdS}_{d+1}} \varphi_s(x; \partial_u) \mathcal{G}_s(x; u) - g \varphi_s(x; \partial_u) J_s(x; u), \quad (5.32)$$

<sup>5</sup>See also [134, 231–234] for earlier works on higher-spin bulk-to-bulk propagators. For results on lower spin propagators, see: [235–242].

<sup>6</sup>See however [152] for the traceless and transverse parts of spinning bulk-to-bulk propagators for any mass in AdS.



where  $\mathcal{G}_s$  is the generalisation of the linearised Einstein tensor to spin- $s$  gauge fields

$$\mathcal{G}_s(x; u) = \left(1 - \frac{1}{4} u^2 \partial_u \cdot \partial_u\right) \mathcal{F}_s(x; u, \nabla, \partial_u) \varphi_s(x, u), \quad (5.33)$$

with Fronsdal operator [81, 246]

$$\mathcal{F}_s(x, u, \nabla, \partial_u) = \square - m_s^2 - u^2(\partial_u \cdot \partial_u) - (u \cdot \nabla) \left( (\nabla \cdot \partial_u) - \frac{1}{2}(u \cdot \nabla)(\partial_u \cdot \partial_u) \right), \quad (5.34)$$

$$m_s^2 R^2 = (s + d - 2)(s - 2) - s,$$

In order to satisfy the Bianchi identity

$$(\partial_u \cdot \nabla) \mathcal{G}_s(x, u) = 0, \quad (5.35)$$

we require the field  $\varphi_s$  to be double-traceless,  $(\partial_u \cdot \partial_u) \varphi_s(x, u) = 0$ . The above action is invariant under the linearised spin- $s$  gauge transformations

$$\delta \varphi_s(x, u) = (u \cdot \nabla) \varepsilon_{s-1}(x, u), \quad (5.36)$$

where  $\varepsilon_{s-1}(x, u)$  is a generating function for a rank- $(s-1)$  symmetric and traceless gauge parameter

$$\varepsilon_{s-1}(x, u) \equiv \frac{1}{(s-1)!} \varepsilon_{\mu_1 \mu_2 \dots \mu_{s-1}} u^{\mu_1} u^{\mu_2} \dots u^{\mu_{s-1}}, \quad (\partial_u \cdot \partial_u) \varepsilon_{s-1}(x, u) = 0. \quad (5.37)$$

The field is given in response to the source by

$$\varphi_s(x, u) = g \int_{\text{AdS}} \Pi_s(x, u; x', \partial_{u'}) J_s(x'; u'), \quad (5.38)$$

where the bulk-to-bulk propagator satisfies the equation<sup>7</sup>

$$\begin{aligned} \left(1 - \frac{1}{4}u_1^2\partial_{u_1} \cdot \partial_{u_1}\right)\mathcal{F}_s(x_1, u_1, \nabla_1, \partial_{u_1})\Pi_s(x_1, u_1, x_2, u_2) = \\ - \{\{(u_1 \cdot u_2)^s\}\}\delta(x_1, x_2) + (u_2 \cdot \nabla_2)\Lambda_{s,s-1}(x_1, u_1, x_2, u_2). \end{aligned} \quad (5.40)$$

The term  $\Lambda_{s,s-1}$  is pure gauge, whose effect is immaterial when integrating against a conserved current (5.38).<sup>8</sup> We thus have the following freedom in the definition of the propagator

$$\Pi_s(x_1, u_1, x_2, u_2) \sim \Pi_s(x_1, u_1, x_2, u_2) + (u_2 \cdot \nabla_2)\mathcal{E}_{2,s,s-1}(x_1, u_1, x_2, u_2), \quad (5.42)$$

where  $\mathcal{E}_2$  is defined by  $\Lambda$ .

However, fixing  $\Lambda$  does not specify the propagator uniquely. Due to gauge invariance (5.36), the left hand side of (5.40) is not sensitive to variations of the propagator of the form

$$\Pi_s(x_1, u_1, x_2, u_2) \sim \Pi_s(x_1, u_1, x_2, u_2) + (u_1 \cdot \nabla_1)\mathcal{E}_{1,s-1,s}(x_1, u_1, x_2, u_2),$$

where  $\mathcal{E}_1$  has the same rank and trace properties as  $\mathcal{E}_2$ , but with ‘1’ and ‘2’ interchanged. This freedom is usually fixed by imposing a gauge, and is required to make the operator on the left hand side of (5.40) non-degenerate and thus invertible.

---

<sup>7</sup>The notation  $\{\{\bullet\}\}$  signifies a double-traceless projection:

$$(\partial_u \cdot \partial_u)^2 \{\{f(u, x)\}\} = 0, \quad \text{and} \quad \{\{f(u, x)\}\} = f(u, x) \quad \text{iff} \quad (\partial_u \cdot \partial_u)^2 f(u, x) = 0. \quad (5.39)$$

<sup>8</sup>In more detail, it is a bi-tensor that is traceless in tangent indices at  $x_2$  and double-traceless in tangent indices at  $x_1$ :

$$(u_2 \cdot \partial_{u_2})\Lambda_{s,s-1} = s - 1, \quad (u_1 \cdot \partial_{u_1})\Lambda_{s,s-1} = s, \quad (\partial_{u_2} \cdot \partial_{u_2})\Lambda_{s,s-1} = (\partial_{u_1} \cdot \partial_{u_1})^2\Lambda_{s,s-1} = 0. \quad (5.41)$$

In the following sections, we determine the explicit form of the bulk-to-bulk propagator (5.38) in three different gauges: the de Donder gauge, a traceless gauge and what we define as the “manifest trace gauge”. As the name suggests, the latter makes the trace structure of the propagator manifest. In particular, in the view of expressing exchange diagrams as partial wave expansions, we derive the propagators in a basis of AdS harmonic functions.

### de Donder gauge

It is often useful to eliminate gradients and divergences from the Fronsdal tensor (5.34). This can be achieved by imposing the de Donder gauge

$$(\nabla \cdot \partial_u) \varphi_s(x, u) - \frac{1}{2} (u \cdot \nabla) (\partial_u \cdot \partial_u) \varphi_s(x, u) = 0, \quad (5.43)$$

With this choice, the Fronsdal tensor reads

$$\mathcal{F}_s(x, u, \nabla, \partial_u) \varphi_s(x, u) = (\square - m_s^2) \varphi_s(x, u) - u^2 (\partial_u \cdot \partial_u) \varphi_s(x, u). \quad (5.44)$$

To proceed, we decompose the bulk-to-bulk propagator (5.38) into symmetric and traceless components<sup>9</sup>

$$\begin{aligned} \Pi_s(x_1, u_1, x_2, u_2) &= \Pi_s^{\{0\}}(x_1, u_1, x_2, u_2) + u_1^2 u_2^2 \Pi_{s-2}^{\{1\}}(x_1, u_1, x_2, u_2), \\ (\partial_{u_1} \cdot \partial_{u_1}) \Pi_s^{\{0\}} &= (\partial_{u_2} \cdot \partial_{u_2}) \Pi_s^{\{0\}} = 0 = (\partial_{u_1} \cdot \partial_{u_1}) \Pi_{s-2}^{\{1\}} = (\partial_{u_2} \cdot \partial_{u_2}) \Pi_{s-2}^{\{1\}}. \end{aligned} \quad (5.45)$$

---

<sup>9</sup>Any double-traceless tensor  $(\partial_u \cdot \partial_u)^2 t_s(x, u) = 0$  can be expressed in the form

$$t_s(x, u) = t_s^{\{0\}}(x, u) + u^2 t_{s-2}^{\{1\}}(x, u), \quad (\partial_u \cdot \partial_u) t_s^{\{0\}}(x, u) = (\partial_u \cdot \partial_u) t_s^{\{1\}}(x, u) = 0.$$

The equation (5.40) for the propagator thus splits in two equations for traceless bi-tensors

$$\begin{aligned} (\square_1 - m_s^2) \Pi_s^{\{0\}}(x_1, u_1, x_2, u_2) &= - \{(u_1 \cdot u_2)^s\} \delta(x_1, x_2), \\ (\square_1 - m_t^2) \Pi_{s-2}^{\{1\}}(x_1, u_1, x_2, u_2) &= \frac{s(s-1)}{(d+2s-3)(d+2s-5)} \{(u_1 \cdot u_2)^{s-2}\} \delta(x_1, x_2), \end{aligned} \quad (5.46)$$

where  $m_t^2 = s^2 + (d-1)s - 2$ , and we used

$$\{\{(u_1 \cdot u_2)^s\}\} = \{(u_1 \cdot u_2)^s\} + \frac{s(s-1)}{2(d+2s-3)} u_1^2 u_2^2 \{(u_1 \cdot u_2)^{s-2}\}. \quad (5.47)$$

In decomposing into symmetric and traceless components, we can employ the operator algebra on traceless symmetric tensors (which can be found in §A.2.3). This reduces the problem to an algebraic one: Make an ansatz in a basis of traceless harmonic functions, and apply the latter operator algebra to determine the basis coefficients. This is most streamlined using the ambient formalism (§A), in which we make the ansatz

$$\begin{aligned} \Pi_s^{\{0\}} &= \sum_{\ell=0}^s \int_{-\infty}^{\infty} d\nu f_{s,\ell}^{\{0\}}(\nu) (W_1 \cdot \nabla_1)^\ell (W_2 \cdot \nabla_2)^\ell \Omega_{\nu, s-\ell}(X_1, X_2; W_1, W_2), \\ \Pi_{s-2}^{\{1\}} &= \sum_{\ell=0}^{s-2} \int_{-\infty}^{\infty} d\nu f_{s-2,\ell}^{\{1\}}(\nu) (W_1 \cdot \nabla_1)^\ell (W_2 \cdot \nabla_2)^\ell \Omega_{\nu, s-2-\ell}(X_1, X_2; W_1, W_2), \end{aligned} \quad (5.48)$$

where  $f_{s,\ell}^{\{0\}}$  and  $f_{s-2,\ell}^{\{1\}}$  are to be fixed by (5.46), and  $W_{1,2}$  are null auxiliary vectors which enforce tracelessness. Employing the equation of motion (C.2) for  $\Omega$ , the completeness relation (C.5) and the commutator (A.50), we can conclude that

$$f_{s,\ell}^{\{0\}}(\nu) = \frac{c_{s,\ell}(\nu)}{m_s^2 + \frac{d^2}{4} + \nu^2 + s - \ell + \ell(d+2s - \ell - 1)}, \quad (5.49)$$

$$f_{s-2,\ell}^{\{1\}}(\nu) = - \frac{s(s-1)}{(d+2s-3)(d+2s-5)} \frac{c_{s-2,\ell}(\nu)}{m_t^2 + \frac{d^2}{4} + \nu^2 + s - \ell - 2 + \ell(d+2s - \ell - 5)}, \quad (5.50)$$

where

$$c_{s,\ell}(\nu) = \frac{2^\ell (r - \ell + 1)_\ell \left(\frac{d}{2} + r - \ell - \frac{1}{2}\right)_\ell}{\ell! (d + 2r - 2\ell - 1)_\ell \left(\frac{d}{2} + r - \ell + i\nu\right)_\ell \left(\frac{d}{2} + r - \ell - i\nu\right)_\ell}, \quad (5.51)$$

comes from the completeness relation (C.5).

### Traceless gauge

In a similar way, we can also solve for the propagator by imposing a traceless gauge on the spin- $s$  field. To set the trace of the spin- $s$  Fronsdal field to zero, the gauge parameter  $\varepsilon_s(x, u)$  needs to be chosen such that

$$(\nabla \cdot \partial_u) \varepsilon_{s-1}(x, u) = -(\partial_u \cdot \partial_u) \varphi_s, \quad (5.52)$$

after which  $(\nabla \cdot \partial_u) \varepsilon_{s-1}$  is fixed. In this partial gauge, the field  $\varphi_s(x, u)$  is left with residual gauge symmetry

$$\delta \varphi_s(x, u) = (u \cdot \nabla) \varepsilon_{s-1}(x, u), \quad (\nabla \cdot \partial_u) \varepsilon_{s-1}(x, u) = (\partial_u \cdot \partial_u) \varepsilon_{s-1}(x, u) = 0. \quad (5.53)$$

Subject to the condition  $(\partial_u \cdot \partial_u) \varphi_s = 0$ , the Fronsdal tensor then reads

$$\begin{aligned} \mathcal{F}_s(x, u, \nabla, \partial_u) \varphi_s(x, u) & \quad (5.54) \\ & = (\square - m_s^2) \varphi_s(x, u) - (u \cdot \nabla) (\nabla \cdot \partial_u) \varphi_s(x, u) + \frac{1}{d + 2s - 3} u^2 (\nabla \cdot \partial_u)^2 \varphi_s(x, u), \end{aligned}$$

which is simply its traceless part.

Before using the form (5.54) for the Fronsdal tensor to solve for the propagator, we need to fix the residual gauge symmetry (5.53). This can be done by further demanding that

the propagator is symmetric under  $u_1 \leftrightarrow u_2$ .<sup>10</sup> The equation for the propagator is then

$$\begin{aligned} (\square_1 - m_s^2) \Pi_s(x_1, u_1, x_2, u_2) - (u_1 \cdot \nabla_1) (\nabla_1 \cdot \partial_{u_1}) \Pi_s(x_1, u_1, x_2, u_2) \\ + \frac{1}{d+2s-3} u_1^2 (\nabla_1 \cdot \partial_{u_1})^2 \Pi_s(x_1, u_1, x_2, u_2) = - \{(u_1 \cdot u_2)^s\} \delta(x_1, x_2), \end{aligned} \quad (5.55)$$

with

$$(\partial_{u_1} \cdot \partial_{u_1}) \Pi_s(x_1, u_1, x_2, u_2) = (\partial_{u_2} \cdot \partial_{u_2}) \Pi_s(x_1, u_1, x_2, u_2) = 0, \quad (5.56)$$

$$\Pi_s(x_1, u_1, x_2, u_2) = \Pi_s(x_1, u_2, x_2, u_1). \quad (5.57)$$

In the ambient formalism, using the null auxiliary vectors  $W_{1,2}$  this is

$$(\square_1 - m_s^2) \Pi_s - \frac{2}{d+2s-3} (W_1 \cdot \nabla_1) (\nabla_1 \cdot \hat{D}_{W_1}) \Pi_s = - (W_1 \cdot W_2)^s \delta(X_1, X_2), \quad (5.58)$$

with  $\Pi_s(X_1, X_2; W_1, W_2) = \Pi_s(X_1, X_2; W_2, W_1)$ .

Like for the de Donder gauge, since the propagator is traceless we can make the ansatz

$$\Pi_s = \sum_{\ell=0}^s \int_{-\infty}^{\infty} d\nu f_{s,\ell}(\nu) (W_1 \cdot \nabla_1)^\ell (W_2 \cdot \nabla_2)^\ell \Omega_{\nu, s-\ell}(X_1, X_2; W_1, W_2), \quad (5.59)$$

with  $f_{s,\ell}(\nu)$  arbitrary functions to be determined. By simply employing again a combination of the equation of motion for the  $\Omega$  (C.2), the completeness relation (C.5), and commutators: (A.50) and (A.51), the propagator equation (5.58) determines the basis coefficients to be

$$f_{s,\ell}(\nu) = -c_{s,\ell}(\nu) \frac{d+2s-3}{(\ell-1)(2s+d-\ell-3)} \frac{1}{\nu^2 + (s-2 + \frac{d}{2})^2}. \quad (5.60)$$

where we recall that  $c_{s,\ell}(\nu)$  comes from the completeness relation (C.5).

---

<sup>10</sup>Note that (5.40) does not assume this symmetry.

### Manifest trace gauge

In the previous sections, fixing a gauge for the massless spin- $s$  field allowed us to invert the differential operator in the defining equation (5.38) for the bulk-to-bulk propagator, and solve for its form. However, as already explained, since we ultimately integrate the propagators against a conserved current (5.38), they are defined only up to the addition of terms proportional to gradients.<sup>11</sup>

This freedom can be used to simplify the form of the de Donder- (5.48) and traceless- (5.59) propagators: By removing terms proportional to gradients, it is possible to bring them into the form

$$\Pi_s = \sum_{k=0}^{[s/2]} \int_{-\infty}^{\infty} d\nu g_{s,k}(\nu) (u_1^2)^k (u_2^2)^k \Omega_{\nu,s-2k}, \quad (5.61)$$

i.e. with no differential operators in the explicit expression. We refer to this as the “manifest trace gauge”.

Reaching this gauge, on the other hand, is a non-trivial task owing to the non-commuting covariant derivatives  $\nabla$ . Relegating the details to §E, starting from either the de Donder (5.48) or traceless (5.59) propagators, we find

$$\begin{aligned} g_{s,0}(\nu) &= \frac{1}{(\frac{d}{2} + s - 2)^2 + \nu^2}, \\ g_{s,k}(\nu) &= -\frac{(1/2)_{k-1}}{2^{2k+3} \cdot k!} \frac{(s - 2k + 1)_{2k}}{(\frac{d}{2} + s - 2k)_k (\frac{d}{2} + s - k - 3/2)_k} \\ &\quad \times \frac{\left(\frac{d}{2} + s - 2k + i\nu\right)_{k-1} \left(\frac{d}{2} + s - 2k - i\nu\right)_{k-1}}{\left(\frac{d}{2} + s - 2k + 1 + i\nu\right)_k \left(\frac{d}{2} + s - 2k + 1 - i\nu\right)_k}, \quad k \neq 0. \end{aligned} \quad (5.62)$$

<sup>11</sup>In other words, we are free to add or eliminate terms of the form  $\nabla \cdot f$ , because they drop out after integrating by parts and employing current conservation in (5.38).

### 5.2.2 Review: Boundary-to-bulk propagators

Boundary-to-bulk propagators are limiting cases of bulk-to-bulk propagators, where one of the two bulk points is taken to the boundary. In this section we review existing results for the bosonic boundary-to-bulk propagators for particles of arbitrary spin and mass in the ambient formalism [152].<sup>12</sup>

Generalising the scalar case (5.14) presented in §5.1.1, the boundary-to-bulk propagator for a symmetric and traceless rank- $s$  tensor  $\varphi_{\mu_1 \dots \mu_s}$  of energy  $\Delta$  and spin- $s$  satisfies the wave equation

$$(-\square + \Delta(\Delta - d) - s) K_{\Delta, \mu_1 \dots \mu_s}{}^{i_1 \dots i_s}(z, y; y') = 0, \quad (5.63)$$

$$\lim_{z \rightarrow 0} (z^{\Delta-d+s} K_{\Delta, \mu_1 \dots \mu_s}{}^{i_1 \dots i_s}(z, y; y')) = \frac{\delta_{\{\mu_1 \dots \mu_s\}}^{i_1 \dots i_s}}{2\Delta - d} \delta^d(y - y') \quad (5.64)$$

Finding the explicit form of the propagator is straightforward in the ambient formalism. Within this framework, the simple form of the constraints imposed by the boundary and bulk  $SO(d, 2)$  symmetry:

$$K_{\Delta, s}(X, \alpha_1 W; \lambda P, \alpha_2 Z + \beta P) = \lambda^{-\Delta} (\alpha_1 \alpha_2)^s K_{\Delta, s}(X, W; P, Z), \quad (5.65)$$

imply the unique structure

$$K_{\Delta, s}(X, W; P, Z) = (W \cdot \mathcal{P} \cdot Z)^s \frac{C_{\Delta, s}}{(-2X \cdot P)^\Delta}, \quad (5.66)$$

with projector

$$\mathcal{P}^A{}_B = \delta_B^A - \frac{P^A X_B}{P \cdot X}. \quad (5.67)$$

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<sup>12</sup>See also [81, 134, 247, 248] for earlier formulations of the propagators, which includes the intrinsic and unfolded forms.



The role of the above projector is to ensure transversality of the propagator at both its bulk and boundary points. The normalisation  $C_{\Delta,s}$  is fixed by equation (5.64), where the coefficient of the Dirac delta function in the latter ensures consistency with the boundary limit of the corresponding bulk-to-bulk propagator,

$$K_{\Delta,s}(z, y; y') = \lim_{w \rightarrow 0} \Pi_{\Delta,s}(z, y; w, y'). \quad (5.68)$$

This gives,

$$C_{\Delta,s} = \frac{(s + \Delta - 1) \Gamma(\Delta)}{2\pi^{d/2} (\Delta - 1) \Gamma(\Delta + 1 - \frac{d}{2})}. \quad (5.69)$$

## 5.3 Three-point Witten diagrams

By taking the results for the propagators in the previous section, we can proceed to evaluate Witten diagrams with spinning external legs and exchanged fields. We begin in this section with three-point Witten diagrams, which also (as we shall see) play a significant role for higher-point amplitudes.

When considering spinning external particles, the bulk integrals encountered are more complicated due to the introduction of tensor structures. Before proceeding further, we first introduce some useful tools in order to handle the latter more effectively.

### 5.3.1 Tool kit

The basic idea behind the techniques introduced in this subsection, is to simplify the bulk integrals encountered when evaluating Witten diagrams with spinning external legs by removing the tensor structures from the integrand. This enables amplitudes with spinning external legs to be expressed in terms of those generated by the basic scalar cubic vertex (5.29).

The way this idea is implemented, is to express spinning propagators and their derivatives in terms of a scalar propagator:

Notice that the spin- $s$  propagator (5.65) is essentially the propagator of a scalar field (5.21) of the same dimension  $\Delta$ , with the projector (5.67) simply propagating the indices. This relationship can be expressed as the  $s$ -fold application of an  $X$ -independent differential operator<sup>13</sup>

$$K_{\Delta,s}(X, W|P, Z) = \frac{1}{(\Delta - 1)_s} (\mathcal{D}_P(Z|W))^s K_{\Delta,0}(X; P), \quad (5.70)$$

$$\mathcal{D}_P(Z|W) = (Z \cdot W) \left( Z \cdot \frac{\partial}{\partial Z} - P \cdot \frac{\partial}{\partial P} \right) + (P \cdot W) \left( Z \cdot \frac{\partial}{\partial Z} \right). \quad (5.71)$$

Moreover, we can express the  $n$ -th ambient derivative of the bulk-to-boundary propagator in terms of a scalar propagator of dimension  $\Delta + n$ :

$$(W \cdot \partial_X)^n K_{\Delta,s}(X, P|U, Z) \quad (5.72)$$

$$\begin{aligned} &= \frac{C_{\Delta,s}}{s! C_{\Delta+n,0}} \sum_{i=0}^s \sum_{\omega=0}^i \binom{s}{i} \binom{i}{\omega} \frac{(-1)^s 2^n}{(n - \omega + 1)_\omega} \frac{1}{\Gamma(\Delta + i)} (U \cdot P)^i (U \cdot Z)^{s-i} \\ &\quad \times (Z \cdot W)^\omega (P \cdot W)^{n-\omega} ((Z \cdot \partial_P)^{i-\omega} K_{\Delta+n,0}(X, P)). \end{aligned} \quad (5.73)$$

With (5.70) and (5.73), we can shift the tensor structures from the integrand in a Witten diagram with spinning external legs, allowing the amplitudes to be expressed in terms of a basic scalar amplitude (i.e. generated by an interaction involving only scalars and with no derivatives). As we shall demonstrate in the following, this provides a powerful approach to evaluating bulk amplitudes with external fields of non-zero spin.

### 5.3.2 Two scalars and a spin- $s$

We begin with the simplest case of three-point Witten diagrams involving two external scalars and a bosonic spin- $s$  field.

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<sup>13</sup>These relationships are straightforward to verify using commutators  $[\square, \mathcal{P}] = -\mathcal{P}$  and  $[\square, \mathcal{D}_P] = -\mathcal{D}_P$ .

Any such amplitude can be evaluated at tree-level with the knowledge of that generated by the basic vertex,

$$\hat{\mathcal{V}}_{s,0,0} = \varphi_{\mu_1 \dots \mu_s} \phi_1 \nabla^{\mu_1} \dots \nabla^{\mu_s} \phi_2, \quad (5.74)$$

where the covariant derivatives just act on the scalar  $\phi_2$ . The reason for this is that external legs are on-shell, and there is just a single on-shell non-trivial vertex involving two scalars and a spin- $s$  field [221].

The example that we encounter in later sections is the following: On-shell, a spin- $s$  field is traceless and transverse. Through integrating by parts and using transversality, the following types of vertices can then be identified on-shell with the basic vertex (5.74):

$$\begin{aligned} \nabla^{\mu_1} \varphi_{\mu_1 \dots \mu_s} &\approx 0 \\ \implies \int_{\text{AdS}} \varphi_{\mu_1 \dots \mu_s} \nabla^{\mu_1} \dots \nabla^{\mu_r} \phi_1 \nabla^{\mu_{r+1}} \dots \nabla^{\mu_s} \phi_2 &\approx \int_{\text{AdS}} \hat{\mathcal{V}}_{s,0,0}(x), \quad r \in \{1, \dots, s\}. \end{aligned} \quad (5.75)$$

We therefore spend the remainder of this section on evaluating tree-level three-point amplitudes generated by the basic vertex (5.74). According to the usual recipe, this is given by

$$\begin{aligned} \hat{\mathcal{A}}_{\Delta_1, \Delta_2, \Delta_3, s}(P_1, P_2, P_3 | Z) \\ = \frac{1}{s! \left(\frac{d}{2} - 1\right)_s} \int_{\text{AdS}} K_{\Delta_3, s}(X, \hat{D}_W; P_3, Z) K_{\Delta_1, 0}(X; P_1) (W \cdot \nabla)^s K_{\Delta_2, 0}(X; P_2). \end{aligned} \quad (5.76)$$

As explained in §5.3.1, with the help of (5.70) and (5.73) which express any propagator or its derivatives in terms of a scalar propagator, this amplitude can be generated from that with  $s = 0$ . In this way we can use the result already obtained in §5.1.1, equation (5.30). More explicitly, we have

$$\hat{\mathcal{A}}_{\Delta_1, \Delta_2, \Delta_3, s}(P_1, P_2, P_3 | Z) \quad (5.77)$$

$$\begin{aligned}
&= \frac{1}{s! \left(\frac{d}{2} - 1\right)_s} \frac{\left(\mathcal{D}_{P_3}(Z|\hat{D}_W)\right)^s}{(\Delta_3 - 1)_s} \int_{\text{AdS}} K_{\Delta_3,0}(X; P_3) K_{\Delta_1,0}(X; P_1) (W \cdot \nabla)^s K_{\Delta_2,0}(X; P_2) \\
&= \frac{\left(1 - \frac{d}{2} + \Delta_2\right)_s}{(\Delta_3 - 1)_s} (2 \mathcal{D}_{P_3}(Z|P_2))^s \hat{\mathcal{A}}_{\Delta_1, \Delta_2+s, \Delta_3, 0}(P_1, P_2, P_3).
\end{aligned}$$

This relationship is illustrated in figure 5.3.

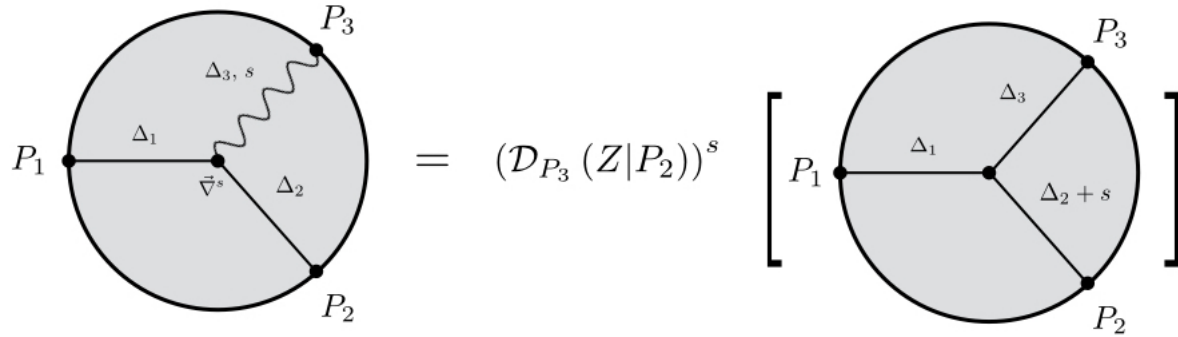


Figure 5.3: Through the techniques given in §5.3.1, the three-point Witten diagram generated by the basic vertex (5.29), with a single spin- $s$  external leg and scaling dimensions  $(\Delta_1, \Delta_2, \Delta_3)$ , may be expressed in terms of that with only external scalars and scaling dimensions  $(\Delta_1, \Delta_2 + s, \Delta_3)$ . The latter we already computed in §5.1.1. The notation  $\vec{\nabla}$  indicates that the covariant derivative acts on the rightmost external scalar field.

The amplitude (5.76) can then be expressed in the more familiar form (3.80) dictated by conformal symmetry by using

$$\begin{aligned}
(\mathcal{D}_{P_3}(Z|P_2))^s & \frac{1}{P_{13}^{\frac{\Delta_1+\Delta_3-\Delta_2-s}{2}} P_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1+s}{2}} P_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3+s}{2}}} \\
&= \left( \frac{\Delta_1 + \Delta_3 - \Delta_2 - s}{2} \right)_s \frac{((Z \cdot P_1) P_{23} - (Z \cdot P_2) P_{13})^s}{P_{13}^{\frac{\Delta_1+\Delta_3-\Delta_2+s}{2}} P_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1+s}{2}} P_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3+s}{2}}}.
\end{aligned} \tag{5.78}$$

We therefore obtain

$$\hat{\mathcal{A}}_{\Delta_1, \Delta_2, \Delta_3, s}(P_1, P_2, P_3) = \mathcal{C}(\Delta_1, \Delta_2, \Delta_3; s) \frac{((Z \cdot P_1) P_{23} - (Z \cdot P_2) P_{13})^s}{P_{13}^{\frac{\Delta_1+\Delta_3-\Delta_2+s}{2}} P_{23}^{\frac{\Delta_2+\Delta_3-\Delta_1+s}{2}} P_{12}^{\frac{\Delta_1+\Delta_2-\Delta_3+s}{2}}}, \tag{5.79}$$

with

$$\begin{aligned}
& \mathcal{C}(\Delta_1, \Delta_2, \Delta_3; s) \tag{5.80} \\
&= \frac{2^s \left(1 - \frac{d}{2} + \Delta_2\right)_s}{(\Delta_3 - 1)_s} \left(\frac{\Delta_1 + \Delta_3 - \Delta_2 - s}{2}\right)_s \mathcal{C}(\Delta_1, \Delta_2 + s, \Delta_3; 0) \\
&= C_{\Delta_1, 0} C_{\Delta_2, 0} C_{\Delta_3, s} \frac{2^s \pi^{\frac{d}{2}} \Gamma\left(\frac{\Delta_1 + \Delta_2 + \Delta_3 - d + s}{2}\right) \Gamma\left(\frac{\Delta_1 + \Delta_2 - \Delta_3 + s}{2}\right) \Gamma\left(\frac{\Delta_1 + \Delta_3 - \Delta_2 + s}{2}\right) \Gamma\left(\frac{\Delta_2 + \Delta_3 - \Delta_1 + s}{2}\right)}{2\Gamma(\Delta_1) \Gamma(\Delta_2) \Gamma(\Delta_3 + s)}.
\end{aligned}$$

### 5.3.3 General case: s1-s2-s3

We now consider the more involved general case of tree-level three-point Witten diagrams with external fields of spin  $(s_1, s_2, s_3)$ . The logic is the same as for the simpler  $(0, 0, s)$  amplitudes considered in the previous section: Reduce the task to the straightforward evaluation of basic three-point Witten diagrams with external scalars.

The most general cubic vertex involving symmetric fields of spin  $(s_1, s_2, s_3)$  in  $\text{AdS}_{d+1}$  is parameterised by six basic contractions, given by (5.83) in the ambient formalism (§A) below. These are the bulk counterparts of the six conformal structures on the boundary (3.85). Using point-splitting, the most general (on-shell) cubic vertex takes the form [109]<sup>14</sup>

$$\mathcal{V}_{s_1, s_2, s_3}(\varphi_i) = \sum_{n_i=0}^{s_i} g_{s_1, s_2, s_3}^{n_1, n_2, n_3} \mathcal{I}_{s_1, s_2, s_3}^{n_1, n_2, n_3}(\varphi_i), \tag{5.81}$$

with structures

$$\begin{aligned}
\mathcal{I}_{s_1, s_2, s_3}^{n_1, n_2, n_3}(\varphi_i) &= \mathcal{Y}_1^{s_1 - n_2 - n_3} \mathcal{Y}_2^{s_2 - n_3 - n_1} \mathcal{Y}_3^{s_3 - n_1 - n_2} \\
&\times \mathcal{H}_1^{n_1} \mathcal{H}_2^{n_2} \mathcal{H}_3^{n_3} \varphi_{s_1}(X_1, U_1) \varphi_{s_2}(X_2, U_2) \varphi_{s_3}(X_3, U_3) \Big|_{X_i=X}, \tag{5.82}
\end{aligned}$$

<sup>14</sup>In  $\text{AdS}_4$ , dimensional dependent identities reduce the number of independent structures, and allow for parity violating vertices. However these are not relevant in the context of this thesis.

and

$$\mathcal{Y}_1 = \partial_{U_1} \cdot \partial_{X_2}, \quad \mathcal{Y}_2 = \partial_{U_2} \cdot \partial_{X_3}, \quad \mathcal{Y}_3 = \partial_{U_3} \cdot \partial_{X_1}, \quad (5.83a)$$

$$\mathcal{H}_1 = \partial_{U_2} \cdot \partial_{U_3}, \quad \mathcal{H}_2 = \partial_{U_3} \cdot \partial_{U_1}, \quad \mathcal{H}_3 = \partial_{U_1} \cdot \partial_{U_2}. \quad (5.83b)$$

Note that in the above we employed the partial derivative  $\partial_A$  (which acts in the ambient  $\mathbb{R}^{d,2}$  space), as opposed to the covariant derivative  $\nabla$  on  $\text{AdS}_{d+1}$ . For obvious reasons, this is for ease of computation e.g. avoiding the non-commutativity of covariant derivatives. However, it is explained in detail in §G how the above expressions can be re-expressed in terms of  $\nabla$ , and this is just a change of basis for representing the vertices in the ambient framework. The above is simply more convenient of packaging the vertices.

To compute the tree-level amplitude generated by the structure (5.82), using (5.73) we again generate it from the result (5.30) for the basic scalar vertex. The final expression for the amplitude is quite involved, but for completeness we include it below

$$\begin{aligned} \mathcal{A}_{(\Delta_1, s_1), (\Delta_2, s_2), (\Delta_3, s_3)}^{n_1, n_2, n_3}(\{P_i\} | \{Z_i\}) & \quad (5.84) \\ &= \int_{\text{AdS}} dX \mathcal{I}_{s_1, s_2, s_3}^{n_1, n_2, n_3}(K_{\Delta_i, s_i}) \\ &= f(\{\partial_{P_i}\}, \{\partial_{P_i}\}, \{Z_i\}) \mathcal{A}_{(\Delta_1 + s_3 - k_3, 0), (\Delta_2 + s_1 - k_1, 0), (\Delta_3 + s_2 - k_2, 0)}^{n_1, n_2, n_3}(\{P_i\}), \end{aligned}$$

where

$$\begin{aligned} f(\{\partial_{P_i}\}, \{\partial_{P_i}\}, \{Z_i\}) & \quad (5.85) \\ &= \mathcal{H}_1^{n_1} \mathcal{H}_2^{n_2} \mathcal{H}_3^{n_3} \left( \prod_{\alpha=1}^3 \sum_{i_\alpha=0}^{s_\alpha - k_\alpha} \sum_{j_\alpha=0}^{k_\alpha} \binom{s_\alpha - k_\alpha}{i_\alpha} \binom{k_\alpha}{j_\alpha} \frac{2^{s_\alpha - k_\alpha} C_{\Delta_\alpha, s_\alpha}}{\Gamma(\Delta_\alpha + s_\alpha) k_\alpha!} (U_\alpha \cdot Z_\alpha)^{k_\alpha - j_\alpha} (U_\alpha \cdot P_\alpha)^{j_\alpha} \right) \\ &\times \left( \sum_{\omega_2=0}^{i_1 + j_1} \sum_{\omega_1=0}^{i_3 + j_3} \sum_{\omega_3=0}^{i_2 + j_2} \binom{i_1 + j_1}{\omega_2} \binom{i_2 + j_2}{\omega_3} \binom{i_3 + j_3}{\omega_1} \frac{(Z_1 \cdot Z_3)^{\omega_2} (Z_3 \cdot P_1)^{s_3 - k_3 - i_3 - \omega_2} (P_1 \cdot P_3)^{i_3}}{C_{\Delta_1 + s_3 - k_3, 0} C_{\Delta_2 + s_1 - k_1, 0} C_{\Delta_3 + s_2 - k_2, 0}} \right) \\ &\times (Z_2 \cdot Z_3)^{\omega_1} (Z_2 \cdot P_3)^{s_2 - k_2 - i_2 - \omega_1} (P_2 \cdot P_3)^{i_2} (Z_1 \cdot Z_2)^{\omega_3} (Z_1 \cdot P_2)^{s_1 - k_1 - i_1 - \omega_3} (P_1 \cdot P_2)^{i_1} \end{aligned}$$

$$\begin{aligned}
& \times (s_1 - k_1 - i_1 - \omega_3 + 1)_{\omega_3} (s_2 - k_2 - i_2 - \omega_1 + 1)_{\omega_1} (s_3 - k_3 - i_3 - \omega_2 + 1)_{\omega_2} \\
& \times (-P_1 \cdot \partial_{P_1} - s_3 + k_3 + \omega_2)_{s_1 - i_1 - j_1} (-P_2 \cdot \partial_{P_2} - s_1 + k_1 + \omega_3)_{s_2 - i_2 - j_2} \\
& \times (-P_3 \cdot \partial_{P_3} - s_2 + k_2 + \omega_1)_{s_3 - i_3 - j_3} (Z_1 \cdot \partial_{P_1})^{i_1 + j_1 - \omega_2} (Z_3 \cdot \partial_{P_3})^{i_3 + j_3 - \omega_1} (Z_1 \cdot \partial_{P_1})^{i_2 + j_2 - \omega_3} \Big),
\end{aligned}$$

and we defined  $k_1 = n_2 + n_3$ ,  $k_2 = n_1 + n_3$  and  $k_3 = n_1 + n_2$ . We illustrate this relation in figure 5.4.

