Moduli Spaces of Parabolic Twisted Generalized Higgs Bundles

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A

Abstract

In this thesis we study moduli spaces of decorated parabolic principal G-bundles on a compact Riemann surface X.
In [Sch08] Alexander Schmitt constructed the moduli space of affine $\tilde{\varrho}$-Higgs bundles[1] (P, φ) consisting of a principal G-bundle P on X and a global section $\varphi \in H^0(X, P_{\tilde{\varrho}} \otimes L)$ as a GIT-quotient. Here L is a line bundle on X and $P_{\tilde{\varrho}}$ is the vector bundle associated to P by a rational representation $\tilde{\varrho}$ of the reductive algebraic group G. $\tilde{\varrho}$-Higgs bundles are generalizations of several well-studied objects, such as G-Higgs bundles, Bradlow pairs or quiver representations.
In this work we generalize this GIT-construction of the moduli space of affine $\tilde{\varrho}$-Higgs bundles to the case of affine parabolic $\tilde{\varrho}$-Higgs bundles. A parabolic structure on P over a fixed finite subset S of punctures x^j of the compact Riemann surface X is given by reductions $s^j : \{x^j\} \to P \times_X \{x^j\}/P^j$; P^j a parabolic subgroup of G. Our main result shows the existence of the resulting moduli space $\mathscr{M}_{\mathrm{par}}$ of decorated parabolic bundles as a quasi-projective scheme over $\mathbb{C}$.
For a suitable choice of $\tilde{\varrho}$, i. e. $\tilde{\varrho}$ the adjoint representation of G on its Lie algebra $\mathfrak{g}$, the moduli space of parabolic G-Higgs bundles (see [Sim94]) is obtained from our construction by slight modifications of the semistability concept. Other important applications include the construction of a (generalized) projective Hitchin morphism from $\mathscr{M}_{\mathrm{par}}$ into an affine scheme $\mathbb{Hit}$ as well as an extension of the results of Nikolai Beck [Be14] on moduli spaces of pointwisely decorated principal bundles.

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[1] An affine Higgs bundle is called "bump" in [Sch08].

ZUSAMMENFASSUNG

In der vorliegenden Dissertation untersuchen wir Modulräume dekorierter parabolischer G-Hauptfaserbündel über einer kompakten Riemannschen Fläche X.
Alexander Schmitt konstruiert in [Sch08] erstmals den Modulraum affiner $\tilde{\varrho}$-Higgsbündel[2] (P, φ) bestehend aus einem G-Hauptfaserbündel P über X sowie einem globalen Schnitt $\varphi \in H^0(X, P_{\tilde{\varrho}} \otimes L)$ als GIT-Quotient. Hierbei bezeichnet L ein Geradenbündel auf X und $P_{\tilde{\varrho}}$ das durch eine rationale Darstellung $\tilde{\varrho}$ einer reduktiven algebraischen Gruppe G zu P assoziierte Vektorbündel. $\tilde{\varrho}$-Higgsbündel enthalten als wichtige Spezialfälle unter anderem G-Higgsbündel, Bradlow-Paare und gewisse Quiverdarstellungen.
In dieser Arbeit erweitern wir diese GIT-Konstruktion des Modulraums affiner $\tilde{\varrho}$-Higgsbündel auf den Fall affiner parabolischer $\tilde{\varrho}$-Higgsbündel. Eine parabolische Struktur auf P über einer vorgegebenen Menge S von Punktierungen der kompakten Riemannschen Fläche X ist gegeben durch Reduktionen $s^j : \{x^j\} \to P \times_X \{x^j\}/P^j$; P^j ist dabei eine parabolische Untergruppe von G. Als Hauptresultat zeigen wir, dass der resultierende Modulraum $\mathscr{M}_{\mathrm{par}}$ dekorierter parabolischer Hauptfaserbündel als quasi-projektives Schema über $\mathbb{C}$ existiert.
Nach kleineren Modifikationen des Semistabilitätsbegriffes ergibt sich der Modulraum parabolischer G-Higgsbündel (siehe [Sim94]) für eine gewisse Wahl von $\tilde{\varrho}$, d. h. für $\tilde{\varrho}$ die adjungierte Darstellung von G auf ihrer Lie Algebra $\mathfrak{g}$, als Spezialfall unserer allgemeinen Konstruktion. Weitere wichtige Anwendungen beinhalten die Konstruktion einer (verallgemeinerten) projektiven Hitchin-Abbildung von $\mathscr{M}_{\mathrm{par}}$ in ein affines Schema $\mathbb{H}\mathrm{it}$ sowie eine Erweiterung der Ergebnisse von Nikolai Beck [Be14] zu Modulräumen punktweise dekorierter G-Hauptfaserbündel.

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[2] Ein affines Higgsbündel wird in [Sch08] mit „bump“ bezeichnet.

C CONTENTS

I

INTRODUCTION

I.1. Parabolic structures of vector bundles on a punctured Riemann surface were probably first defined by Mehta and Seshadri in [MS80]. Their extension of a classical result by Narasimhan and Seshadri [NS65] identifies reductive unitary representations of the orbifold fundamental group $\pi_1^{\mathrm{orb}}(X)$ with semistable vector bundles of parabolic degree 0. The bijection respects the two natural concepts of equivalence, namely conjugation of the representation and isomorphy of vector bundles and thus descends to a homeomorphism on the corresponding moduli spaces. Bhosle [Bho89] extended the result to connected reductive complex algebraic groups.

Carlos Simpson introduced in [Sim90] the concept of parabolic Higgs bundles and related tame semistable parabolic Higgs bundles of degree 0 to certain local systems, flat C^∞-bundles, as well as tame harmonic bundles, i. e. solutions of a Hermitian-Einstein equation. The corresponding moduli spaces (see [Sim94]) possess a rich geometric structure. They occur as a hyperkähler quotient, form a completely integrable Hamiltonian system, where the leaves of the corresponding Lagrangean foliation are just the fibers of the Hitchin morphism, and admit a projectively flat connection.[1] These properties lead to further applications, for example in the Geometric Langlands Program (e. g. [DP09], [GW08]) or as examples of a SYZ duality (e. g. [BD12]).

A rank r Higgs vector bundle is a rank r vector bundle E on X together with a Higgs field $\varphi : E \to E \otimes \omega_X$. The Higgs field amounts to a section $H^0(X, \mathrm{End}(E) \otimes \omega_X) \simeq H^0(X, E_{\mathrm{Ad}} \otimes \omega_X)$ where E_{Ad} is the vector bundle which is associated to the corresponding $\mathrm{Gl}(\mathbb{C}^r)$-bundle E by the adjoint representation $\mathrm{Ad} : \mathrm{Gl}(\mathbb{C}^r) \times \mathfrak{gl}(\mathbb{C}^r) \to \mathfrak{gl}(\mathbb{C}^r)$ on the Lie algebra $\mathfrak{gl}(\mathbb{C}^r) = \mathrm{Lie}(\mathrm{Gl}(\mathbb{C}^r))$. If we replace E by a principal G-bundle P, the adjoint representation by an arbitrary linear representation $\tilde{\varrho}$ of G and ω_X by an arbitrary line bundle on X, we get an

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[1] for the non-parabolic case see e. g. [Hit87], [Hit90], [ADW91] and for the parabolic case see e. g. [Fal93], [ScSc95].

affine $\tilde{\varrho}$-Higgs bundle. The moduli space of affine $\tilde{\varrho}$-Higgs bundles has been constructed by Alexander Schmitt in [Sch08]. Apart from Higgs vector bundles there are several other well-studied objects that occur as instances of $\tilde{\varrho}$-Higgs bundles, for example Bradlow pairs, conic bundles or augmented quiver representations. More examples and details on how these objects look in terms of $\tilde{\varrho}$-Higgs bundles may be found in great detail in [Sch08].
Moreover the Kobayashi-Hitchin correspondence extends to the case of (non-parabolic) $\tilde{\varrho}$-Higgs bundles ([LT06] or [GGM12]). Further applications include for example Kapustins work on mirror symmetry [Kap06].

Main Results

I.2. Let X be a compact Riemann surface, S a finite subset thereof and G a reductive algebraic group over $\mathbb{C}$. Let Y be a scheme of finite type over $\mathbb{C}$ and $P^j \subset G$ parabolic subgroups for each $x^j \in S$. A Y-family of parabolic G-bundles is a principal G-bundle $\mathscr{P}_Y$ over $Y \times X$ together with reductions $s^j : Y \times \{x^j\} \to \mathscr{P}_Y \times_X (Y \times \{x^j\})/P^j$. By a result of Drinfeld and Simpson[2] we may assume that $\mathscr{P}_Y$ is locally trivial w. r. t. the product of the étale topology on Y and the Zariski topology on X.
Given a representation $\varrho : G \to \mathrm{Gl}(W)$ and a line bundle L on X a Y-family of (affine) parabolic ϱ-Higgs bundles (or ϱ-bumps) is a Y-family of parabolic G-bundles together with a homomorphism $\varphi : \mathscr{P}_{Y,\varrho} \to \pi_X^*(L)$.
The **main result 3.19** of this thesis is the construction of a quasi-projective coarse moduli space for the functors[3]

$$\begin{array}{rcl}\mathbf{M}^{s(s)} : \mathbf{Sch}_{\mathbb{C}} & \to & \mathbf{Sets} \\ Y & \mapsto & \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ Y\text{-families of (semi)stable} \\ \text{affine parabolic } \varrho\text{-Higgs bundles} \end{array} \right\}. \end{array}$$

Moreover a projective moduli space for projective ς-Higgs bundles[4] $(P,\ (s^j)_{j[|S|]},\ \varphi,\ L)$ is constructed in 2.40 for every homogeneous representation ς and non-trivial φ. The results will be applied to obtain among others:

- the moduli space of parabolic G-Higgs bundles in the special case when ϱ is the coadjoint representation of G on the dual $\mathfrak{g}^\vee$ of its Lie algebra $\mathfrak{g}$;
- the projective moduli space of parabolic Hitchin pairs;
- an extension of the construction of Nikolai Beck (see [Be14]) of moduli spaces of pointwisely decorated principal bundles;
- a generalized Hitchin morphism.

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[2] [DS95].

[3] $\mathbf{Sch}_{\mathbb{C}}$ denotes the category of schemes of finite type over $\mathbb{C}$.

[4] cf. the notational remarks below.

Synopsis

I.3. An affine parabolic ϱ-Higgs bundle $(P,\ \varphi : P_\varrho \to L,\ (s^j)_{j[|S|]})$ gives rise to a projective parabolic ς-Higgs bundle $(P,\ \varphi_\varsigma : P_\varsigma \to \tilde{L},\ (s^j)_{j[|S|]})$ where ς is a homogeneous representation of G constructed from ϱ and $\tilde{L}$ is a line bundle on X depending on L. Projective parabolic ς-Higgs bundles on the other hand identify under a to be constructed closed embedding $\iota : G \hookrightarrow \bigtimes_{a\in A} \mathrm{Gl}(\mathbb{C}^{r_a})$ with projective $\underline{\varsigma}$-Higgs bundles for a homogeneous representation $\underline{\varsigma}$ of $\bigtimes_{a\in A} \mathrm{Gl}(\mathbb{C}^{r_a})$. The properties of the homogeneous representation ς finally define a resulting finite tuple $E = (E_a)_{a[|A|]}$ of vector bundles E_a together with the global homomorphism $\varphi : (E^{\otimes u})^{\oplus v} \to \det(E)^{\otimes w} \otimes L$ for certain integers $u,\ v,\ w$. We call these objects Higgs tuples. A parabolic Higgs tuple additionally admits parabolic filtrations, i. e. vector space filtrations of the stalks $E|_{x^j}$ over the punctures $x^j \in S$. These are of particular interest to us since the moduli problem for parabolic Higgs tuples can be solved by classical Geometric Invariant Theory as originally introduced by Mumford in [MFK]. We extend the approach of [Sch08] to the parabolic situation. The first two sections are devoted to the construction of a parameter scheme for Higgs tuples. If we wish to construct a parameter scheme for Higgs tuples we first have to show that those vector bundles E_a (of fixed rank r_a and degree d_a) underlying a Higgs tuple live in a bounded family. While this is not the case in general, the subfamily of vector bundles underlying a semistable Higgs tuple is in fact bounded. Now the vector bundle part of a Higgs tuple is parametrized by a tuple of (open subsets of) Quot schemes $\mathfrak{Q}_a$ and we are able to build a parameter scheme $\mathfrak{T}$ for non-parabolic Higgs tuples. Adding Graßmannian varieties for every puncture results in a parameter scheme $\mathfrak{T}_{\mathrm{par}}$ for parabolic Higgs tuples. In section three we check the universal properties of $\mathfrak{T}_{\mathrm{par}}$ necessary to construct moduli spaces as quotients of the given parameter scheme. $\mathfrak{Q}_a$ comes with a natural group action $\mathcal{G}_A$ that extends to $\mathfrak{T}_{\mathrm{par}}$. In order to realize the moduli space of parabolic Higgs tuples as $\mathfrak{T}_{\mathrm{par}}^{(s)s} /\!\!/ \mathcal{G}_A$ we construct an equivariant morphism **Gies** from $\mathfrak{T}_{\mathrm{par}}^{(s)s}$ into some projective space $\mathbb{P}$ and try to pull back an existing projective GIT-Quotient $\mathbb{P}^{(s)s} /\!\!/ \mathcal{G}_A$. This will work under two conditions: first of all the morphism **Gies** should be at least finite, secondly **Gies** should be compatible with the intrinsically defined semistability concepts on $\mathfrak{T}_{\mathrm{par}}$ and the GIT-semistability on $\mathbb{P}$ given by a properly chosen linearization in some ample line bundle on $\mathbb{P}$. While the first condition is satisfied by 1.59, the second one follows in 1.52 after some rather lengthy calculations in the sections 1.6 to 1.9. The existence of the coarse moduli space of stable Higgs pairs is proved in 1.64. We postpone the discussion of S-equivalence to chapter 4. The last two sections deal with slightly modified moduli problems. In particular, we prove the existence of the moduli space of pointwisely decorated Higgs tuples formerly constructed by Nikolai Beck in his dissertation [Be14] for a different choice of stability parameters.