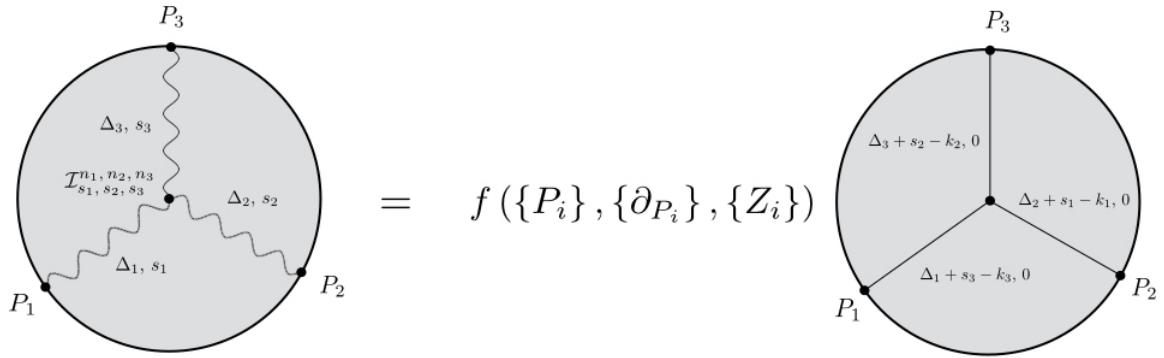


Figure 5.4: Employing the tools given in §5.3.1, we may relate the tree-level Witten diagram with external fields of spins  $(s_1, s_2, s_3)$  and scaling dimensions  $(\Delta_1, \Delta_2, \Delta_3)$  to that with only external scalars and dimensions  $(\Delta_1 + s_3 - n_1 - n_2, \Delta_2 + s_1 - n_2 - n_3, \Delta_3 + s_2 - n_1 - n_3)$ .

Using the explicit result (5.30) for the scalar Witten diagram, after evaluating all derivatives and some lengthy algebra, we can see how the amplitude generated by each individual bulk structure (5.82) decomposes into conformal structures (3.85) on the boundary:

$$\begin{aligned}
\mathcal{A}_{(\Delta_1, s_1), (\Delta_2, s_2), (\Delta_3, s_3)}^{n_1, n_2, n_3}(\{P_i\} | \{Z_i\}) &= \mathcal{E}_3 \left( \prod_{\alpha=1}^3 \sum_{i_\alpha=0}^{s_\alpha - k_\alpha} \sum_{j_\alpha=0}^{k_\alpha} \sum_{\delta_\alpha=0}^{n_\alpha} \sum_{\omega_\alpha=0}^{i_\alpha - 1 + j_\alpha - 1} \right) \sum_{\gamma_3=0}^{i_2 + j_2 - \omega_3} \sum_{\gamma_1=0}^{i_3 + j_3 - \omega_1} \sum_{\gamma_2=0}^{i_1 + j_1 - \omega_2 - \gamma_3} \\
& \times \left( \frac{n_1! n_2! n_3! (i_1 + j_1)! (i_2 + j_2)! (i_3 + j_3)! (s_3 - n_1 - n_2)! (s_2 - n_1 - n_3)! (s_1 - n_2 - n_3)!}{\gamma_1! \gamma_2! \gamma_3! \delta_1! \delta_2! \delta_3! i_1! i_2! i_3! \omega_1! \omega_2! \omega_3! \Gamma(j_1 - n_2 + \delta_2 + 1) \Gamma(j_2 - n_3 + \delta_3 + 1) \Gamma(j_3 - n_1 + \delta_1 + 1)} \right) \\
& \times \frac{(i_1 + j_1 + \Delta_1)_{s_1 - \gamma_2 - n_2 + \delta_{12} - \omega_2 - \Delta_1} (i_2 + j_2 + \Delta_2)_{s_2 - \gamma_3 - n_3 + \delta_{23} - \omega_3 - \Delta_2} (i_3 + j_3 + \Delta_3)_{s_3 - \gamma_1 - n_1 + \delta_{31} - \omega_1 - \Delta_3}}{\Gamma(i_1 + j_1 - \gamma_2 - \gamma_3 - \omega_2 + 1) \Gamma(i_2 + j_2 - \gamma_1 - \gamma_3 - \omega_3 + 1) \Gamma(i_3 + j_3 - \gamma_1 - \gamma_2 - \omega_1 + 1)} \\
& \times \frac{1}{\Gamma(-i_1 - n_2 - n_3 + s_1 - \omega_3 + 1) \Gamma(-i_2 - n_1 - n_3 + s_2 - \omega_1 + 1) \Gamma(-i_3 - n_1 - n_2 + s_3 - \omega_2 + 1)} \\
& \times \frac{1}{\Gamma(-j_1 + n_2 + n_3 - \delta_2 - \delta_3 + 1) \Gamma(-j_2 + n_1 + n_3 - \delta_1 - \delta_3 + 1) \Gamma(-j_3 + n_1 + n_2 - \delta_1 - \delta_2 + 1)}
\end{aligned}$$

$$\begin{aligned} & \times (-1)^{-\delta_1 - \delta_2 - \delta_3 + i_1 + i_2 + i_3 + j_1 + j_2 + j_3 + n_1 + n_2 + n_3 + s_1 + s_2 + s_3} \mathcal{G}^{-\gamma_1 - \gamma_2 - \gamma_3 - \delta_1 - \delta_2 - \delta_3 - n_1 - n_2 - n_3 + s_1 + s_2 + s_3 - \omega_1 - \omega_2 - \omega_3} \\ & \times \mathbf{H}_1^{\gamma_1 + \delta_1 + \omega_1} \mathbf{H}_2^{\gamma_2 + \delta_2 + \omega_2} \mathbf{H}_3^{\gamma_3 + \delta_3 + \omega_3} \mathbf{Y}_1^{s_1 - \gamma_1 - \gamma_2 - \gamma_3 - \delta_2 - \delta_3 - \omega_2 - \omega_3} \mathbf{Y}_2^{s_2 - \gamma_1 - \gamma_3 - \delta_1 - \delta_3 - \omega_1 - \omega_3} \mathbf{Y}_3^{s_3 - \gamma_1 - \gamma_2 - \delta_1 - \delta_2 - \omega_1 - \omega_2} \end{aligned} \Big),$$

with the prefactor

$$\begin{aligned} \mathcal{E}_3 = & \frac{\pi^{-d} \Gamma(\Delta_1 - 1) \Gamma(\Delta_2 - 1) \Gamma(\Delta_3 - 1) (\Delta_1 + s_1 - 1) (\Delta_2 + s_2 - 1) (\Delta_3 + s_3 - 1)}{16 \Gamma(\Delta_1 + 1 - \frac{d}{2}) \Gamma(\Delta_2 + 1 - \frac{d}{2}) \Gamma(\Delta_3 + 1 - \frac{d}{2})} \\ & \times \Gamma\left(\frac{\tau_1 + \tau_2 + \tau_3 - d}{2} + s_1 + s_2 + s_3 - n_1 - n_2 - n_3\right) \frac{1}{(P_{12}^2)^{\delta_{12}} (P_{23}^2)^{\delta_{23}} (P_{31}^2)^{\delta_{31}}}, \end{aligned}$$

and

$$\delta_{ij} = \frac{1}{2}(\tau_i + \tau_j - \tau_k), \quad \tau_i = \Delta_i - s_i. \quad (5.86)$$

Although this result is quite involved, it will play an instrumental role in §6.2.1 to fix the cubic action of higher-spin gauge fields in AdS, by matching with the dual three-point correlators in the free scalar  $O(N)$  vector model. As a double-check, with the above result we recover the same coefficient as for the  $(s, 0, 0)$  case (5.80) when any two of the  $s_i$  are set to zero.

## 5.4 Four-point Witten diagrams

In this section we consider four-point Witten diagrams with identical parity even external scalar fields. At tree level, there are two possible processes: A contact interaction or the exchange of a field between two pairs of the scalars. These two possibilities are illustrated in figure 5.5.

In the present context of higher-spin gauge theories on AdS, the exchanged fields are bosonic gauge fields of integer spin, in addition to the bulk scalar itself. The former have dimension  $\Delta_s = s + d - 2$ , while the parity even scalar of the type A theory has dimension  $\Delta = d - 2$ .



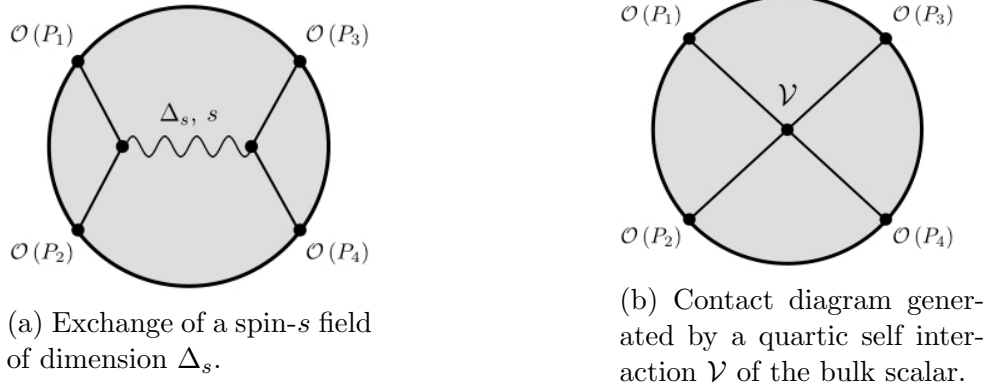


Figure 5.5: Possible tree-level bulk processes with four identical external scalars.

Ultimately, our goal is to identify the above four-point Witten diagrams in the type A minimal bosonic higher-spin theory with the dual scalar operator four-point function in the free scalar  $O(N)$  vector model. This is in order to extract the on-shell quartic self-interaction of the bulk scalar from its corresponding contact diagram (figure 5.5b).

To compare the Witten diagrams with their dual CFT correlator, we put them both on an equal footing by decomposing them into conformal blocks. For the correlator of the scalar operator  $\mathcal{O}$  in the free scalar  $O(N)$  model, this was carried out in §4.2.3. In the following we introduce a method to decompose four-point Witten diagrams into their constituent conformal blocks, which draws upon the results for the three-point Witten diagrams obtained in the previous sections.

### 5.4.1 Conformal partial wave expansion

The key idea behind our method to determine the conformal block expansion of four-point Witten diagrams, is to decompose them into partial waves of the isometry group  $SO(d+1, 1)$ . I.e. in terms of AdS harmonic functions labelled by energy and

spin,

$$\begin{aligned}
 & \text{exchange or contact interaction} = \sum_{\ell} \int_{-\infty}^{\infty} d\nu p_{\ell}(\nu) \Omega_{\nu, \ell} \\
 & \text{O}(P_1) \text{ O}(P_2) \text{ O}(P_3) \text{ O}(P_4) \quad \text{O}(P_1) \text{ O}(P_2) \text{ O}(P_3) \text{ O}(P_4)
 \end{aligned}
 \tag{5.87}$$

Since  $SO(d+1, 1)$  is also the  $d$ -dimensional Euclidean conformal group, this way of decomposing Witten diagrams is equivalent to a conformal block expansion on the boundary. We make this identification more concrete in the following, which draws upon a particular feature of harmonic functions, known as their *split representation*.

### Split representation

As projectors onto angular momentum and energy components, harmonic functions admit the factorisation [134, 152, 231, 232, 249]

$$\Omega_{\nu, \ell}(x_1, w_1; x_2, w_2) = \frac{\nu^2}{\pi \ell! \left(\frac{d}{2} - 1\right)_{\ell}} \int_{\partial \text{AdS}} d^d y K_{\frac{d}{2} + i\nu, \ell}(x_1, w_1; y, \hat{\partial}_w) K_{\frac{d}{2} - i\nu, \ell}(y, w; x_2, w_2),
 \tag{5.88}$$

which is depicted in figure 5.6.

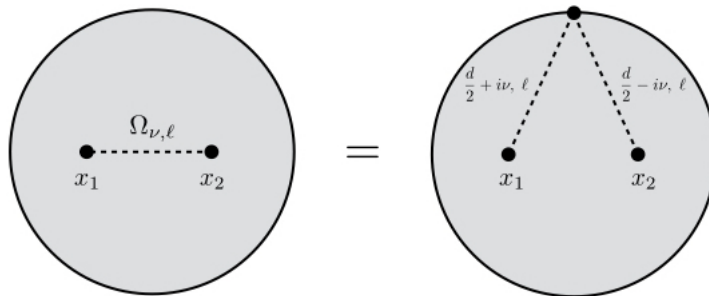


Figure 5.6: Factorisation of an AdS harmonic function  $\Omega_{\nu, \ell}$  into a product of two spin- $\ell$  boundary-to-bulk propagators of dimension  $\frac{d}{2} \pm i\nu$ , integrated over their common boundary point.

Drawing comparisons with the conformal block expansion in CFT, the split represen-

tation (5.88) makes manifest that harmonic functions  $\Omega_{\nu,\ell}$  are the bulk analogue of the boundary projector (3.104) onto a given conformal multiplet. Indeed, inserting this form for the harmonic functions into the partial wave expansion (5.87) of a four-point Witten diagram causes each partial wave to factorise into a product of two three-point Witten diagrams,

The diagrammatic equation (5.89) shows a four-point Witten diagram on the left, which is a circle with four external legs labeled  $\mathcal{O}(P_1)$ ,  $\mathcal{O}(P_2)$ ,  $\mathcal{O}(P_3)$ , and  $\mathcal{O}(P_4)$ . The interior of the circle is labeled "exchange or contact interaction". This is equated to a sum over  $\ell$  of an integral from  $-\infty$  to  $\infty$  of  $d\nu p_\ell(\nu)$  multiplied by a diagram on the right. The right diagram is a circle with the same four external legs, but with two internal vertices connected by a dashed line. The two internal vertices are connected to the external legs by solid lines. The top-left internal vertex is labeled  $\frac{d}{2} + i\nu, \ell$  and the top-right internal vertex is labeled  $\frac{d}{2} - i\nu, \ell$ .

$$(5.89)$$

Evaluating the two integrals over AdS in each bulk partial wave gives the unique structure of a product of two three-point correlation functions with two scalar operator insertions, as in the analogous representation of boundary conformal partial waves (3.114). The bulk partial wave expansion (5.87) thus induces a conformal partial wave expansion on the boundary, in the contour integral form (3.117).

In the following we apply this technology to derive the conformal partial wave expansions of four-point Witten diagrams with identical external scalars (figure 5.5). In particular, for the exchange diagrams we focus on the exchange of bosonic gauge fields of arbitrary spin.

### 5.4.2 Exchange diagrams

We first determine the conformal partial wave expansion of the Witten diagram corresponding to the tree-level exchange of a spin- $s$  gauge field between two pairs of real scalars in the s-channel (figure 5.5a for  $\Delta_s = s + d - 2$ ).<sup>15</sup> As previously explained, the

<sup>15</sup>Since the external scalars are identical, expansions for the t- and u- channel exchanges can be obtained from the s-channel result by permuting the external legs.

cubic vertex mediating this exchange is unique on-shell, and for consistency with the gauge symmetry is required to be of the Noether type

$$\mathcal{V}_{s,0,0}(x) = g_s s! \varphi_s(x; \partial_u) J_s(x, u), \quad (\nabla \cdot \partial_u) J_s(x, u) \approx 0, \quad (5.90)$$

with coupling  $g_s$ , and where [104]

$$J_s(x, u) = \sum_{k=0}^s \frac{(-1)^k}{s! (s-k)!} (u \cdot \nabla)^k \phi(x) (u \cdot \nabla)^{s-k} \phi(x) + \Lambda u^2 (\dots). \quad (5.91)$$

The second term that is proportional to the cosmological constant  $\Lambda$  is pure trace and vanishes in the flat-space limit,  $\Lambda \rightarrow 0$ .

Putting the ingredients together, the four-point amplitude takes the form

$$\begin{aligned} \mathcal{A}_s^{\text{exch.}}(y_1, y_2; y_3, y_4) & \quad (5.92) \\ &= g_s^2 \int_{\text{AdS}} d^{d+1}x_1 \int_{\text{AdS}} d^{d+1}x_2 \Pi_s(x_1, \partial_{u_1}; x_2, \partial_{u_2}) J_s(x_1, u_1; y_1; y_2) J_s(x_2, u_2; y_3; y_4), \end{aligned}$$

where  $\Pi_s$  is the spin- $s$  bulk-to-bulk propagator, and the notation  $J_s(x, u; y_i; y_j)$  denotes the insertion of scalar boundary-to-bulk propagators anchored at the boundary points  $y_i$  and  $y_j$ .

In §5.2.1, we derived bulk-to-bulk propagators gauge fields of arbitrary integer spin in a basis of harmonic functions. This way of representing the propagators therefore turned out to be beneficial, as it readily establishes the partial wave expansion of the bulk amplitude and the subsequent factorisation of each partial wave into three-point Witten diagrams. In practice, it is most convenient to employ the propagator in the manifest trace gauge (5.61), as it eliminates the need to deal directly with derivatives.

The exchange then takes the form

$$\mathcal{A}_s^{\text{exch.}}(y_1, y_2; y_3, y_4) = \sum_{k=0}^{\lfloor s/2 \rfloor} \int_{-\infty}^{\infty} d\nu g_{s,k}(\nu) \bar{\mathcal{A}}_{s,s-2k}^{\text{exch.}}(y_1, y_2; y_3, y_4), \quad (5.93)$$

with factorised partial amplitudes

$$\begin{aligned} \bar{\mathcal{A}}_{s,s-2k}^{\text{exch.}}(y_1, y_2; y_3, y_4) &= \frac{\nu^2}{\pi (s-2k)! \left(\frac{d}{2} - 1\right)_{s-2k}} \int_{\partial\text{AdS}} d^d y \quad (5.94) \\ &\quad g_s s! \int_{\text{AdS}} d^{d+1} x_1 K_{\frac{d}{2}+i\nu, s-2k}(x_1, \partial_{u_1}; y, \hat{\partial}_z) J_s^{(k)}(x_1, u_1; y_1; y_2) \\ &\quad \times g_s s! \int_{\text{AdS}} d^{d+1} x_2 K_{\frac{d}{2}-i\nu, s-2k}(x_2, \partial_{u_2}; y, z) J_s^{(k)}(x_2, u_2; y_3; y_4), \\ &= \frac{\nu^2}{\pi (s-2k)! \left(\frac{d}{2} - 1\right)_{s-2k}} \int_{\partial\text{AdS}} d^d y \\ &\quad \times \mathcal{A}_{\Delta, \Delta, \frac{d}{2}+i\nu, s-2k}(y_1, y_2, y | \hat{\partial}_z) \mathcal{A}_{\Delta, \Delta, \frac{d}{2}-i\nu, s-2k}(y, y_3, y_4 | z) \end{aligned}$$

where  $J_s^{(k)}$  denotes the  $k$ -th trace of  $J_s$ . Each of the above partial amplitudes is a product of three-point Witten diagrams  $\mathcal{A}_{\Delta, \Delta, \frac{d}{2} \pm i\nu, s-2k}$ , generated by cubic vertices of the form

$$\bar{\mathcal{V}}_{\frac{d}{2} \pm i\nu, s-2k} = g_s \varphi_{\frac{d}{2} \pm i\nu, s-2k}(x, \partial_u) J_s^{(k)}(x, u), \quad (5.95)$$

where  $\varphi_{\frac{d}{2} \pm i\nu, s-2k}$  is a spin- $(s-2k)$  field of dual scaling dimension  $\frac{d}{2} \pm i\nu$ .

The next step is to evaluate the amplitudes  $\mathcal{A}_{\Delta, \Delta, \frac{d}{2} \pm i\nu, s-2k}$  in (5.94). As previously noted, cubic vertices involving two scalars and a (massive or massless) spinning field are unique on-shell. This has the implication that the vertex (5.95) above can be expressed in terms of the basic vertex (5.74), modulo terms which do not contribute to the tree-level three-point amplitudes. Relegating the details to §F, this relation is

$$\bar{\mathcal{V}}_{\frac{d}{2}\pm i\nu, s-2k} \approx g_s 2^{s-2k} \tau_{s,k}(\nu) \varphi_{\frac{d}{2}\pm i\nu, s-2k}(x, \partial_u) \varphi_0(x) (u \cdot \nabla)^{s-2k} \varphi_0(x), \quad (5.96)$$

with

$$\tau_{s,k}(\nu) = \sum_{m=0}^k \frac{2^{2k} k!}{m!(k-m)!} \left(\frac{d}{2} - k + m - \frac{3}{2}\right)_{k-m} \binom{\frac{d}{2} + s - 2m + 1 + i\nu}{2}_m \binom{\frac{d}{2} + s - 2m + 1 - i\nu}{2}_m.$$

In this way we may we the result (5.79) for the three-point amplitude of the basic vertex (5.74), which gives<sup>16</sup>

$$\begin{aligned} \mathcal{A}_{\Delta, \Delta, \frac{d}{2}\pm i\nu, s-2k}(y_1, y_2, y|z) & \quad (5.98) \\ &= \frac{g_s 2^{s-2k} \tau_{s,k}(\nu)}{C_{\Delta,0}} \hat{\mathcal{A}}_{\Delta, \Delta, \frac{d}{2}\pm i\nu, s-2k}(y_1, y_2, y|z) \\ &= \frac{g_s 2^{s-2k} \tau_{s,k}(\nu)}{C_{\Delta,0}} C(\Delta, \Delta, \frac{d}{2} \pm i\nu; s-2k) \langle\langle \mathcal{O}(y_1) \mathcal{O}(y_2) \mathcal{O}_{\frac{d}{2}\pm i\nu, s-2k}(y|z) \rangle\rangle, \end{aligned}$$

where the  $\langle\langle \bullet \rangle\rangle$  notation was introduced in §3.6.1.<sup>17</sup>

With this way of decomposing the exchange amplitude, we may readily apply the techniques introduced in §3.6.1 to establish the conformal partial wave expansion. By simply recalling that

$$W_{\frac{d}{2}+i\nu, s-2k}(y_1, y_2; y_3, y_4) + W_{\frac{d}{2}-i\nu, s-2k}(y_1, y_2; y_3, y_4) \quad (5.100)$$

<sup>16</sup>We divide by the bulk-to-boundary propagator normalisation  $C_{\Delta,0}$ , equation (5.69), to normalise the two-point functions canonically:

$$\langle \mathcal{O}(y_1) \mathcal{O}(y_2) \rangle = \frac{1}{(y_{12}^2)^\Delta}. \quad (5.97)$$

<sup>17</sup>Note that

$$C(\Delta, \Delta, \frac{d}{2} - i\nu, s-2k) = \frac{iC(\Delta, \Delta, \frac{d}{2} + i\nu, s-2k)}{\nu C_{\frac{d}{2}+i\nu, s-2k}} \frac{\Gamma\left(\frac{\frac{d}{2}-i\nu+s-2k}{2}\right)^2}{2\Gamma\left(\frac{\frac{d}{2}+i\nu+s-2k}{2}\right)^2} \frac{\kappa_{\frac{d}{2}-i\nu, s-2k}}{(2\pi)^{\frac{d}{4}}}, \quad (5.99)$$

with  $\kappa_{\frac{d}{2}-i\nu, s-2k}$  defined in (3.110).

$$= \frac{\kappa_{\frac{d}{2}-i\nu, s-2k}}{(2\pi)^{\frac{d}{4}}} \frac{\Gamma\left(\frac{\frac{d}{2}-i\nu+s-2k}{2}\right)^2}{\Gamma\left(\frac{\frac{d}{2}+i\nu+s-2k}{2}\right)^2} \int d^d y \langle\langle \mathcal{O}(y_1) \mathcal{O}(y_2) \mathcal{O}_{\frac{d}{2}+i\nu, s-2k}(y) \rangle\rangle \langle\langle \tilde{\mathcal{O}}_{\frac{d}{2}-i\nu, s-2k}(y) \mathcal{O}(y_3) \mathcal{O}(y_4) \rangle\rangle,$$

where  $W_{\frac{d}{2}\pm i\nu, s-2k}$  denote conformal partial waves for  $[\frac{d}{2} \pm i\nu, s - 2k]$  conformal multiplets, we may write down the conformal partial wave expansion for the spin- $s$  exchange diagram in the contour integral form:

$$\mathcal{A}_s^{\text{exch.}}(y_1, y_2; y_3, y_4) = \sum_{k=0}^s \int_{-\infty}^{\infty} d\nu b_{s-2k}(\nu) W_{\frac{d}{2}+i\nu, s-2k}(y_1, y_2; y_3, y_4), \quad (5.101)$$

with

$$b_{s-2k}(\nu) = \frac{i\nu}{\pi} g_{s,k}(\nu) \tau_{s,k}^2(\nu) \frac{(g_s 2^{s-2k} \mathcal{C}(\Delta, \Delta, \frac{d}{2} + i\nu, s - 2k))^2}{C_{\frac{d}{2}+i\nu, s-2k} C_{\Delta, 0}^2}. \quad (5.102)$$

We recall (§3.6.1) that the poles of the function  $b_{s-2k}(\nu)$  in the lower-half complex plane encode the contributions to the conformal block expansion: A pole located at  $\frac{d}{2} + i\nu = \Delta$  corresponds to the contribution from a conformal multiplet with quantum numbers  $[\Delta, s - 2k]$ . We discuss these contributions in detail for the spin- $s$  exchange in the following.

### Conformal block expansion contributions

It is illuminating to understand the contributions to the conformal block expansions of Witten diagrams, for they give a means to quantify the properties of the process that occurs in the bulk. For example, the properties of the intermediate states.

Focusing first on the spin- $s$  contribution, (5.102) for  $k = 0$ , explicitly we have

$$b_s(\nu) = \frac{i\nu}{\pi} \frac{1}{\nu^2 + (\Delta + s - \frac{d}{2})^2} \frac{(g_s 2^s \mathcal{C}(\Delta, \Delta, \frac{d}{2} + i\nu, s))^2}{C_{\frac{d}{2}+i\nu, s} C_{\Delta, 0}^2}. \quad (5.103)$$

This encodes two types of contribution: Single trace and double-trace of spin- $s$ .

- **Single-trace**

The factor  $g_{s,0}(\nu) = \frac{1}{\nu^2 + (\Delta + s - \frac{d}{2})^2}$  coming from the on-shell (traceless and transverse) part of the spin- $s$  bulk-to-bulk propagator contains a pole for  $\text{Im}(\nu) < 0$  at  $\frac{d}{2} + i\nu = \Delta + s = s + d - 2$ . This corresponds to a contribution from a conformal multiplet of a spin- $s$  single-trace conserved current  $\mathcal{J}_s$ . It is consistent with the notion that a single-trace operator in the CFT corresponds to a single-particle state in the bulk: The exchange of a single-trace conserved current in the boundary four-point function corresponds to the exchange of the dual gauge field in AdS.

Furthermore, recalling that  $g_s 2^s \mathcal{C}(\Delta, \Delta, \Delta + s, s)$  is the overall coefficient of the dual  $(0, 0, s)$  CFT correlator as generated by the bulk cubic vertex (5.90), we may identify the following combination with the OPE coefficient  $c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}$  at leading order in  $N_{\text{dof}}$ :<sup>18</sup>

$$\frac{(g_s 2^s \mathcal{C}(\Delta, \Delta, \Delta + s, s))^2}{C_{\Delta+s,s} C_{\Delta,0}^2} = c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}^2. \quad (5.104)$$

In applying Cauchy's residue theorem to (5.101) about the pole  $\frac{d}{2} + i\nu = \Delta + s$ , the spin- $s$  single-trace contribution to the spin- $s$  exchange is thus

$$\mathcal{A}_s^{\text{exch.}}(y_1, y_2; y_3, y_4) \Big|_{\text{spin-}s \text{ single-trace}} = c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}^2 W_{\Delta+s,s}(y_1, y_2; y_3, y_4). \quad (5.105)$$

This precisely coincides with the form of the spin- $s$  single-trace contribution in the (12)(34) channel expansion of the dual CFT correlator.

- **Double-trace**

The remaining contributions come from the factor

$$\mathcal{C}(\Delta, \Delta, \frac{d}{2} + i\nu, s) \quad (5.106)$$

---

<sup>18</sup>The definition of the OPE coefficient is given in terms of those of three- and two-point functions (3.103). Recall also that the bulk-to-boundary propagator coefficients  $C_{\Delta,s}$  give the normalisations of the two-point functions from the bulk side.



$$= \frac{2^{-\frac{d}{2}-i\nu-2}\Gamma\left(\frac{d}{2}+i\nu-1\right)\Gamma\left(\frac{d+2s+2i\nu}{4}\right)\Gamma\left(-\frac{d}{4}+\frac{s}{2}+\Delta+\frac{i\nu}{2}\right)\Gamma\left(\frac{s+2\Delta-\frac{d}{2}-\frac{i\nu}{2}}{2}\right)}{\pi^{d-\frac{1}{2}}\Gamma(i\nu+1)\Gamma\left(-\frac{d}{2}+\Delta+1\right)^2\Gamma\left(\frac{d+2s+2i\nu-2}{4}\right)},$$

in which the Gamma function  $\Gamma\left(\frac{s+2\Delta-\frac{d}{2}-\frac{i\nu}{2}}{2}\right)$  generates an infinite string of poles at  $\frac{d}{2}+i\nu=2\Delta+2n+s$ , for each  $n=0,1,2,3,\dots$ <sup>19</sup> These correspond to contributions from the family of spin- $s$  double-trace operators (4.11).

Double-trace contributions originate from contact terms in the bulk-to-bulk propagator (note the Dirac delta function on the RHS of its equation of motion (5.40)), and therefore arise from the limit of coincidence between the two-points where the spin- $s$  particle is exchanged in AdS. This is consistent with the interpretation that double-trace operators are dual to two-particle states of the scalar in the bulk.

As we shall discuss below, for the exchange in AdS<sub>4</sub> we need only consider the spin- $s$  contributions above. However let us comment that the spin  $s-2k$  contributions in (5.102) for  $k\neq 0$  take the form

$$b_{s-2k}(\nu) = (\text{no poles in } \text{Im}(\nu) < 0) \times \Gamma\left(\frac{2\Delta+s-2k-\frac{d}{2}-i\nu}{2}\right)^2,$$

and therefore give rise to contributions from double-trace operators of scaling dimension  $2\Delta+2n+s-2k$  and spin  $s-2k$ . In the same way as for the contributions from the spin- $s$  double-trace operators above, these also arise from limit of coincidence between the two-points in the bulk.

### Exchange in AdS<sub>4</sub>

In §6.2.2 we will employ the above result for the conformal block expansion of the spin- $s$  exchange to extract the quartic self interaction of the scalar in the type A minimal bosonic higher-spin theory on AdS<sub>4</sub>. That we work in AdS<sub>4</sub> is no coincidence, for in

<sup>19</sup>Recall that the gamma function  $\Gamma(x)$  has poles at  $x=0,-1,-2,-3,\dots$

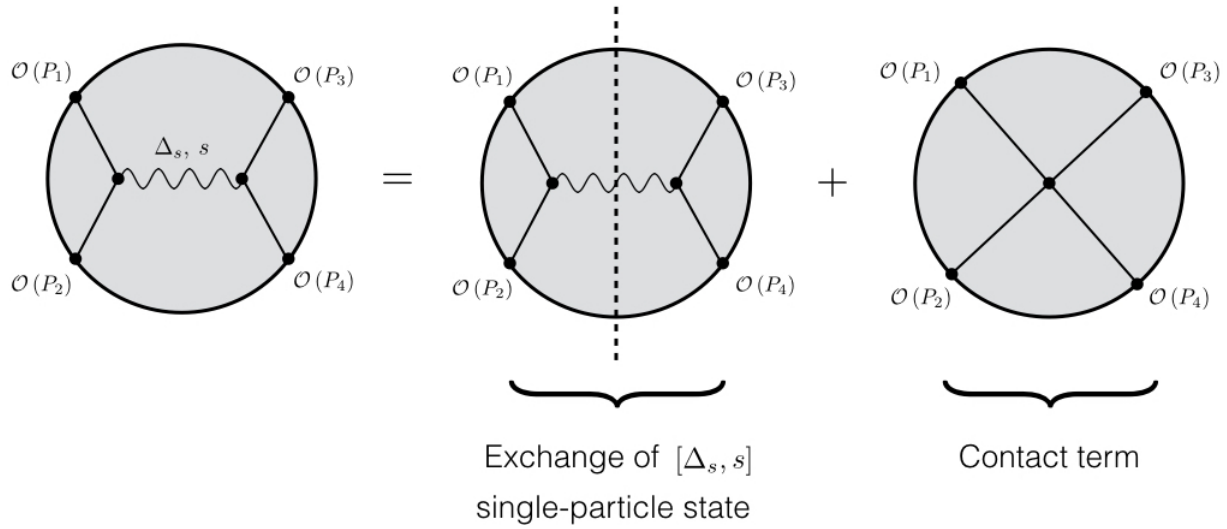


Figure 5.7: An exchange diagram can be divided into two types of contributions: When the points are separated (represented by a dashed line), there is a genuine exchange of a single-particle state. At coinciding points, contact terms arise from the off-shell bulk-to-bulk propagator:  $\square \Pi_s(x_1, x_2) + \dots = \delta(x_1, x_2)$ . The latter generate double-trace contributions in the conformal block expansion, while the exchanged single-particle state generates the contribution from its dual single-trace operator.

this case the bulk scalar is conformally coupled. This provides certain simplifications, which we discuss below.

A particular consequence of having a conformally coupled scalar, is that the spin- $s$  conserved currents (5.91) built from it can be taken to be traceless. Since the scalar is conformal, its energy momentum tensor can be made traceless. The currents of higher-spin sit in the same higher-spin symmetry multiplet as the energy momentum tensor, and they are also traceless as a result. Choosing a traceless conserved current to enter the cubic vertex (5.90), means that the contributions to the conformal block expansion of spin less than  $s$  are vanishing:  $\bar{\mathcal{A}}_{s, s-2k}^{\text{exch.}} \equiv 0$ . This is a dramatic simplification, for then only spin- $s$  conformal blocks contribute to the spin- $s$  exchange; there is no sum over  $k$  in (5.101). Moreover, these contributions are encoded in a single meromorphic function  $b_s(\nu)$ , equation (5.103).

### 5.4.3 Contact diagrams

In this section we determine the conformal partial wave expansion of a general four-point contact Witten diagram with identical external scalars. For similar reasons as described in the previous section, we focus our attention on contact amplitudes in  $\text{AdS}_4$ . As will be explained, since the scalar is conformal in  $\text{AdS}_4$  a particularly useful basis for their quartic interaction may be chosen.

A generic quartic self-interaction of a scalar  $\varphi_0$  in AdS space has the form

$$\square^{n_1} (\nabla \dots \nabla \varphi_0) \square^{n_2} (\nabla \dots \nabla \varphi_0) \square^{n_3} (\nabla \dots \nabla \varphi_0) \square^{n_4} (\nabla \dots \nabla \varphi_0), \quad n_i \in \mathbb{N}, \quad (5.107)$$

i.e. each  $\varphi_0$  is dressed with covariant derivatives  $\nabla$ , which are contracted amongst each other. Not all such vertices are independent: First of all, since the covariant derivatives do not commute there is a choice of ordering. There is also freedom to distribute derivatives amongst the four scalars by integrating by parts. Going on-shell, as relevant for four-point contact Witten diagrams, vertices can also be related using the equation of motion for  $\varphi_0$ .

An example of a complete basis of on-shell quartic self-interactions can be constructed from the spin- $s$  conserved currents (5.91), which are bi-linear in the bulk scalar

$$J_{\mu_1 \dots \mu_s}(x) = \varphi_0(x) \nabla_{(\mu_1} \dots \nabla_{\mu_s)} \varphi_0(x) + \dots \quad (5.108)$$

In §H it is shown that an example of such a basis is given by the collection of structures

$$\hat{\mathcal{V}}_{s,m} = J_s(x) \cdot \square^m(J_s(x)), \quad s = 2k, \quad k \geq m \geq 0, \quad k, m \in \mathbb{N}, \quad (5.109)$$

and accounts for all independent quartic self-interactions of the scalar in  $\text{AdS}_{d+1}$ . As will become clear, this choice of basis allows us to re-cycle some of the steps taken in

computing the partial wave expansion of the exchange diagrams in the previous section. While on the one hand this reduces the amount of extra work, it also facilitates a comparison between the features of exchanges and contact diagrams.

A generic quartic self-interaction in  $\text{AdS}_{d+1}$  can therefore be expressed as the linear combination

$$\mathcal{V}_{0,0,0,0} = \sum_{s,m} a_{s,m} \hat{\mathcal{V}}_{s,m}, \quad a_{s,m} \in \mathbb{R}, \quad (5.110)$$

for some coefficients  $a_{s,m}$ . To determine the conformal partial wave expansion of a generic four-point contact diagram, it is therefore sufficient to determine the conformal partial wave expansions of the contact amplitudes generated by the basis vertices (5.109).