Chapter 2 is devoted to the study of projective parabolic ς-Higgs bundles. We first define a semistability concept for parabolic fiber bundles. Given our faithful representation ι the subsequently defined concept of a pseudo parabolic $(\underline{\varsigma} \circ \iota)$-Higgs bundle helps us relate semistable parabolic ς-Higgs bundles to semistable Higgs tuples, where $\underline{\varsigma}$ is a homogeneous representation chosen such that $\varsigma \subset \underline{\varsigma} \circ \iota$. This one-to-one correspondence allows us to deduce the existence of the projective moduli space $\mathfrak{P}^{(s)s}_{\underline{\varsigma}\circ\iota} /\!\!/ \mathcal{G}_A$ of pseudo non-parabolic ς-Higgs bundles like in [Sch08] from the the existence of $\mathfrak{T}^{(s)s} /\!\!/ \mathcal{G}_A$ using again a finite morphism to pull back the GIT-quotient. It turns out later that $\mathfrak{P}_{\underline{\varsigma}\circ\iota}$ contains a parameter scheme $\mathfrak{P}_{\varsigma}$ for non-parabolic ς-Higgs bundles as a closed subscheme. Hence the moduli space $\mathfrak{P}_{\varsigma} /\!\!/ \mathcal{G}_A$ exists as a projective scheme, too. The final section 2.4 of chapter 2 constructs a parameter scheme $\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}$ for pseudo parabolic $(\underline{\varsigma} \circ \iota)$-Higgs bundles as a fiber bundle over the parameter scheme $\mathfrak{P}_{\underline{\varsigma}\circ\iota}$ of pseudo non-parabolic $(\underline{\varsigma} \circ \iota)$-Higgs bundles. We show that the finite morphism constructed in the non-parabolic case can be lifted to a finite equivariant morphism between the parameter schemes for parabolic objects. This morphism moreover preserves parabolic semistability. The moduli space of pseudo parabolic $(\underline{\varsigma} \circ \iota)$-Higgs bundles $\mathfrak{P}^{(s)s}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}} /\!\!/ \mathcal{G}_A$ exists as a projective scheme.

The third chapter starts with a discussion of asymptotic semistability. We give a new proof for the boundedness of the family of vector bundles underlying a ε-semistable pseudo (non-parabolic or parabolic) $(\underline{\varsigma} \circ \iota)$-Higgs bundle for any choice of a stability parameter $\varepsilon > 0$. This result allows us to show that semistable $(\underline{\varsigma} \circ \iota)$-Higgs bundles correspond to semistable pseudo $(\underline{\varsigma} \circ \iota)$-Higgs bundles, as claimed in chapter 2. Finally we are in the situation to address the existence of a moduli space for the functors given in I.2. Isomorphism classes of semistable affine ϱ-Higgs bundles map finite-to-one to isomorphism classes of asymptotically semistable projective ς-Higgs bundles. The constructions of chapter 2 may be used now to prove our main result: the existence of a quasi-projective moduli space $\mathfrak{A}^{(s)s}_{\mathrm{par}} /\!\!/ \mathcal{G}_A$ of affine parabolic ϱ-Higgs bundles in 3.19.
In section 3.4 it turns out that the semistability concept used so far does not allow any stable objects to exist if G is not semisimple. In particular it fails to extend the known stability criteria for G-bundles (cf. [Ram96i]) or G-Higgs bundles (e. g. in [GGM12]) in the general reductive case. Using a central isogeny this deficit will be overcome. The last section of chapter 3 extends the Hitchin morphism constructed in the non-parabolic case by [Sch08] to a projective morphism **Hit** from the moduli space $\mathfrak{A}^{(s)s}_{\mathrm{par}} /\!\!/ \mathcal{G}_A$ into an affine scheme $\mathbb{Hit}$.

We decided to put the treatment of S-equivalence into a separate chapter 4. This allows us to define S-equivalence for all occurring objects at once and relate the concepts immediately. Note that the existence of a moduli space of semistable objects is only really established once S-equivalence is treated.

The final chapter 5 rewrites the semistability concept in terms more suited to the formulation of the Kobayashi-Hitchin correspondence in 5.2. We will recover the concept of a tame parabolic Higgs bundle as originally defined in [Sim90]. The moduli space of tame parabolic Higgs bundles is constructed as a closed subscheme of the moduli space of affine parabolic Ad-Higgs bundles. Furthermore the moduli space of Hitchin pairs exists as a projective scheme.