For the remainder of this section, we therefore consider the tree-level four-point contact diagram associated to the vertex  $\hat{\mathcal{V}}_{s,m}$ . Furthermore, for simplicity we restrict to the  $\text{AdS}_4$  case, where we may instead use the traceless improvement  $\tilde{J}_s$  in place of  $J_s$ . That this still provides a complete basis is shown in §H.

The first step in determining the conformal partial wave expansion of a given contact diagram, is to decompose the corresponding interaction into into AdS harmonic functions. This can be achieved by simply inserting the completeness relation (C.5) for harmonic functions between the two conserved currents in the vertex (5.109), through which we obtain

$$\begin{aligned} \hat{\mathcal{V}}_{s,m}(x_1) &= (\hat{\partial}_{w_1} \cdot \hat{\partial}_{w_2})^s \tilde{J}_s(x_1, w_1) \square^m(\tilde{J}_s(x_1, w_2)) \\ &= \sum_{\ell=0}^s \int_{-\infty}^{\infty} d\nu c_{s,\ell}(\nu) \int_{\text{AdS}} d^4x_2 (\hat{\partial}_{w_1} \cdot \nabla)^\ell (\hat{\partial}_{w_2} \cdot \nabla)^\ell \Omega_{\nu,s-\ell}(x_1, \hat{\partial}_{w_1}; x_2, \hat{\partial}_{w_2}) \\ &\quad \times \tilde{J}_s(x_1, w_1) \square^m(\tilde{J}_s(x_2, w_2)) \\ &\approx \int_{-\infty}^{\infty} d\nu (-1)^m (\nu^2 + s + \frac{9}{4})^m \int_{\text{AdS}} d^4x_2 \Omega_{\nu,s}(x_1, \hat{\partial}_{w_1}; x_2, \hat{\partial}_{w_2}) \end{aligned} \quad (5.111)$$

$$\times \tilde{J}_s(x_1, w_1) \tilde{J}_s(x_2, w_2),$$

where in the second equality we used conservation of the spin- $s$  current, and the equation of motion for the harmonic function  $\Omega_{\nu,s}$ . The function  $c_{s,\ell}(\nu)$  originates from the completeness relation (C.5) for the harmonic functions. Owing to the tracelessness of the currents, only the term  $\ell = s$  contributes.

With the above form for the vertex, we can determine the partial wave expansion for the corresponding bulk amplitude in the same way as for the exchange diagrams in the previous section. For a given configuration of external legs, we have

$$\begin{aligned} \mathcal{A}_{m,s}^{\text{cont.}}(y_1, y_2; y_3, y_4) &= \int_{-\infty}^{\infty} d\nu (-1)^m \left(\nu^2 + s + \frac{9}{4}\right)^m \frac{\nu^2}{\pi} \int_{\partial\text{AdS}} d^3y \frac{1}{s! \left(\frac{1}{2}\right)_s} \\ &\quad \times \int_{\text{AdS}} \sqrt{|g|} d^4x_1 \tilde{J}_s(x_1, \partial_{u_1}; y_1, y_2) K_{\frac{3}{2}+i\nu,s}(x_1, u_1; y, \hat{\partial}_z) \\ &\quad \times \int_{\text{AdS}} \sqrt{|g|} d^4x_2 \tilde{J}_s(x_2, \partial_{u_2}; y_3, y_4) K_{\frac{3}{2}-i\nu,s}(x_2, u_2; y, z). \end{aligned} \quad (5.112)$$

The utility of the basis (5.109) becomes manifest when one notices that the product of three-point Witten diagrams in the above is precisely the same as that appearing in the corresponding result for the spin- $s$  exchange (5.94). This allows us to immediately write down the conformal partial wave expansion

$$\begin{aligned} \mathcal{A}_{m,s}^{\text{cont.}}(y_1, y_2; y_3, y_4) & \\ &= \int_{-\infty}^{\infty} d\nu (-1)^m \left(\nu^2 + s + \frac{d^2}{4}\right)^m \frac{i\nu}{\pi} \frac{(2^s \mathbf{C}(\Delta, \Delta, \frac{d}{2} + i\nu, s))^2}{C_{\frac{d}{2}+i\nu,s} C_{\Delta,0}^2} W_{\frac{d}{2}+i\nu,s}(u, v), \end{aligned} \quad (5.113)$$

where we must keep in mind that  $d = 3$ .

In this case, since the only poles with  $\text{Im}(\nu) < 0$  come from the factor  $\mathbf{C}(\Delta, \Delta, \frac{d}{2} + i\nu, s)$ , the only contributions to the conformal block expansion come from spin- $s$  double-trace operators. The absence of any single-trace contribution is consistent, since there is no

exchange of a single-particle state in a local contact interaction.

With the above result (5.113) for a given basis vertex, the conformal block expansion of a contact diagram generated by a generic quartic scalar self interaction follows by expressing it in the basis (5.109).

Let us note that while in this section we restricted for simplicity to the  $\text{AdS}_4$  case, the same procedure applies in general dimensions. The difference is that generally it is not possible to use traceless currents in the basis (5.109). In general dimensions, the use of *traceful* currents introduces contributions for all  $\ell \leq s$  in (5.111). This gives rise to contributions from double-trace operators for all spins  $\ell \leq s$  in the subsequent conformal block expansion (5.113). As a final comment, in applying the above approach it is not necessary to use a basis of quartic vertices built from conserved currents. It applies to any quartic vertex, and the choice (5.109) was made for ease of comparison with the exchange diagrams.

# Holographic reconstruction

This chapter is a culmination of the results established in the previous, in which we fix the complete cubic action and the quartic scalar self-interaction in the type A minimal bosonic higher-spin theory on AdS. We do so by employing its conjectured duality with the free scalar  $O(N)$  vector model, by matching three- and four-point functions of single-trace operators in the latter (computed in §4.2) with their dual Witten diagrams in the higher-spin theory (computed in §5). In extracting the quartic scalar interaction, we employ the conformal partial wave expansion in CFT §3.6 and §4.2.3, and for four-point Witten diagrams §5.4.

We begin by reviewing the standard non-holographic Noether approach to constructing Lagrangian interactions in higher-spin theories, exhibiting the existing results upon which we build in this work.

## 6.1 Interactions in higher-spin theory

One of the main tools for constructing interactions in higher-spin theories is the Noether method. Any theory whose non-linear form is determined by a gauge principle can be constructed by a Noether procedure, which was first understood in the context of gravity in [250–254] and proved highly successful in the construction of supergravity

theories [255–259]. This is a perturbative approach whose aim is to give a systematic prescription for constructing interacting theories, order-by-order in the fields, as deformations of free theories.

The starting point is a given spectrum of free fields, which we denote collectively by  $\varphi$ , on a fixed background. These are described by a quadratic action  $S^{(2)}[\varphi]$ , which is invariant under the linear gauge transformations  $\delta_\xi^{(0)}\varphi$ . One then postulates the existence of a full non-linear action with corresponding non-linear gauge transformations, and expands them both in a weak field expansion around the background in powers of the fields

$$S[\varphi] = S^{(2)}[\varphi] + S^{(3)}[\varphi] + S^{(4)}[\varphi] + \dots, \quad (6.1)$$

$$\delta_\xi\varphi = \delta^{(0)}\varphi + \delta^{(1)}\varphi + \delta^{(2)}\varphi + \dots \quad (6.2)$$

The superscript ( $n$ ) means that the corresponding term is power  $n$  in the fields. The requirement  $\delta_\xi S[\varphi] = 0$  that the action is gauge-invariant provides a set of recursive equations

$$\delta^{(0)}S^{(2)}[\varphi] = 0, \quad (6.3a)$$

$$\delta^{(0)}S^{(3)}[\varphi] + \delta^{(1)}S^{(2)}[\varphi] = 0, \quad (6.3b)$$

$$\delta^{(0)}S^{(4)}[\varphi] + \delta^{(1)}S^{(3)}[\varphi] + \delta^{(2)}S^{(2)}[\varphi] = 0, \quad (6.3c)$$

$$\dots, \quad (6.3d)$$

whose solution would iteratively fix the full non-linear action.

In this way, all possible cubic vertices involving symmetric gauge fields of arbitrary integer spin on an  $\text{AdS}_{d+1}$  background that are consistent with the first order constraint (6.3b) have been determined [30, 90, 96, 97, 104, 108–114, 260].



The approach to establishing consistent cubic vertices is similar to that of constraining correlation functions involving conserved currents of arbitrary spin (§4.2.2): Working in the ambient framework [30, 90, 109–112, 114], for a given triplet of spins  $\{s_1, s_2, s_3\}$ , one considers the most general cubic vertex

$$\mathcal{V}_{s_1, s_2, s_3}(\varphi_i) = \sum_{n_i=0}^{s_i} g_{s_1, s_2, s_3}^{n_1, n_2, n_3} \mathcal{I}_{s_1, s_2, s_3}^{n_1, n_2, n_3}(\varphi_i), \quad (6.4)$$

where

$$\begin{aligned} \mathcal{I}_{s_1, s_2, s_3}^{n_1, n_2, n_3}(\varphi_i) &= \mathcal{Y}_1^{s_1 - n_2 - n_3} \mathcal{Y}_2^{s_2 - n_3 - n_1} \mathcal{Y}_3^{s_3 - n_1 - n_2} \\ &\quad \times \mathcal{H}_1^{n_1} \mathcal{H}_2^{n_2} \mathcal{H}_3^{n_3} \varphi_{s_1}(X_1, U_1) \varphi_{s_2}(X_2, U_2) \varphi_{s_3}(X_3, U_3) \Big|_{X_i=X}, \end{aligned} \quad (6.5)$$

built from the six basic contractions (5.83),

$$\mathcal{Y}_1 = \partial_{U_1} \cdot \partial_{X_2}, \quad \mathcal{Y}_2 = \partial_{U_2} \cdot \partial_{X_3}, \quad \mathcal{Y}_3 = \partial_{U_3} \cdot \partial_{X_1}, \quad (6.6a)$$

$$\mathcal{H}_1 = \partial_{U_2} \cdot \partial_{U_3}, \quad \mathcal{H}_2 = \partial_{U_3} \cdot \partial_{U_1}, \quad \mathcal{H}_3 = \partial_{U_1} \cdot \partial_{U_2}. \quad (6.6b)$$

This general cubic interaction (6.4) is the analogue of (3.86), for a correlation function involving operators of spin  $\{s_1, s_2, s_3\}$  in CFT. To constrain the coefficients  $g_{s_1, s_2, s_3}^{n_1, n_2, n_3}$ , one first considers the first non-trivial constraint (6.3b), which leads to the requirement

$$\delta^{(0)} \mathcal{S}^{(3)} \approx 0, \quad (6.7)$$

since the rightmost term in (6.3b) is proportional to the free equations of motion. By inserting the ansatz (6.4) into the above, it is possible to identify all independent cubic interactions consistent with the linear gauge symmetries  $\delta^{(0)} \varphi_{s_i}$ .<sup>1</sup>

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<sup>1</sup>Each such structure is a linear combination of the  $\mathcal{I}_{s_1, s_2, s_3}^{n_1, n_2, n_3}$ . The number of independent structures is equal to:  $1 + \min(s_1, s_2, s_3)$ , which (as required by the duality) coincides with the number demanded by current conservation on the CFT three-point correlator  $\langle \mathcal{J}_{s_1} \mathcal{J}_{s_2} \mathcal{J}_{s_3} \rangle$ , with  $\partial \cdot \mathcal{J}_{s_i} \approx 0$ .

For example, in the case of spin-1 gauge fields there are two independent interaction terms of the schematic form

$$S^{(3)} = \mathbf{b}_{1,1,1}^1 \int A A F + \mathbf{b}_{1,1,1}^0 \int F F F. \quad (6.8)$$

The first is the one-derivative Yang-Mills vertex, while the second is the three-derivative Born Infeld vertex. For spin-2 gauge fields, there are three independent interactions

$$S^{(3)} = \mathbf{b}_{2,2,2}^2 \int h \nabla h \nabla h + \mathbf{b}_{2,2,2}^1 \int R R + \mathbf{b}_{2,2,2}^0 \int R R R, \quad (6.9)$$

where the first is the two-derivative gravitational minimal coupling, while the other two come from the expansions of  $(\text{Riemann})^2$  and  $(\text{Riemann})^3$ . The latter involve four and six derivatives, respectively. More generally, solutions to (6.7) are vertices involving fields of spin  $(s_1, s_2, s_3)$ , with different number of derivatives associated to the coupling constants  $\mathbf{b}_{s_1, s_2, s_3}^n$ , where  $0 \leq n \leq \min(s_1, s_2, s_3)$ .

Let us note that the coupling constants  $\mathbf{b}_{s_1, s_2, s_3}^n$  are not fixed at this level: equation (6.3b) is linear in the deformation, and so any linear combination of solutions is also a solution. This freedom left over in the cubic action can in principle be fixed by considering higher-order consistency conditions. This has led to results [96, 108, 117, 261] for the relative coefficients between a certain class of cubic vertices, known as non-abelian vertices.<sup>2</sup> However, due to the increasing complexity of higher order conditions in the Noether procedure, neither the complete cubic action nor higher-order interactions have been determined in a standard Lagrangian form.<sup>3</sup>

In this chapter we push the above results further by taking an alternative route, using

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<sup>2</sup>As their name suggests, such vertices deform the linearised gauge transformations and give rise to the non-zero structure constants of the higher-spin algebra. The relative coefficients between such vertices can be constrained by demanding that the linearised theory carries a representation of the global symmetry algebra. This is known as *admissibility* and is a second order constraint.

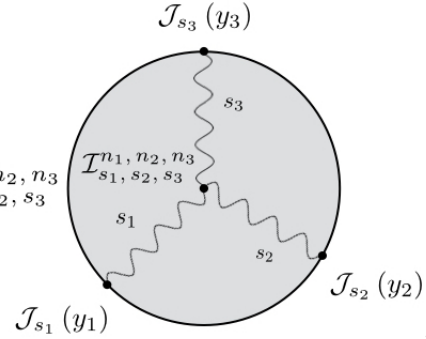
<sup>3</sup>See however [262] for results for higher-order vertices in the so-called unfolded formulation, which employs an infinite set of auxiliary fields.

the conjectured holographic duality between higher-spin theories in  $\text{AdS}_{d+1}$  and free CFTs in  $d$ -dimensions. As we saw in §2.2, the generating function of connected correlators in a free CFT can be identified with the on-shell classical action of the dual higher-spin theory in AdS,

$$\langle \mathcal{O}_1(y_1) \dots \mathcal{O}_n(y_n) \rangle_{\text{conn.}} = \frac{\delta}{\delta \bar{\varphi}_1(y_1)} \dots \frac{\delta}{\delta \bar{\varphi}_n(y_n)} S_{\text{HS AdS}} [\varphi_i|_{\partial \text{AdS}} = \bar{\varphi}_i] \Big|_{\bar{\varphi}=0}. \quad (6.10)$$

With the result for the correlators of single-trace operators in the free CFT, one should then in principle be able to determine the on-shell interactions of their dual higher-spin fields in AdS by requiring that (6.10) holds.

Here, we work in the context of the conjectured duality between the type A minimal bosonic higher-spin theory on AdS and the free scalar  $O(N)$  vector model. In §4.2.2, we determined the three-point functions of all single-trace operators in the latter. With these results, combined with those for tree-level 3-point Witten diagrams with spinning external fields in §5.3.3, by employing the holographic requirement (6.10) we are able to extract the explicit form of *all* coefficients in the cubic vertex (6.4) in the higher-spin theory on  $\text{AdS}_{d+1}$ . In more detail, we solve the equation shown diagrammatically below for  $g_{s_1, s_2, s_3}^{n_1, n_2, n_3}$ , for any triplet of even integer spins  $\{s_1, s_2, s_3\}$

$$\langle \mathcal{J}_{s_1}(y_1) \mathcal{J}_{s_2}(y_2) \mathcal{J}_{s_3}(y_3) \rangle = \sum_{n_i} g_{s_1, s_2, s_3}^{n_1, n_2, n_3} \mathcal{I}_{s_1, s_2, s_3}^{n_1, n_2, n_3}$$


With the action fixed up to cubic order, we take the holographic reconstruction to

quartic order.<sup>4</sup> The simplest example to begin with is given by solving for the quartic self interaction of the scalar, as it minimises the number of tensor structures that one needs to confront. The scalar quartic self-interaction contributes to the holographic computation of the scalar single-trace operator 4-pt function §4.2.3, together with the bulk exchanges mediated by the 0-0- $s$  cubic vertices (6.4). The latter we determine at lower order in the holographic reconstruction procedure. The equation we solve for the quartic interaction is shown in figure 6.1.

$$\langle \mathcal{O}(y_1) \mathcal{O}(y_2) \mathcal{O}(y_3) \mathcal{O}(y_4) \rangle_{\text{conn.}}$$

$$= \sum_s \left( \begin{array}{c} \mathcal{O}(y_1) \quad \mathcal{O}(y_3) \\ \diagdown \quad \diagup \\ \mathcal{V}_{s,0,0} \quad \text{---} \quad \mathcal{V}_{s,0,0} \\ \diagup \quad \diagdown \\ \mathcal{O}(y_2) \quad \mathcal{O}(y_4) \end{array} \right) + \left( \begin{array}{c} \mathcal{O}(y_1) \quad \mathcal{O}(y_3) \\ \diagdown \quad \diagup \\ \mathcal{V}_{0,0,0,0} \\ \diagup \quad \diagdown \\ \mathcal{O}(y_2) \quad \mathcal{O}(y_4) \end{array} \right) + \text{t- and u- channels}$$

Figure 6.1: Assuming validity of the HS / free vector model duality, the four-point function of the boundary scalar single trace operator  $\mathcal{O}$  can be computed from the sum of all four-point exchanges of the spin- $s$  gauge fields between two pairs of scalars and the four-point contact diagram generated by the quartic scalar self-interaction  $\mathcal{V}_{0,0,0,0}$ .

To extract the scalar quartic self-interaction, we proceed in the same way as at cubic order by making the most general ansatz

$$\mathcal{V}_{0,0,0,0} = \sum_{r,m} a_{r,m} (\varphi_0 \nabla_{\mu_1} \dots \nabla_{\mu_r} \varphi_0 + \dots) \square^m (\varphi_0 \nabla^{\mu_1} \dots \nabla^{\mu_r} \varphi_0 + \dots) \quad (6.11)$$

where a completion of the  $\dots$  is given for example by the basis of quartic vertices (5.109). For technical reasons that were explained in the previous section, we restrict to extracting the quartic vertex in  $\text{AdS}_4$ . To solve the holographic equation shown in figure 6.1 for the coefficients  $a_{r,m}$  in the derivative expansion, it is effective to decompose

<sup>4</sup>In general, the tree-level Witten diagram computation of  $n$ -point CFT correlators involves bulk interactions of orders  $n, n-1, \dots, 3$ . The holographic reconstruction therefore provides an alternative (and more efficient) iterative procedure to the Noether procedure in constructing interactions.

the bulk and boundary amplitudes into conformal partial waves. We therefore employ the results §4.2.3 for the expansion of the scalar single-trace four-point function, and those established in §5.4.2 and §5.4.3 for tree-level four-point Witten diagrams with identical parity even external scalars.

## 6.2 Holographic reconstruction

Before we proceed, let us note that in order for the holographic relation (6.10) which we use to extract the interactions to be meaningful, it must be ensured that the two-point functions have the same normalisation from both the bulk and boundary perspectives. In all that follows, we choose the unit normalisation

$$\langle \mathcal{J}_s(y_1|z_1) \mathcal{J}_s(y_2|z_2) \rangle = \frac{(\mathbf{H}_3)^s}{(y_{12}^2)^\Delta}. \quad (6.12)$$

### 6.2.1 Cubic couplings

#### Simple illustrative example: s-0-0 coupling

As customary, we begin the simplest example: Fixing holographically the bulk 0-0- $s$  interactions, which are of the Noether type

$$\mathcal{V}_{s,0,0} = g_{s,0,0} \varphi_s(x, \partial_u) \cdot J_s(x, u), \quad \nabla \cdot \partial_u J_s(x, u) \approx 0, \quad (6.13)$$

where  $J_s$  is a conserved current that is bi-linear in the bulk scalar  $\varphi_0$  (e.g. (5.91)), and  $g_{s,0,0}$  is the coupling we would like to determine. As we already saw, for each  $s$  the form of these vertices is unique on-shell, and the structure is on-shell equivalent to the basic structure (5.74)

$$\mathcal{V}_{s,0,0} = g_{s,0,0} \varphi_s(x, \partial_u) \cdot J_s(x, u) \approx g_{s,0,0} 2^s \hat{\mathcal{V}}_{s,0,0}, \quad (6.14)$$

where

$$\hat{\mathcal{V}}_{s,0,0} = \varphi_s(x, \partial_u) \varphi_0(x) (u \cdot \nabla)^s \varphi_0(x). \quad (6.15)$$

The vertex (6.14) in the type A minimal bosonic theory is the unique bulk structure responsible for the holographic computation of the 0-0- $s$  correlator (4.19) in the free scalar  $O(N)$  vector model. Using AdS/CFT dictionary (6.10), we have

$$\begin{aligned} \langle \mathcal{O}(y_1) \mathcal{O}(y_2) \mathcal{J}_s(y_3|z) \rangle &= \frac{\delta}{\delta \bar{\varphi}_0(y_1)} \frac{\delta}{\delta \bar{\varphi}_0(y_2)} \frac{\delta}{\delta \bar{\varphi}_s(y_3)} S_{\text{HS AdS}} [\varphi_i|_{\partial \text{AdS}} = \bar{\varphi}_i] \\ &= g_{s,0,0} 2^s \hat{\mathcal{A}}_{\Delta, \Delta, \Delta+s, s}(y_1, y_2, y_3|z), \end{aligned} \quad (6.16)$$

where  $\hat{\mathcal{A}}_{\Delta, \Delta, \Delta+s, s}$  is the tree-level 3-pt amplitude (5.79) generated by the basic vertex (6.15). Together with the result (4.19) for the dual vector model correlator on the LHS, normalising the two-point functions consistently (6.12), we can extract  $g_{s,0,0}$  from (6.16):

$$g_{s,0,0} = \frac{2^{\frac{3d-s-1}{2}} \pi^{\frac{d-3}{4}} \Gamma\left(\frac{d-1}{2}\right) \sqrt{\Gamma\left(s + \frac{d}{2} - \frac{1}{2}\right)}}{\sqrt{N} \sqrt{s!} \Gamma(d+s-3)}. \quad (6.17)$$

In particular, for  $d = 3$  we have

$$g_{s,0,0} = \frac{2^{4-\frac{s}{2}}}{\sqrt{N} \Gamma(s)}. \quad (6.18)$$

This is consistent with the known vanishing of the cubic scalar self coupling ( $s = 0$ ) in AdS<sub>4</sub> [73], which itself provided an early check of the holographic duality by comparing with earlier CFT results in [263].

In §6.2.2 we shall employ the result (6.18) to complete the computation of the tree-level spin- $s$  exchange diagram (5.92), mediated by the  $(0, 0, s)$  cubic vertices (6.13) in the type A minimal bosonic higher-spin theory on AdS<sub>4</sub>.

### Full cubic action: s1-s2-s3 couplings

Let us now consider the problem in full generality. Our goal is to determine holographically the complete set of cubic couplings in the action of the type A minimal bosonic higher-spin theory on  $\text{AdS}_{d+1}$ . As discussed in the preceding section, this takes the form

$$S_{\text{HS AdS}}[\varphi] = \sum_{s_i} \int_{\text{AdS}_{d+1}} \frac{1}{2} \varphi_{s_i} (\square - m_{s_i}^2 + \dots) \varphi_{s_i} + \sum_{s_i, n_i} g_{s_1, s_2, s_3}^{n_1, n_2, n_3} \int_{\text{AdS}_{d+1}} \mathcal{I}_{s_1, s_2, s_3}^{n_1, n_2, n_3}(\varphi_{s_i}) + \dots,$$

for some coefficients  $g_{s_1, s_2, s_3}^{n_1, n_2, n_3}$  to be determined. We also display the kinetic terms, to emphasise that the couplings we determine correspond to the canonical normalisation of the former. This is ensured by the normalisation of the boundary-to-bulk propagators.

In this general case, we have to solve the holographic equation in the figure below

$$\langle \mathcal{J}_{s_1}(y_1) \mathcal{J}_{s_2}(y_2) \mathcal{J}_{s_3}(y_3) \rangle = \sum_{n_i} g_{s_1, s_2, s_3}^{n_1, n_2, n_3} \mathcal{I}_{s_1, s_2, s_3}^{n_1, n_2, n_3} \left( \begin{array}{c} \mathcal{J}_{s_3}(y_3) \\ \text{---} s_3 \text{---} \\ \mathcal{I}_{s_1, s_2, s_3}^{n_1, n_2, n_3} \\ \text{---} s_1 \text{---} \quad \text{---} s_2 \text{---} \\ \mathcal{J}_{s_1}(y_1) \quad \mathcal{J}_{s_2}(y_2) \end{array} \right), \quad (6.19)$$

for the coefficients  $g_{s_1, s_2, s_3}^{n_1, n_2, n_3}$ . All the ingredients are at our disposal: In §5.3.3 we computed the tree-level 3-pt amplitudes generated by each vertex structure (6.5), and in §4.2.2 we computed all three-point correlation functions of the single-trace conserved currents in the free scalar  $O(N)$  vector model. For convenience, we recall here that the latter are given by (4.31)

$$\langle \mathcal{J}_{s_1}(y_1|z_1) \mathcal{J}_{s_2}(y_2|z_2) \mathcal{J}_{s_3}(y_3|z_3) \rangle$$

$$= N \left( \prod_{i=1}^3 c_{s_i} q_i^{\frac{1}{2}-\frac{\Delta}{4}} \Gamma\left(\frac{\Delta}{2}\right) J_{\frac{\Delta-2}{2}}(\sqrt{q_i}) \right) \frac{Y_1^{s_1} Y_2^{s_2} Y_3^{s_3}}{(y_{12}^2)^{\Delta/2} (y_{23}^2)^{\Delta/2} (y_{31}^2)^{\Delta/2}}, \quad (6.20)$$

with

$$q_1 = 2H_1 \partial_{Y_2} \partial_{Y_3}, \quad q_2 = 2H_2 \partial_{Y_3} \partial_{Y_1}, \quad q_3 = 2H_3 \partial_{Y_1} \partial_{Y_2}. \quad (6.21)$$

The  $c_{s_i}$  are given by

$$c_{s_i}^2 = \frac{\sqrt{\pi} 2^{-\Delta-s_i+3} \Gamma(s_i + \frac{\Delta}{2}) \Gamma(s_i + \Delta - 1)}{N s_i! \Gamma(s_i + \frac{\Delta-1}{2}) \Gamma(\frac{\Delta}{2})^2}, \quad (6.22)$$

for canonical normalisation (6.12) of the two-point functions.

To solve equation (6.19) for the  $g_{s_1, s_2, s_3}^{n_1, n_2, n_3}$ , the crucial observation is that the 3-pt amplitude generated by the simplest vertex structure (6.5) with  $n_i = 0$  and  $\tau_i = d - 2$  can be re-summed. The resulting expression precisely produces the free scalar conformal structure (6.20) above, up to an overall coefficient

$$\begin{aligned} \int_{\text{AdS}_{d+1}} \mathcal{I}_{s_1, s_2, s_3}^{0,0,0}(K_{s_i+d-2, s_i}) &= \frac{\pi^{\frac{3}{2}-d} (-1)^{s_1+s_2+s_3} 2^{-3d-s_1-s_2-s_1+8} \Gamma(d+s_1+s_2+s_1-3)}{(y_{12}^2)^{d/2-1} (y_{23}^2)^{d/2-1} (y_{31}^2)^{d/2-1}} \\ &\times \frac{\Gamma(d-3+s_1) \Gamma(d-3+s_2) \Gamma(d-3+s_1)}{\Gamma(\frac{d-3}{2}+s_1) \Gamma(\frac{d-3}{2}+s_2) \Gamma(\frac{d-3}{2}+s_1)} \prod_{i=1}^3 \left( q_i^{1-\frac{d}{4}} J_{\frac{d-2}{2}}(\sqrt{q_i}) \right) Y_1^{s_1} Y_2^{s_2} Y_3^{s_3}. \end{aligned} \quad (6.23)$$

Since the mapping between consistent cubic vertices in the bulk and conserved 3-pt structures in the dual CFT is one-to-one (cf. footnote 1 of this chapter), the above vertex is the only one which can possibly generate the CFT result (6.20).<sup>5</sup> This immediately gives  $g_{s_1, s_2, s_3}^{n_1, n_2, n_3} = 0$  for all  $n_i$  except when  $n_1 = n_2 = n_3 = 0$ .

---

<sup>5</sup>Other vertex structures (6.5) may generate the free scalar conserved structure (6.20) on the boundary, but since the mapping is one-to-one there will also be contributions from the other conserved structures in (4.25) for  $l \neq 0$ , which cannot be removed by any linear combination of bulk cubic vertices. The vertex with  $n_i = 0$  is thus singled out when demanding agreement with the result (6.20) in the free scalar  $O(N)$  model.



The final step is then to determine  $g_{s_1, s_2, s_3}^{0,0,0}$ . Normalising the two-point functions in both the bulk and boundary computations consistently (6.12), and combining the CFT (6.20) and bulk (6.23) results, we obtain the following coupling constants

$$g_{s_1, s_2, s_3} := g_{s_1, s_2, s_3}^{0,0,0} = \frac{1}{\sqrt{N}} \frac{\pi^{\frac{d-3}{4}} 2^{\frac{3d-1+s_1+s_2+s_3}{2}}}{\Gamma(d+s_1+s_2+s_3-3)} \prod_{i=1}^3 \sqrt{\frac{\Gamma(s_i + \frac{d-1}{2})}{\Gamma(s_i + 1)}}. \quad (6.24)$$

The simplest form for the above coupling constants manifests itself in AdS<sub>4</sub>, where the spin-dependence coincides with the one obtained in [91, 92] from a flat space quartic analysis

$$g_{s_1, s_2, s_3} = \frac{2^{\frac{s_1+s_2+s_3}{2}+4}}{\sqrt{N} \Gamma(s_1+s_2+s_3)}, \quad (6.25)$$

confirming a conjecture [264] made for the coefficient of the highest derivative part of the  $s_1$ - $s_2$ - $s_3$  vertex in AdS<sub>4</sub>, which is the part of a vertex which survives when taking the flat limit of AdS. This is a non-trivial consistency check of our results, and also a check of the higher-spin / vector model duality itself: Upon taking the flat limit  $\Lambda \rightarrow 0$ , we recover the known results in flat space.

### Discussion / Back to intrinsic

As mentioned earlier in §5.3.3, for ease of computation we established the cubic couplings in the ambient formalism. In this framework, complete bulk cubic coupling for given  $(s_1, s_2, s_3)$  in ambient space reads

$$\mathcal{V}_{s_1, s_2, s_3} = \sum_{s_1, s_2, s_3} g_{s_1, s_2, s_3} \mathcal{I}_{s_1, s_2, s_3}^{0,0,0}, \quad (6.26)$$

where explicitly

$$\begin{aligned} & \mathcal{I}_{s_1, s_2, s_3}^{0,0,0}(\varphi_i) \\ &= (\partial_{U_1} \cdot \partial_{X_2})^{s_1} (\partial_{U_2} \cdot \partial_{X_3})^{s_2} (\partial_{U_3} \cdot \partial_{X_1})^{s_3} \varphi_{s_1}(X_1, U_1) \varphi_{s_2}(X_2, U_2) \varphi_{s_3}(X_3, U_3) \Big|_{X_i=X}. \end{aligned} \quad (6.27)$$

In particular, we employed the partial derivative  $\partial_A$  which acts in the flat  $\mathbb{R}^{d,2}$  ambient space, as opposed to the covariant derivative  $\nabla$  that is intrinsic to the  $\text{AdS}_{d+1}$  manifold. In §G we explain how to obtain the intrinsic expressions for the interactions (6.26), i.e. in terms of  $\nabla$ .

One should note that although the interaction (6.27) contains  $s_1 + s_2 + s_3$  *ambient* partial derivatives, when re-expressing it in terms intrinsic to AdS one picks up lower derivative terms. In particular, the quasi-minimal interactions with the lowest number  $s_1 + s_2 - s_3$  of derivatives ( $s_1 \geq s_2 \geq s_3$ ). In the spin-2 case for example, one recovers the Einstein-Hilbert terms supplemented by those with higher-derivatives.

### 6.2.2 Quartic couplings

In the previous section, we fixed holographically the complete cubic action for the type A minimal bosonic higher-spin theory on  $\text{AdS}_{d+1}$ . With this result, we may now extend this reconstruction of bulk interactions to the quartic order. The simplest case is given by extracting the scalar self-interaction, whose four-point contact diagram contributes to the holographic computation of the scalar single-trace operator four-point function (4.35), with the total contributing Witten diagrams shown in figure 6.2.

In §5.4.3 we saw that any bulk quartic scalar self-interaction can be expressed in the form

$$\begin{aligned} \mathcal{V}_{0,0,0,0} &= \sum_{r,m} a_{r,m} \hat{\mathcal{V}}_{r,m} \\ &= \sum_{r,m} a_{r,m} (\varphi_0 \nabla_{\mu_1} \dots \nabla_{\mu_r} \varphi_0 + \dots) \square^m (\varphi_0 \nabla_{\mu_1} \dots \nabla_{\mu_r} \varphi_0 + \dots), \end{aligned} \tag{6.28}$$

built from the complete set of vertices  $\hat{\mathcal{V}}_{r,m}$  defined in (5.109). In this section we determine the coefficients  $a_{r,m}$  for the higher-spin theory on  $\text{AdS}_4$ . As discussed in §5.4, working in  $\text{AdS}_4$  allowed for certain simplifications in computing the relevant Witten

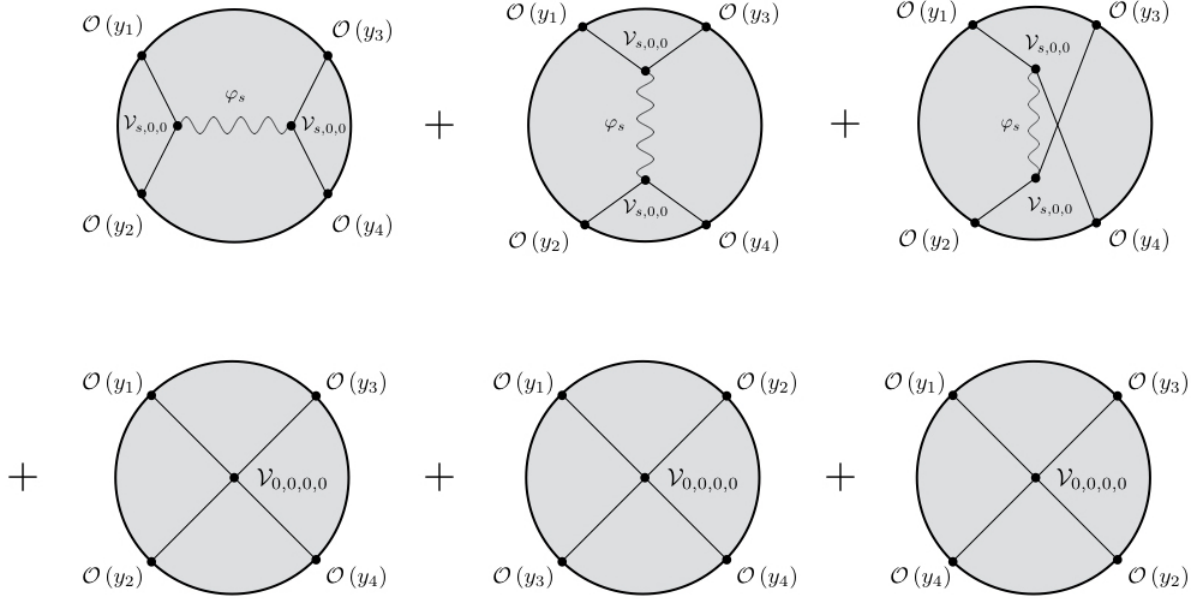


Figure 6.2: Total contributing four-point Witten diagrams to the holographic computation of  $\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle_{\text{conn.}}$ . The four-point exchange diagrams (top line), were computed in §5.4.2 as a conformal partial wave expansion. The analogous result for tree-level contact diagrams for a general scalar quartic self-interaction (second line) was computed in §5.4.3. All permutations of the external legs are taken into account.

diagrams, which will also manifest themselves in extracting the quartic vertex in this section. The coefficients are fixed by requiring that its four-point contact diagram correctly reproduces the dual vector model correlator (4.35) when supplemented with the exchange diagrams (5.101) (figure 6.2). In other words, they must satisfy the equation,

$$\begin{aligned} \langle \mathcal{O}(y_1)\mathcal{O}(y_2)\mathcal{O}(y_3)\mathcal{O}(y_4) \rangle_{\text{conn.}} & \quad (6.29) \\ &= \sum_s \mathcal{A}_s^{\text{exch.}}(y_1, y_2; y_3, y_4) + \sum_{s,m} a_{s,m} \hat{\mathcal{A}}_{s,m}^{\text{cont.}}(y_1, y_2; y_3, y_4) + y_2 \leftrightarrow y_3, y_4, \end{aligned}$$

which we solve in the following. The role of  $y_2 \leftrightarrow y_3, y_4$  is to include all permutations of the external legs – i.e. all diagrams in figure 6.2.