Notation. A scheme (if not specified differently) is assumed to be a scheme of finite type over $\mathbb{C}$. A vector bundle is assumed to be algebraic. A reductive group G is assumed to be **connected**. However all results extend as in remark 2.7.5.4, [Sch08] to non-connected reductive groups.
If a semistability criterion is checked against one-parameter subgroups or filtrations, we assume those to be **non-trivial**. In some theorems or definitions we will use brackets to treat several (slightly differing) versions at once. For example there are some theorems that work for both parabolic and non-parabolic objects, i. e. [parabolic] G-bundles $(P,\ [(s^j)_{j[|S|]}])$. Most prominent example is the definition of (semi)stability. The symbol $(\leq)$ stands for $\leq$ in the semistable version of the definition, and for $<$ in the stable version.
π_Y will (if not otherwise defined) denote the projection onto Y where Y is a component of some cartesian (or fiber) product.
As in [LP97], 5.3 the vector subbundle generated by a coherent subsheaf $F \subset E$ is the inverse image of $\mathrm{Tor}(E/F)$ under the projection $E \to E/F$.
If $\varrho : G \times W \to W$ is a representation of G and P a principal G-bundle, P_ϱ denotes the associated fiber bundle. $\mathrm{Gl}(W)$-bundles and their associated vector bundles are identified throughout the text.
We denote vectors and matrices as $(v^i)_{i[m]} := (v^i)_{1\leq i\leq m}$ or $(A_{ij})_{i[m]j[n]} := (A_{ij})_{\substack{1\leq i\leq m,\\ 1\leq j\leq n}}$. If it is obvious over which range an index varies we will often shorten the notation by writing simply $(v^i)_i$ or $(A_{ij})_{ij}$.
Weights will usually be denoted by the letters $\alpha^*_\star$, $\beta^*_\star$, $\gamma^*_\star$, $\delta^*_\star$, ranks and degrees of coherent sheaves by $r^*_\star$ resp. $d^*_\star$, where $^*_\star$ stands for a possible indexing. We write $E^{\oplus v} := \bigoplus_{i=1}^{v} E$ as well as $E^{\otimes u} := \bigotimes_{i=1}^{u} E$.

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1

THE MODULI SPACE OF HIGGS TUPLES

The main objects of this first chapter are parabolic Higgs tuples. The construction of their moduli space is the central application of Mumford's Geometric Invariant Theory in this thesis and also marks the starting point for all further constructions to follow in the upcoming chapters. Before we can however start with the actual construction we need to state basic facts about one-parameter subgroups, parabolic filtrations and their weights. The techniques presented in the following are crucial for every numerical calculation performed later on.

1.1. Parabolic Vector Bundles. A punctured Riemann surface $(X,\ S)$ is a compact Riemann surface X together with a finite set of punctures $S \subset X$.
A quasi-parabolic vector bundle over the punctured Riemann surface $(X,\ S)$ is an algebraic vector bundle $E \to X$ together with filtrations of the stalks

$$0 \subsetneq E^{1j} \subsetneq \cdots \subsetneq E^{s^j j} = E|_{x^j}, \quad x^j \in S.$$

A parabolic vector bundle additionally contains the information of rational numbers

$$0 < \beta^{s^j j} < \cdots < \beta^{1j} < 1.$$

The parabolic degree of E is defined to be the rational number par-deg$(E) := \deg(E) + \sum_{j:x^j \in S} \sum_{i=1}^{s^j} \beta^{ij}(\dim(E^{ij}) - \dim(E^{i-1,j}))$.[1]

1.2. Higgs Tuples. Let A be a finite set and $\kappa_a \in \mathbb{Z}_+$ for every $a \in A$. Let $u,\ v,\ w \in \mathbb{N}$. A parabolic Higgs tuple $(\mathbf{E},\ \varphi,\ L)$ is a tuple of (quasi-)parabolic vector bundles $\mathbf{E} = (E_a,\ (E_a^{ij})_{i[|s_a^j]j[|S|]})_{a[|A|]}$ plus a non-trivial homomorphism $\varphi : (E^{\otimes u})^{\oplus v} \to \det(E)^{\otimes w} \otimes L$ where L is a line bundle on X and $E = \bigoplus_{a \in A} E_a^{\oplus \kappa_a}$.

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[1] For equivalent definitions see 1.8.

1.3. Weights. Let $\gamma^k \in \mathbb{Z},\ 1 \le k \le m$ be a tuple of integers and $r \in \mathbb{Z} \setminus \{0\}$. Define

$$\Gamma_{m,r} := \frac{1}{r} \begin{pmatrix} -1 & 1 & & & 0 \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ 0 & & & & -1 \end{pmatrix} \in \mathbb{C}^{m \times m}.$$

Then $\Gamma_{m,r}$ is invertible with inverse $\Gamma_{m,r}^{-1} =: \begin{pmatrix} \gamma_{r,m}^1 & \dots & \gamma_{r,m}^m \end{pmatrix}$.[2] Let $\alpha^i := (\Gamma_{m,r}(\gamma^k)_{k[m]})_i = \sum_{j=1}^m \Gamma_{m,r}^{ik} \gamma^k$. Analogously a tuple $(\alpha^i)_{i[m]}$ induces by multiplication with $\Gamma_{m,r}^{-1}$ a tuple $(\gamma^k)_{k[m]}$. If the γ^k are ordered $\gamma^1 \le \gamma^2 \le \ldots \le \gamma^m$, then $\alpha^k \ge 0,\ \forall\ 1 \le k \le m-1$ and vice versa. Given $r^k \in \mathbb{Z},\ 1 \le k \le m-1$ such that $\sum_{k=1}^m \gamma^k (r^k - r^{k-1}) = 0$ with $r^0 = 0$ and $r^m = r$, then

$$\frac{1}{r}(r^k - r^{k-1})_{k[m]}^t \Gamma_{m,r}^{-1} (\alpha^k)_{k[m]} = (-r^k)_{k[m]}^t (\alpha^k)_{k[m]} = -\sum_{k=1}^m \alpha^k r^k = 0.$$

On the other hand if $\alpha^m = -\frac{\sum_{k=1}^{m-1} \alpha^k r^k}{r^m}$ for r^k as above, then $\sum_{k=1}^m \gamma^k (r^k - r^{k-1}) = 0$.