First, notice that the Witten diagram computation of the dual CFT correlator is au-

tomatically crossing symmetric (as can be seen from the RHS of (6.29)). Using the manifestly crossing symmetric form (4.47) for the scalar single-trace four-point function, we may thus reduce the above problem to solving the equation

$$\mathcal{G}(y_1, y_2; y_3, y_4) = \sum_s \mathcal{A}_s^{\text{exch.}}(y_1, y_2; y_3, y_4) + \sum_{s, m} a_{s, m} \hat{\mathcal{A}}_{s, m}^{\text{cont.}}(y_1, y_2; y_3, y_4), \quad (6.30)$$

where we recall that

$$\langle \mathcal{O}(y_1) \mathcal{O}(y_2) \mathcal{O}(y_3) \mathcal{O}(y_4) \rangle_{\text{conn.}} = \mathcal{G}(y_1, y_2; y_3, y_4) + y_2 \leftrightarrow y_3, y_4. \quad (6.31)$$

If a vertex that solves (6.30) is found, then the definition of  $\mathcal{G}$  above ensures that equation (6.29) is automatically satisfied. Since the non-trivial quartic vertex is completely determined by (6.29), any solution of (6.30) would give the only solution of (6.29). The two problems are thus equivalent.

To proceed, we solve for the coefficients  $a_{s, m}$  by matching the contributions in the conformal block expansion of both sides of equation (6.30). Solving equation (6.30) is therefore much simpler, as it eliminates the issue of comparing conformal block expansions in different channels. This is essentially the key technical difficulty behind the conformal bootstrap. Another crucial simplification is provided by the conformally coupled scalar in AdS<sub>4</sub>: As we saw in §5.4, in this case each bulk amplitude for fixed  $s$  in equation (6.30) only generates spin- $s$  contributions. We may thus solve the equation spin-by-spin:

$$\begin{aligned} \mathcal{A}_s^{\text{exch.}}(y_1, y_2; y_3, y_4) + \sum_m a_{s, m} \hat{\mathcal{A}}_{s, m}^{\text{cont.}}(y_1, y_2; y_3, y_4) \\ = a \cdot c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}^2 W_{\Delta+s, s}(y_1, y_2; y_3, y_4) + c \cdot \sum_n c_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{s, n}}^2 W_{2\Delta+2n+s, s}(y_1, y_2; y_3, y_4), \end{aligned} \quad (6.32)$$

where we used the conformal block expansion for  $\mathcal{G}$  derived in §4.2.3, which we repeat

for convenience below:

$$\begin{aligned} \mathcal{G}(y_1, y_2; y_3, y_4) & \tag{6.33} \\ & = a \cdot \sum_s c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}^2 W_{\Delta+s,s}(y_1, y_2; y_3, y_4) + c \cdot \sum_{s,n} c_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{s,n}}^2 W_{2\Delta+2n+s,s}(y_1, y_2; y_3, y_4), \end{aligned}$$

with  $a$  and  $c$  constrained by  $2a + c = 1$ . As previously explained, this freedom in the choice of  $a$  and  $c$  represents the ambiguity in the definition (6.31) of  $\mathcal{G}$ , in which we are at liberty to add terms  $\mathcal{G}_{\text{trivial}}$  with the property

$$\mathcal{G}_{\text{trivial}}(y_1, y_2; y_3, y_4) + \mathcal{G}_{\text{trivial}}(y_1, y_3; y_2, y_4) + \mathcal{G}_{\text{trivial}}(y_1, y_4; y_3, y_2) = 0. \tag{6.34}$$

In the bulk, this translates to the freedom of adding trivial vertices to the scalar self interaction (6.28), which give a vanishing total contact amplitude

$$\mathcal{A}_{\text{trivial}}^{\text{cont.}}(y_1, y_2; y_3, y_4) + \mathcal{A}_{\text{trivial}}^{\text{cont.}}(y_1, y_3; y_2, y_4) + \mathcal{A}_{\text{trivial}}^{\text{cont.}}(y_1, y_4; y_3, y_2) = 0. \tag{6.35}$$

In other words, the different solutions for the quartic vertex which arise from the ambiguity in the definition of  $\mathcal{G}$  differ by trivial vertices which vanish on-shell.

A particularly illuminating choice is  $a = -c = 1$ :

$$\begin{aligned} \mathcal{G}(y_1, y_2; y_3, y_4) & \tag{6.36} \\ & = \sum_s c_{\mathcal{O}\mathcal{O}\mathcal{J}_s}^2 W_{\Delta+s,s}(y_1, y_2; y_3, y_4) - \sum_{s,n} c_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{s,n}}^2 W_{2\Delta+2n+s,s}(y_1, y_2; y_3, y_4). \end{aligned}$$

Recalling (equation (5.105)) that the spin- $s$  exchange generates precisely the spin- $s$  single-trace contribution in the (12)(34) channel expansion of the dual CFT correlator, the above choice of  $a$  and  $c$  exhibits the explicit cancellation of the single-trace contributions in equation (6.32). This leaves only double-trace contributions, to be matched by the contact amplitude of the quartic vertex.

Solving for the coefficients is simplified by using the contour-integral form of the conformal block expansion (4.50), as it eliminates the sum over  $n$ .<sup>6</sup> In this way, the coefficients  $a_{s,m}$  are given implicitly by

$$\sum_m a_{s,m} (-1)^m \left(\nu^2 + s + \frac{9}{4}\right)^m = \frac{2^{8-s}}{N} \frac{1}{\nu^2 + (s - \frac{1}{2})^2} \left[ \frac{\pi}{\Gamma\left(\frac{2s-2i\nu+1}{4}\right)^2 \Gamma\left(\frac{2s+2i\nu+1}{4}\right)^2} - \frac{1}{\Gamma(s)^2} \right] \quad (6.37)$$

$$- \frac{1}{N} \frac{(-1)^{\frac{s}{2}} \pi^{\frac{3}{2}} 2^{s+5} \Gamma\left(s + \frac{3}{2}\right) \Gamma\left(\frac{s}{2} + \frac{1}{2}\right)}{\sqrt{2} \Gamma\left(\frac{s}{2} + 1\right) \Gamma(s+1) \Gamma\left(\frac{3}{4} - \frac{i\nu}{2}\right) \Gamma\left(\frac{3}{4} + \frac{i\nu}{2}\right) \Gamma\left(s + \frac{1}{2} + i\nu\right) \Gamma\left(s + \frac{1}{2} - i\nu\right)},$$

which we obtain by inserting the contour integral forms (4.50), (5.101) and (5.113) into equation (6.32) for  $a = -c = 1$ . Through setting  $z = -\left(\nu^2 + s + \frac{9}{4}\right)$ , one can establish a generating function

$$a_s(z) \equiv \sum_m a_{s,m} z^m, \quad (6.38)$$

for the coefficients for each  $s$ :

$$a_s(z) = \frac{\Lambda^s}{N} \frac{\left(\pi e^{\gamma(2s+1)} \Gamma(s)^2 \left((2s+1)^2 + (\Lambda z + s + \frac{9}{4})^2\right)^2 P\left(s + \frac{1}{2}; z\right)^2 - 4^2\right)}{2^{s-4} \Gamma(s)^2 \left((\Lambda z + s + \frac{9}{4})^2 + (s - \frac{1}{2})^2\right)}$$

$$- \frac{i^s \pi^2}{2\sqrt{2}N} \frac{\Gamma\left(s + \frac{3}{2}\right)}{\Gamma\left(\frac{s}{2} + 1\right)^2} e^{\gamma(2s+\frac{5}{2})} P\left(s + \frac{1}{2}; z\right) P\left(\frac{3}{4}; z\right)$$

$$\times \left((2s+1)^2 + (2\Lambda z + 2s + \frac{9}{2})^2\right) \left(9 + (2\Lambda z + 2s + \frac{9}{2})^2\right),$$

where we introduced

$$P(a|z) = \prod_{k=0}^{\infty} \left[ \left(1 + \frac{a}{k}\right)^2 + \left(\frac{4z + 4s + 9}{4k}\right)^2 \right] e^{-2a/k}, \quad (6.39)$$

---

<sup>6</sup>This is because the contributions from double-trace operators of the same spin are packaged in a single gamma function.

and  $\gamma$  is the Euler-Mascheroni constant. In particular, we employed the Weierstrass infinite product representation of the Gamma function and reinserted the dependence on the cosmological constant.

This determines the quartic vertex (6.28) of the parity even scalar in the minimal bosonic higher-spin theory on AdS<sub>4</sub>, in the form

$$\mathcal{V}_{0,0,0,0} = \sum_{s \in 2\mathbb{N}} \tilde{J}_s(x, \partial_u) a_s(\square) \tilde{J}_s(x, u), \quad (6.40)$$

where the  $\tilde{J}_s$  is the traceless improvement of the spin- $s$  conserved currents (5.91), bilinear in the bulk scalar  $\varphi_0$ .

We conclude this chapter with some comments:

- To solve for the vertex (6.40), we relaxed the constraint  $m \leq k$  in the basis (5.109). I.e. we used a redundant set of vertices, which can be re-expressed purely in terms of those for  $m \leq k$  using the freedom of integration by parts and the free equations of motion.
- Since we solve for the coefficients  $a_{s,m}$  at the level of the contour integral integrand in (6.37), naïvely it may seem that the vertex is not defined uniquely: One is free to add functions of  $\nu$  that are entire within the contour. In doing so, this would change the form of  $a_{s,m}$  and thus the solution for the quartic vertex. However, the addition of such terms does not change the four-point amplitude and therefore the different vertices obtained in this manner would differ only by trivial terms which vanish on the free mass shell.
- Let us also comment on the presence of anomalous scaling dimensions for the double-trace operators in the partial wave decompositions of Witten diagrams: These manifest themselves by the presence of *double-poles* in (5.102) and (5.113) for  $\frac{d}{2} + i\nu = 2\Delta + 2n + s$ ,  $n = 0, 1, 2, \dots$ . Owing to the double-poles, evaluating

the  $\nu$ -integral using Cauchy's theorem produces not only the “non-anomalous” double-trace conformal blocks  $G_{\Delta_{n,s},s}$ , but also their derivatives with respect to the dimension  $(\partial/\partial\Delta_{n,s})G_{\Delta_{n,s},s}$ . These two terms together originate from the conformal block  $G_{\Delta_{n,s}+\gamma_{n,s},s}$ , with an anomalous scaling dimension  $\gamma_{n,s}$  for the double-trace operator. Indeed, this can be seen by expanding as a Taylor series in  $\gamma_{n,s}$ ,

$$G_{\Delta_{n,s}+\gamma_{n,s},s} = G_{\Delta_{n,s},s} + \gamma_{n,s} \frac{\partial}{\partial\Delta_{n,s}} G_{\Delta_{n,s},s} + \mathcal{O}(1/N^2). \quad (6.41)$$

The presence of anomalous dimensions for a given Witten diagram is natural, since the former are dual to the binding energies of two-particle states in the bulk. At the same time however, the boundary theory is free and the double-trace operators (4.11) do not receive any anomalous dimensions. This fact manifests itself in the contour integral representation (4.50) for the dual CFT four-point function, which contains only single poles at  $\frac{d}{2} + i\nu = 2\Delta + 2n + s$ ,  $n = 0, 1, 2, \dots$ . The holographic duality between higher spin theories and free CFTs therefore requires a delicate cancellation of the anomalous conformal blocks generated by each individual Witten diagram. That the quartic vertex that we found solves (6.29), implies that this cancellation indeed takes place.

In this respect, the contour integral representation for the conformal block decomposition turned out to be a powerful tool: First of all, it allowed us to solve for the quartic interaction by performing simple algebraic manipulations with the functions of  $\nu$ . Moreover, this representation allowed us to treat conformal blocks with anomalous dimensions on the same footing as non-anomalous ones, simply by controlling the degree of the pole in  $\nu$  at the associated point.

- As a final point: One motivation for studying higher-spin interactions using holography, was owing due to the technical difficulties encountered when restricted to



standard methods of approach. Indeed, using the Noether procedure it has not yet been possible to fix all cubic couplings, or to determine quartic or higher-order vertices in a standard Lagrangian form. In comparison, the holographic approach we took seems to be more efficient. Furthermore, solving for the quartic scalar self-interaction (7.2) holographically even skips steps which are necessary in the Noether procedure. The first Noether-consistency condition in which the quartic action enters is (6.3d)

$$\delta^{(2)}S^{(2)} + \delta^{(1)}S^{(3)} + \delta^{(0)}S^{(4)} = 0. \quad (6.42)$$

However, since the scalar is invariant under  $\delta^{(0)}$  (it is not a gauge field itself), to extract the quartic scalar self-interaction (7.2) using the Noether method one needs to (at least) consider the quintic-order condition

$$\delta^{(3)}S^{(2)} + \delta^{(2)}S^{(3)} + \delta^{(1)}S^{(4)} + \delta^{(0)}S^{(5)} = 0. \quad (6.43)$$

Furthermore, in principle one may require the knowledge of other quartic interactions to solve (6.43) for the scalar quartic self-interaction.



# Summary and discussion

## 7.1 Summary of results

In this thesis, employing the conjectured duality between higher-spin gauge theories on anti-de Sitter space and free conformal field theories, we studied higher-spin interactions on an AdS background.

In the context of the duality between the type A minimal bosonic higher-spin theory on AdS and the free scalar  $O(N)$  vector model, we determined the complete on-shell cubic action (on  $\text{AdS}_{d+1}$ ) and the quartic self interaction of the scalar (on  $\text{AdS}_4$ ). These interactions were previously unknown in a metric-like Lagrangian form, without auxiliary fields. For this we drew on the equivalence between correlation functions of single-trace operators in CFT and Witten diagrams in the dual theory on AdS in the large  $N$  limit. The approach is therefore particularly appealing, as correlation functions are straightforward to compute in free CFTs.

We repeat the explicit results for the interactions in the higher-spin action (expanded

about AdS) below:

$$S[\varphi] = \sum_s \int_{\text{AdS}} \frac{1}{2} \varphi_s (\square - m_s^2 + \dots) \varphi_s + \sum_{s_i} \int_{\text{AdS}} \mathcal{V}_{s_1, s_2, s_3}(\varphi) + \int_{\text{AdS}} \mathcal{V}_{0,0,0,0}(\varphi_0) + \dots$$

The cubic interaction for each triplet of spins takes the schematic form

$$\mathcal{V}_{s_1, s_2, s_3}(\varphi) = g_{s_1, s_2, s_3} \left[ \nabla^{\mu_1} \dots \nabla^{\mu_{s_3}} \varphi_{\nu_1 \dots \nu_{s_1}} \nabla^{\nu_1} \dots \nabla^{\nu_{s_1}} \varphi_{\rho_1 \dots \rho_{s_2}} \nabla^{\rho_1} \dots \nabla^{\rho_{s_2}} \varphi_{\mu_1 \dots \mu_{s_1}} + \mathcal{O}(\sqrt{|\Lambda|}) \right],$$

where the terms  $\mathcal{O}(\sqrt{|\Lambda|})$  and higher in the bracket are descending in the number of derivatives, and are fixed uniquely. The overall coupling is given by

$$g_{s_1, s_2, s_3} = \frac{1}{\sqrt{N}} \frac{\pi^{\frac{d-3}{4}} 2^{\frac{3d-1}{2}} \sqrt{-2\Lambda}^{s_1+s_2+s_3}}{\Gamma(d+s_1+s_2+s_3-3)} \sqrt{\frac{\Gamma(s_1 + \frac{d-1}{2}) \Gamma(s_2 + \frac{d-1}{2}) \Gamma(s_3 + \frac{d-1}{2})}{\Gamma(s_1+1) \Gamma(s_2+1) \Gamma(s_3+1)}}. \quad (7.1)$$

The quartic self-interaction of the scalar takes the schematic form

$$\mathcal{V}_{0,0,0,0}(\varphi_0) = \sum_{r,m} a_{r,m} (\varphi_0 \nabla_{\mu_1} \dots \nabla_{\mu_r} \varphi_0 + \dots) \square^m (\varphi_0 \nabla^{\mu_1} \dots \nabla^{\mu_r} \varphi_0 + \dots), \quad (7.2)$$

where the  $\dots$  denote a finite number of terms with no more than  $r$  derivatives. In §6.2.2 we gave a generating function (6.39) for the coefficients  $a_{r,m}$ .

Establishing the above results involved a number of non-trivial intermediate steps. We list those which may be of interest for further applications below. Afterwards (§7.2), in the context of the above results we discuss the issue of locality in higher-spin theories.

### Three-point Witten diagrams with external fields of arbitrary spin

In order to evaluate tree-level three-point Witten diagrams involving external fields of arbitrary integer spin, we developed an approach which allowed them to be re-expressed in terms of a differential operator acting on three-point diagrams involving only external scalars. This was an instrumental result, in particular because the latter scalar ampli-

tudes are well-known and straightforward to compute. This method is summarised in the figure below.

In this work we focused on symmetric fields, while for broader applications it may be useful to extend this result to more general representations of the isometry group. For instance, those of the anti-symmetric or mixed-symmetry type, which can be found for example in string theory.

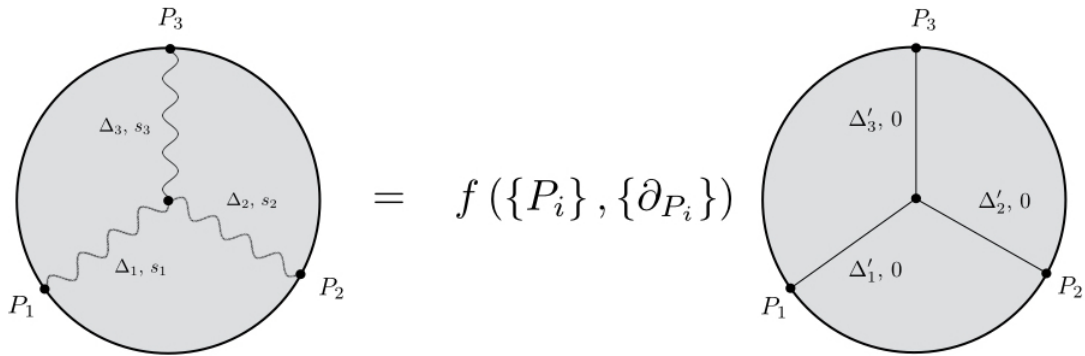


Figure 7.1: Tree-level three-point Witten diagrams with external fields of arbitrary integer spin and scaling dimensions, can be expressed in terms of tree-level three-point diagrams involving only external scalars. This is achieved through the application of a differential operator  $f(\{P_i\}, \{\partial_{P_i}\})$ , which encodes the tensor structure of the original amplitude.

### Conformal partial wave expansion of four-point Witten diagrams

In order to extract the quartic self-interaction of the scalar (7.2), we developed tools to determine conformal block expansions of tree-level four-point Witten diagrams. This was in order to match with the corresponding decomposition of the dual CFT correlator. The idea was centred upon the decomposition of four-point Witten diagrams into partial

waves,

$$\begin{aligned}
 & \text{Diagram with four external legs } \mathcal{O}(P_1), \mathcal{O}(P_2), \mathcal{O}(P_3), \mathcal{O}(P_4) \text{ and internal interaction} \\
 &= \sum_{\ell} \int_{-\infty}^{\infty} d\nu p_{\ell}(\nu) \text{Diagram with two internal vertices and legs } \frac{d}{2} + i\nu, \ell \text{ and } \frac{d}{2} - i\nu, \ell
 \end{aligned}
 \tag{7.3}$$

whose factorised form could be identified with the integral representation (3.114) of conformal blocks,

$$\begin{aligned}
 W_{\mathcal{O}_{\Delta_i, \ell}}(y_1, y_2; y_3, y_4) &= \frac{(\Delta_i - \frac{d}{2})}{2\pi} \int_{-\infty}^{\infty} d\nu \frac{1}{\nu^2 + (\Delta_i - \frac{d}{2})^2} \frac{\kappa_{\frac{d}{2} - i\nu, \ell}}{(2\pi)^{\frac{d}{4}}} \frac{\Gamma\left(\frac{\frac{d}{2} - i\nu + \ell}{2}\right)^2}{\Gamma\left(\frac{\frac{d}{2} + i\nu + \ell}{2}\right)^2} \\
 &\times \int d^d y \langle\langle \mathcal{O}(y_1) \mathcal{O}(y_2) \mathcal{O}_{\frac{d}{2} + i\nu, \ell}(y) \rangle\rangle \langle\langle \tilde{\mathcal{O}}_{\frac{d}{2} - i\nu, \ell}(y) \mathcal{O}(y_3) \mathcal{O}(y_4) \rangle\rangle.
 \end{aligned}
 \tag{7.4}$$

In particular, for exchange diagrams we expressed the bulk-to-bulk propagators for gauge fields of arbitrary spin in a basis of AdS harmonic functions, shown below. This form for the propagator is also applicable in extending this approach to conformal block expansions of diagrams with loops. This would allow, for example, further tests of the higher-spin AdS/CFT duality at loop-level.

$$\begin{aligned}
 & \text{Diagram with two external legs } x_1, x_2 \text{ and internal wavy line } \Pi_s \\
 &= \int_{-\infty}^{\infty} d\nu \sum_{\ell=0}^s g_{\ell}(\nu) \text{Diagram with two external legs } x_1, x_2 \text{ and internal dashed line with legs } \frac{d}{2} + i\nu, \ell \text{ and } \frac{d}{2} - i\nu, \ell
 \end{aligned}
 \tag{7.5}$$

While in this work we focused on four-point Witten diagrams involving external scalars,

the above method is applicable to more general scenarios. For example, to determine the conformal block expansions of four-point Witten diagrams with spinning external legs. The latter result would be crucial to extract quartic vertices involving fields with non-zero spin. Moreover, it would utilise the result for three-point Witten diagrams mentioned above.

## 7.2 Discussion: Locality

The most distinguishable feature of our quartic vertex (7.2) is that it is unbounded in its number of derivatives. In a sense, this is to be expected: In string theory there is an infinite number of higher-derivative  $\alpha'$  corrections, which play a key role in the soft behaviour of high energy scattering amplitudes. However, the implications of non-localities for higher-spin gauge theories are still to be fully understood. In this section, we discuss how the AdS/CFT correspondence provides a framework in which these issues may be elucidated. But let us begin by revisiting the concept of locality in the more familiar framework of QFT.

In QFT, locality is a crucial requirement for causality. In non-gravitational theories, where the space-time manifold is fixed and non-dynamical, causality means that the theory has localisable observables which commute at space-like separations,

$$[\mathcal{O}(x), \mathcal{O}(y)] = 0, \quad (x - y)^2 > 0, \quad (7.6)$$

since space-like separated measurements should not interfere. This follows directly from locality, since observables are functions of local fields that, as quantum operators, commute at space-like separations (up to a gauge transformation) as a consequence of Lorentz invariance and locality of interactions. However in gravitational theories there are no local gauge (diffeomorphism) invariant observables, which leaves open the possi-

bility for fields to interact in a non-local way. As we shall discuss below, in such theories these issues can be translated instead into analyticity properties of the S-matrix.

A well known example of a consistent theory that is not local in the standard QFT sense is string theory. Indeed, the covariant string field theory action governing the dynamics of the infinite set of fields contains space-time derivatives of all orders. For example, one finds interactions of the form

$$\mathcal{V}_3 = \left[ e^{\frac{1}{2}\alpha'\partial^2} \phi(x) \right]^3, \quad (7.7)$$

where we can write

$$e^{\frac{1}{2}\alpha'\partial^2} \phi(x) = \frac{1}{2\pi\alpha'} \int dy e^{\frac{1}{2\alpha'}(x-y)^2} \phi(y). \quad (7.8)$$

Naïvely, this seems to be problematic: The field  $\phi$  at the point  $x$  not only couples to itself, but also to its values arbitrarily far in the future, past, and even at space-like separations. Most troubling is the presence of an infinite number of time derivatives. The issues that generally accompany theories containing such interactions are well-known, and include [265–275]: Possible violations of unitarity and causality, problems in setting up an initial value problem, and difficulties quantising the theory and/or finding a stable Hamiltonian. However, for string theory there is evidence which indicates that the higher-derivative structure is not problematic. Indeed, when passing to the light-cone gauge it is possible to make field redefinitions which render the theory completely local in light-cone time [276–278].

### Locality in higher-spin theories, and Mellin amplitudes

Our understanding of these issues in higher-spin gauge theories is comparably limited. Recently, classes of admissible field redefinitions (i.e. those which do not alter physical observables) have been proposed in [279–281], however their implications are yet to



be fully realised. The point is whether the field re-definitions required to reduce the number of time derivatives are physical.

More generally, questions about locality can be translated into certain analytical properties of the S-matrix [282]. It is well-known that local interactions give rise to amplitudes which are polynomial in Mandelstam variables. In turn, amplitudes for exchange diagrams, which are distinct from local contact interactions, contain simple poles in the Mandelstam variables. So one criterion for consistency in higher-spin theories may be to demand that amplitudes of contact interactions be free from poles, for such non-localities cannot be tamed by admissible field redefinitions. In fact, already in higher-spin theory on flat space it has been observed that gauge invariance requires the quartic contact amplitudes to generate poles in the Mandelstam variables [28]. I.e. they are of the form

$$\varphi^2 \frac{1}{\square - m^2} \varphi^2, \quad (7.9)$$

and the corresponding class of field-redefinitions is unphysical, as they would change the form of the on-shell cubic vertices.

On the other hand, it is well known that higher-spin theories in flat space are plagued with their own issues, which are summarised by the number of no-go theorems stacked up against their existence. So one might hope for a better picture when turning to higher-spin theories on AdS. While the standard notion of an S-matrix in AdS is ill defined,<sup>1</sup> we can create and annihilate particles in AdS by changing the boundary conditions at the time-like boundary. Via the AdS/CFT correspondence, the transition amplitudes between such states are identified with the correlation functions of the dual CFT, which suggests an interpretation of CFT correlators as AdS scattering amplitudes

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<sup>1</sup>AdS space-time has a time-like conformal boundary, and thus does not admit in and out states; particles in AdS live in a box and interact forever.

[283–286]. In fact, it has been put forward that the Mellin representation of a CFT correlator of Mack [216, 287] may provide a definition for an S-matrix of the dual gravity theory in AdS [146, 147, 150, 216, 287]:

Consider the example of a Euclidean four-point CFT correlator with identical scalar fields,

$$\langle \mathcal{O}(y_1) \mathcal{O}(y_2) \mathcal{O}(y_3) \mathcal{O}(y_4) \rangle = \frac{g(u, v)}{(y_{12}^2 y_{34}^2)^\Delta}. \quad (7.10)$$

The Mellin representation of (7.10) is given by inverting the Mellin transform

$$\langle \mathcal{O}(y_1) \mathcal{O}(y_2) \mathcal{O}(y_3) \mathcal{O}(y_4) \rangle = \frac{\mathcal{N}}{(2\pi i)^2} \int d\delta_{ij} M(\delta_{ij}) \prod_{i<j}^4 (y_{ij}^2)^{-\delta_{ij}}, \quad (7.11)$$

with some normalisation  $\mathcal{N}$ , and where the integration contour runs parallel to the imaginary axis with  $\text{Re}(\delta_{ij}) > 0$ . By conformal symmetry, the Mellin variables  $\delta_{ij}$  are constrained by

$$\sum_j \delta_{ij} = \Delta, \quad \delta_{ii} = 0, \quad (7.12)$$

The object  $\mathcal{M}(\delta_{ij})$  defined by

$$M(\delta_{ij}) = \mathcal{M}(\delta_{ij}) \prod_{i<j}^4 \Gamma(\delta_{ij}), \quad (7.13)$$

is known as the *Mellin amplitude*, and has been proposed to be understood as the AdS scattering amplitude [146]. To help motivate this identification, let us first solve the conditions (7.12) in a suggestive way by introducing auxiliary “momenta”  $k_i$ , satisfying

$$\sum_{i=4}^4 k_i = 0, \quad k_i^2 = -\Delta. \quad (7.14)$$

One can then define analogues of Mandelstam variables as

$$s = -(k_1 + k_2)^2 = 2\Delta - 2\delta_{12} \quad (7.15)$$

$$t = -(k_1 + k_3)^2 = 2\Delta - 2\delta_{13} \quad (7.16)$$

$$u = -(k_1 + k_4)^2 = 2\Delta - 2\delta_{14}, \quad (7.17)$$

satisfying the relation  $s + t + u = 4\Delta$ . Let us now consider the structure of the Mellin amplitude in terms of these variables. As observed by Mack, in order to reproduce the contribution of a conformal multiplet with quantum numbers  $[\Delta_k, J]$  in the conformal partial wave expansion of (7.10) in, say, the (12)(34) channel, the Mellin representation (7.13)

$$M(s, t, u) = \mathcal{M}(s, t, u) \Gamma\left(\Delta - \frac{s}{2}\right)^2 \Gamma\left(\Delta - \frac{t}{2}\right)^2 \Gamma\left(\Delta - \frac{u}{2}\right)^2, \quad (7.18)$$

must contain simple poles at

$$s = \Delta_k - J + 2m, \quad m = 0, 1, 2, 3, \dots \quad (7.19)$$

The contributions from double-trace operators are thus taken care of by the Gamma function factors in (7.18), leaving the Mellin amplitude  $\mathcal{M}(s, t, u)$  to take care of the single-trace contributions. Under the AdS/CFT correspondence, single-trace CFT operators are identified with single-particle bulk states §2.1.1, and therefore the Mellin amplitude of a four-point exchange Witten diagram contains the corresponding simple poles in the Mandelstam variables, analogous to flat space S-matrix amplitudes for exchanges.<sup>2</sup> Furthermore, local bulk contact interactions give rise to polynomial Mellin amplitudes [146], in perfect analogy with flat space scattering amplitudes. More substance to this identification of the Mellin amplitude as an S-matrix for AdS was given by the observation that the standard flat space scattering amplitude is recovered in the

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<sup>2</sup>This was verified explicitly in: [146] for the scalar and graviton exchange.

appropriate flat limit [146, 288] (see also: [289]).

With the above analogy in place, let us explore its implications in the context of higher-spin holography. According to the above prescription, we may interpret the Mellin amplitude of the connected four-point correlator (4.35) in the free scalar  $O(N)$  vector model

$$\langle \mathcal{O}(y_1) \mathcal{O}(y_2) \mathcal{O}(y_3) \mathcal{O}(y_4) \rangle_{\text{conn.}} = \frac{4}{N} \frac{1}{(y_{12}^2 y_{34}^2)^\Delta} \left[ u^{\frac{\Delta}{2}} + \left(\frac{u}{v}\right)^{\frac{\Delta}{2}} + u^{\frac{\Delta}{2}} \left(\frac{u}{v}\right)^{\frac{\Delta}{2}} \right], \quad (7.20)$$

as a scattering amplitude with external scalar legs in the type A minimal bosonic higher-spin theory on AdS. While the Mellin amplitudes of correlators in strongly coupled CFTs are well understood [290], there are some subtleties in applying the definition (7.11) to free (or weakly coupled) CFTs. To see this more clearly, consider the latter in terms of the ‘‘Mandelstam variables’’,

$$\langle \mathcal{O}(y_1) \mathcal{O}(y_2) \mathcal{O}(y_3) \mathcal{O}(y_4) \rangle = \frac{\mathcal{N}}{(y_{12}^2 y_{34}^2)^\Delta} \int \frac{ds}{2\pi i} \int \frac{dt}{2\pi i} u^{\frac{s}{2}} v^{\Delta - \frac{s+t}{2}} M(s, t). \quad (7.21)$$

The main issue is that a power function (which is characteristic of free CFT correlators as a result of Wick contractions)

$$f(w) = w^\alpha, \quad \alpha \in \mathbb{R}, \quad (7.22)$$

does not have a well defined Mellin transform [291]

$$M(z) = \int_0^\infty dw w^z f(w) \frac{dw}{w}. \quad (7.23)$$

One possibility is to understand the Mellin transform in this case as a distribution [292]

$$M(z) = \delta(z + \alpha). \quad (7.24)$$

With this interpretation, the Mellin transform of the vector model correlator (7.20) is given by [3, 281]

$$M(s, t, u) = \frac{4}{N} \delta\left(\frac{s}{2} - \frac{\Delta}{2}\right) \delta\left(\frac{t}{2} - \frac{\Delta}{2}\right) \delta\left(\frac{u}{2} - \Delta\right) + \text{cycl.} \quad (7.25)$$

Owing to its distributional nature, there may be subtleties in applying the definition (7.18) to obtain the corresponding Mellin amplitude (and hence scattering amplitude in AdS higher-spin theory). The problem is that each value in the support of the LHS in (7.18) corresponds to a pole of one of the Gamma functions on the RHS. Formally, treating the Mellin amplitude as a distributional product one would conclude that it must *vanish* identically.

Assuming the validity of the above observation, let us briefly explore its implications in the context of the results obtained in this thesis. Recall (§5.4.2) that the single-trace contributions from the bulk exchanges<sup>3</sup>

$$\begin{aligned} \mathcal{A}_{\text{total}}^{\text{exch.}}(y_1, y_2; y_3, y_4) &= \sum_J \mathcal{A}_J^{\text{exch.}}(y_1, y_2; y_3, y_4) \\ &= \sum_J c_{\mathcal{O}\mathcal{O}J}^2 W_{\Delta+J,J}(y_1, y_2; y_3, y_4) + \text{double-trace,} \end{aligned} \quad (7.26)$$

precisely account for all single-trace contributions in the conformal block expansion of the dual CFT correlator (4.41),

$$\mathcal{G}(y_1, y_2; y_3, y_4) = \sum_J c_{\mathcal{O}\mathcal{O}J}^2 W_{\Delta+J,J}(y_1, y_2; y_3, y_4) + \text{double-trace.} \quad (7.27)$$

The Witten diagram generated by the quartic vertex (7.2) therefore should not generate single-trace contributions. Recalling that single-trace contributions are the only ones

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<sup>3</sup>To avoid confusion with the Mandelstam variable “ $s$ ”, in this section we use the label  $J$  to denote spin.

which can generate simple poles in the corresponding Mellin amplitude,<sup>4</sup> one is led to conclude that the scattering amplitude generated by the quartic vertex (7.2) does not give rise to simple poles in the “Mandelstam variables”.

While it seems that there are no simple poles in the four-point contact amplitude in AdS, we encounter the following issue. Should the re-summation (7.28) of the Mellin transforms of the bulk exchanges be an analytic function, with simple poles corresponding to the respective intermediate states (7.19), the above discussion may be in contradiction with the form (7.25) of the dual CFT correlator’s Mellin transform. In this case, the absence of these poles in the full amplitude (7.25) would then imply that the bulk contact amplitude contains simple poles to compensate. On the other hand, the appearance of Dirac delta functions in the Mellin transform (7.25) indicates possible issues of convergence for the conformal partial wave expansion of the scalar four-point function when summing over the spin. Therefore, one subtlety may lie in subtracting, term-by-term, a divergent sum over bulk exchanges from a divergent series. In this delicate scenario, to study the Mellin amplitude generated by the quartic vertex (7.2), the most rigorous way to proceed may be to instead re-sum the bulk exchanges (sum over spins) and *then* subtract them from the re-summed CFT amplitude (7.25). Further clarity may also be afforded by expressing the quartic vertex (7.2) in a non-redundant basis.

As a final comment, let us note that potential divergent behaviour of higher-spin exchanges discussed above has been observed in the case of conformal higher-spin gauge

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<sup>4</sup>For clarity, the Mellin transform of the sum over exchanges (7.26) takes the form,

$$M_{\text{total}}^{\text{exch.}}(s, t) = \Gamma(\Delta - \frac{s}{2})^2 \Gamma(\Delta - \frac{t}{2})^2 \Gamma(\frac{s+t}{2} - \Delta)^2 \sum_{J=0}^{\infty} \mathcal{M}_J^{\text{exch.}}(s, t), \quad (7.28)$$

$$\mathcal{M}_J^{\text{exch.}}(s, t) = \sum_{m=0}^{\infty} g_J \frac{\mathcal{Q}_{J,m}(t)}{s - \Delta - 2m} + \text{Pol}_J(s, t) \quad (7.29)$$

where  $\mathcal{Q}_{J,m}(t)$  is a kinematical polynomial [216],  $\mathcal{Q}_{J,m}(t) = t^J + \mathcal{O}(t^{J-1})$ , and  $\text{Pol}_J(s, t)$  is a possible degree  $J - 1$  polynomial in  $s$  and  $t$ .

theories in flat space [293]. There, the regularised sum over the spin- $J$  exchanges gives

$$\mathcal{M}_{\text{total exch.}}(s, t, u) = \frac{1}{(-s)^{\frac{d-4}{2}}} \sum_{J=0}^{\infty} \left(J + \frac{d-3}{2}\right) C_J^{\left(\frac{d-3}{2}\right)} \left(\frac{t-u}{t+u}\right) \quad (7.30)$$

$$\sim \frac{1}{(-s)^{\frac{d-4}{2}}} \frac{(-1)^{d-4}}{(d-4)!} \delta^{[d-4]} \left(-\frac{2u}{t+u}\right), \quad (7.31)$$

where  $\delta^{[d-4]}(z)$  denotes the  $(d-4)$ -th derivative of the Dirac delta function, and we recall that  $C_J^{\left(\frac{d-3}{2}\right)}(z)$  is a Gegenbauer polynomial.