1.4. Filtrations of Tuples. Let $(E_a)_{a[|A|]}$ be a tuple of coherent $\mathcal{O}_X$-modules and $(F_a^k)_{k[m_a]}$ a filtration by coherent submodules of E_a with weights $\gamma_a^1 \le \cdots \le \gamma_a^{m_a}$, $\forall a \in A$ such that $\gamma_a^k = \gamma_a^{k+1} \Leftrightarrow F_a^k = F_a^{k+1}$ for $1 \le k \le m_a - 1$. We call a pair of a filtration and suitable weights (as above) a *weighted filtration* of $(E_a)_{a[|A|]}$. Let $\{\gamma^k :\ k = 1, \ldots, m\} = \{\gamma_a^i : a \in A,\ 1 \le i \le m_a\}$ s. t. $\gamma^k \le \gamma^{k+1},\ 1 \le k \le m-1$ and $F^k = \bigoplus_{a \in A} (F_a^{k_a})^{\kappa_a}$,

$$k_a := \begin{cases} \max\{i \in \{1, \ldots, m_a\} |\ \gamma_a^i \le \gamma^k\} & \exists i \in \{1, \ldots, m_a\} :\ \gamma_a^i \le \gamma^k \\ 0 & \text{otherwise.} \end{cases}$$

Then $(F^k)_{k[m]}$ is a filtration of E. If $(F_a^k)_{k[m_a]}$ is proper for all $a \in A$, so is $(F^k)_{k[m]}$. On the other hand given a filtration $F^k = \bigoplus_{a \in A} (F_a^k)^{\kappa_a}$ and weights $\gamma^k \le \gamma^{k+1}$ such that $\gamma^k = \gamma^{k+1} \Leftrightarrow F^k = F^{k+1}$ for $1 \le k \le m-1$. Then $(F_a^k)_{k[m]}$ is a filtration of E_a. Setting $\gamma_a^i := \min\{\gamma^k |\ F_a^i = F_a^k\}$ leads us back to ascending weights $(\gamma_a^i)_{i[m]}$ such that $\gamma_a^k = \gamma_a^{k+1} \Leftrightarrow F_a^k = F_a^{k+1}$ for $1 \le k \le m-1$. Observe that $(F_a^k, \gamma_a^k)_{k[m_a]}$ leads by the previous two constructions to some $(F_a'^k, \gamma_a'^k)_{k[m]}$. Note that by removing improper inclusions the two weighted filtrations become the same.

Remark. We say that the weights $(\gamma_a^i)_{i[m]a[|A|]}$ are induced by $(F^k)_{k[m]}$ from $(\gamma^k)_{k[m]}$. If additionally there are weights $(\hat{\gamma}_a^i)_{i[m]a[|A|]}$ induced by another filtration $(\hat{F}^k)_{k[m]}$

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[2] If $m = r$ we define $\Gamma_r := \Gamma_{r,r}$ and $\gamma_r^j := \gamma_{r,r}^j$ for all $1 \le j \le r$.

from the same $(\gamma^k)_{k[m]}$, then we will call the weights $(\gamma_a^i)_{i[m]a[|A|]}$ coarser than the weights $(\hat{\gamma}_a^i)_{i[m]a[|A|]}$ if $\{\gamma_a^i : 1 \leq i \leq m\} \subset \{\hat{\gamma}_a^i : 1 \leq i \leq m\}$, $\forall a \in A$.
Given $r_a^k \in \mathbb{Z}$ and increasing weights $\gamma^k \in \mathbb{Z}$, $1 \leq k \leq m$, we call the weight vector $(\gamma_a^k)_{k[m]}$ with $\gamma_a^k = \min\{\gamma^j : r_a^k = r_a^j\}$ induced by $(r_a^k)_{k[m]}$ from $(\gamma^k)_{k[m]}$.

1.5. Let $0 = F^0 \subset F^1 \subset \cdots \subset F^m = E = \bigoplus_{a \in A} E_a^{\oplus \kappa_a}$ be a filtration of a rank r locally free sheaf E. For $(\gamma^k)_{k[m]}$ ascending integer weights as in 1.4 let $(\gamma_a^k)_{k[m]a[|A|]}$ be the induced weights and $(F_a^k)_{k[m]}$ the induced filtrations of the locally free sheaves E_a with $\operatorname{rk}(F_a^k) = r_a^k$ for $0 \leq k \leq m$ and $\operatorname{rk}(E_a) = r_a$. Note that we have $\sum_{a \in A} \kappa_a r_a^k = r^k$, $\forall 1 \leq k \leq m$. As in 1.3 we get weights $(\alpha^k)_{k[m]} = (\Gamma_{m,r}(\gamma_a^j)_{j[m]})$ and $(\alpha_a^k)_{k[m]} = (\Gamma_{m,r_a}(\gamma_a^j)_{j[m]})$, $\forall a \in A$. Then

$$\begin{aligned} r\sum_{k=1}^{m} \alpha^k r^k &= \sum_{k=1}^{m-1} \left(\gamma^{k+1} - \gamma^k\right) r^k - \gamma^r r = \sum_{k=1}^{m} \gamma^k \left(r^{k-1} - r^k\right), \\ &= \sum_{a \in A} \sum_{k=1}^{m} \gamma_a^k \kappa_a \left(r_a^{k-1} - r_a^k\right) = \sum_{a \in A} \kappa_a r_a \sum_{k=1}^{m} \alpha_a^k r_a^k. \end{aligned}$$

1.6. Semistability. Let $\delta > 0$, $\xi_a \in \mathbb{Q}$ for $a \in A$. A Higgs tuple $(\mathbf{E}, \varphi, L)$ is $(\xi_a, \delta)-$(semi)stable if and only if

$$M_{\text{par}}^{\kappa,\xi}(F^k, \alpha^k) + \delta \cdot \mu(F^k, \alpha^k, \varphi) \;(\geq)\; 0$$

holds for all $(F^k, \alpha^k)_{k[r]}$ where $(\alpha^k)_{k[r-1]} \in \mathbb{Q}_{\geq 0}^{r-1}$ and $(F^k)_{k[r]}$ is a filtration of E such that $F^k := \bigoplus_{a \in A} (F_a^k)^{\oplus \kappa_a}$ with subbundles[3]

$$0 \subset F_a^1 \subset \cdots \subset F_a^r = E_a \quad \text{and} \quad \alpha^r := -\operatorname{rk}(E)^{-1} \sum_{k=1}^{r-1} \alpha^k \operatorname{rk}(F^k).$$

Define

$$\begin{aligned} M_{\text{par}}^{\kappa,\xi}(F^k, \alpha^k) :=& \sum_{k=1}^{r} \alpha^k \cdot \Bigg(\text{par-deg}(E) \operatorname{rk}(F^k) - \text{par-deg}(F^k) \operatorname{rk}(E) \\ & + \sum_{a \in A} \xi_a (\operatorname{rk}(E_a) \operatorname{rk}(F^k) - \operatorname{rk}(F_a^k) \operatorname{rk}(E)) \Bigg), \\ \mu(F^k, \alpha^k, \varphi) :=& -\min \left\{ \sum_{j=1}^{u} \gamma^{k_j} \,\middle|\, (k_j)_{j[u]} \in \{1, \ldots, r\}^u : \varphi|_{(\bigotimes_{j=1}^{u} F^{k_j})^{\oplus v}} \not\equiv 0 \right\}, \end{aligned}$$

where γ^i is defined as in 1.3, i. e. $\gamma^i := -\operatorname{rk}(E) \sum_{k=i}^{r} \alpha^k$.

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[3] For the transition $(F_a^k, \alpha_a^k)_{k[r]a[|A|]}$ to $(F^k, \alpha^k)_{k[r]}$ see 1.4.

1.7. Let $\xi'_a = \xi_a + l \cdot \kappa_a$. Then

$$\sum_{a \in A} \xi'_a (\operatorname{rk}(E_a) \operatorname{rk}(F) - \operatorname{rk}(F_a) \operatorname{rk}(E) = \sum_{a \in A} \xi_a (\operatorname{rk}(E_a) \operatorname{rk}(F) - \operatorname{rk}(F_a) \operatorname{rk}(E))$$

for every subbundle $F \subset E$. Thus for every weighted filtration $(F^k, \alpha^k)_{k[r]}$ we get $M^{\kappa,\xi}_{[\mathrm{par}]}(F^k, \alpha^k) = M^{\kappa,\xi'}_{[\mathrm{par}]}(F^k, \alpha^k)$, i. e. the (semi)stability concept is independent of the choice of a representative within $\{(\xi_a + l \cdot \kappa_a)_{a[|A|]},\ l \in \mathbb{R}\}$. Hence we may choose $l = -\frac{\sum_{a \in A} \xi_a r_a}{\sum_{a \in A} \kappa_a r_a} \Rightarrow \sum_{a \in A} \xi'_a r_a = 0$.

1.8. Equivalent Definitions of the Parabolic Degree. We want to apply the general transition described in 1.3 to the parabolic degree.
Let $(E^{ij}, \beta^{ij})_{i[s^j]}$ be a parabolic filtration of $E|_{x^j}$ and $q^{ij} : E|_{x^j} \to V^{ij}$ quotients onto vector spaces V^{ij} such that $\ker(q^{ij}) = E^{ij},\ \ker(q^{0j}) = E^{0j} = 0,\ \ker(q^{s^j j}) = E^{s^j j} = E|_{x^j}$.
Elementary properties of linear maps imply

$$\sum_{j: x^j \in S} \sum_{i=1}^{s^j} \beta^{ij} (\dim E^{ij} - \dim E^{i-1,j}) = \ - \sum_{j: x^j \in S} \sum_{i=1}^{s^j} \beta^{ij} (\dim \operatorname{im} q^{ij} - \dim \operatorname{im} q^{i-1,j}).$$

Set $\delta^{ij} := \beta^{ij} - \beta^{i+1,j}$ for $0 \le i < s^j,\ \delta^{s^j j} := \beta^{s^j j}$

$$\begin{aligned}
&\sum_{j: x^j \in S} \sum_{i=1}^{s^j} \delta^{ij} (\dim \operatorname{im} q^{ij}) \\
= \ &\sum_{j: x^j \in S} \sum_{i=1}^{s^j - 1} (\beta^{ij} - \beta^{i+1,j}) (\dim \operatorname{im} q^{ij}) \\
= \ &\sum_{j: x^j \in S} \left(\sum_{i=1}^{s^j - 1} \beta^{ij} (\dim \operatorname{im} q^{ij}) - \sum_{i=2}^{s^j} \beta^{ij} (\dim \operatorname{im} q^{i-1,j}) \right) \\
= \ &\sum_{j: x^j \in S} \sum_{i=1}^{s^j} \beta^{ij} (\dim \operatorname{im} q^{ij} - \dim \operatorname{im} q^{i-1,j}) + \beta^{1,j} \underbrace{\dim \operatorname{im} q^{0,j}}_{=r}.
\end{aligned}$$

On the other hand for $\beta^{1j} = \sum_{k=1}^{s^j} \delta^{kj},\ \beta^{ij} := \beta^{1j} - \sum_{k=1}^{i-1} \delta^{kj},\ i > 1$ we have $\beta^{ij} - \beta^{i+1,j} = \delta^{ij}$ whenever $i < s^j$ and the calculation above works the other way round. Note that using 1.3 we have $(\delta^{ij})_{i[s^j]} = \Gamma_{s^j,-1} (\beta^{kj})_{k[s^j]}$ and $(\beta^{ij})_{i[s^j]} = \Gamma^{-1}_{s^j,-1} (\delta^{kj})_{k[s^j]}$. Moreover

$$\sum_{j: x^j \in S} \sum_{i=1}^{s^j} \delta^{ij} (\dim \ker q^{ij}) = \sum_{j: x^j \in S} \sum_{i=1}^{s^j} \beta^{ij} (\dim \ker q^{ij} - \dim \ker q^{i-1,j}).$$

Finally for admissible weights δ^{ij}, i. e. $\delta^{ij} > 0$, $\sum_{i=1}^{s^j} \delta^{ij} < 1$ we get for the weights β^{ij}

$$0 < \beta^{s^j j} < \ldots < \beta^{1j} < 1 \tag{1}$$

and for β^{ij} satisfying (1) the δ^{ij} are admissible:

$$\delta^{ij} := \beta^{ij} - \beta^{i+1,j} > 0, \quad \sum_{i=1}^{s^j} \delta^{ij} = \beta^{1j} < 1.$$