## Ambient formalism

### A.1 Ambient Space

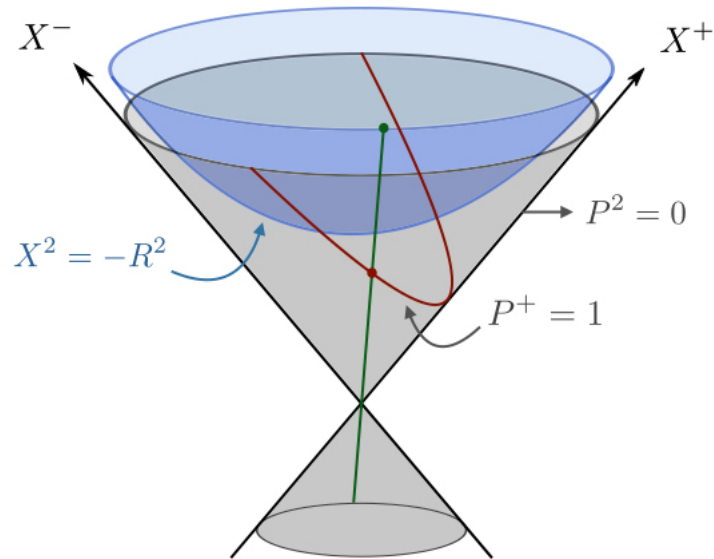


Figure A.1: Euclidean AdS and its boundary in ambient space. This figure displays the AdS surface  $X^2 = -R^2 = -1$  and the identification of a (green) boundary point with a (green) light ray of the light cone  $P^2 = 0$ , which intersects the Poincaré section (A.12) on a (red) point.

The basic idea behind the ambient formalism is to regard AdS space as a one-sheeted

hyperboloid  $H_{d+1}$  with curvature radius  $R$

$$H_{d+1} : \quad X^2 = -R^2, \quad X^0 > 0, \quad (\text{A.1})$$

embedded into an ambient  $(d+2)$ -dimensional flat space-time, parameterised by Cartesian co-ordinates  $X^A$  with  $A = 0, 1, \dots, d+1$ .

To be more precise, denoting the intrinsic coordinates on  $H_{d+1}$  by  $x^\mu$ , we are considering the isometric smooth embedding

$$i : \quad H_{d+1} \hookrightarrow \mathbb{R}^{d+2} : \quad x^\mu \longmapsto X^A(x^\mu). \quad (\text{A.2})$$

The ambient space itself is endowed with a flat metric  $\eta_{AB}$ , which defines the quadratic form  $X^2 = -R^2$ . The choice of signature dictates whether one describes Euclidean or Lorentzian AdS:

$$\text{Lorentzian signature: } \eta_{AB} = (- + + \dots +) \quad \rightarrow \quad \text{Euclidean AdS} \quad (\text{A.3})$$

$$\text{Conformal signature: } \eta_{AB} = (- + \dots + -) \quad \rightarrow \quad \text{Lorentzian AdS}, \quad (\text{A.4})$$

where in both cases we adopt the ‘‘mostly plus’’ convention.

Towards the boundary of AdS, the hyperboloid  $H_{d+1}$  asymptotes to the light cone  $X^2 = 0$ . The boundary is thus identified with the ambient projective cone of light rays, which can be identified by setting

$$P^A \equiv \epsilon X^A, \quad (\text{A.5})$$

in the limit  $\epsilon \rightarrow 0$ . Since  $X^2$  is fixed, these ambient co-ordinates satisfy

$$P^2 = 0, \quad P \sim \lambda P, \quad \lambda \neq 0, \quad (\text{A.6})$$

where the equivalence relation expresses the fact that we are dealing with rays. This quotienting of the light cone allows us to describe the  $d$ -dimensional boundary. We can identify this projective null cone with the  $d$ -dimensional space by “gauge-fixing” this rescaling. We give an example of this below.

A particular virtue of the ambient formalism is that the mutual  $SO(d, 2)$  (or  $SO(d + 1, 1)$  in the Euclidean case) symmetry of AdS and its conformal boundary is made manifest, by which we mean that the symmetry transformations act linearly. Simply put, the ambient formalism makes the kinematics in AdS or its conformal boundary as simple as Lorentz-invariant kinematics. For co-ordinates, we have

$$X'^A = e^{i\omega^{CD} J_{CD}} X^A, \quad iJ_{CD} = X_C \frac{\partial}{\partial X_D} - X_D \frac{\partial}{\partial X_C}, \quad (\text{A.7})$$

with  $SO(d, 2)$  generators  $J_{CD}$ , which satisfy the commutation relations (3.8).

### Example: Euclidean AdS in Poincaré co-ordinates

A simple example is given by Euclidean AdS in Poincaré co-ordinates  $x^\mu = (z, y^i)$ ,  $i = 1, \dots, d$ . The points on AdS are parameterised by

$$X^0 = \frac{z^2 + y^2 + R^2}{2z} \quad (\text{A.8})$$

$$X^{d+1} = \frac{R^2 - z^2 - y^2}{2z} \quad (\text{A.9})$$

$$X^i = \frac{R}{z} y^i \quad (\text{A.10})$$

Pulling the ambient metric  $\eta_{AB}$  with signature (A.3) back onto the AdS manifold one recovers

$$ds^2 = \left( \frac{\partial X^A}{\partial x^\mu} \frac{\partial X^B}{\partial x^\nu} \eta_{AB} \right) dx^\mu dx^\nu = \frac{R^2}{z^2} (dz^2 + \delta_{ij} dy^i dy^j). \quad (\text{A.11})$$

With the gauge choice  $P^+ = P^0 + P^{d+1} = 1$ , the boundary can be parameterised by

$$P^0 = \frac{1}{2}(1 + y^2), \quad P^{d+1} = \frac{1}{2}(1 - y^2), \quad P^i = y^i. \quad (\text{A.12})$$

This corresponds to a Poincaré section of the light cone, and is illustrated in figure A.1 for the Euclidean conformal group  $SO(d+1, 1)$ . Conformal transformations relate this section to others with  $dP^+ = 0$ .

## A.2 Fields in the ambient formalism

To describe fields in the bulk or on the boundary of AdS in this formalism, they need to be identified with the appropriate representatives in the ambient space. Uplifting to the higher-dimensional ambient space introduces extra degrees of freedom, and thus the representatives of bulk and boundary fields must be constrained to keep the number of degrees of freedom constant. In the following subsections we detail the ambient description of scalar and tensorial bulk and boundary fields. For definiteness, we work with the Euclidean AdS isometry group / conformal group  $SO(d+1, 1)$ .

### A.2.1 Bulk fields

A smooth rank- $r$  covariant tensor field  $t_{\mu_1 \dots \mu_r}(x)$  on  $\text{AdS}_{d+1}$  is represented in ambient space  $\mathbb{R}^{d+1,1}$  by a  $SO(d+1, 1)$ -tensor  $T_{A_1 \dots A_r}(X)$ , whose pullback onto the AdS manifold satisfies

$$i^* : T_{A_1 \dots A_r}(X) \longmapsto t_{\mu_1 \dots \mu_r}(x) = \frac{\partial X^{A_1}(x)}{\partial x^{\mu_1}} \dots \frac{\partial X^{A_r}(x)}{\partial x^{\mu_r}} T_{A_1 \dots A_r}(X(x)). \quad (\text{A.13})$$

They are not represented uniquely: The pullback is surjective, so every such tensor on the AdS manifold has an ambient representative, but it is not injective. Indeed, since for  $H_{d+1}$

$$X^2 = -R^2 \implies \frac{\partial X}{\partial x^\mu} \cdot X \Big|_{H_{d+1}} = 0, \quad (\text{A.14})$$

the kernel of the pullback (A.13) contains “pure gauge” tensors with components normal to the AdS manifold, which have no influence in the theory defined on AdS.

To obtain a unique representation of AdS tensors in ambient space, the extra components need to be eliminated [134]. This is achieved by demanding:

- **Tangentiality** to surfaces of constant radial co-ordinate  $\rho = \sqrt{-X^2}$ ,

$$X^{A_i} T_{A_1 \dots A_i \dots A_r} \Big|_{\rho=\text{const.}} = 0. \quad (\text{A.15})$$

Explicitly, we can apply the projection operator

$$\mathcal{P}_A^B = \delta_A^B - \frac{X_A X^B}{X^2}, \quad (\text{A.16})$$

which acts on ambient tensors as

$$(\mathcal{P}T)_{A_1 \dots A_r} := \mathcal{P}_{A_1}^{B_1} \dots \mathcal{P}_{A_r}^{B_r} T_{B_1 \dots B_r}, \quad X^{A_i} (\mathcal{P}T)_{A_1 \dots A_i \dots A_r} = 0. \quad (\text{A.17})$$

- **Homogeneity** in  $X^A$ . I.e. prescribed extension along the extra radial direction  $\rho$

$$T_{A_1 \dots A_r}(\lambda X) = \lambda^{-\mu} T_{A_1 \dots A_r}(X), \quad (\text{A.18})$$

where  $\mu$  is fixed by the representation of  $SO(d+1, 1)$ .

### Example I: AdS metric

A straightforward application of the above is to the intrinsic AdS metric

$$g_{\mu\nu} = \frac{\partial X^A}{\partial x^\mu} \frac{\partial X^B}{\partial x^\nu} \eta_{AB}. \quad (\text{A.19})$$

The induced AdS metric is given by

$$G_{AB} = \mathcal{P}_A^C \mathcal{P}_B^D \eta_{CD} = \eta_{AB} + \frac{X_A X_B}{R^2},$$

confirming its role as a projector.

### Example II: Covariant derivative

Following the above prescription, in ambient representative of the covariant derivative is

$$\nabla_A = \mathcal{P}_A^B \partial_B, \quad (\text{A.20})$$

and its action is given by

$$\nabla = \mathcal{P} \circ \partial \circ \mathcal{P}. \quad (\text{A.21})$$

This ensures that the result is tangent to the AdS manifold, and crucially that the partial derivative is acting on an object which already represents an AdS tensor.

For example,

$$\nabla_B T_{A_1 \dots A_r} = \mathcal{P}_B^C \mathcal{P}_{A_1}^{C_1} \dots \mathcal{P}_{A_r}^{C_r} \frac{\partial}{\partial X^C} (PT)_{C_1 \dots C_r} (X). \quad (\text{A.22})$$

### Example III: Symmetric and traceless spin- $s$ fields

An important example for this work is the ambient description of symmetric and traceless spin- $s$  fields in  $\text{AdS}_{d+1}$ . Such fields satisfy the Fierz system (generalised to AdS)

$$\left(-\nabla^\mu \nabla_\mu + m_s^2\right) \varphi_{\mu_1 \dots \mu_s} (x) = 0, \quad \nabla^{\mu_1} \varphi_{\mu_1 \dots \mu_s} (x) = 0, \quad \varphi_{\mu_1 \dots \mu_{s-1}}{}^{\mu_{s-1}} (x) = 0, \quad (\text{A.23})$$

where  $\nabla_\mu$  is the intrinsic covariant derivative on AdS and  $m_s^2 R^2 = \Delta(\Delta - d) - s$ , with  $[\Delta, s]$  labelling representation of  $SO(d, 2)$ :  $\Delta$  is the lowest energy eigenvalue, and  $s$  the

spin.

The ambient description  $\varphi_{A_1 \dots A_s}(X)$  of the above intrinsic system, is naturally given by

$$\left(-\nabla^A \nabla_A + m_s^2\right) \varphi_{A_1 \dots A_s}(X) = 0, \quad \nabla^{A_1} \varphi_{A_1 \dots A_s}(X) = 0, \quad \varphi_{A_1 \dots A_{s-1}}{}^{A_{s-1}}(X) = 0, \quad (\text{A.24})$$

where  $\nabla^A$  is the ambient representative of the covariant derivative. These are supplemented by the tangentiality and homogeneity conditions

$$X^{A_1} \varphi_{A_1 \dots A_s}(X) = 0, \quad \varphi_{A_1 \dots A_s}(\lambda X) = \lambda^{-\Delta} \varphi_{A_1 \dots A_s}(X), \quad (\text{A.25})$$

with the requirement that  $\varphi_{A_1 \dots A_s}$  carries the same representation of  $SO(d, 2)$  fixing the degree of homogeneity to be  $\Delta$ .

A nice observation is that the above is equivalent to the flat ambient Fierz system

$$\partial^2 \varphi_{A_1 \dots A_s}(X) = 0, \quad \partial^{A_1} \varphi_{A_1 \dots A_s}(X) = 0, \quad \eta^{A_1 A_2} \varphi_{A_1 \dots A_s}(X) = 0. \quad (\text{A.26})$$

when supplemented with the homogeneity and tangentiality conditions (A.25), and employing the definition (A.20) of the ambient representative of the covariant derivative.

### A.2.2 Boundary fields

In this thesis we consider only symmetric and traceless primary fields in CFT, though the ambient description can be extended to other representations of  $SO(d)$ .

A spin- $r$  primary field  $f_{i_1 \dots i_r}(y)$  of dimension  $\Delta$  is represented in the ambient formalism by a  $SO(d+1, 1)$ -tensor  $F_{A_1 \dots A_r}(P)$  which lives on the light cone  $P^2 = 0$ . This ambient

tensor is also symmetric and traceless, as well as homogeneous of degree  $-\Delta$

$$F_{A_1 \dots A_r}(\lambda P) = \lambda^{-\Delta} F_{A_1 \dots A_r}(P), \quad \lambda > 0. \quad (\text{A.27})$$

To be tangent to the light cone,  $F_{A_1 \dots A_r}(P)$  must satisfy

$$P^{A_1} F_{A_1 \dots A_r} = 0. \quad (\text{A.28})$$

We can define  $f_{i_1 \dots i_r}(y)$  is related to  $F_{A_1 \dots A_r}(P)$  by a projection onto the Euclidean section (A.12)<sup>1</sup>

$$f_{i_1 \dots i_r}(y) = \frac{\partial P^{A_1}(y)}{\partial y^{i_1}} \dots \frac{\partial P^{A_r}(y)}{\partial y^{i_r}} F_{A_1 \dots A_r}(P(y)), \quad (\text{A.30})$$

whose kernel contains arbitrary tensors proportional to  $P^A$ .

To check that the number of degrees of freedom is unchanged, note that the transversality condition (A.28) eliminates one of the two extra components per ambient index. Since we are on the light cone  $P^2 = 0$ , unlike for the bulk case considered in the previous section, the transversality condition (A.28) is not sufficient to fix  $F_{A_1 \dots A_r}$  uniquely. This ‘‘gauge invariance’’ eliminates the residual extra degrees of freedom.

### A.2.3 Operations with ambient tensors

Tensor operations such as contractions can be greatly simplified by encoding tensors in generating functions. In  $d$ -dimensional intrinsic space, we can encode symmetric

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<sup>1</sup>Note that this relation preserves tracelessness: To compute the trace of  $f_{i_1 \dots i_r}(y)$ , we need the contraction

$$\delta^{ij} \frac{\partial P^A}{\partial y^i} \frac{\partial P^B}{\partial y^j} = \eta^{AB} + P^A Q^B + P^B Q^A, \quad \text{where } Q^A = (1, 0, -1). \quad (\text{A.29})$$

This gives vanishing trace of  $f_{i_1 \dots i_r}$  owing to the tracelessness and transversality (A.28) of  $F_{A_1 \dots A_r}$ .



tensors in polynomials

$$t_{\mu_1 \dots \mu_r}(x) \longrightarrow t(x, u) = \frac{1}{r!} t_{\mu_1 \dots \mu_r}(x) u^{\mu_1} \dots u^{\mu_r}, \quad (\text{A.31})$$

where we have introduced the constant  $d$ -dimensional auxiliary vector  $u^\mu$ . Furthermore, *traceless* tensors can be encoded as

$$t_{\mu_1 \dots \mu_r}(x) \longrightarrow t(x, w) = \frac{1}{r!} t_{\mu_1 \dots \mu_r}(x) w^{\mu_1} \dots w^{\mu_r}, \quad (\text{A.32})$$

where now the auxiliary vector is *null*,  $w^2 = 0$ .

In terms of generating functions, tensor operations are then translated into an operator calculus, which simplifies manipulations significantly. For example, the contraction between two symmetric rank- $r$  tensors is implemented via

$$t_{\mu_1 \dots \mu_r}(x) s^{\mu_1 \dots \mu_r}(x) = r! t(x, \partial_u) s(x, u) \Big|_{u=0} = r! s(x, \partial_u) t(x, u) \Big|_{u=0}. \quad (\text{A.33})$$

The covariant derivative also gets modified when acting on functions of the intrinsic auxiliary variables,

$$\nabla_\mu \rightarrow \nabla_\mu + \omega_\mu^{ab} u_a \frac{\partial}{\partial u^b}, \quad (\text{A.34})$$

with  $\omega_\mu^{ab}$  the spin-connection. With this, the following further operations can be represented:

$$\text{divergence: } \nabla \cdot \partial_u, \quad \text{symmetrised gradient: } u \cdot \nabla, \quad \text{trace: } \partial_u \cdot \partial_u, \quad (\text{A.35})$$

The symmetric metric  $g_{\mu\nu}$  is denoted simply by  $u^2$ , and thus terms proportional to  $u^2$  are pure trace.

For symmetric and *traceless* tensors, one instead uses the Thomas derivative [294],

which we denote by  $\hat{\partial}_w$ .<sup>2</sup> This takes into account  $w^2 = 0$ ,

$$t_{\mu_1 \dots \mu_r}(x) s^{\mu_1 \dots \mu_r}(x) = \frac{r!}{\left(\frac{d}{2} - 1\right)_r} t(x, \partial_w) s(x, w) \Big|_{w=0} = \frac{r!}{\left(\frac{d}{2} - 1\right)_r} s(x, \partial_w) t(x, w) \Big|_{w=0}, \quad (\text{A.36})$$

where

$$\hat{\partial}_{w^\mu} = \left( \frac{d}{2} - 1 + w \cdot \frac{\partial}{\partial w} \right) \frac{\partial}{\partial w^\mu} - \frac{1}{2} w^\mu \frac{\partial^2}{\partial w \cdot \partial w}. \quad (\text{A.37})$$

This formalism can be extended to ambient representatives of bulk and boundary tensor fields discussed in the previous section, which we outline in the following.

### Ambient AdS tensors

The ambient counterpart of the intrinsic generating function (A.31) for symmetric rank- $r$  tensors is,

$$T_{A_1 \dots A_r}(X) \longrightarrow T(X, U) = \frac{1}{r!} T_{A_1 \dots A_r}(X) U^{A_1} \dots U^{A_r}, \quad X \cdot U = 0, \quad (\text{A.38})$$

where  $T_{A_1 \dots A_r}$  is the ambient representative of  $t_{\mu_1 \dots \mu_r}$ . In this case, the auxiliary vector  $U^A$  is constrained to ensure that we are working modulo components which drop out after projection onto the tangent space of  $H_{d+1}$ .

Contraction between symmetric tensors slightly modified in contrast to the intrinsic case. The generalisation of (A.33) being

$$T_{A_1 \dots A_r}(X) S^{A_1 \dots A_r}(X) = r! T(X, D_U) S(X, U) \Big|_{U=0} = r! S(X, D_U) T(X, U) \Big|_{U=0}, \quad (\text{A.39})$$

where the derivative  $D_U$  is given by,

$$D_U^A = \mathcal{P}_A^B \frac{\partial}{\partial U^B} = \frac{\partial}{\partial U^A} - \frac{X_A}{X^2} \left( X \cdot \frac{\partial}{\partial U} \right), \quad (\text{A.40})$$

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<sup>2</sup>This is also known in the literature as the Todorov differential [295].

and accommodates for the condition  $X \cdot U = 0$ .

The ambient representative of the covariant derivative in the generating formalism takes the form

$$\nabla^A = \partial_X^A - \frac{1}{X^2} (X^A X \cdot \partial_X + U^A X \cdot \partial_U - U \cdot X \partial_U^A), \quad (\text{A.41})$$

which satisfies the useful operator algebra

$$[X \cdot \partial_U, \nabla_A] = 0, \quad [\partial_U \cdot \partial_U, \nabla_A] = 0, \quad [\nabla_A, X^2] = 0, \quad X \cdot \nabla = 0, \quad (\text{A.42})$$

together with

$$[D_U^A, \nabla^B] = \frac{X^A}{X^2} D_U^B. \quad (\text{A.43})$$

For symmetric and traceless tensors, the ambient counterpart of the intrinsic generating function (A.32) is

$$T_{A_1 \dots A_r}(X) \longrightarrow T(X, W) = \frac{1}{r!} T_{A_1 \dots A_r}(X) W^{A_1} \dots W^{A_r}, \quad X \cdot W = W^2 = 0. \quad (\text{A.44})$$

Like for the symmetric ambient tensors above, the Thomas derivative (A.37) that implements contractions between symmetric traceless tensors is slightly modified, to account for the transversality condition  $X \cdot W = 0$ . We denote it by  $\hat{D}_W$ , and it is given explicitly by

$$\begin{aligned} \hat{D}_{W^A} &= \left( \frac{d+1}{2} - 1 + W \cdot \frac{\partial}{\partial W} \right) \mathcal{P}_A^B \frac{\partial}{\partial W^B} - \frac{1}{2} W_A (\mathcal{P} \circ \mathcal{P})^{CD} \frac{\partial}{\partial W^C} \frac{\partial}{\partial W^D} \\ &= \frac{d-1}{2} \left( \frac{\partial}{\partial W^A} - \frac{X_A}{X^2} \left( X \cdot \frac{\partial}{\partial W} \right) \right) + \left( W \cdot \frac{\partial}{\partial W} \right) \frac{\partial}{\partial W^A} \\ &\quad - \frac{X_A}{X^2} \left( W \cdot \frac{\partial}{\partial W} \right) \left( X \cdot \frac{\partial}{\partial W} \right) - \frac{1}{2} W_A \left( \frac{\partial^2}{\partial W \cdot \partial W} - \frac{1}{X^2} \left( X \cdot \frac{\partial}{\partial W} \right) \left( X \cdot \frac{\partial}{\partial W} \right) \right). \end{aligned} \quad (\text{A.45})$$

The ambient generalisation of (A.36) is then

$$\begin{aligned} T_{A_1 \dots A_r}(X) S^{A_1 \dots A_r}(X) &= \frac{r!}{\left(\frac{d}{2} - \frac{1}{2}\right)_r} T(X, \hat{D}_W) S(X, W) \Big|_{W=0} \\ &= \frac{r!}{\left(\frac{d}{2} - \frac{1}{2}\right)_r} S(X, \hat{D}_W) T(X, W) \Big|_{W=0}. \end{aligned} \quad (\text{A.46})$$

When working modulo traces, we have the operations

$$\text{divergence: } \nabla \cdot \hat{D}_W, \quad \text{symmetrised gradient: } W \cdot \nabla, \quad \text{Laplacian: } \nabla^2, \quad (\text{A.47})$$

and  $W \cdot \partial_W$ , which returns the spin of a tensor. These satisfy the operator algebra

$$\begin{aligned} [\nabla \cdot \hat{D}_W, W \cdot \nabla] & \quad (\text{A.48}) \\ &= \left(\frac{d}{2} - \frac{1}{2} + W \cdot \partial_W\right) \nabla^2 - \left((W \cdot \partial_W)^2 + 3\left(\frac{d}{2} - \frac{1}{2}\right) W \cdot \partial_W + \left(\frac{d}{2} - \frac{1}{2}\right)^2\right) W \cdot \partial_W, \end{aligned}$$

$$[\nabla^2, W \cdot \nabla] = -2\left(\frac{d}{2} - 1 + W \cdot \partial_W\right) W \cdot \nabla. \quad (\text{A.49})$$

The following commutators will come in use,

$$[\nabla^2, (W \cdot \nabla)^n] = -n(d - 1 + 2W \cdot \partial_W - n)(W \cdot \nabla)^n, \quad (\text{A.50})$$

and

$$\begin{aligned} [\nabla \cdot \hat{D}_W, (W \cdot \nabla)^n] & \quad (\text{A.51}) \\ &= \frac{n}{2} (W \cdot \nabla)^{n-1} (d + n + 2W \cdot \partial_W - 2)(1 - n - (n + W \cdot \partial_W - 1)(d + n + W \cdot \partial_W - 2) + \nabla^2). \end{aligned}$$

### Ambient boundary tensors

The ambient representative (A.27) of a symmetric traceless spin- $r$  primary field  $f_{i_1 \dots i_r}$  can be encoded in the generating polynomial

$$F_{A_1 \dots A_r}(P) \longrightarrow F(P, Z) = \frac{1}{r!} F_{A_1 \dots A_r}(P) Z^{A_1} \dots Z^{A_r}, \quad Z^2 = 0. \quad (\text{A.52})$$

Tangentiality to the light cone,

$$\left( P \cdot \frac{\partial}{\partial Z} \right) F(P, Z) = 0, \quad (\text{A.53})$$

can be enforced by requiring  $F(P, Z + \alpha P) = F(P, Z)$  for any  $\alpha$ . The “gauge freedom” is represented by the orthogonality condition  $Z \cdot P = 0$ .

Contractions between ambient representatives of primary fields can be implemented via the Thomas derivative (A.37), acting in the  $(d + 2)$ -dimensional ambient space

$$\hat{\partial}_Z^A = \left( \frac{d}{2} - 1 + Z \cdot \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial Z_A} - \frac{1}{2} Z^A \frac{\partial^2}{\partial Z \cdot \partial Z}. \quad (\text{A.54})$$



## Appendix B

# Appendix of integrals

### B.1 Bulk integrals

In using the Schwinger parameterised form (B.2) of boundary-to-bulk propagators in the ambient formalism, a key integral often encountered is of the form

$$\begin{aligned} \int_0^{+\infty} \prod_{i=1}^n \left( \frac{dt_i}{t_i} t_i^{\Delta_i} \right) \int_{\text{AdS}} dX \exp \left( 2 \sum_{i=1}^n t_i P_i \cdot X \right) \\ = \pi^{d/2} \Gamma \left( -\frac{d}{2} + \frac{1}{2} \sum_{i=1}^n \Delta_i \right) \int_0^{+\infty} \prod_{i=1}^n \left( \frac{dt_i}{t_i} t_i^{\Delta_i} \right) \exp \left( - \sum_{i < k} t_i t_k P_{ik} \right). \end{aligned} \quad (\text{B.1})$$

In this appendix we explain its application to tree-level contact diagrams, and give a proof. We also discuss its utility in establishing Mellin amplitudes of Witten diagrams.

#### **Application: Tree-level contact diagrams**

A straightforward application of the above integral is in evaluating tree-level Witten diagrams. Recall that the Schwinger parameterised form of a scalar boundary-to-bulk propagator of dimension  $\Delta$  is

$$K_{\Delta,0}(X, P) = \frac{C_{\Delta,0}}{(-2X \cdot P)^\Delta} = \frac{C_{\Delta,0}}{\Gamma(\Delta)} \int_0^\infty \frac{dt}{t} t^\Delta \exp(2tP \cdot X). \quad (\text{B.2})$$

Considering for example the tree-level  $n$ -point contact diagram

$$\begin{aligned} \mathcal{A}^{\text{cont}}(P_1, \dots, P_n) &= \int_{\text{AdS}} dX K_{\Delta_1,0}(X, P_1) \dots K_{\Delta_n,0}(X, P_n) \\ &= \left( \prod_{i=1}^n C_{\Delta_i,0} \right) \int_0^{+\infty} \prod_{i=1}^n \left( \frac{dt_i}{t_i} t_i^{\Delta_i} \right) \int_{\text{AdS}} dX \exp \left( 2 \sum_{i=1}^n t_i P_i \cdot X \right), \end{aligned} \quad (\text{B.3})$$

we see that the integral (B.1) naturally arises and can be employed to evaluate the amplitude. This is particularly straightforward for three-point diagrams, for which no integration is required. This will be shown in the subsequent, after proving the formula (B.1).

**Proof of (B.1):**

To prove (B.1), we first evaluate the AdS integral. Defining  $T = \sum_{i=1}^n t_i P_i$ , by Lorentz invariance we can make the simple choice  $T = |T|(1, 1, 0)$ , where we recall that we parameterise  $\text{AdS}_{d+1}$  in the  $(d+2)$ -dimensional ambient space by

$$X = (X^+, X^-, X^\mu) = \frac{1}{z} (1, z^2 + y^2, y^i). \quad (\text{B.4})$$

In this way we obtain

$$\int_{\text{AdS}} dX \exp \left( 2 \sum_{i=1}^n t_i P_i \cdot X \right) = \int_0^{+\infty} \frac{dz}{z} z^{-d} \int d^d x e^{-(1+z^2+y^2)|T|/z} \quad (\text{B.5})$$

$$= \pi^{d/2} \int_0^{+\infty} \frac{dz}{z} z^{-d/2} e^{-z+T^2/z}, \quad (\text{B.6})$$

where in the second line we evaluated the Gaussian integral over  $x$ . Returning to the RHS of the full expression (B.1), and rescaling  $t_i \rightarrow t_i/\sqrt{z}$ ,

$$\begin{aligned} \pi^{d/2} \int_0^{+\infty} \prod_{i=1}^n \left( \frac{dt_i}{t_i} t_i^{\Delta_i} \right) \int_0^{+\infty} \frac{dz}{z} z^{-d/2} e^{-z+T^2/z} \\ = \pi^{d/2} \int_0^{+\infty} \prod_{i=1}^n \left( \frac{dt_i}{t_i} t_i^{\Delta_i} \right) e^{T^2} \int_0^{+\infty} \frac{dz}{z} z^{-\frac{d}{2} + \frac{1}{2} \sum_{i=1}^n \Delta_i} e^{-z} \end{aligned} \quad (\text{B.7})$$



$$= \pi^{d/2} \Gamma \left( -\frac{d}{2} + \frac{1}{2} \sum_{i=1}^n \Delta_i \right) \int_0^{+\infty} \prod_{i=1}^n \left( \frac{dt_i}{t_i} t_i^{\Delta_i} \right) e^{T^2},$$

where in the second equality we simply employ the integral representation of the Gamma function. Noting that  $T^2 = -\sum_{i<k} P_{ik}$ , the above establishes (B.1).

### The Symanzik star formula

To establish the Mellin amplitude of Witten diagrams, it is illuminating to employ the Symanzik star formula [296]

$$\int_0^{+\infty} \prod_{i=1}^n \left( \frac{dt_i}{t_i} t_i^{\Delta_i} \right) \exp \left( -\sum_{i<k} t_i t_k P_{ik} \right) = \frac{\pi^{d/2}/2}{(2\pi i)^{\frac{1}{2}n(n-3)}} \int d\delta_{ij} \prod_{1 \leq i < j \leq n} \Gamma(\delta_{ij}) (P_{ij})^{-\delta_{ij}}, \quad (\text{B.8})$$

which allows to massage Witten diagrams in the form of a Mellin transform [146, 216]. The integration contour runs parallel to the imaginary axis with  $\text{Re } \delta_{ij} > 0$ , with the integration variables constrained by

$$\sum_{j \neq i}^n \delta_{ij} = \Delta_i, \quad (\text{B.9})$$

so that the integrand is conformally covariant with scaling dimension  $\Delta_i$  at the point  $P_i$ . This gives  $n(n-3)/2$  integration variables.

In particular, this implies that the integration variables are completely fixed for three-point integrals, allowing the straightforward evaluation of three-point Witten diagrams.



## Appendix C

# Harmonic functions in AdS

In this appendix we give an overview of the harmonic decomposition of tensorial functions of geodesic distance in  $\text{AdS}_{d+1}$ .

Consider bi-tensors  $t_{\mu_1 \dots \mu_r, \nu_1 \dots \nu_r}(x_1, x_2)$ , which depend only on the geodesic distance between  $x_1, x_2 \in \text{AdS}_{d+1}$ , and are square integrable

$$\int_{\text{AdS}_{d+1}} |t(x_1, x_2)|^2 < \infty. \quad (\text{C.1})$$

This includes bulk-to-bulk propagators  $\Pi_{\Delta, r}(x_1, x_2)$ .

Such tensors can be decomposed in a basis of regular eigenfunctions of the AdS Laplacian,

$$\left( \square_{\text{AdS}} + \left( \frac{d}{2} + i\nu \right) \left( \frac{d}{2} - i\nu \right) + \ell \right) \Omega_{\nu, \ell}(x_1, x_2) \equiv 0, \quad \nu \in \mathbb{R}, \quad \ell \in \mathbb{N} \quad (\text{C.2})$$

$$\nabla \cdot \Omega_{\nu, \ell} = 0, \quad (\text{C.3})$$

which are symmetric and traceless spin- $\ell$  irreducible representations of  $SO(d+1, 1)$ , with  $\nu$  labelling the energy.

Crucially, they satisfy orthogonality and (traceless) completeness relations [152]

$$\int_{\text{AdS}} d^{d+1}x' \Omega_{\bar{\nu},\ell}(x_1; x') \cdot \Omega_{\nu,\ell}(x'; x_2) = \frac{1}{2} [\delta(\nu + \bar{\nu}) + \delta(\nu - \bar{\nu})] \Omega_{\nu,\ell}(x_1; x_2), \quad (\text{C.4})$$

$$\sum_{\ell=0}^s \int_{-\infty}^{\infty} d\nu c_{r,\ell}(\nu) (w_1 \cdot \nabla_1)^\ell (w_2 \cdot \nabla_2)^\ell \Omega_{\nu,r-\ell}(x_1, w_1; x_2, w_2) = (w_1 \cdot w_2)^r \delta^{d+1}(x_1, x_2), \quad (\text{C.5})$$

where auxiliary variables  $w_i^2 = 0$  enforce tracelessness, and

$$c_{r,\ell}(\nu) = \frac{2^\ell (r - \ell + 1)_\ell \left(\frac{d}{2} + r - \ell - \frac{1}{2}\right)_\ell}{\ell! (d + 2r - 2\ell - 1)_\ell \left(\frac{d}{2} + r - \ell + i\nu\right)_\ell \left(\frac{d}{2} + r - \ell - i\nu\right)_\ell}. \quad (\text{C.6})$$

The set of tensors

$$\left\{ (w_1 \cdot \nabla_1)^\ell (w_2 \cdot \nabla_2)^\ell \Omega_{\nu,r-\ell}(x_1, w_1; x_2, w_2) \mid w_i^2 = 0, \nu \in \mathbb{R}, \ell = 0, 1, \dots, r \right\}, \quad (\text{C.7})$$

thus form a basis for symmetric traceless bi-tensors in AdS,

$$t(x_1, w_1; x_2, w_2) = \sum_{\ell=0}^r \int_{-\infty}^{\infty} f_{r,\ell}(\nu) (w_1 \cdot \nabla_1)^\ell (w_2 \cdot \nabla_2)^\ell \Omega_{\nu,r-\ell}(x_1, w_1; x_2, w_2), \quad (\text{C.8})$$

for some  $f_{r,\ell}(\nu) = f_{r,\ell}(-\nu)$ .

As corollary, any symmetric bi-tensor can be expressed in the basis (C.7) supplemented with products of the metric; any symmetric bi-tensor  $t(x_1, u_1; x_2, u_2)$  admits the trace decomposition of the schematic form,<sup>1</sup>

$$t(x_1, u_1; x_2, u_2) \sim t^{\{0\}}(x_1, u_1; x_2, u_2) + u_1^2 u_2^2 t^{\{1\}}(x_1, u_1; x_2, u_2) + \dots + (u_1^2 u_2^2)^{\lfloor \frac{r}{2} \rfloor} t^{\{\lfloor \frac{r}{2} \rfloor\}}(x_1, u_1; x_2, u_2), \quad (\text{C.9})$$

<sup>1</sup>I.e. omitting for concision the relative coefficients between the terms.

where here  $t^{\{n\}}$  denotes the traceless part of the  $n$ -th trace of  $t$ . Each of the  $t^{\{n\}}$  can then be decomposed in the basis (C.7).



## Appendix D

# Double trace operators

In this appendix we construct primary operators bi-linear in two scalar primary operators  $\mathcal{O}_1$  and  $\mathcal{O}_2$  of scaling dimensions  $\Delta_1$  and  $\Delta_2$ . These have the schematic form

$$[\mathcal{O}_1\mathcal{O}_2]_{n,s} = \mathcal{O}_1\partial_{\mu(s)}\square^n\mathcal{O}_2 + \dots, \quad (\text{D.1})$$

where the ellipsis denote terms which ensure that  $[\mathcal{O}_1\mathcal{O}_2]_{n,s}$  is primary. The double trace operator  $[\mathcal{O}_1\mathcal{O}_2]_{n,s}$  has spin  $s$ , and in a free theory (or otherwise at leading order in the  $1/N$  expansion) has scaling dimension  $\Delta_{n,s} = \Delta_1 + \Delta_2 + 2n + s$ .

Our goal is to specify implicit terms in (D.1). To this end, we make the most general ansatz<sup>1</sup>

$$[\mathcal{O}_1\mathcal{O}_2]_{n,s} = \sum_{s_1,b_1,b_2} a_{n,s}(s_1, s_2; b_1, b_2, b_{12})\partial_{\mu(s_1)}\square^{b_1}\partial^{\nu(b_{12})}\mathcal{O}_1\partial_{\mu(s_2)}\square^{b_2}\partial_{\nu(b_{12})}\mathcal{O}_2 - \text{traces}, \quad (\text{D.2})$$

and specify  $a_{n,s}$  from the requirement (3.20) that  $[\mathcal{O}_1\mathcal{O}_2]_{n,s}$  is primary. In other words, we solve for the coefficients by imposing the condition

$$K_\mu[\mathcal{O}_1\mathcal{O}_2]_{n,s} = 0. \quad (\text{D.3})$$

---

<sup>1</sup>There are three independent summations in the above sum, since  $s = s_1 + s_2$  and  $n = b_1 + b_2 + b_{12}$ .

### The action of the special conformal generator

For ease of computation, we contract the free indices with the auxiliary traceless symmetric tensors  $z^\mu$  to ensure that  $[\mathcal{O}_1\mathcal{O}_2]_{n,s}$  is traceless and symmetric. A typical term in the sum (D.2) then takes the form

$$T_{n,s}(s_1, s_2; b_1, b_2, b_{12}) = z^{\mu(s_1)\nu(s_2)}(\eta^{\mu\nu})^{b_1}(\eta^{\mu\nu})^{b_{12}}(\eta^{\nu\nu})^{b_2}P_{\mu(s_1+2b_1+b_{12})}\mathcal{O}_1P_{\nu(s_2+2b_2+b_{12})}\mathcal{O}_2, \quad (\text{D.4})$$

where we employ the notation  $u^{\mu(s)} = u^{\mu_1}\dots u^{\mu_s}$ .