Sometimes in literature the order of the β^{ij} is reversed, i. e. $0 < \beta^{1j} < \ldots < \beta^{s^j j} < 1$ to $E|_{x^j} \supsetneq E^{1j} \supsetneq \ldots \supsetneq E^{s^j j} = 0$.
Furthermore we will take a look at parabolic tuple filtrations. By 1.4 we see that a parabolic filtration of a tuple induces a filtration $(E_a^{ij},\ \beta_a^{ij})_{i[s^j]a[|A|]}$ and we already know that both add the same parabolic contribution

$$\sum_{j:x^j \in S} \sum_{i=1}^{s^j} \beta_a^{ij} (\dim E_a^{ij} - \dim E_a^{i-1,j})$$

to the parabolic degree par-deg(E_a). Denote by $(E^{ij}, \beta^{ij})_{i[s^j]}$ the corresponding filtration of $E|_{x^j}$ for every $x^j \in S$. Then we get

$$\sum_{j:x^j \in S} \sum_{i=1}^{s^j} \beta^{ij} (\dim E^{ij} - \dim E^{i-1,j}) = \sum_{a \in A} \sum_{j:x^j \in S} \sum_{i=1}^{s^j} \beta_a^{ij} \kappa_a (\dim E_a^{ij} - \dim E_a^{i-1,j}),$$

where we used that for $E_a^{ij} \neq E_a^{i-1,j} \Leftrightarrow \beta^{ij} = \beta_a^{ij} \neq \beta_a^{i-1,j}$. Thus par-deg$(E) = \sum_{a \in A} \kappa_a$ par-deg(E_a).

Remark. Up to a scalar factor the transition from $(\gamma^k)_{k[m]}$ to $(\alpha^k)_{k[m]}$ is the same as from $(\beta^{ij})_{i[s^j]}$ to $(\delta^{ij})_{i[s^j]}$. We will often switch between the different kinds of weights to simplify some of the calculations ahead. Additionally we will often simplify the notation by using trivially extended filtrations as above or in 1.4.

1.1. BOUNDED FAMILIES OF VECTOR BUNDLES

The goal of the next two sections is the construction of a scheme parametrizing (at least) all semistable parabolic Higgs tuples. This can be done stepwise starting with the parametrization of those vector bundles that occur in semistable parabolic Higgs tuples. If we can show that this family of vector bundles E_a is bounded then we already know that there is a natural number n_0 such that for all $n \geq n_0$: $E_a(n)$

is globally generated and $H^1(E_a(n)) = 0$.[4] This on the other hand implies that E_a may be written as a quotient $q_a : H^0(E_a(n)) \otimes \mathcal{O}_X(-n) \to E_a$ and such quotients are parametrized by a suitable Grothendieck Quot scheme.

Definition. A family of vector bundles $\mathfrak{F}$ is bounded, if there is a scheme Y of finite type over $\mathbb{C}$ and a universal bundle $\mathscr{E}_{Y\times X}$ on $Y \times X$ such that each element of $\mathfrak{F}$ is isomorphic to $\mathscr{E}_{Y\times X}|_{\{y\}\times X}$ for some $y \in Y$.

We will mainly use the following criterion for boundedness of families of vector bundles:

1.9. Lemma. ([Sch08], 2.2.3.7.) *A family $\mathfrak{F}$ of isomorphy classes of vector bundles of a certain rank r and certain degree d is bounded, if and only if there is a $c \in \mathbb{R}$ such that for every vector bundle E with $[E] \in \mathfrak{F}$:*

$$\max\left\{\mu(F) := \frac{\deg(F)}{\operatorname{rk}(F)} \,\middle|\, \{0\} \subsetneq F \subset E \textit{ subbundle}\right\} \leq \mu(E) + c.$$

1.10. Proposition. (Harder-Narasimhan Filtration) *Let E be a vector bundle. Then there is a unique filtration*

$$0 = E^0 \subsetneq \ldots \subsetneq E^m = E$$

such that $E_k := E^k/E^{k-1}$ is semistable for all $1 \leq k \leq m$ and $\mu(E_{k-1}) > \mu(E_k)$ holds for all $2 \leq k \leq m$. Denote by $\mu_{max}(E) = \mu(E_1) = \mu(E^1)$ and by $\mu_{min}(E) = \mu(E_m)$.

Remark. There is a version of the Harder-Narasimhan filtration for parabolic bundles ([Ses82]) as well as for principal G-bundles with a reductive structure group (see e. g. Biswas, Holla [BH04]).

1.11. We would like to extend the Harder-Narasimhan filtration to tuples of vector bundles. Let $(E_a)_{a[A]}$ be such a tuple and $(E_a^k)_{k[m_a]}$ the Harder-Narasimhan filtration of E_a. Let $\mu_1 > \ldots > \mu_{|M|}$ denote the pairwise distinct weights in $M = \{\mu(E_a^k/E_a^{k-1}) : 1 \leq k \leq m_a,\ a \in A\}$. Now define $\operatorname{HN}(E)_0 = 0$ and $\operatorname{HN}(E)_j = \bigoplus_{a\in A} E_a^{k_{j,a}}$ with $E_a^{k_{j,a}}$ such that $\mu(E_a^{k_{j,a}}/E_a^{k_{j,a}-1}) \geq \mu_j > \mu(E_a^{k_{j,a}+1}/E_a^{k_{j,a}})$ or 0 if no such index exists. We claim that $0 \subset \operatorname{HN}(E)_1 \subset \ldots \subset E$ is the unique Harder-Narasimhan filtration of $E = \bigoplus_{a\in A} E_a$. By definition of the filtration, $\operatorname{HN}(E)_j/\operatorname{HN}(E)_{j-1}$ is isomorphic to the direct sum of all those E_a^k/E_a^{k-1} for which $\mu(E_a^k/E_a^{k-1}) = \mu_j$. In particular $\mu(\operatorname{HN}(E)_j/\operatorname{HN}(E)_{j-1}) = \mu_j$. Thus it remains to check that the direct sum of semistable vector bundles with the same slope is again semistable. Suppose that there is a $0 \neq G \subset \tilde{E}$ with $\mu(G) > \mu(\tilde{E}) = \mu(\tilde{E}_i)$, $\forall i \in I$

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[4] [Ha77], III 12.11.