Using the conformal algebra (3.6), one can show that

$$K_\nu P_{\mu(n)}\mathcal{O}_1 = 2n(\Delta_1 + n - 1)\eta_{\mu\nu}P_{\mu(n-1)}\mathcal{O}_1 - n(n - 1)\eta_{\mu\mu}P_\nu P_{\mu(n-3)}\mathcal{O}_1, \quad (\text{D.5})$$

which use to determine the action of  $K_\mu$  on each term in the ansatz (D.2). We find

$$\begin{aligned} K_\nu T_{n,s}(s_1, s_2; b_1, b_2, b_{12}) &= 2[\Delta_1 + s_1 + 2b_1 + b_{12} - 1]s_1 C_z(s_1 - 1, s_2; b_1, b_2, b_{12}) \\ &\quad + 2b_1[2\Delta_1 + 2b_1 - d]C_1(s_1 + 1, s_2; b_1 - 1, b_2, b_{12}) \\ &\quad - b_{12}(b_{12} - 1)C_1(s_1 + 1, s_2; b_1, b_2 + 1, b_{12} - 2) \\ &\quad - 2s_1 b_{12} C_1(s_1, s_2 + 1; b_1, b_2, b_{12} - 1) \\ &\quad + 2[\Delta_1 + s_1 + 2b_1 + b_{12} - 1]b_{12} C_2(s_1, s_2 + 1; b_1, b_2, b_{12} - 1) \\ &\quad + 2[\Delta_2 + s_2 + 2b_2 + b_{12} - 1]s_2 C_z(s_1, s_2 - 1; b_1, b_2, b_{12}) \\ &\quad + 2b_2[2\Delta_2 + 2b_2 - d]C_2(s_1, s_2 + 1; b_1, b_2 - 1, b_{12}) \\ &\quad - b_{12}(b_{12} - 1)C_2(s_1, s_2 + 1; b_1 + 1, b_2, b_{12} - 2) \\ &\quad - 2s_2 b_{12} C_2(s_1 + 1, s_2; b_1, b_2, b_{12} - 1) \\ &\quad + 2[\Delta_2 + s_2 + 2b_2 + b_{12} - 1]b_{12} C_1(s_1 + 1, s_2; b_1, b_2, b_{12} - 1), \end{aligned} \quad (\text{D.6})$$



where

$$\begin{aligned}
C_z(s_1 - 1, s_2; b_1, b_2, b_{12}) &= V^{\nu\mu(s_1-1)\sigma(s_2)} (\eta^{\mu\mu})^{b_1} (\eta^{\mu\sigma})^{b_2} (\eta^{\sigma\sigma})^{b_2} P_{\mu(s_1+2b_1+b_{12}-1)} \mathcal{O}_1 P_{\sigma(s_2+2b_2+b_{12})} \mathcal{O}_2, \\
C_1(s_1 + 1, s_2; b_1 - 1, b_2, b_{12}) &= z^{\mu(s_1)\sigma(s_2)} (\eta^{\mu\mu})^{b_1-1} (\eta^{\mu\sigma})^{b_{12}} (\eta^{\sigma\sigma})^{b_2} P_{\nu} P_{\mu(s_1+2b_1+b_{12}-2)} \mathcal{O}_1 P_{\sigma(s_2+2b_2+b_{12})} \mathcal{O}_2, \\
C_2(s_1, s_2 + 1; b_1, b_2, b_{12} - 1) &= z^{\mu(s_1)\sigma(s_2)} (\eta^{\mu\mu})^{b_1} (\eta^{\mu\sigma})^{b_{12}-1} (\eta^{\sigma\sigma})^{b_2} P_{\mu(s_1+2b_1+b_{12}-1)} \mathcal{O}_1 P_{\nu} P_{\sigma(s_2+2b_2+b_{12}-1)} \mathcal{O}_2.
\end{aligned} \tag{D.7}$$

### Imposing the primary condition

Having understood how the special conformal generator  $K_\nu$  acts on each term of the ansatz (D.2), we act with  $K_\nu$  on the entire expression. Using (D.6), the condition

$$K_\nu[\mathcal{O}_1\mathcal{O}_2]_{n,s} = 0, \tag{D.8}$$

can be expressed in terms of the three independent structures  $C_z$ ,  $C_1$  and  $C_2$  defined in (D.7). By setting the prefactors of each of these structures to zero we obtain three conditions on  $a_{n,s}$ .

The equation corresponding to  $C_z$  reads

$$\begin{aligned}
a_{n,s}(s_1, s_2; b_1, b_2, b_{12}) 2[\Delta_1 + s_1 + 2b_1 + b_{12} - 1] s_1 \\
+ a_{n,s}(s_1 - 1, s_2 + 1; b_1, b_2, b_{12}) 2[\Delta_2 + s_2 + 2b_2 + b_{12}] (s_2 + 1) = 0.
\end{aligned} \tag{D.9}$$

It can be used to fix the dependence of  $a_{n,s}$  on  $s_1$  for fixed  $b_1$  and  $b_2$

$$a_{n,s}(s_1, s_2; b_1, b_2, b_{12}) = (-1)^{s_2} \frac{s!}{s_2! s_1!} \frac{(\Delta_1 + 2b_1 + b_{12} + s_1)_{s_2}}{(\Delta_2 + 2b_2 + b_{12})_{s_2}} a_{n,s}(s, 0; b_1, b_2, b_{12}). \tag{D.10}$$

A second equation, which sets the prefactor of  $C_1$  to zero is

$$\begin{aligned}
& a_{n,s}(s_1, s_2; b_1, b_2, b_{12})2b_1(2\Delta_1 + 2b_1 - d) \\
& - a_{n,s}(s_1, s_2; b_1 - 1, b_2 - 1, b_{12} + 2)(b_{12} + 2)(b_{12} + 1) \\
& - a_{n,s}(s_1 + 1, s_2 - 1; b_1 - 1, b_2, b_{12} + 1)2(s_1 + 1)(b_{12} + 1) \\
& + a_{n,s}(s_1, s_2; b_1 - 1, b_2, b_{12} + 1)2(\Delta_2 + s_2 + 2b_2 + b_{12})(b_{12} + 1) = 0. \quad (\text{D.11})
\end{aligned}$$

The third term contains  $a_{n,s}(s_1 + 1, s_2 - 1; b_1 - 1, b_2, b_{12} + 1)$ , which can be expressed in terms of  $a_{n,s}(s_1, s_2; b_1 - 1, b_2, b_{12} + 1)$  by (D.9). This results in

$$\begin{aligned}
& a_{n,s}(s_1, s_2; b_1, b_2, b_{12})2b_1(2\Delta_1 + 2b_1 - d) \\
& - a_{n,s}(s_1, s_2; b_1 - 1, b_2 - 1, b_{12} + 2)(b_{12} + 2)(b_{12} + 1) \\
& + a_{n,s}(s_1, s_2; b_1 - 1, b_2, b_{12} + 1)2(b_{12} + 1)(\Delta_2 + s_2 + 2b_2 + b_{12}) \\
& \quad \times \frac{\Delta_1 + s + 2b_1 + b_{12} - 1}{\Delta_1 + s_1 + 2b_1 + b_{12} - 1} = 0, \quad (\text{D.12})
\end{aligned}$$

which relates  $a_{n,s}$ 's for the same values of arguments  $s_1$  and  $s_2$  and different values of  $b_1$ ,  $b_2$  and  $b_{12}$ .

In a similar way, the equation corresponding to  $C_2$  reads

$$\begin{aligned}
& a_{n,s}(s_1, s_2; b_1, b_2, b_{12})2b_2(2\Delta_2 + 2b_2 - d) \\
& - a_{n,s}(s_1, s_2; b_1 - 1, b_2 - 1, b_{12} + 2)(b_{12} + 2)(b_{12} + 1) \\
& + a_{n,s}(s_1, s_2; b_1, b_2 - 1, b_{12} + 1)2(b_{12} + 1)(\Delta_1 + s_1 + 2b_1 + b_{12}) \\
& \quad \times \frac{\Delta_2 + s + 2b_2 + b_{12} - 1}{\Delta_2 + s_2 + 2b_2 + b_{12} - 1} = 0. \quad (\text{D.13})
\end{aligned}$$

Together with (D.12), this defines  $b$ -dependence of  $a_{n,s}$  for fixed  $s_1$  and  $s_2$ .

### Dependence on contracted derivatives

In this section we solve the recurrence equations (D.12), (D.13) subject to the boundary condition

$$a_{n,s}(s_1, s_2; b_1, b_2, b_{12}) = 0, \quad \text{for } b_1 < 0 \text{ or } b_2 < 0 \text{ or } b_1 + b_2 > n. \quad (\text{D.14})$$

More precisely, we express the unknown coefficients in terms of  $a_{n,s}(s_1, s_2; 0, 0, n)$ . For brevity, in the following we will keep only the arguments  $b_1$  and  $b_2$  of  $a_{n,s}$  explicit.

First, we consider (D.12) for  $b_2 = 0$ . In this case the recursion relation simplifies, since the boundary condition causes the second term to drop out. We can then express  $a(b_1, 0)$  with arbitrary  $b_1$  in terms of  $a(0, 0)$

$$a(b_1, 0) = (-1)^{b_1} \frac{n!}{b_1! b_2!} \frac{(\Delta_2 + s_2 + n - b_1)_{b_1} (\Delta_1 + s + n)_{b_1}}{2^{b_1} (\Delta_1 + 1 - h)_{b_1} (\Delta_1 + s_1 + n)_{b_1}} a(0, 0). \quad (\text{D.15})$$

To establish the  $b_2$ -dependence, we use (D.13): First, for any  $b_1$  we express  $a(b_1, 1)$  in terms of  $a(b_1, 0)$  and  $a(b_1 - 1, 0)$ , which are known. We then solve for  $a(b_1, 2)$  in terms of  $a(b_1, 1)$  and  $a(b_1 - 1, 1)$ , which were determined in the previous step. This process can be continued to express  $a(b_1, b_2)$  for any  $b_2$  in terms of  $a(0, 0)$ . We find

$$\begin{aligned} a(b_1, b_2) &= \left(-\frac{1}{2}\right)^{b_1+b_2} \frac{n!}{b_1! b_2! b_{12}!} \\ &\times \frac{(\Delta_1 + s + n)_{b_1} (\Delta_2 + s + n - b_1)_{b_2}}{(\Delta_1 + 1 - h)_{b_1} (\Delta_2 + 1 - h)_{b_2} (\Delta_1 + s_1 + n)_{b_1 - b_2} (\Delta_2 + s_2 + n)_{b_2 - b_1}} \\ &\times \sum_{k=0}^{b_2} \frac{b_2!}{k! (b_2 - k)!} \frac{(b_1 - k + 1)_k (\Delta_1 + b_1 - h - k + 1)_k}{(\Delta_2 + s + n - b_1)_k (\Delta_1 + s + n + b_1 - k)_k} a(0, 0). \end{aligned} \quad (\text{D.16})$$

It can be checked that (D.16) satisfies (D.12) and (D.13).

Combining the  $s$ - and  $b$ -dependences, we obtain the final result

$$\begin{aligned}
a_{n,s}(s_1, s_2; b_1, b_2, b_{12}) &= \frac{(-1)^{s_2+b_1+b_2}}{2^{b_1+b_2}} \frac{s!}{s_1!s_2!} \frac{(\Delta_1 + s_1 + 2b_1 + b_{12})_{s_2}}{(\Delta_2 + 2b_2 + b_{12})_{s_2}} \\
&\times \frac{n!}{b_1!b_2!b_{12}!} \frac{(\Delta_1 + s + n)_{b_1} (\Delta_2 + s + n - b_1)_{b_2}}{(\Delta_1 + 1 - h)_{b_1} (\Delta_2 + 1 - h)_{b_2} (\Delta_1 + s_1 + n)_{b_1 - b_2} (\Delta_2 + s_2 + n)_{b_2 - b_1}} \\
&\times \sum_{k=0}^{b_2} \frac{b_2!}{k!(b_2 - k)!} \frac{(b_1 - k + 1)_k (\Delta_1 + b_1 - h - k + 1)_k}{(\Delta_2 + s + n - b_1)_k (\Delta_1 + s + n + b_1 - k)_k} a_{n,s}(s, 0; 0, 0, n),
\end{aligned} \tag{D.17}$$

where  $a_{n,s}(s, 0; 0, 0, n)$  is an arbitrary factor.

## Appendix E

# Trace structure of spinning propagators

With the freedom to remove gradients within the propagators (5.48) and (5.59) already derived, it should be possible to bring them into the form

$$\Pi_s = \sum_{k=0}^{[s/2]} \int_{-\infty}^{\infty} d\nu g_{s,k}(\nu) (u_1^2)^k (u_2^2)^k \Omega_{\nu,s-2k}, \quad (\text{E.1})$$

by making gauge transformations. This is the goal of this section. We refer to this as the “manifest trace gauge”, because here the propagator is presented as a sum of terms, each being essentially a product of certain number of background metrics and a harmonic function  $\Omega$ , which is traceless and transverse.

To reach the form (E.1), naively one might expect that, for example in (5.59), one can gauge away all the terms except the one for which  $\ell = 0$ . However, closer inspection reveals that this is not the case: According to our conventions, contractions with  $W$  implicitly make a projection onto the traceless part, so a generic term in (5.59) is of the form

$$\left\{ (u_1 \cdot \nabla_1)^\ell (u_2 \cdot \nabla_2)^\ell \Omega_{\nu,s-\ell}(u_1, x_1, u_2, x_2) \right\}. \quad (\text{E.2})$$

For a general rank- $s$  tensor  $T_s$ , the explicit action of the operator that implements the traceless projection  $\{\bullet\}$ , is

$$\{T_s(x, u)\} = \sum_{j=0}^{\lfloor s/2 \rfloor} \frac{(-1)^j}{4^j j! \binom{d}{2} + s - 3/2}_j (u^2)^j (\partial_u \cdot \partial_u)^j T_s(x, u), \quad (\text{E.3})$$

which makes various contractions of its argument  $T_s$ . It is then clear that for non-zero  $\ell$  there are terms in (E.2) which cannot be gauged away. For example, terms in which all  $\nabla$ 's are contracted because then no gradients will be present. In fact, the story is even more complicated because for other terms in the projection the commutators of derivatives that appear yield extra lower derivative terms, some of which are also not pure gauge. To summarise, eliminating pure gradient terms is non-trivial, and our aim in the following is to compute (E.2) explicitly modulo such gradient terms. We do this by studying the details of the contractions under (E.3).

It is straightforward to see that when  $\ell$  is odd, (E.2) can be gauged away since it is a gradient. Let us then consider examples for when  $\ell$  is even:

- $\ell = 2$

Using (E.3), the explicit form of the traceless projection for  $\ell = 2$  is

$$\begin{aligned} & \{(u_1 \cdot \nabla_1)^2 (u_2 \cdot \nabla_2)^2 \Omega_{\nu, s-2}(u_1, u_2)\} \\ &= \left(1 - \frac{u_1^2 (\partial_{u_1} \cdot \partial_{u_1})}{2(d+2s-3)}\right) \left(1 - \frac{u_2^2 (\partial_{u_2} \cdot \partial_{u_2})}{2(d+2s-3)}\right) (u_1 \cdot \nabla_1)^2 (u_2 \cdot \nabla_2)^2 \Omega_{\nu, s-2}(u_1, u_2). \end{aligned} \quad (\text{E.4})$$

Dropping gradient terms, it is straightforward to compute that

$$(\partial_{u_1} \cdot \partial_{u_1})(u_1 \cdot \nabla_1)^2 \Omega_{\nu, n}(u_1, u_2) \sim 2(\square_1 - n(d+n-1)) \Omega_{\nu, n}(u_1, u_2), \quad (\text{E.5})$$

where we used the fact that  $\Omega$  traceless and divergence-less.<sup>1</sup> Therefore the only

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<sup>1</sup>The dropping of gradient terms is denoted by “ $\sim$ ”. Note that in appendix F.2 we also use this notation to instead indicate that equalities hold modulo gradients *and* traces.

terms that are not pure gradient in (E.4) come from the product of the second terms in each of the brackets:

$$\begin{aligned} & \frac{u_1^2(\partial_{u_1} \cdot \partial_{u_1})}{2(d+2s-3)} (u_1 \cdot \nabla_1)^2 \frac{u_2^2(\partial_{u_2} \cdot \partial_{u_2})}{2(d+2s-3)} (u_2 \cdot \nabla_2)^2 \Omega_{\nu, s-2}(u_1, u_2) \\ & \sim u_1^2 u_2^2 \left( \frac{\square_1 - (s-2)(d+s-3)}{d+2s-3} \right)^2 \Omega_{\nu, s-2}(u_1, u_2), \end{aligned} \quad (\text{E.6})$$

Finally, employing the equation of motion (C.2) for  $\Omega$  we find

$$\left\{ (u_1 \cdot \nabla_1)^2 (u_2 \cdot \nabla_2)^2 \Omega_{\nu, s-2}(u_1, u_2) \right\} \sim u_1^2 u_2^2 \left( \frac{\nu^2 + \left(\frac{d}{2} + s - 2\right)^2}{d+2s-3} \right)^2 \Omega_{\nu, s-2}(u_1, u_2). \quad (\text{E.7})$$

- $\underline{\ell = 4}$  Explicitly, traceless projection in this case is

$$\begin{aligned} & \left\{ (u_1 \cdot \nabla_1)^4 (u_2 \cdot \nabla_2)^4 \Omega_{\nu, s-4}(u_1, u_2) \right\} \\ & = \left( 1 - \frac{u_1^2(\partial_{u_1} \cdot \partial_{u_1})}{2(d+2s-3)} + \frac{u_1^4(\partial_{u_1} \cdot \partial_{u_1})^2}{8(d+2s-3)(d+2s-5)} \right) (1 \leftrightarrow 2) \\ & \quad \times (u_1 \cdot \nabla_1)^4 (u_2 \cdot \nabla_2)^4 \Omega_{\nu, s-4}(u_1, u_2). \end{aligned} \quad (\text{E.8})$$

In order to evaluate the traces in the above, a tedious but straightforward computation shows that

$$\begin{aligned} & (\partial_{u_1} \cdot \partial_{u_1})(u_1 \cdot \nabla_1)^4 \Omega_{\nu, n}(u_1, u_2) = 2 \left( 6(u_1 \cdot \nabla_1)^2 \square_1 \right. \\ & \quad \left. - 2(3n^2 + 3dn + 4d + 5n + 2)(u_1 \cdot \nabla_1)^2 + 4u_1^2 \square_1 - 4n(d+n-1)u_1^2 \right) \Omega_{\nu, n}(u_1, u_2), \end{aligned} \quad (\text{E.9})$$

and

$$(\partial_{u_1} \cdot \partial_{u_1})^2 (u_1 \cdot \nabla_1)^4 \Omega_{\nu, n}(u_1, u_2) \sim 4 \left( 6\square_1^2 - 4(d+3dn-n-3n^2)\square_1 \right)$$

$$+ 2n(d+n-1)(2d+n+3dn+n^2)\big)\Omega_{\nu,n}(u_1, u_2). \quad (\text{E.10})$$

Modulo gradient terms, each bracket in (E.8) therefore produces a second order polynomial in  $\square$ . Using the equations of motion (C.2), this eventually gives

$$\begin{aligned} & \{(u_1 \cdot \nabla_1)^4 (u_2 \cdot \nabla_2)^4 \Omega_{\nu,s-4}(u_1, u_2)\} \\ & \sim (u_1^2)^2 (u_2^2)^2 3^2 \left( \frac{\nu^2 + (\frac{d}{2} + s - 2)^2}{d + 2s - 3} \right)^2 \left( \frac{\nu^2 + (\frac{d}{2} + s - 4)^2}{d + 2s - 5} \right)^2 \Omega_{\nu,s-4}(u_1, u_2). \end{aligned} \quad (\text{E.11})$$

- $\ell = 2k$

After studying explicitly cases the  $\ell = 2k$  with  $k = 1, 2$  above, we conjecture that

$$\begin{aligned} & \{(u_1 \cdot \nabla_1)^{2k} (u_2 \cdot \nabla_2)^{2k} \Omega_{\nu,s-2k}(u_1, x_1, u_2, x_2)\} \quad (\text{E.12}) \\ & = (N(k, s))^2 (u_1^2)^k (u_2^2)^k \left( 4^k \left( \frac{\frac{d}{2} + s - 2k + i\nu}{2} \right)_k \left( \frac{\frac{d}{2} + s - 2k - i\nu}{2} \right)_k \right)^2 \\ & \quad + (u_1 \cdot \nabla_1)(\dots) + (u_2 \cdot \nabla_2)(\dots), \end{aligned}$$

where

$$N(k, s) = \frac{(2k)!}{4^k k! (\frac{d}{2} + s - k - 1/2)_k} \quad (\text{E.13})$$

and  $(a)_r = \Gamma(a+r)/\Gamma(a)$  is the rising Pochhammer symbol.

Let us note that in spite of the fact that equation (E.12) is a conjecture, the pre-factor  $N(k, s)$  is determined exactly. We explain how in the following. The combinatorial factor  $N(k, s)$  can be derived by studying only the contractions that produce the maximal power of  $\square$ . The issue of non-commutativity of covariant derivatives for this computation is irrelevant. Indeed, let us consider an analogous computation in the flat space.



Then in (E.5) one has only a  $\square$ -term so that instead of (E.6) we find

$$\left\{ (u_1 \cdot \nabla_1)^2 (u_2 \cdot \nabla_2)^2 \Omega_{\nu, s-2}(u_1, u_2) \right\} \Big|_{flat} \sim u_1^2 u_2^2 \left( \frac{\square}{d+2s-3} \right)^2 \Omega_{\nu, s-2}(u_1, u_2). \quad (\text{E.14})$$

Analogously, for the  $\ell = 4$  flat case one finds

$$\begin{aligned} \left\{ (u_1 \cdot \nabla_1)^4 (u_2 \cdot \nabla_2)^4 \Omega_{\nu, s-4}(u_1, u_2) \right\} \Big|_{flat} \\ \sim (u_1^2)^2 (u_2^2)^2 3^2 \left( \frac{\square^2}{(d+2s-3)(d+2s-5)} \right)^2 \Omega_{\nu, s-4}(u_1, u_2). \end{aligned} \quad (\text{E.15})$$

This computation can be easily generalised to the case of any  $\ell = 2k$ , and the result is

$$\left\{ (u_1 \cdot \nabla_1)^{2k} (u_2 \cdot \nabla_2)^{2k} \Omega_{\nu, s-2k}(u_1, u_2) \right\} \Big|_{flat} \sim (u_1^2)^k (u_2^2)^k N(k, s) \square^{2k} \Omega_{\nu, s-2k}(u_1, u_2), \quad (\text{E.16})$$

with  $N(k, s)$  given in (E.13). In fact, this justifies its explicit form.

On the other hand, in AdS the  $\square^{2k}$ -term will receive lower derivative corrections, which originate from the non-commutativity of the covariant derivatives. By generalising the  $k = 1, 2$  cases, what we conjecture is that these lower derivative terms are such that after evaluating  $\square$  on  $\Omega_{\nu, s-2k}$  one finds

$$\left( (\nu^2 + (\frac{d}{2} + s - 2)^2)(\nu^2 + (\frac{d}{2} + s - 4)^2) \dots (\nu^2 + (\frac{d}{2} + s - 2k)^2) \right)^2.$$

Combining this with the pre-factor found previously, one obtains (E.12).



## Appendix F

# Higher-spin conserved currents in AdS

### F.1 Single trace of the currents

In this appendix, we show how a single trace of the spin- $s$  current

$$J_s(x, u) = \sum_{k=0}^s \frac{(-1)^k}{s! (s-k)!} (u \cdot \nabla)^k \varphi_0(x) (u \cdot \nabla)^{s-k} \varphi_0(x) + \Lambda u^2(\dots), \quad (\text{F.1})$$

can be expressed in terms of currents of the same form, but with lower spin. For simplicity we work in ambient space, with the necessary ingredients reviewed below.

#### Scalar fields in AdS

Since the currents (F.1) are bi-linear in the bulk scalar, we first recall the representation of scalar fields in AdS and their ambient formulation. The lowest weight unitary irreducible scalar representations of  $so(d, 2)$

$$\square \varphi_0(x) - m^2 \varphi_0(x) = 0, \quad m^2 \equiv \Delta(\Delta - d) \quad (\text{F.2})$$

can be realised as the evaluation  $\varphi_0(x)$  on  $\text{AdS}_{d+1}$  of ambient homogeneous harmonic functions  $\Phi_0(X)$

$$(\partial_X \cdot \partial_X)\Phi_0(X) = 0, \quad \Phi_0(X) = (X^2)^{-\frac{\Delta}{2}} \varphi_0(x). \quad (\text{F.3})$$

For later use, let us denote

$$\Phi_0^\dagger(X) = (X^2)^{-\frac{\Delta_-}{2}} \varphi_0(x), \quad (\text{F.4})$$

where  $\Delta_- = d - \Delta$ , and we assume  $\Delta \geq \Delta_-$ . Throughout,  $\Delta$  will often be referred to as  $\Delta_+$ .

### The currents

Towards an ambient representation of the currents (F.1) on AdS, it is instructive to consider analogous conserved currents in the flat ambient space. I.e. bi-linear currents conserved with respect to the flat ambient derivative.

It is straightforward to verify that for any ambient massive scalar fields  $\Phi_1(X)$  and  $\Phi_2(X)$  of the same mass  $M$

$$(\partial_X \cdot \partial_X - M^2)\Phi_1(X) = 0 = (\partial_X \cdot \partial_X - M^2)\Phi_2(X), \quad (\text{F.5})$$

the currents, given by the generating function [104, 297]

$$I(X, U) = \Phi_1(X + U)\Phi_2(X - U), \quad (\text{F.6})$$

are conserved with respect to flat ambient space derivative

$$(\partial_X \cdot \partial_U)I(X, U) = 0. \quad (\text{F.7})$$

Explicitly, the rank- $s$  current generating function is given by

$$I_s(X, U) = \sum_{k=0}^s \frac{(-1)^k}{k!(s-k)!} (U \cdot \partial_X)^{s-k} \Phi_1(X) (U \cdot \partial_X)^k \Phi_2(X), \quad (\text{F.8})$$

which is obtained by extracting from (F.6) the  $\mathcal{O}(U^s)$  coefficient. I.e.

$$I(X, U) = \sum_s \frac{1}{s!} I_{A_1 \dots A_s} U^{A_1} \dots U^{A_s}, \quad I_s(X, U) = \frac{1}{s!} I_{A_1 \dots A_s} U^{A_1} \dots U^{A_s}. \quad (\text{F.9})$$

However, a conserved current in the flat ambient space does not necessarily define a conserved current on AdS. In other words, the pull-back of (F.6) onto the AdS manifold is not in general conserved with respect to the covariant derivative on AdS. To do so, it must satisfy some additional constraints, which we specify in the following.

We require (F.6) to be conserved with respect to the ambient representative (A.20) of the covariant derivative,  $\nabla = \mathcal{P} \circ \partial \circ \mathcal{P}$ . First, the projection of the ambient current onto AdS is of the form

$$\mathcal{P} I(X, U) = I(X, U) + (X \cdot U) L(X, U) \quad (\text{F.10})$$

for some  $L(X, U)$ . Further, the commutation relation

$$[(\partial_X \cdot \partial_U), X \cdot U] = X \cdot \partial_X + U \cdot \partial_U + d + 2$$

implies that

$$(\partial_X \cdot \partial_U) \mathcal{P} I(X, U) = (X \cdot U) (\partial_X \cdot \partial_U) L(X, U) + (X \cdot \partial_X + U \cdot \partial_U + d + 2) L(X, U). \quad (\text{F.11})$$

The first term in (F.11) is transversal to  $H_{d+1}$ , so it drops out upon application of the second projection in the covariant derivative (A.20). We then demand that  $I(X, U)$  is

conserved in AdS,

$$(\nabla \cdot \partial_U) I(X, U) = 0, \quad (\text{F.12})$$

which yields the condition:

$$(X \cdot \partial_X + U \cdot \partial_U + d + 2) L(X, U) = 0.$$

This can be satisfied by imposing the following homogeneity condition on the current

$$(X \cdot \partial_X + U \cdot \partial_U + d) I(X, U) = 0. \quad (\text{F.13})$$

The demonstrate that the current (F.6) is conserved in AdS if it obeys (F.13). In particular for  $\Phi_1 = \Phi$ , and  $\Phi_2 = \Phi^\dagger$  as introduced in (F.4), the current

$$J(X, U) = \Phi(X + U)\Phi^\dagger(X - U) \quad (\text{F.14})$$

is conserved on AdS. Note that the more familiar form (F.1), intrinsic to AdS, can be obtained from the above by expressing the ambient partial derivatives in terms of AdS covariant derivatives and the metric.

### Trace of the currents

We are now ready to proceed in ambient space to express the trace of the current (F.1) in terms of those of lower ranks. The formula we want to establish generalises to AdS the expression

$$(\partial_u \cdot \partial_u) I(x, u) = (-\square + 4M^2) I(x, u), \quad (\text{F.15})$$

for the trace in flat space.

On AdS, the trace is given by

$$(\partial_u \cdot \partial_u) J(X, U) \equiv (\partial_U \cdot \partial_U + (X \cdot \partial_U)^2) J(X, U)$$

$$= (-\partial_X \cdot \partial_X + (X \cdot \partial_U)^2) J(X, U), \quad (\text{F.16})$$

where on the first line we expressed the intrinsic AdS trace in terms of ambient quantities, and in the second equality we used that

$$(\partial_X \cdot \partial_X + \partial_U \cdot \partial_U)J(X, U) = 0, \quad (\text{F.17})$$

taking into account that the ambient scalar  $\Phi_0$  is massless (F.3).

The next step is to rewrite the RHS of (F.16),

$$(\partial_u \cdot \partial_u)J(X, U) = (-\partial_X \cdot \partial_X + (X \cdot \partial_U)^2) J(X, U),$$

in terms of the ambient representatives of AdS covariant operators: The Laplacian  $\square$ , rank of the current  $(U \cdot \partial_U)$ , multiplication by the metric  $U^2$  and the covariant gradient  $(U \cdot \nabla)$ :

1. We first focus on the term  $(X \cdot \partial_U)^2 J(X, U)$ . We may eliminate  $(X \cdot \partial_U)$  in favour of  $(U \cdot \partial_X)$ , using that

$$(X \cdot \partial_U) + (U \cdot \partial_X) = X^+ \partial_+ - X^- \partial_-.$$

This gives<sup>1</sup>

$$(X \cdot \partial_U)J = (\Delta_+ - \Delta_- - (U \cdot \partial_X))J. \quad (\text{F.18})$$

Applying this twice, we find

$$\begin{aligned} (X \cdot \partial_U)^2 J &= (X \cdot \partial_U)(\Delta_+ - \Delta_- - (U \cdot \partial_X))J \\ &= (\Delta_+ - \Delta_- - (U \cdot \partial_X))(X \cdot \partial_U)J - [(X \cdot \partial_U), (U \cdot \partial_X)]J \\ &= (\Delta_+ - \Delta_- - (U \cdot \partial_X))^2 J - (X \cdot \partial_X - U \cdot \partial_U)J. \end{aligned}$$

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<sup>1</sup>Let us note that the identity above can be used only when  $(X \cdot \partial_U)$  acts directly on the current.

The  $(X \cdot \partial_X)$  can be expressed in terms of  $(U \cdot \partial_U)$  via (F.13),

$$(X \cdot \partial_X)J = -((U \cdot \partial_U) + d)J. \quad (\text{F.19})$$

As a final step, we express  $(U \cdot \partial_X)$  in terms of covariant gradients  $(U \cdot \nabla)$ , using

$$(U \cdot \nabla) = U^2(X \cdot \partial_U) + (U \cdot \partial_X).$$

Together with (F.18), this yields

$$(U \cdot \partial_X)J = \frac{1}{1-U^2} ((U \cdot \nabla) - U^2(\Delta_+ - \Delta_-)) J, \quad (\text{F.20})$$

where the fraction should be understood as a power series

$$\frac{1}{1-U^2} = 1 + U^2 + U^4 + \dots$$

We finally arrive to,

$$(U \cdot \partial_X)^2 J = \left( \frac{1}{1-U^2} ((U \cdot \nabla) - U^2(\Delta_+ - \Delta_-)) \right)^2 J - \frac{U^2}{1-U^2} (X \partial_X - U \partial_U) J. \quad (\text{F.21})$$

2. To express  $(\partial_X \cdot \partial_X) J(X, U)$  in terms of AdS covariant quantities, we need only employ

$$\square = \partial_X \cdot \partial_X + (X \cdot \partial_X) ((X \cdot \partial_X) + d - 1) - (U \cdot \partial_U) + 2(U \cdot \partial_X)(X \cdot \partial_U) + U^2(X \cdot \partial_U)^2, \quad (\text{F.22})$$

in combination with the above results.

The generalisation of (F.15) to AdS is thus

$$(\partial_u \cdot \partial_u) J = \left( -\square + (u \cdot \partial_u + 1)(u \cdot \partial_u + d) \right)$$



$$+ \frac{1}{1-u^2} \left( (\Delta_+ - \Delta_-)^2 - (u \cdot \nabla)^2 \right) J. \quad (\text{F.23})$$

Recalling that

$$J(X, U) = \sum_{s=0}^{\infty} J_s(X, U), \quad (\text{F.24})$$

i.e. it is the generating function of conserved currents, equation (F.23) serves as a starting point to express the trace of a current of given spin in terms of other currents.

## F.2 Multiple traces of the currents

In this appendix we compute  $\tau_{s,k}(\nu)$  in (5.96). To this end, we take repeated traces of (F.23). This allows us to express a  $k$ -fold trace of rank- $s$  current  $J_s$  in terms of lower degree traces of lower-spin currents

$$(\partial_u \cdot \partial_u)^k J_s \quad \rightarrow \quad (\partial_u \cdot \partial_u)^{k-1} J_{s-2}, \quad (\partial_u \cdot \partial_u)^{k-2} J_{s-4}, \quad \dots, \quad J_{s-2k} \quad (\text{F.25})$$

and terms of the form

$$(u \cdot \nabla)(\dots) \quad \text{or} \quad u^2 \cdot (\dots). \quad (\text{F.26})$$

The relation (F.25) can then be iteratively applied to eliminate all the traces, thereby expressing  $(\partial_u \cdot \partial_u)^k J_s$  in terms of  $J_{s-2k}$  and terms of the form (F.26). Our main goal is to compute the four-point spin- $s$  exchange §5.4.2, so it is enough to know  $(\partial_u \cdot \partial_u)^k J_s$  modulo terms which vanish upon contraction against the traceless and divergence-free  $\Omega_{s-2k}$ . These are precisely the terms in (F.26). We will often drop such terms where they are unimportant. Equalities that hold modulo these terms will be denoted by “ $\sim$ ”.

We shall employ the following useful identities

$$(\partial_u \cdot \partial_u)^i (u^2)^k J_l = \sum_{n=0}^i 4^{i-n} \frac{i!}{n!(i-n)!} (k-i+n+1)_{i-n} \\ \times \left(\frac{d+1}{2} + k + l - i\right)_{i-n} (u^2)^{k-i+n} (\partial_u \cdot \partial_u)^n J_l, \quad (\text{F.27})$$

and

$$(\partial_u \cdot \partial_u)^m (u \cdot \nabla)^2 J_k = 2m \cdot \frac{k-2m+2}{k-2m+3} (\partial_u \cdot \nabla)(u \cdot \nabla)(\partial_u \cdot \partial_u)^{m-1} J_k \\ + 2m \square (\partial_u \cdot \partial_u)^{m-1} J_k + (u \cdot \nabla)^2 (\partial_u \cdot \partial_u)^m J_k. \quad (\text{F.28})$$

Here and throughout we use that the current is conserved. To commute divergence and gradient in (F.28), we employ

$$\frac{1}{n+1} (\partial_u \cdot \nabla)(u \cdot \nabla) J_n = \frac{1}{n} (u \cdot \nabla)(\partial_u \cdot \nabla) J_n - (n+d) J_n + \frac{1}{n} u^2 (\partial_u \cdot \partial_u) J_n, \quad (\text{F.29})$$

which entails

$$(\partial_u \cdot \partial_u)^m (u \cdot \nabla)^2 J_k \sim 2m (\square - (k-2m-2)(k-2m+d+1)) (\partial_u \cdot \partial_u)^{m-1} J_k. \quad (\text{F.30})$$

With these formulas at hand, we are prepared to compute the  $i$ -fold trace  $(\partial_u \cdot \partial_u)^i$  of both sides of (F.23). Using (F.27), keeping only terms with  $k-i+n=0$ , we obtain

$$(\partial_u \cdot \partial_u)^{i+1} J_{s+2} \sim -\square (\partial_u \cdot \partial_u)^i J_s + (s+1)(s+d)(\partial_u \cdot \partial_u)^i J_s \\ + \sum_{k=0}^i \frac{i!}{(i-k)!} 4^k ((d+1)/2 + s - i - k)_k (\partial_u \cdot \partial_u)^{i-k} ((\Delta_+ - \Delta_-)^2 J_{s-2k} - (u \cdot \nabla)^2 J_{s-2k-2}).$$

Which, employing (F.30), becomes

$$(\partial_u \cdot \partial_u)^{i+1} J_{s+2} \sim -\square (\partial_u \cdot \partial_u)^i J_s + (s+1)(s+d)(\partial_u \cdot \partial_u)^i J_s$$

$$\begin{aligned}
& + \sum_{k=0}^i (\partial_u \cdot \partial_u)^{i-k} (\Delta_+ - \Delta_-)^2 J_{s-2k} \frac{i!}{(i-k)!} 4^k \left(\frac{d+1}{2} + s - i - k\right)_k \\
& - \sum_{k=0}^{i-1} (\square - (s-2i)(s-2i+d-1)) (\partial_u \cdot \partial_u)^{i-k-1} J_{s-2k-2} \\
& \quad \times \frac{i!}{(i-k-1)!} 4^{k+1} \left(\frac{d+1}{2} + s - i - k\right)_{k+1}. \tag{F.31}
\end{aligned}$$

The crucial observation is that applying (F.31) to the right combination of  $(\partial_u \cdot \partial_u)^{i+1} J_{s+2}$  and  $(\partial_u \cdot \partial_u)^i J_s$ , the tails of terms on the right hand side cancel. To wit,

$$\begin{aligned}
& (\partial_u \cdot \partial_u)^{i+1} J_{s+2} - 2i(d+2s-2i-1)(\partial_u \cdot \partial_u)^i J_s \tag{F.32} \\
& \sim ((\Delta_+ - \Delta_-)^2 - \square + (s+1)(s+d)) (\partial_u \cdot \partial_u)^i J_s \\
& \quad - 2i(\square - (s-2i)(s-2i+d-1)) (\partial_u \cdot \partial_u)^{i-1} J_{s-2} \\
& \quad - 2i(d+2s-2i-1)(\square - (s-1)(s+d-2)) (\partial_u \cdot \partial_u)^{i-1} J_{s-2}.
\end{aligned}$$

To deal with the  $\square$  terms, from this point onwards we assume that both sides of (F.32) are integrated against the traceless and divergenceless harmonic function  $\Omega_{s-2k}$ . Then one can integrate by parts all  $\square$ 's that appear on the right hand side of (F.32), thus making them act on  $\Omega_{s-2k}$ . We can then use the equation of motion (C.2) for  $\Omega_{s-2k}$  to eliminate all  $\square$ 's. This establishes the following iterative equation

$$(\partial_u \cdot \partial_u)^{i+1} J_{s+2} \sim f(i, s)(\partial_u \cdot \partial_u)^i J_s + g(i, s)(\partial_u \cdot \partial_u)^{i-1} J_{s-2}, \tag{F.33}$$

where

$$\begin{aligned}
f(i, s) &= (2\Delta - d - 1)(2\Delta - d + 1) + (\nu^2 + (\frac{d}{2} + (s-2i) + 1)^2) + 8i(\frac{d}{2} + (s-2i) + i), \\
g(i, s) &= -4i(\frac{d}{2} + (s-2i) + i - 1) \left( (\nu^2 + (\frac{d}{2} + (s-2i) + 1)^2) + 4(i-1)(\frac{d}{2} + (s-2i) + i) \right).
\end{aligned}$$

As ‘‘boundary conditions’’ we take  $(\partial_u \cdot \partial_u)^{-1} J_{s-2k-2} \equiv 0$  and assume that  $J_{s-2k}$  is given. Then the iterative equation (F.33) allows to express  $(\partial_u \cdot \partial_u)^k J_s$  in terms of  $J_{s-2k}$ . The

result is

$$(\partial_u \cdot \partial_u)^k J_s \sim \tau_{s,k}(\nu) J_{s-2k},$$

with

$$\tau_{s,k}(\nu) = \sum_{m=0}^k \frac{2^{2k} \cdot k!}{m!(k-m)!} (\Delta - \frac{d}{2} - k + m + 1/2)_{k-m} \left( \frac{\frac{d}{2} + s - 2m + 1 + i\nu}{2} \right)_m \left( \frac{\frac{d}{2} + s - 2m + 1 - i\nu}{2} \right)_m.$$

## Appendix G

### Intrinsic expressions for vertices

In this appendix we give the recipe of how to express vertices in ambient space in intrinsic AdS terms. In §6.2.1 we used holography to fix the cubic vertices of minimal bosnic higher-spin theory on  $\text{AdS}_{d+1}$  in terms of the ambient structures

$$\begin{aligned} \mathcal{I}_{s_1, s_2, s_3}^{n_1, n_2, n_3}(\varphi_{s_i}) &= \mathcal{Y}_1^{s_1 - n_2 - n_3} \mathcal{Y}_2^{s_2 - n_3 - n_1} \mathcal{Y}_3^{s_3 - n_1 - n_2} \\ &\quad \times \mathcal{H}_1^{n_1} \mathcal{H}_2^{n_2} \mathcal{H}_3^{n_3} \varphi_{s_1}(X_1, U_1) \varphi_{s_2}(X_2, U_2) \varphi_{s_3}(X_3, U_3) \Big|_{X_i=X}. \end{aligned} \quad (\text{G.1})$$

where we used point splitting, and introduced

$$\mathcal{Y}_1 = \partial_{U_1} \cdot \partial_{X_2}, \quad \mathcal{Y}_2 = \partial_{U_2} \cdot \partial_{X_3}, \quad \mathcal{Y}_3 = \partial_{U_3} \cdot \partial_{X_1}, \quad (\text{G.2})$$

$$\mathcal{H}_1 = \partial_{U_2} \cdot \partial_{U_3}, \quad \mathcal{H}_2 = \partial_{U_3} \cdot \partial_{U_1}, \quad \mathcal{H}_3 = \partial_{U_1} \cdot \partial_{U_2}. \quad (\text{G.3})$$

While this representation allowed for a more straightforward manipulation of tensor structures, since they are vertices for a theory on AdS naturally one would like to express them in terms intrinsic to the AdS manifold. In other words, in terms of contractions of the form

$$\tilde{\mathcal{Y}}_1 = \partial_{U_1} \cdot \nabla_2, \quad \tilde{\mathcal{Y}}_2 = \partial_{U_2} \cdot \nabla_3, \quad \tilde{\mathcal{Y}}_3 = \partial_{U_3} \cdot \nabla_1. \quad (\text{G.4})$$

In order to achieve this, we consider the structure

$$I_{l_1, l_2, l_3, k_1, k_2, k_3}^{n_1, n_2, n_3, m_1, m_2, m_3}(\varphi_{s_i}) \equiv \mathcal{H}_1^{n_1} \mathcal{H}_2^{n_2} \mathcal{H}_3^{n_3} \mathcal{Y}_1^{l_1} \mathcal{Y}_2^{l_2} \mathcal{Y}_3^{l_3} \tilde{\mathcal{Y}}_1^{k_1} \tilde{\mathcal{Y}}_2^{k_2} \tilde{\mathcal{Y}}_3^{k_3} \\ \times (X_1^2)^{-m_1} (X_2^2)^{-m_2} (X_3^2)^{-m_3} \varphi_{s_1}(X_1, U_1) \varphi_{s_2}(X_2, U_2) \varphi_{s_3}(X_3, U_3), \quad (\text{G.5})$$

where

$$I_{s_1 - n_2 - n_3, s_2 - n_1 - n_3, s_3 - n_1 - n_2, 0, 0, 0}^{n_1, n_2, n_3, 0, 0, 0}(\varphi_{s_i}) = \mathcal{I}_{s_1, s_2, s_3}^{n_1, n_2, n_3}(\varphi_{s_i}). \quad (\text{G.6})$$

Employing the commutation relations (A.42), one can establish the following recursion relations<sup>1</sup>

$$I_{l_1, l_2, l_3, 0, 0, k_3}^{n_1, n_2, n_3, m_1, 0, 0} = I_{l_1, l_2, l_3 - 1, 0, 0, k_3 + 1}^{n_1, n_2, n_3, m_1, 0, 0} - l_2(\Delta_1 - k_3 - n_2 - 2m_1) I_{l_1, l_2 - 1, l_3 - 1, 0, 0, k_3}^{n_1 + 1, n_2, n_3, m_1 + 1, 0, 0} \\ - n_3(l_3 - 1) I_{l_1, l_2, l_3 - 2, 0, 0, k_3}^{n_1 + 1, n_2 + 1, n_3 - 1, m_1 + 1, 0, 0} \\ - l_1(\Delta_2 - l_1 - n_3 - l_3 + 2) I_{l_1 - 1, l_2, l_3 - 1, 0, 0, k_3}^{n_1, n_2 + 1, n_3, m_1 + 1, 0, 0} \\ + \lambda l_1 l_2 (l_3 - 1) I_{l_1 - 1, l_2 - 1, l_3 - 2, 0, 0, k_3}^{n_1 + 1, n_2 + 1, n_3, m_1 + 1, 0, 0}, \quad (\text{G.8a})$$

$$I_{l_1, l_2, 0, 0, k_2, k_3}^{n_1, n_2, n_3, m_1, 0, m_3} = I_{l_1, l_2 - 1, 0, 0, k_2 + 1, k_3}^{n_1, n_2, n_3, m_1, 0, m_3} - l_1(\Delta_3 - k_2 - n_1 - 2m_3) I_{l_1 - 1, l_2 - 1, l_3, 0, k_2, k_3}^{n_1, n_2, n_3 + 1, m_1, 0, m_3 + 1} \\ - n_2(l_2 - 1) I_{l_1, l_2 - 2, l_3, 0, k_2, k_3}^{n_1 + 1, n_2 - 1, n_3 + 1, m_1, 0, m_3 + 1} \quad (\text{G.8b})$$

$$I_{l_1, 0, 0, k_1, k_2, k_3}^{n_1, n_2, n_3, m_1, m_2, m_3} = I_{l_1 - 1, 0, 0, k_1 + 1, k_2, k_3}^{n_1, n_2, n_3, m_1, m_2, m_3} - n_1(l_1 - 1) I_{l_1 - 2, 0, 0, k_1, k_2, k_3}^{n_1 - 1, n_2 + 1, n_3 + 1, m_1, m_2 + 1, m_3}, \quad (\text{G.8c})$$

which can be solved in terms of the intrinsic structures

$$I_{0, 0, 0, k_1, k_2, k_3}^{n_1, n_2, n_3, 0, 0, 0}(\varphi_{s_i}) = \mathcal{H}_1^{n_1} \mathcal{H}_2^{n_2} \mathcal{H}_3^{n_3} \tilde{\mathcal{Y}}_1^{k_1} \tilde{\mathcal{Y}}_2^{k_2} \tilde{\mathcal{Y}}_3^{k_3} \varphi_{s_1}(X_1, U_1) \varphi_{s_2}(X_2, U_2) \varphi_{s_3}(X_3, U_3). \quad (\text{G.9})$$

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<sup>1</sup>Recall that  $\Delta_i$  is the homogeneity degree of the field  $\varphi_{s_i}$ , and  $\lambda$  is an auxiliary variable to be replaced at the very end of the recursion procedure as follows:

$$\lambda^n \equiv (-1)^n (\Delta + d)(\Delta + d - 2) \dots (\Delta + d - 2n + 2), \quad (\text{G.7})$$

where  $\Delta$  is the total degree of homogeneity of the given term of the vertex. The reason  $\lambda$  appears is that in order to establish the recursion relations one has to integrate by parts in ambient space (see e.g. [30]).

Starting with the original basis (G.2) for the vertices, with the above recursions their intrinsic forms can therefore be established.

Below we give some examples obtained by implementing the above recursion relations in Mathematica for massless fields ( $\Delta_i = 2 - d - s_i$ )

$$\mathcal{Y}_1 \mathcal{Y}_2 = \tilde{\mathcal{Y}}_1 \tilde{\mathcal{Y}}_2 - (d-2) \mathcal{H}_3, \quad (\text{G.10a})$$

$$\mathcal{Y}_1 \mathcal{Y}_2 \mathcal{Y}_3 = \tilde{\mathcal{Y}}_1 \tilde{\mathcal{Y}}_2 \tilde{\mathcal{Y}}_3 - (d-1) (\mathcal{H}_1 \tilde{\mathcal{Y}}_1 + \mathcal{H}_2 \tilde{\mathcal{Y}}_2 + \mathcal{H}_3 \tilde{\mathcal{Y}}_3),$$

$$\begin{aligned} \mathcal{Y}_1^2 \mathcal{Y}_2^2 \mathcal{Y}_3^2 &= \tilde{\mathcal{Y}}_1^2 \tilde{\mathcal{Y}}_2^2 \tilde{\mathcal{Y}}_3^2 - 2(2d+1) \mathcal{H}_1 \tilde{\mathcal{Y}}_1^2 \tilde{\mathcal{Y}}_2 \tilde{\mathcal{Y}}_3 - 2(2d+3) \mathcal{H}_2 \tilde{\mathcal{Y}}_1 \tilde{\mathcal{Y}}_2^2 \tilde{\mathcal{Y}}_3 - 2(2d+1) \mathcal{H}_3 \tilde{\mathcal{Y}}_1 \tilde{\mathcal{Y}}_2 \tilde{\mathcal{Y}}_3^2 \\ &\quad + 2d(d+2) \mathcal{H}_1^2 \tilde{\mathcal{Y}}_1^2 + 2d(d+2) \mathcal{H}_3^2 \tilde{\mathcal{Y}}_3^2 + 2d(d+2) \mathcal{H}_2^2 \tilde{\mathcal{Y}}_2^2 \\ &\quad + 8(d^2 + 2d + 2) \mathcal{H}_1 \mathcal{H}_2 \tilde{\mathcal{Y}}_1 \tilde{\mathcal{Y}}_2 + 2(4d^2 + 4d - 1) \mathcal{H}_1 \mathcal{H}_3 \tilde{\mathcal{Y}}_1 \tilde{\mathcal{Y}}_3 \\ &\quad + 4(d+1)(2d+1) \mathcal{H}_2 \mathcal{H}_3 \tilde{\mathcal{Y}}_2 \tilde{\mathcal{Y}}_3 - 8(d^3 + 2d^2 + d + 1) \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3, \end{aligned} \quad (\text{G.10b})$$

$$\begin{aligned} \mathcal{Y}_1^3 \mathcal{Y}_2^3 \mathcal{Y}_3^3 &= \tilde{\mathcal{Y}}_1^3 \tilde{\mathcal{Y}}_2^3 \tilde{\mathcal{Y}}_3^3 \\ &\quad - 9(d+2) \mathcal{H}_1 \tilde{\mathcal{Y}}_1^3 \tilde{\mathcal{Y}}_2^2 \tilde{\mathcal{Y}}_3^2 - 9(d+4) \mathcal{H}_2 \tilde{\mathcal{Y}}_1^2 \tilde{\mathcal{Y}}_2^3 \tilde{\mathcal{Y}}_3^2 \\ &\quad - 9(d+2) \mathcal{H}_3 \tilde{\mathcal{Y}}_1^2 \tilde{\mathcal{Y}}_2^2 \tilde{\mathcal{Y}}_3^3 + 18(d^2 + 5d + 5) \mathcal{H}_1^2 \tilde{\mathcal{Y}}_1^3 \tilde{\mathcal{Y}}_2 \tilde{\mathcal{Y}}_3 \\ &\quad + 27(2d^2 + 12d + 21) \mathcal{H}_1 \mathcal{H}_2 \tilde{\mathcal{Y}}_1^2 \tilde{\mathcal{Y}}_2^2 \tilde{\mathcal{Y}}_3 \\ &\quad + 18(d^2 + 7d + 11) \mathcal{H}_2^2 \tilde{\mathcal{Y}}_1 \tilde{\mathcal{Y}}_2^3 \tilde{\mathcal{Y}}_3 + 18(d^2 + 5d + 5) \mathcal{H}_3^2 \tilde{\mathcal{Y}}_1 \tilde{\mathcal{Y}}_2 \tilde{\mathcal{Y}}_3^3 \\ &\quad + 54(d+1)(d+3) \mathcal{H}_1 \mathcal{H}_3 \tilde{\mathcal{Y}}_1^2 \tilde{\mathcal{Y}}_2 \tilde{\mathcal{Y}}_3^2 + 27(d+2)(2d+7) \mathcal{H}_2 \mathcal{H}_3 \tilde{\mathcal{Y}}_1 \tilde{\mathcal{Y}}_2^2 \tilde{\mathcal{Y}}_3^2 \\ &\quad - 54(d+3)(d^2 + 6d + 10) \mathcal{H}_1^2 \mathcal{H}_2 \tilde{\mathcal{Y}}_1^2 \tilde{\mathcal{Y}}_2 - 54(d+3)(d^2 + 6d + 10) \mathcal{H}_1 \mathcal{H}_2^2 \tilde{\mathcal{Y}}_1 \tilde{\mathcal{Y}}_2^2 \\ &\quad - 54(2d+7)(2d^2 + 8d + 9) \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3 \tilde{\mathcal{Y}}_1 \tilde{\mathcal{Y}}_2 \tilde{\mathcal{Y}}_3 - 54(d+4)(d^2 + 3d + 1) \mathcal{H}_1 \mathcal{H}_3^2 \tilde{\mathcal{Y}}_1 \tilde{\mathcal{Y}}_3^2 \\ &\quad - 27(2d^3 + 14d^2 + 28d + 11) \mathcal{H}_1^2 \mathcal{H}_3 \tilde{\mathcal{Y}}_1^2 \tilde{\mathcal{Y}}_3 - 27(2d^3 + 16d^2 + 38d + 25) \mathcal{H}_2^2 \mathcal{H}_3 \tilde{\mathcal{Y}}_2^2 \tilde{\mathcal{Y}}_3 \\ &\quad - 54(d^3 + 8d^2 + 21d + 17) \mathcal{H}_2 \mathcal{H}_3^2 \tilde{\mathcal{Y}}_2 \tilde{\mathcal{Y}}_3^2 - 6(d+1)(d+3)(d+5) \mathcal{H}_1^3 \tilde{\mathcal{Y}}_1^3 \\ &\quad - 6(d+1)(d+3)(d+5) \mathcal{H}_2^3 \tilde{\mathcal{Y}}_2^3 - 6(d+1)(d+3)(d+5) \mathcal{H}_3^3 \tilde{\mathcal{Y}}_3^3 \\ &\quad + 54(d+3)(2d^3 + 15d^2 + 34d + 27) \mathcal{H}_1^2 \mathcal{H}_2 \mathcal{H}_3 \tilde{\mathcal{Y}}_1 \\ &\quad + 54(d+3)(2d^3 + 14d^2 + 28d + 19) \mathcal{H}_1 \mathcal{H}_2^2 \mathcal{H}_3 \tilde{\mathcal{Y}}_2 \\ &\quad + 54(d+3)(2d^3 + 14d^2 + 29d + 19) \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3^2 \tilde{\mathcal{Y}}_3. \end{aligned} \quad (\text{G.10c})$$





## Appendix H

### Basis of quartic vertices

In this appendix we demonstrate completeness of the basis (5.109) of quartic scalar self-interactions.

A given vertex can be brought into different forms via integration by parts. For our purposes, we need only find a complete set of quartic vertices which are independent on-free-shell. We therefore need to take into account all possible relations between such vertices via integration by parts and using the free equations of motion. In counting the number of independent on-shell vertices, it is simpler to consider the equivalent problem of counting the associated four-point amplitudes. This is how we proceed in the following. For simplicity, following the approach of [298] we carry out this analysis in flat space.

As shown in [298], a complete non-redundant basis of flat space four-point amplitudes is given by the set monomials of the form

$$s^k t^k u^m \quad \text{with integers } k \geq m \geq 0, \quad (\text{H.1})$$

where  $s$ ,  $t$  and  $u$  are the Mandelstam variables

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_2 + p_3)^2.$$

For massless particles  $p_i^2 = 0$ , one has

$$s = 2 p_1 \cdot p_2 = 2 p_3 \cdot p_4, \quad t = 2 p_1 \cdot p_3 = 2 p_2 \cdot p_4, \quad u = 2 p_2 \cdot p_3 = 2 p_1 \cdot p_4. \quad (\text{H.2})$$

Our goal is to replace the basis (H.1) with an equivalent one, that is better suited to our purposes - i.e. in terms of amplitudes generated by quartic vertices built from conserved currents. Let us first consider such vertices in position space

$$\square_{12}^m J_{\mu_1 \dots \mu_\ell}(\varphi_0(x_1), \varphi_0(x_2)) J^{\mu_1 \dots \mu_\ell}(\varphi_0(x_3), \varphi_0(x_4)), \quad (\text{H.3})$$

with  $\square_{12} = (\partial_{x_1} + \partial_{x_2})^2$  and

$$J_{\mu_1 \dots \mu_\ell}(\varphi_0(x_1), \varphi_0(x_2)) \equiv \varphi_0(x_1) \overleftrightarrow{\partial}_{\mu_1} \dots \overleftrightarrow{\partial}_{\mu_\ell} \varphi_0(x_2), \quad (\text{H.4})$$

are the conserved currents of [299], with  $\overleftrightarrow{\partial} = \overleftarrow{\partial}_{x_1} - \overrightarrow{\partial}_{x_2}$ . The amplitude in momentum space associated to the vertex (H.3) reads

$$(p_1 + p_2)^m [(p_1 - p_2) \cdot (p_3 - p_4)]^\ell = s^m (t - u)^\ell. \quad (\text{H.5})$$

For  $\ell = 2k$ , this amplitude contains a term  $s^m t^k u^k$ . Up to a Bose symmetry transformation, this reproduces the form of the vertices (H.1). We therefore conclude that vertices (H.3) with  $\ell = 2k$  and  $k \geq m \geq 0$  generate the basis of quartic vertices.

For our purposes it is more convenient to use the traceless improvement of (H.4), given

in [300]<sup>1</sup>

$$\begin{aligned} & \tilde{J}_{\mu_1 \dots \mu_\ell}(\varphi_0(x_1), \varphi_0(x_2)) \\ &= \frac{\sqrt{\pi} \Gamma(d+s-2)}{2^{s+d-3} \Gamma\left(\frac{d-1}{2}\right) \Gamma\left(s+\frac{d}{2}-1\right)} \sum_{k=0}^{\ell} a_k \partial_{\mu_1} \dots \partial_{\mu_k} \varphi_0(x_1) \partial_{\mu_{k+1}} \dots \partial_{\mu_\ell} \varphi_0(x_2) - \text{traces}, \end{aligned} \quad (\text{H.6})$$

where

$$a_k = (-1)^k \frac{l!}{k!(l-k)!} \frac{\left(\frac{d}{2}-1\right)_l}{\left(\frac{d}{2}-1\right)_k \left(\frac{d}{2}-1\right)_{l-k}}.$$

The improvement (H.6) can always be obtained from (H.4) when the scalar field is conformal.

Analogously to (H.3), the flat space amplitude corresponding to a vertex

$$\mathcal{V}_{m,\ell} \equiv \square_{12}^m \tilde{J}_{\mu_1 \dots \mu_\ell}(\varphi_0(x_1), \varphi_0(x_2)) \tilde{J}^{\mu_1 \dots \mu_\ell}(\varphi_0(x_3), \varphi_0(x_4)) \quad (\text{H.7})$$

with  $\ell = 2k$  generates a term  $s^m t^k u^k$ . Thus, vertices (H.7) with  $\ell = 2k$  and  $k \geq m \geq 0$  may be used as a basis for quartic interactions.

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<sup>1</sup>Note that the normalisation here is such that it is consistent with the normalisation of (H.4).



# Bibliography

- [1] Xavier Bekaert, Johanna Erdmenger, Dmitry Ponomarev, and Charlotte Sleight. Towards holographic higher-spin interactions: Four-point functions and higher-spin exchange. *JHEP*, 1503:170, 2015.
- [2] Xavier Bekaert, Johanna Erdmenger, Dmitry Ponomarev, and Charlotte Sleight. Quartic AdS Interactions in Higher-Spin Gravity from Conformal Field Theory. *JHEP*, 11:149, 2015.
- [3] Xavier Bekaert, Johanna Erdmenger, Dmitry Ponomarev, and Charlotte Sleight. Bulk quartic vertices from boundary four-point correlators. In *International Workshop on Higher Spin Gauge Theories Singapore, Singapore, November 4-6, 2015*, 2016.
- [4] Charlotte Sleight and Massimo Taronna. Higher-spin Interactions from CFT: The Complete Cubic Couplings. 2016.
- [5] S. L. Glashow. Partial Symmetries of Weak Interactions. *Nucl. Phys.*, 22:579–588, 1961.
- [6] Steven Weinberg. A Model of Leptons. *Phys. Rev. Lett.*, 19:1264–1266, 1967.
- [7] A. Salam. N. Svartholm, ed. Elementary Particle Physics: Relativistic Groups and Analyticity. Eighth Nobel Symposium. *Stockholm: Almqvist and Wiksell*, page 367, 1968.

- 
- [8] Albert Einstein. The Foundation of the General Theory of Relativity. *Annalen Phys.*, 49:769–822, 1916. [Annalen Phys.14,517(2005)].
- [9] Marc H. Goroff and Augusto Sagnotti. The Ultraviolet Behavior of Einstein Gravity. *Nucl. Phys.*, B266:709, 1986.
- [10] Marc H. Goroff and Augusto Sagnotti. Quantum gravity at two loops. *Phys. Lett.*, B160:81, 1985.
- [11] Michael B. Green, Jorge G. Russo, and Pierre Vanhove. Ultraviolet properties of maximal supergravity. *Phys. Rev. Lett.*, 98:131602, 2007.
- [12] Z. Bern, J. J. Carrasco, Lance J. Dixon, H. Johansson, and R. Roiban. The Ultraviolet Behavior of  $\mathcal{N} = 8$  Supergravity at Four Loops. *Phys. Rev. Lett.*, 103:081301, 2009.
- [13] Niklas Beisert, Henriette Elvang, Daniel Z. Freedman, Michael Kiermaier, Alejandro Morales, and Stephan Stieberger.  $E_{7(7)}$  constraints on counterterms in  $N=8$  supergravity. *Phys. Lett.*, B694:265–271, 2011.
- [14] Renata Kallosh.  $E_{7(7)}$  Symmetry and Finiteness of  $\mathcal{N} = 8$  Supergravity. *JHEP*, 03:083, 2012.
- [15] Xian O. Camanho, Jose D. Edelstein, Juan Maldacena, and Alexander Zhiboedov. Causality Constraints on Corrections to the Graviton Three-Point Coupling. *JHEP*, 02:020, 2016.
- [16] G. Veneziano. Construction of a crossing - symmetric, Regge behaved amplitude for linearly rising trajectories. *Nuovo Cim.*, A57:190–197, 1968.
- [17] D. Amati, M. Ciafaloni, and G. Veneziano. Superstring Collisions at Planckian Energies. *Phys. Lett.*, B197:81, 1987.
- [18] David J. Gross and Paul F. Mende. The High-Energy Behavior of String Scattering Amplitudes. *Phys. Lett.*, B197:129, 1987.

- 
- [19] David J. Gross and Paul F. Mende. String Theory Beyond the Planck Scale. *Nucl. Phys.*, B303:407, 1988.
- [20] D. Amati, M. Ciafaloni, and G. Veneziano. Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions. *Int. J. Mod. Phys.*, A3:1615–1661, 1988.
- [21] Daniele Amati, Marcello Ciafaloni, and Gabriele Veneziano. Can space time be probed below the string size? *Phys. Lett. B*, 216(CERN-TH-5207-88):41–47. 15 p, Oct 1988.
- [22] I. J. Muzinich and M. Soldate. High-energy unitarity of gravitation and strings. *Phys. Rev. D*, 37:359–367, Jan 1988.
- [23] B. Sundborg. High-energy Asymptotics: The One Loop String Amplitude and Resummation. *Nucl. Phys.*, B306:545–566, 1988.
- [24] Nicolas Moeller and Peter C. West. Arbitrary four string scattering at high energy and fixed angle. *Nucl. Phys.*, B729:1–48, 2005.
- [25] David J. Gross. High-Energy Symmetries of String Theory. *Phys. Rev. Lett.*, 60:1229, 1988.
- [26] David J. Gross and J. L. Manes. The High-energy Behavior of Open String Scattering. *Nucl. Phys.*, B326:73, 1989.
- [27] Gregory W. Moore. Symmetries of the bosonic string S matrix. 1993.
- [28] Massimo Taronna. *Higher Spins and String Interactions*. PhD thesis, Pisa U., 2010.
- [29] A. Sagnotti and M. Taronna. String Lessons for Higher-Spin Interactions. *Nucl. Phys.*, B842:299–361, 2011.

- 
- [30] Massimo Taronna. *Higher-Spin Interactions: three-point functions and beyond*. PhD thesis, Pisa, Scuola Normale Superiore, 2012.
- [31] Matthias R Gaberdiel and Rajesh Gopakumar. Stringy Symmetries and the Higher Spin Square. *J. Phys.*, A48(18):185402, 2015.
- [32] Matthias R Gaberdiel and Rajesh Gopakumar. String Theory as a Higher Spin Theory. 2015.
- [33] Mitsuhiro Kato and Kaku Ogawa. Covariant Quantization of String Based on BRS Invariance. *Nucl. Phys.*, B212:443, 1983.
- [34] Warren Siegel. Classical Superstring Mechanics. *Nucl. Phys.*, B263:93, 1986.
- [35] Warren Siegel and Barton Zwiebach. Gauge String Fields. *Nucl. Phys.*, B263:105, 1986.
- [36] Tom Banks and Michael E. Peskin. Gauge Invariance of String Fields. *Nucl. Phys.*, B264:513, 1986.
- [37] Nobuyoshi Ohta. Covariant Quantization of Superstrings Based on BRS Invariance. *Phys. Rev.*, D33:1681, 1986.
- [38] Nobuyoshi Ohta. Covariant Second Quantization of Superstrings. *Phys. Rev. Lett.*, 56:440, 1986. [Erratum: *Phys. Rev. Lett.*56,1316(1986)].
- [39] A. Neveu, H. Nicolai, and P.C. West. Gauge covariant local formulation of free strings and superstrings. *Nuclear Physics B*, 264:573 – 587, 1986.
- [40] A. Neveu and P.C. West. Gauge covariant local formulation of bosonic strings. *Nuclear Physics B*, 268(1):125 – 150, 1986.
- [41] Edward Witten. Noncommutative Geometry and String Field Theory. *Nucl. Phys.*, B268:253, 1986.



- [42] Barton Zwiebach. Closed string field theory: Quantum action and the B-V master equation. *Nucl. Phys.*, B390:33–152, 1993.
- [43] Theodore Erler, Yuji Okawa, and Tomoyuki Takezaki. Complete Action for Open Superstring Field Theory with Cyclic  $A_\infty$  Structure. 2016.
- [44] Juan Martin Maldacena. The Large  $N$  limit of superconformal field theories and supergravity. *Int. J. Theor. Phys.*, 38:1113–1133, 1999. [Adv. Theor. Math. Phys.2,231(1998)].
- [45] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. Gauge theory correlators from noncritical string theory. *Phys. Lett.*, B428:105–114, 1998.
- [46] Edward Witten. Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253–291, 1998.
- [47] Martin Ammon and Johanna Erdmenger. *Gauge/gravity duality*. Cambridge Univ. Pr., Cambridge, UK, 2015.
- [48] Horatiu Nastase. *Introduction to the AdS/CFT Correspondence*. Cambridge University Press, Cambridge, 2015.
- [49] Parviz Haggi-Mani and Bo Sundborg. Free large  $N$  supersymmetric Yang-Mills theory as a string theory. *JHEP*, 04:031, 2000.
- [50] Bo Sundborg. Stringy gravity, interacting tensionless strings and massless higher spins. *Nucl. Phys. Proc. Suppl.*, 102:113–119, 2001. [,113(2000)].
- [51] E Witten. Stringy gravity, interacting tensionless strings and massless higher spins. 2001. Conf. in Honor of John Schwarz 60th Birthday, California Institute of Technology, Pasadena, CA, USA, <http://theory.caltech.edu/jhs60/witten/1.html>.
- [52] Arkady A. Tseytlin. On limits of superstring in  $AdS_5 \times S^5$ . *Theor. Math. Phys.*, 133:1376–1389, 2002. [Teor. Mat. Fiz.133,69(2002)].

- 
- [53] Andreas Karch. Light cone quantization of string theory duals of free field theories. 2002.
- [54] Rajesh Gopakumar. From free fields to AdS. *Phys. Rev.*, D70:025009, 2004.
- [55] Giulio Bonelli. On the covariant quantization of tensionless bosonic strings in AdS space-time. *JHEP*, 11:028, 2003.
- [56] Rajesh Gopakumar. From free fields to ads space. ii. *Phys. Rev. D*, 70:025010, Jul 2004.
- [57] Rajesh Gopakumar. Free field theory as a string theory? *Comptes Rendus Physique*, 5:1111–1119, 2004.
- [58] Rajesh Gopakumar. From free fields to ads. iii. *Phys. Rev. D*, 72:066008, Sep 2005.
- [59] Ofer Aharony, Zohar Komargodski, and Shlomo S. Razamat. On the worldsheet theories of strings dual to free large N gauge theories. *JHEP*, 05:016, 2006.
- [60] Itamar Yaakov. Open and closed string worldsheets from free large N gauge theories with adjoint and fundamental matter. *JHEP*, 11:065, 2006.
- [61] Ofer Aharony, Justin R. David, Rajesh Gopakumar, Zohar Komargodski, and Shlomo S. Razamat. Remarks on worldsheet theories dual to free large  $n$  gauge theories. *Phys. Rev. D*, 75:106006, May 2007.
- [62] Massimo Bianchi, Jose F. Morales, and Henning Samtleben. On stringy  $\text{AdS}_5 \times S^5$  and higher spin holography. *JHEP*, 07:062, 2003.
- [63] N. Beisert, M. Bianchi, J. F. Morales, and H. Samtleben. On the spectrum of AdS / CFT beyond supergravity. *JHEP*, 02:001, 2004.
- [64] M. Bianchi and V. Didenko. Massive higher spin multiplets and holography. In *Higher spin gauge theories: Proceedings, 1st Solway Workshop*, pages 1–20, 2004.

- [65] Niklas Beisert, Massimo Bianchi, Jose F. Morales, and Henning Samtleben. Higher spin symmetry and  $\mathcal{N} = 4$  SYM. *JHEP*, 07:058, 2004.
- [66] Massimo Bianchi. Higher spins and stringy  $\text{AdS}_5 \times S^5$ . *Fortsch. Phys.*, 53:665–691, 2005.
- [67] Massimo Bianchi, Paul J. Heslop, and Fabio Riccioni. More on La Grande Bouffe. *JHEP*, 08:088, 2005.
- [68] Massimo Bianchi and Fabio Riccioni. Massive higher spins and holography. *J. Phys. Conf. Ser.*, 33:49–56, 2006.
- [69] Niklas Beisert et al. Review of AdS/CFT Integrability: An Overview. *Lett. Math. Phys.*, 99:3–32, 2012.
- [70] E. Sezgin and P. Sundell. Massless higher spins and holography. *Nucl. Phys.*, B644:303–370, 2002. [Erratum: *Nucl. Phys.*B660,403(2003)].
- [71] I. R. Klebanov and A. M. Polyakov. AdS dual of the critical  $O(N)$  vector model. *Phys. Lett.*, B550:213–219, 2002.
- [72] Robert G. Leigh and Anastasios C. Petkou. Holography of the  $\mathcal{N} = 1$  higher spin theory on  $\text{AdS}(4)$ . *JHEP*, 06:011, 2003.
- [73] E. Sezgin and P. Sundell. Holography in 4D (super) higher spin theories and a test via cubic scalar couplings. *JHEP*, 0507:044, 2005.
- [74] Simone Giombi, Shiraz Minwalla, Shiroman Prakash, Sandip P. Trivedi, Spenta R. Wadia, and Xi Yin. Chern-Simons Theory with Vector Fermion Matter. *Eur. Phys. J.*, C72:2112, 2012.
- [75] Dionysios Anninos, Thomas Hartman, and Andrew Strominger. Higher Spin Realization of the dS/CFT Correspondence. 2011.

- [76] Chi-Ming Chang, Shiraz Minwalla, Tarun Sharma, and Xi Yin. ABJ Triality: from Higher Spin Fields to Strings. *J. Phys.*, A46:214009, 2013.
- [77] S. E. Konstein, M. A. Vasiliev, and V. N. Zaikin. Conformal higher spin currents in any dimension and AdS / CFT correspondence. *JHEP*, 12:018, 2000.
- [78] O. V. Shaynkman and M. A. Vasiliev. Higher spin conformal symmetry for matter fields in (2+1)-dimensions. *Theor. Math. Phys.*, 128:1155–1168, 2001. [Teor. Mat. Fiz.128,378(2001)].
- [79] E. Sezgin and P. Sundell. Doubletons and 5-D higher spin gauge theory. *JHEP*, 09:036, 2001.
- [80] M. A. Vasiliev. Conformal higher spin symmetries of 4-d massless supermultiplets and  $osp(L, 2M)$  invariant equations in generalized (super)space. *Phys. Rev.*, D66:066006, 2002.
- [81] Andrei Mikhailov. Notes on higher spin symmetries. 2002.
- [82] Matthias R Gaberdiel and Rajesh Gopakumar. Higher Spins & Strings. *JHEP*, 11:044, 2014.
- [83] Steven Weinberg. Photons and gravitons in  $s$ -matrix theory: Derivation of charge conservation and equality of gravitational and inertial mass. *Phys. Rev.*, 135:B1049–B1056, Aug 1964.
- [84] Sidney R. Coleman and J. Mandula. All Possible Symmetries of the S Matrix. *Phys. Rev.*, 159:1251–1256, 1967.
- [85] C. Aragone and Stanley Deser. Consistency Problems of Hypergravity. *Phys. Lett.*, B86:161, 1979.
- [86] Stanley Deser and Z. Yang. Inconsistency of Spin 4 - Spin-2 Gauge Field Couplings. *Class. Quant. Grav.*, 7:1491–1498, 1990.

- 
- [87] Steven Weinberg and Edward Witten. Limits on Massless Particles. *Phys. Lett.*, B96:59, 1980.
- [88] M. Porrati. Universal Limits on Massless High-Spin Particles. *Phys. Rev.*, D78:065016, 2008.
- [89] Xavier Bekaert, Nicolas Boulanger, and Serge Leclercq. Strong obstruction of the Berends-Burgers-van Dam spin-3 vertex. *J. Phys.*, A43:185401, 2010.
- [90] Euihun Joung and Massimo Taronna. Cubic-interaction-induced deformations of higher-spin symmetries. *JHEP*, 03:103, 2014.
- [91] R. R. Metsaev. Poincare invariant dynamics of massless higher spins: Fourth order analysis on mass shell. *Mod. Phys. Lett.*, A6:359–367, 1991.
- [92] R. R. Metsaev. S matrix approach to massless higher spins theory. 2: The Case of internal symmetry. *Mod. Phys. Lett.*, A6:2411–2421, 1991.
- [93] Charlotte Sleight and Massimo Taronna. Higher-Spin Algebras, Holography and Flat Space. 2016.
- [94] Dmitry Ponomarev and E. D. Skvortsov. Light-Front Higher-Spin Theories in Flat Space. 2016.
- [95] Xavier Bekaert, Nicolas Boulanger, and Per Sundell. How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples. *Rev. Mod. Phys.*, 84:987–1009, 2012.
- [96] E. S. Fradkin and Mikhail A. Vasiliev. Cubic Interaction in Extended Theories of Massless Higher Spin Fields. *Nucl. Phys.*, B291:141, 1987.
- [97] E. S. Fradkin and Mikhail A. Vasiliev. On the Gravitational Interaction of Massless Higher Spin Fields. *Phys. Lett.*, B189:89–95, 1987.

- 
- [98] Ruben Manvelyan and Werner Ruhl. Conformal coupling of higher spin gauge fields to a scalar field in  $\text{AdS}_4$  and generalized Weyl invariance. *Phys. Lett.*, B593:253–261, 2004.
- [99] Angelos Fotopoulos, Nikos Irges, Anastasios C. Petkou, and Mirian Tsulaia. Higher-Spin Gauge Fields Interacting with Scalars: The Lagrangian Cubic Vertex. *JHEP*, 10:021, 2007.
- [100] Nicolas Boulanger, Serge Leclercq, and Per Sundell. On The Uniqueness of Minimal Coupling in Higher-Spin Gauge Theory. *JHEP*, 08:056, 2008.
- [101] Yu. M. Zinoviev. On spin 3 interacting with gravity. *Class. Quant. Grav.*, 26:035022, 2009.
- [102] Ruben Manvelyan and Karapet Mkrtchyan. Conformal invariant interaction of a scalar field with the higher spin field in  $\text{AdS}_D$ . *Mod. Phys. Lett.*, A25:1333–1348, 2010.
- [103] Angelos Fotopoulos and Mirian Tsulaia. Current Exchanges for Reducible Higher Spin Modes on AdS. In *19th International Colloquium on Integrable Systems and Quantum Symmetries (ISQS-19) Prague, Czech Republic, June 17-19, 2010*, 2010.
- [104] Xavier Bekaert and Elisa Meunier. Higher spin interactions with scalar matter on constant curvature spacetimes: conserved current and cubic coupling generating functions. *JHEP*, 11:116, 2010.
- [105] Konstantin Alkalaev. FV-type action for  $\text{AdS}_5$  mixed-symmetry fields. *JHEP*, 03:031, 2011.
- [106] Nicolas Boulanger, E. D. Skvortsov, and Yu. M. Zinoviev. Gravitational cubic interactions for a simple mixed-symmetry gauge field in AdS and flat backgrounds. *J. Phys.*, A44:415403, 2011.

- [107] Nicolas Boulanger and E. D. Skvortsov. Higher-spin algebras and cubic interactions for simple mixed-symmetry fields in AdS spacetime. *JHEP*, 09:063, 2011.
- [108] M. A. Vasiliev. Cubic Vertices for Symmetric Higher-Spin Gauge Fields in (A)dS<sub>d</sub>. *Nucl. Phys.*, B862:341–408, 2012.
- [109] Euihun Joung and Massimo Taronna. Cubic interactions of massless higher spins in (A)dS: metric-like approach. *Nucl. Phys.*, B861:145–174, 2012.
- [110] Euihun Joung, Luca Lopez, and Massimo Taronna. On the cubic interactions of massive and partially-massless higher spins in (A)dS. *JHEP*, 07:041, 2012.
- [111] Euihun Joung, Luca Lopez, and Massimo Taronna. Solving the Noether procedure for cubic interactions of higher spins in (A)dS. *J. Phys.*, A46:214020, 2013.
- [112] Euihun Joung, Luca Lopez, and Massimo Taronna. Generating functions of (partially-)massless higher-spin cubic interactions. *JHEP*, 01:168, 2013.
- [113] Nicolas Boulanger, Dmitry Ponomarev, and E.D. Skvortsov. Non-abelian cubic vertices for higher-spin fields in anti-de Sitter space. *JHEP*, 1305:008, 2013.
- [114] Euihun Joung, Massimo Taronna, and Andrew Waldron. A Calculus for Higher Spin Interactions. *JHEP*, 07:186, 2013.
- [115] Frits A. Berends, G. J. H. Burgers, and H. van Dam. On the Theoretical Problems in Constructing Interactions Involving Higher Spin Massless Particles. *Nucl. Phys.*, B260:295, 1985.
- [116] E. S. Fradkin and Mikhail A. Vasiliev. Candidate to the Role of Higher Spin Symmetry. *Annals Phys.*, 177:63, 1987.
- [117] Nicolas Boulanger, Dmitry Ponomarev, E.D. Skvortsov, and Massimo Taronna. On the uniqueness of higher-spin symmetries in AdS and CFT. *Int.J.Mod.Phys.*, A28:1350162, 2013.

- 
- [118] Mikhail A. Vasiliev. Equations of Motion of Interacting Massless Fields of All Spins as a Free Differential Algebra. *Phys. Lett.*, B209:491–497, 1988.
- [119] Mikhail A. Vasiliev. Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions. *Phys.Lett.*, B243:378–382, 1990.
- [120] Mikhail A. Vasiliev. Properties of equations of motion of interacting gauge fields of all spins in (3 + 1)-dimensions. *Class. Quant. Grav.*, 8:1387–1417, 1991.
- [121] Mikhail A. Vasiliev. More on equations of motion for interacting massless fields of all spins in (3 + 1)-dimensions. *Phys. Lett.*, B285:225–234, 1992.
- [122] Mikhail A. Vasiliev. Equations of motion for  $d = 3$  massless fields interacting through Chern-Simons higher spin gauge fields. *Mod. Phys. Lett.*, A7:3689–3702, 1992.
- [123] Mikhail A. Vasiliev. Higher spin gauge theories in four-dimensions, three-dimensions, and two-dimensions. *Int. J. Mod. Phys.*, D5:763–797, 1996.
- [124] S. F. Prokushkin and Mikhail A. Vasiliev. Higher spin gauge interactions for massive matter fields in  $3 - D$  AdS space-time. *Nucl. Phys.*, B545:385, 1999.
- [125] Mikhail A. Vasiliev. Higher spin gauge theories: Star product and AdS space. 1999.
- [126] J. Engquist, E. Sezgin, and P. Sundell. On  $\mathcal{N} = 1$ ,  $\mathcal{N} = 2$ ,  $\mathcal{N} = 4$  higher spin gauge theories in four-dimensions. *Class. Quant. Grav.*, 19:6175–6196, 2002.
- [127] M. A. Vasiliev. Nonlinear equations for symmetric massless higher spin fields in (A)dS $_d$ . *Phys. Lett.*, B567:139–151, 2003.
- [128] Nicolas Boulanger and Per Sundell. An action principle for Vasiliev’s four-dimensional higher-spin gravity. *J. Phys.*, A44:495402, 2011.



- 
- [129] Nicolas Boulanger, Nicolo Colombo, and Per Sundell. A minimal BV action for Vasiliev's four-dimensional higher spin gravity. *JHEP*, 10:043, 2012.
- [130] V. E. Didenko, N. G. Misuna, and M. A. Vasiliev. Charges in nonlinear higher-spin theory. 2015.
- [131] Roberto Bonezzi, Nicolas Boulanger, Ergin Sezgin, and Per Sundell. An Action for Matter Coupled Higher Spin Gravity in Three Dimensions. 2015.
- [132] Cesar Arias, Roberto Bonezzi, Nicolas Boulanger, Ergin Sezgin, Per Sundell, Alexander Torres-Gomez, and Mauricio Valenzuela. Action principles for higher and fractional spin gravities. In *International Workshop on Higher Spin Gauge Theories Singapore, Singapore, November 4-6, 2015*, 2016.
- [133] C. Fronsdal. Elementary particles in a curved space. ii. *Phys. Rev.*, D10:589–598, 1974.
- [134] Christian Fronsdal. Singletons and Massless, Integral Spin Fields on de Sitter Space (Elementary Particles in a Curved Space. 7. *Phys. Rev.*, D20:848–856, 1979.
- [135] R. R. Metsaev. Massless mixed symmetry bosonic free fields in  $d$ -dimensional anti-de Sitter space-time. *Phys. Lett.*, B354:78–84, 1995.
- [136] R. R. Metsaev. Arbitrary spin massless bosonic fields in  $d$ -dimensional anti-de Sitter space. *Lect. Notes Phys.*, 524:331–340, 1999.
- [137] Itzhak Bars and Costas Kounnas. Theories with two times. *Phys. Lett.*, B402:25–32, 1997.
- [138] Tirthabir Biswas and W. Siegel. Radial dimensional reduction: Anti-de Sitter theories from flat. *JHEP*, 07:005, 2002.

- 
- [139] X. Bekaert, I. L. Buchbinder, A. Pashnev, and M. Tsulaia. On higher spin theory: Strings, BRST, dimensional reductions. *Class. Quant. Grav.*, 21:S1457–1464, 2004.
- [140] Karl Hallowell and Andrew Waldron. Constant curvature algebras and higher spin action generating functions. *Nucl. Phys.*, B724:453–486, 2005.
- [141] Glenn Barnich and Maxim Grigoriev. Parent form for higher spin fields on anti-de Sitter space. *JHEP*, 08:013, 2006.
- [142] Angelos Fotopoulos, Kamal L. Panigrahi, and Mirian Tsulaia. Lagrangian formulation of higher spin theories on AdS space. *Phys. Rev.*, D74:085029, 2006.
- [143] Konstantin B. Alkalaev and Maxim Grigoriev. Unified BRST description of AdS gauge fields. *Nucl. Phys.*, B835:197–220, 2010.
- [144] Lorenzo Cornalba, Miguel S. Costa, and Joao Penedones. Deep Inelastic Scattering in Conformal QCD. *JHEP*, 03:133, 2010.
- [145] Steven Weinberg. Six-dimensional Methods for Four-dimensional Conformal Field Theories. *Phys. Rev.*, D82:045031, 2010.
- [146] Joao Penedones. Writing CFT correlation functions as AdS scattering amplitudes. *JHEP*, 03:025, 2011.
- [147] Miguel F. Paulos. Towards Feynman rules for Mellin amplitudes. *JHEP*, 10:074, 2011.
- [148] Miguel S. Costa, Joao Penedones, David Poland, and Slava Rychkov. Spinning Conformal Correlators. *JHEP*, 1111:071, 2011.
- [149] Miguel S. Costa, Joao Penedones, David Poland, and Slava Rychkov. Spinning Conformal Blocks. *JHEP*, 11:154, 2011.

- 
- [150] A. Liam Fitzpatrick, Jared Kaplan, Joao Penedones, Suvrat Raju, and Balt C. van Rees. A Natural Language for AdS/CFT Correlators. *JHEP*, 11:095, 2011.
- [151] David Simmons-Duffin. Projectors, Shadows, and Conformal Blocks. *JHEP*, 04:146, 2014.
- [152] Miguel S. Costa, Vasco Gonçalves, and João Penedones. Spinning AdS Propagators. *JHEP*, 1409:064, 2014.
- [153] Peter Breitenlohner and Daniel Z. Freedman. Positive Energy in anti-De Sitter Backgrounds and Gauged Extended Supergravity. *Phys. Lett.*, B115:197, 1982.
- [154] M. A. Vasiliev. Higher spin gauge theories in various dimensions. *Fortsch. Phys.*, 52:702–717, 2004. [,137(2004)].
- [155] Dmitri Sorokin. Introduction to the classical theory of higher spins. *AIP Conf. Proc.*, 767:172–202, 2005. [,172(2004)].
- [156] X. Bekaert, S. Cnockaert, Carlo Iazeolla, and M. A. Vasiliev. Nonlinear higher spin theories in various dimensions. In *Higher spin gauge theories: Proceedings, 1st Solvay Workshop*, pages 132–197, 2004.
- [157] Andrea Campoleoni. Metric-like Lagrangian Formulations for Higher-Spin Fields of Mixed Symmetry. *Riv. Nuovo Cim.*, 33:123–253, 2010.
- [158] Matthias R. Gaberdiel and Rajesh Gopakumar. Minimal Model Holography. *J. Phys.*, A46:214002, 2013.
- [159] Simone Giombi and Xi Yin. The Higher Spin/Vector Model Duality. *J. Phys.*, A46:214003, 2013.
- [160] A. Sagnotti. Notes on Strings and Higher Spins. *J. Phys.*, A46:214006, 2013. [,183(2013)].
- [161] Rakibur Rahman. Higher Spin Theory - Part I. *PoS*, ModaveVIII:004, 2012.

- [162] Gustavo Lucena Gómez. *Aspects of Higher-Spin Theory with Fermions*. PhD thesis, Brussels U., 2014.
- [163] V. E. Didenko and E. D. Skvortsov. *Elements of Vasiliev theory*. 2014.
- [164] Rakibur Rahman and Massimo Taronna. *From Higher Spins to Strings: A Primer*. 2015.
- [165] Sumit R. Das and Antal Jevicki. Large  $N$  collective fields and holography. *Phys. Rev.*, D68:044011, 2003.
- [166] Michael R. Douglas, Luca Mazzucato, and Shlomo S. Razamat. Holographic dual of free field theory. *Phys. Rev.*, D83:071701, 2011.
- [167] Robert de Mello Koch, Antal Jevicki, Kewang Jin, and Joao P. Rodrigues. AdS<sub>4</sub>/CFT<sub>3</sub> Construction from Collective Fields. *Phys. Rev.*, D83:025006, 2011.
- [168] Antal Jevicki, Kewang Jin, and Qibin Ye. Collective Dipole Model of AdS/CFT and Higher Spin Gravity. *J. Phys.*, A44:465402, 2011.
- [169] Antal Jevicki, Kewang Jin, and Qibin Ye. Bi-local Model of AdS/CFT and Higher Spin Gravity. In *Proceedings, 11th Workshop on Non-Perturbative Quantum Chromodynamics*, 2011.
- [170] Robert de Mello Koch, Antal Jevicki, Kewang Jin, Joao P. Rodrigues, and Qibin Ye.  $S = 1$  in  $O(N)$ /HS duality. *Class. Quant. Grav.*, 30:104005, 2013.
- [171] Leopoldo A. Pando Zayas and Cheng Peng. Toward a Higher-Spin Dual of Interacting Field Theories. *JHEP*, 10:023, 2013.
- [172] Ivo Sachs. Higher spin versus renormalization group equations. *Phys. Rev.*, D90(8):085003, 2014.
- [173] Robert G. Leigh, Onkar Parrikar, and Alexander B. Weiss. Holographic geometry of the renormalization group and higher spin symmetries. *Phys. Rev.*, D89(10):106012, 2014.

- 
- [174] Robert G. Leigh, Onkar Parrikar, and Alexander B. Weiss. Exact renormalization group and higher-spin holography. *Phys. Rev.*, D91(2):026002, 2015.
- [175] Robert de Mello Koch, Antal Jevicki, Joo P. Rodrigues, and Junggi Yoon. Canonical Formulation of  $O(N)$  Vector/Higher Spin Correspondence. *J. Phys.*, A48(10):105403, 2015.
- [176] Eric Mintun and Joseph Polchinski. Higher Spin Holography, RG, and the Light Cone. 2014.
- [177] Kewang Jin, Robert G. Leigh, and Onkar Parrikar. Higher Spin Fronsdal Equations from the Exact Renormalization Group. *JHEP*, 06:050, 2015.
- [178] Simone Giombi and Xi Yin. Higher Spin Gauge Theory and Holography: The Three-Point Functions. *JHEP*, 09:115, 2010.
- [179] Simone Giombi and Xi Yin. Higher Spins in AdS and Twistorial Holography. *JHEP*, 04:086, 2011.
- [180] Simone Giombi and Igor R. Klebanov. One Loop Tests of Higher Spin AdS/CFT. *JHEP*, 12:068, 2013.
- [181] Simone Giombi, Igor R. Klebanov, and Benjamin R. Safdi. Higher Spin  $\text{AdS}_{d+1}/\text{CFT}_d$  at One Loop. *Phys. Rev.*, D89(8):084004, 2014.
- [182] Simone Giombi, Igor R. Klebanov, and Arkady A. Tseytlin. Partition Functions and Casimir Energies in Higher Spin  $\text{AdS}_{d+1}/\text{CFT}_d$ . *Phys. Rev.*, D90(2):024048, 2014.
- [183] Matteo Beccaria and Arkady A. Tseytlin. Higher spins in  $\text{AdS}_5$  at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT. *JHEP*, 11:114, 2014.
- [184] Jin-Beom Bae, Euihun Joung, and Shailesh Lal. One-loop test of free  $SU(N)$  adjoint model holography. *JHEP*, 04:061, 2016.

- 
- [185] Matteo Beccaria and Arkady A. Tseytlin. Vectorial AdS<sub>5</sub>/CFT<sub>4</sub> duality for spin-one boundary theory. *J. Phys.*, A47(49):492001, 2014.
- [186] Matteo Beccaria, Guido Macorini, and Arkady A. Tseytlin. Supergravity one-loop corrections on AdS<sub>7</sub> and AdS<sub>3</sub>, higher spins and AdS/CFT. *Nucl. Phys.*, B892:211–238, 2015.
- [187] Matteo Beccaria and A. A. Tseytlin. On higher spin partition functions. *J. Phys.*, A48(27):275401, 2015.
- [188] M. Beccaria and A. A. Tseytlin. Iterating free-field AdS/CFT: higher spin partition function relations. 2016.
- [189] S. Fubini, A. J. Hanson, and R. Jackiw. New approach to field theory. *Phys. Rev. D*, 7:1732–1760, Mar 1973.
- [190] David Simmons-Duffin. TASI Lectures on the Conformal Bootstrap. 2016.
- [191] Juan Maldacena and Alexander Zhiboedov. Constraining Conformal Field Theories with A Higher Spin Symmetry. *J. Phys.*, A46:214011, 2013.
- [192] Vasyl Alba and Kenan Diab. Constraining conformal field theories with a higher spin symmetry in d ≤ 3 dimensions. 2015.
- [193] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge University Press, 2007.
- [194] Duccio Pappadopulo, Slava Rychkov, Johnny Espin, and Riccardo Rattazzi. OPE Convergence in Conformal Field Theory. *Phys. Rev.*, D86:105043, 2012.
- [195] Paul A. M. Dirac. Wave equations in conformal space. *Annals Math.*, 37:429–442, 1936.
- [196] S. Ferrara, A. F. Grillo, and R. Gatto. Tensor representations of conformal algebra and conformally covariant operator product expansion. *Annals Phys.*, 76:161–188, 1973.

- [197] A. M. Polyakov. Non-Hamiltonian approach to conformal quantum field theory. *Soviet Journal of Experimental and Theoretical Physics*, 39:10, July 1974.
- [198] Riccardo Rattazzi, Vyacheslav S. Rychkov, Erik Tonni, and Alessandro Vichi. Bounding scalar operator dimensions in 4D CFT. *JHEP*, 12:031, 2008.
- [199] F. A. Dolan and H. Osborn. Conformal four point functions and the operator product expansion. *Nucl. Phys.*, B599:459–496, 2001.
- [200] F. A. Dolan and H. Osborn. Conformal partial waves and the operator product expansion. *Nucl. Phys.*, B678:491–507, 2004.
- [201] F.A. Dolan and H. Osborn. Conformal Partial Waves: Further Mathematical Results. 2011.
- [202] Joo Penedones, Emilio Trevisani, and Masahito Yamazaki. Recursion Relations for Conformal Blocks. 2015.
- [203] Matthijs Hogervorst and Slava Rychkov. Radial Coordinates for Conformal Blocks. *Phys. Rev.*, D87:106004, 2013.
- [204] Matthijs Hogervorst, Hugh Osborn, and Slava Rychkov. Diagonal Limit for Conformal Blocks in  $d$  Dimensions. *JHEP*, 08:014, 2013.
- [205] Luca Iliesiu, Filip Kos, David Poland, Silviu S. Pufu, David Simmons-Duffin, and Ran Yacoby. Bootstrapping 3D Fermions. *JHEP*, 03:120, 2016.
- [206] Alejandro Castedo Echeverri, Emtinan Elkhidir, Denis Karateev, and Marco Serone. Deconstructing Conformal Blocks in 4D CFT. *JHEP*, 08:101, 2015.
- [207] S. Ferrara and G. Parisi. Conformal covariant correlation functions. *Nucl. Phys.*, B42:281–290, 1972.
- [208] S. Ferrara, A. F. Grillo, and G. Parisi. Nonequivalence between conformal covariant wilson expansion in euclidean and minkowski space. *Lett. Nuovo Cim.*, 5S2:147–151, 1972. [Lett. Nuovo Cim.5,147(1972)].

- [209] S. Ferrara, A. F. Grillo, G. Parisi, and R. Gatto. The shadow operator formalism for conformal algebra. vacuum expectation values and operator products. *Lett. Nuovo Cim.*, 4S2:115–120, 1972. [Lett. Nuovo Cim.4,115(1972)].
- [210] S. Ferrara, A. F. Grillo, G. Parisi, and Raoul Gatto. Covariant expansion of the conformal four-point function. *Nucl. Phys.*, B49:77–98, 1972. [Erratum: Nucl. Phys.B53,643(1973)].
- [211] Miguel S. Costa and Tobias Hansen. Conformal correlators of mixed-symmetry tensors. *JHEP*, 02:151, 2015.
- [212] Fernando Rejon-Barrera and Daniel Robbins. Scalar-Vector Bootstrap. *JHEP*, 01:139, 2016.
- [213] S. Ferrara, A. F. Grillo, G. Parisi, and R. Gatto. Covariant expansion of the conformal four -point function. *Nuclear Physics B*, 49:77–98, December 1972.
- [214] V. K. Dobrev, V. B. Petkova, S. G. Petrova, and I. T. Todorov. Dynamical derivation of vacuum operator-product expansion in euclidean conformal quantum field theory. *Phys. Rev. D*, 13:887–912, Feb 1976.
- [215] V. K. Dobrev, G. Mack, V. B. Petkova, S. G. Petrova, and I. T. Todorov, editors. *Harmonic Analysis: on the  $n$ -Dimensional Lorentz Group and Its Application to Conformal Quantum Field Theory*, volume 63 of *Lecture Notes in Physics*, Berlin Springer Verlag, 1977.
- [216] Gerhard Mack. D-independent representation of Conformal Field Theories in D dimensions via transformation to auxiliary Dual Resonance Models. Scalar amplitudes. 2009.
- [217] N. S. Craigie, V. K. Dobrev, and I. T. Todorov. Conformally Covariant Composite Operators in Quantum Chromodynamics. *Annals Phys.*, 159:411–444, 1985.



- 
- [218] H. Osborn and A. C. Petkou. Implications of conformal invariance in field theories for general dimensions. *Annals Phys.*, 231:311–362, 1994.
- [219] Yassen S. Stanev. Correlation Functions of Conserved Currents in Four Dimensional Conformal Field Theory. *Nucl. Phys.*, B865:200–215, 2012.
- [220] Alexander Zhiboedov. A note on three-point functions of conserved currents. 2012.
- [221] R. R. Metsaev. Cubic interaction vertices of massive and massless higher spin fields. *Nucl. Phys.*, B759:147–201, 2006.
- [222] Ruben Manvelyan, Karapet Mkrtchyan, and Werner Rühl. General trilinear interaction for arbitrary even higher spin gauge fields. *Nucl. Phys.*, B836:204–221, 2010.
- [223] A. Fotopoulos and Mirian Tsulaia. On the Tensionless Limit of String theory, Off - Shell Higher Spin Interaction Vertices and BCFW Recursion Relations. *JHEP*, 11:086, 2010.
- [224] Simone Giombi, Shiroman Prakash, and Xi Yin. A Note on CFT Correlators in Three Dimensions. *JHEP*, 07:105, 2013.
- [225] Nicolo Colombo and Per Sundell. Higher Spin Gravity Amplitudes From Zero-form Charges. 2012.
- [226] V. E. Didenko and E. D. Skvortsov. Exact higher-spin symmetry in CFT: all correlators in unbroken Vasiliev theory. *JHEP*, 04:158, 2013.
- [227] J. Erdmenger and H. Osborn. Conserved currents and the energy momentum tensor in conformally invariant theories for general dimensions. *Nucl. Phys.*, B483:431–474, 1997.
- [228] Danilo E. Diaz and Harald Dorn. On the AdS higher spin /  $O(N)$  vector model correspondence: Degeneracy of the holographic image. *JHEP*, 0607:022, 2006.

- [229] Luis F. Alday and Alexander Zhiboedov. Conformal Bootstrap With Slightly Broken Higher Spin Symmetry. 2015.
- [230] Daniel Z. Freedman, Samir D. Mathur, Alec Matusis, and Leonardo Rastelli. Correlation functions in the  $CFT_d / AdS_{d+1}$  correspondence. *Nucl. Phys.*, B546:96–118, 1999.
- [231] Thorsten Leonhardt, Ruben Manvelyan, and Werner Rühl. The Group approach to AdS space propagators. *Nucl. Phys.*, B667:413–434, 2003.
- [232] Thorsten Leonhardt, Werner Rühl, and Ruben Manvelyan. The Group approach to AdS space propagators: A Fast algorithm. *J. Phys.*, A37:7051, 2004.
- [233] Ruben Manvelyan and Werner Rühl. The Off-shell behaviour of propagators and the Goldstone field in higher spin gauge theory on  $AdS_{d+1}$  space. *Nucl. Phys.*, B717:3–18, 2005.
- [234] Ruben Manvelyan, Karapet Mkrtchyan, and Werner Rühl. Ultraviolet behaviour of higher spin gauge field propagators and one loop mass renormalization. *Nucl. Phys.*, B803:405–427, 2008.
- [235] C. P. Burgess and C. A. Lutken. Propagators and Effective Potentials in Anti-de Sitter Space. *Phys. Lett.*, B153:137, 1985.
- [236] Christopher J.C Burges, Daniel Z Freedman, S Davis, and G.W Gibbons. Supersymmetry in anti-de sitter space. *Annals of Physics*, 167(2):285 – 316, 1986.
- [237] Bruce Allen and Theodore Jacobson. Vector Two Point Functions in Maximally Symmetric Spaces. *Commun. Math. Phys.*, 103:669, 1986.
- [238] Bruce Allen and Michael Turyn. An Evaluation of the Graviton Propagator in De Sitter Space. *Nucl. Phys.*, B292:813, 1987.
- [239] Michael Turyn. The Graviton Propagator in Maximally Symmetric Spaces. *J. Math. Phys.*, 31:669, 1990.

- 
- [240] Eric D'Hoker, Daniel Z. Freedman, Samir D. Mathur, Alec Matusis, and Leonardo Rastelli. Graviton and gauge boson propagators in  $\text{AdS}_{d+1}$ . *Nucl. Phys.*, B562:330–352, 1999.
- [241] Eric D'Hoker and Daniel Z. Freedman. General scalar exchange in  $\text{AdS}_{d+1}$ . *Nucl. Phys.*, B550:261–288, 1999.
- [242] Eric D'Hoker, Daniel Z. Freedman, Samir D. Mathur, Alec Matusis, and Leonardo Rastelli. Graviton exchange and complete four point functions in the AdS / CFT correspondence. *Nucl. Phys.*, B562:353–394, 1999.
- [243] N. Bouatta, G. Compere, and A. Sagnotti. An Introduction to free higher-spin fields. In *Higher spin gauge theories: Proceedings, 1st Solvay Workshop*, pages 79–99, 2004.
- [244] Dario Francia. Geometric Lagrangians for massive higher-spin fields. *Nucl. Phys.*, B796:77–122, 2008.
- [245] Dario Francia. On the Relation between Local and Geometric Lagrangians for Higher spins. *J. Phys. Conf. Ser.*, 222:012002, 2010.
- [246] R.R. Metsaev. Light cone form of field dynamics in Anti-de Sitter space-time and AdS / CFT correspondence. *Nucl. Phys.*, B563:295–348, 1999.
- [247] V. K. Dobrev. Intertwining operator realization of the AdS / CFT correspondence. *Nucl. Phys.*, B553:559–582, 1999.
- [248] V. E. Didenko and E. D. Skvortsov. Towards higher-spin holography in ambient space of any dimension. *J. Phys.*, A46:214010, 2013.
- [249] C. Fronsdal. Elementary particles in a curved space. ii. *Phys. Rev. D*, 10:589–598, Jul 1974.
- [250] A. Papapetrou. Einstein's theory of gravitation and flat space. *Proc. Roy. Irish Acad. (Sect. A)*, 52A:11–23, 1948.

- 
- [251] S. N. Gupta. Quantization of Einstein's gravitational field: general treatment. *Proc. Phys. Soc.*, A65:608–619, 1952.
- [252] Robert H. Kraichnan. Special-Relativistic Derivation of Generally Covariant Gravitation Theory. *Phys. Rev.*, 98:1118–1122, 1955.
- [253] Richard P Feynman, Fernando B Morinigo, and William G Wagner. Feynman lectures on gravitation. *European Journal of Physics*, 24(3):330, 2003.
- [254] Stanley Deser. Selfinteraction and gauge invariance. *Gen. Rel. Grav.*, 1:9–18, 1970.
- [255] Daniel Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara. Progress Toward a Theory of Supergravity. *Phys. Rev.*, D13:3214–3218, 1976.
- [256] Stanley Deser and B. Zumino. Consistent Supergravity. *Phys. Lett.*, B62:335, 1976.
- [257] K. S. Stelle and Peter C. West. Minimal Auxiliary Fields for Supergravity. *Phys. Lett.*, B74:330–332, 1978.
- [258] S. Ferrara and P. van Nieuwenhuizen. The Auxiliary Fields of Supergravity. *Phys. Lett.*, B74:333, 1978.
- [259] E. Cremmer, B. Julia, and Joel Scherk. Supergravity Theory in Eleven-Dimensions. *Phys. Lett.*, B76:409–412, 1978.
- [260] K. B. Alkalaev and M. A. Vasiliev.  $\mathcal{N}=1$  supersymmetric theory of higher spin gauge fields in  $\text{AdS}_5$  at the cubic level. *Nucl. Phys.*, B655:57–92, 2003.
- [261] Pan Kessel, Gustavo Lucena Gómez, E.D. Skvortsov, and Massimo Taronna. Higher Spins and Matter Interacting in Dimension Three. 2015.
- [262] Mikhail A. Vasiliev. Dynamics of Massless Higher Spins in the Second Order in Curvatures. *Phys. Lett.*, B238:305–314, 1990.

- [263] Anastasios Petkou. Conserved currents, consistency relations and operator product expansions in the conformally invariant  $O(N)$  vector model. *Annals Phys.*, 249:180–221, 1996.
- [264] E. D. Skvortsov. On (Un)Broken Higher-Spin Symmetry in Vector Models. 2015.
- [265] M. Ostrogradsky. *Mem. Ac. St. Petersbourg*, VI 4:385, 1850.
- [266] D. A. Eliezer and R. P. Woodard. Instability of Higher Difference Initial Value Theories. *Phys. Rev.*, D40:465, 1989.
- [267] D. A. Eliezer and R. P. Woodard. The Problem of Nonlocality in String Theory. *Nucl. Phys.*, B325:389, 1989.
- [268] F. Mller-Hoissen. Higher derivative versus second order field equations. *Annalen der Physik*, 503(8):543–557, 1991.
- [269] Jonathan Z. Simon. Higher Derivative Lagrangians, Nonlocality, Problems and Solutions. *Phys. Rev.*, D41:3720, 1990.
- [270] Hans-Jurgen Schmidt. Stability and Hamiltonian formulation of higher derivative theories. *Phys. Rev.*, D49:6354, 1994. [Erratum: *Phys. Rev.*D54,7906(1996)].
- [271] Takao Nakamura and Shinji Hamamoto. Higher derivatives and canonical formalisms. *Prog. Theor. Phys.*, 95:469–484, 1996.
- [272] D. L. Bennett, Holger Bech Nielsen, and R. P. Woodard. The Initial value problem for maximally nonlocal actions. *Phys. Rev.*, D57:1167–1170, 1998.
- [273] R. P. Woodard. A Canonical formalism for Lagrangians with nonlocality of finite extent. *Phys. Rev.*, A62:052105, 2000.
- [274] K. Bering. A Note on nonlocality and Ostrogradski’s construction. 2000.
- [275] Tai-Chung Cheng, Pei-Ming Ho, and Mao-Chuang Yeh. Perturbative approach to higher derivative and nonlocal theories. *Nucl. Phys.*, B625:151–165, 2002.

- 
- [276] Michio Kaku and K. Kikkawa. Field theory of relativistic strings. i. trees. *Phys. Rev. D*, 10:1110–1133, Aug 1974.
- [277] M Kaku and K Kikkawa. The field theory of relativistic strings, pt 2; loops and Pomerons. Technical report, May 1974.
- [278] Theodore Erler and David J. Gross. Locality, causality, and an initial value formulation for open string field theory. 2004.
- [279] M. A. Vasiliev. Star-Product Functions in Higher-Spin Theory and Locality. *JHEP*, 06:031, 2015.
- [280] E. D. Skvortsov and Massimo Taronna. On Locality, Holography and Unfolding. *JHEP*, 11:044, 2015.
- [281] Massimo Taronna. Pseudo-local Theories: A Functional Class Proposal. In *International Workshop on Higher Spin Gauge Theories Singapore, Singapore, November 4-6, 2015*, 2016.
- [282] R. J. Eden, P. V. Landshoff, and D. I. Olive. *The analytic S-matrix*. 1966.
- [283] Joseph Polchinski. S matrices from AdS space-time. 1999.
- [284] Leonard Susskind. Holography in the flat space limit. pages 98–112, 1998. [AIP Conf. Proc.493,98(1999)].
- [285] Steven B. Giddings. The Boundary S matrix and the AdS to CFT dictionary. *Phys. Rev. Lett.*, 83:2707–2710, 1999.
- [286] Steven B. Giddings. Flat space scattering and bulk locality in the AdS / CFT correspondence. *Phys. Rev.*, D61:106008, 2000.
- [287] Gerhard Mack. D-dimensional Conformal Field Theories with anomalous dimensions as Dual Resonance Models. *Bulg. J. Phys.*, 36:214–226, 2009.

- 
- [288] A. Liam Fitzpatrick and Jared Kaplan. Analyticity and the Holographic S-Matrix. *JHEP*, 10:127, 2012.
- [289] A. Liam Fitzpatrick and Jared Kaplan. Unitarity and the Holographic S-Matrix. *JHEP*, 10:032, 2012.
- [290] Miguel S. Costa, Vasco Gonçalves, and Joao Penedones. Conformal Regge theory. *JHEP*, 1212:091, 2012.
- [291] Philippe Flajolet, Xavier Gourdon, and Philippe Dumas. Mellin transforms and asymptotics: Harmonic sums. *Theoretical Computer Science*, 144(12):3 – 58, 1995.
- [292] Bertrand P. Bertrand, J. and J.-P. Ovarlez. “The Mellin transform”, in A. D. Poularikas (ed.), *Transforms and applications handbook (CRC Press, 2000) chapter 11*. CRC Press, 2000.
- [293] E. Joung, S. Nakach, and A. A. Tseytlin. Scalar scattering via conformal higher spin exchange. *JHEP*, 02:125, 2016.
- [294] Tracy Yerkes Thomas. On conformal geometry. *Proceedings of the National Academy of Sciences*, 12(5):352–359, 1926.
- [295] V. K. Dobrev, V. B. Petkova, S. G. Petrova, and I. T. Todorov. Dynamical Derivation of Vacuum Operator Product Expansion in Euclidean Conformal Quantum Field Theory. *Phys. Rev.*, D13:887, 1976.
- [296] K. Symanzik. On Calculations in conformal invariant field theories. *Lett. Nuovo Cim.*, 3:734–738, 1972.
- [297] Xavier Bekaert, Euihun Joung, and Jihad Mourad. On higher spin interactions with matter. *JHEP*, 05:126, 2009.
- [298] Idse Heemskerk, Joao Penedones, Joseph Polchinski, and James Sully. Holography from Conformal Field Theory. *JHEP*, 0910:079, 2009.

- [299] Frits A. Berends, G. J. H. Burgers, and H. van Dam. Explicit Construction of Conserved Currents for Massless Fields of Arbitrary Spin. *Nucl. Phys.*, B271:429, 1986.
- [300] Damiano Anselmi. Higher spin current multiplets in operator product expansions. *Class. Quant. Grav.*, 17:1383–1400, 2000.



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