and $\tilde{E} = \bigoplus_{i \in I} \tilde{E}_i$ a finite direct sum of semistable vector bundles $\tilde{E}_i$. Then the morphism $\mathrm{pr}_i : \tilde{E} \to \tilde{E}_i$ must be trivial on G and since $\tilde{E} = \bigoplus_{i \in I} \tilde{E}_i$, G is trivial in contradiction to our assumption. In general if $\varphi : G \to F$ is a morphism of semistable vector bundles and $\mu(G) > \mu(F)$, then φ is trivial: Consider the short exact sequence of coherent sheaves $0 \to \ker(\varphi) \to G \to \varphi(G) \to 0$, then the given conditions tell us for $\varphi(G) \neq 0$ that $\mu(\varphi(G)) \leq \mu(F) < \mu(G)$ and furthermore by $\deg(G) = \deg(\ker(\varphi)) + \deg(\varphi(G))$, $\mathrm{rk}(G) = \mathrm{rk}(\ker(\varphi)) + \mathrm{rk}(\varphi(G))$ we must have $\mu(\ker(\varphi)) > \mu(G)$. Therefore G can be semistable only if $\varphi(G) = 0$.
This shows that $\mathrm{HN}(E)_j / \mathrm{HN}(E)_{j-1}$ is in fact semistable and the uniqueness of the Harder-Narasimhan filtration already implies that $(\mathrm{HN}(E)_k)_{k[|M|]}$ is the desired filtration. Note in particular that $\mu_{max}(E) = \max\{\mu_{max}(E_a) : a \in A\}$, $\mu_{min}(E) = \min\{\mu_{min}(E_a) : a \in A\}$.

1.12. The tensor product of semistable sheaves is semistable ([HL10], Theorem 3.1.4.). Moreover $\mu(E \otimes F) = \mu(E) + \mu(F)$ since $\deg(E \otimes F) = \mathrm{rk}(E)\deg(F) + \deg(E)\,\mathrm{rk}(F)$. Hence $\mu_{max}(E \otimes F) = \mu_{max}(E) + \mu_{max}(F)$ and $\mu_{min}(E \otimes F) = \mu_{min}(E) + \mu_{min}(F)$.

Remark. For $Q = E/F$, $\deg(E) = \deg(F) + \deg(Q)$. Thus if $\{\mu(F) : \ F \subset E\}$ is not bounded from above, $\{\mu(Q) : \ Q \text{ quotient of } E\}$ is not bounded from below. Hence instead of searching for an upper bound for the slope of subbundles, we may establish boundedness equivalently if we find a lower bound for the slope of quotients.

We can now apply the previous definitions to our family of vector bundles underlying a semistable [parabolic] Higgs tuple:

1.13. Lemma. *Fix r_a, d_a, l. The family of vector bundles E_b such that there is a semistable [parabolic] Higgs tuples $(\mathbf{E},\ \varphi,\ L)$ with $\mathbf{E} = (E_a,\ (E_a^{ij})_{i[s_a^j]j[|S|]})_{a[|A|]}$ and $E_a \simeq E_b$ for some $a \in A$, is bounded.*

Proof. First note that by 1.11 we get $\mu_{max}(E) = \max\{\mu_{max}(E_a) : a \in A\}$. By lemma 1.9 it is enough to show that all $\mu_{max}(E_a)$ are bounded. Therefore upper bounds on $\mu_{max}(E_a)$ for all $a \in A$ will in particular bound $\mu_{max}(E)$. We consider an arbitrary subbundle $F_a \subset E_a$ and denote by $F = 0 \oplus F_a^{\oplus \kappa_a} \oplus 0 \subset E$ the trivial extension to a subbundle of E. Using 1.8 we get

$$\deg(F) \leq \text{par-deg}(F) \leq \deg(F) + |S| \cdot \mathrm{rk}(F)$$

for every subbundle $F \subset E$. Consider the weighted filtration $0 \subsetneq F^1 = \cdots = F^{\mathrm{rk}(F)} = F \subsetneq F^{\mathrm{rk}\,F+1} = \cdots = F^{\mathrm{rk}(E)} = E$ with a non-zero weight $\alpha^{\mathrm{rk}(F)} = 1$. 1.3 implies

$$\sum_{j=1}^{u} \gamma^{i_j} = u \sum_{k=1}^{r-1} \alpha^k r^k - r \sum_{k=1}^{r-1} \alpha^k \#\{i_j |\ k \geq i_j,\ 1 \leq j \leq u\} \geq u \sum_{k=1}^{r-1} \alpha^k (k - r)$$

$$\Rightarrow \mu(F^k,\alpha^k,\varphi) \le u\sum_{k=1}^{r-1}\alpha^k(r-k) = u\cdot\alpha^{\mathrm{rk}(F)}\cdot(\mathrm{rk}(E)-\mathrm{rk}(F)) \le u\cdot\mathrm{rk}(E).$$

Semistability tells us further that

$$\begin{aligned}0 \le\ & M^{\kappa,\xi}_{[\mathrm{par}]}(F^k,\alpha^k)+\delta\cdot\mu(F^k,\alpha^k,\varphi)\\ \le\ & \deg(E)\cdot\mathrm{rk}(F)+|S|\cdot\mathrm{rk}(E)\cdot\mathrm{rk}(F)-\deg(F)\cdot\mathrm{rk}(E)\\ & +\mathrm{rk}(E)\cdot\mathrm{rk}(F)\cdot\left(\sum_{a\in A}|\xi_a|\right)+\delta\cdot u\cdot\mathrm{rk}(E)\end{aligned}$$

and therefore

$$\frac{\deg(F_a)}{\mathrm{rk}(F_a)}=\frac{\deg(F)}{\mathrm{rk}(F)}\le\frac{\deg(E)}{\mathrm{rk}(E)}+|S|+\sum_{a\in A}|\xi_a|+\delta\cdot u.$$

□

1.2. CONSTRUCTION OF THE PARAMETER SPACE

Before we can define a parameter scheme we need to state the following central technical lemma.

1.14. Lemma. ([GS00], Lemma 3.1.) *Let Y be a scheme of finite type over $\mathbb{C}$, $\mathscr{F}_Y$ a Y-flat and $\mathscr{E}_Y$ an arbitrary coherent $\mathcal{O}_{Y\times X}$-module and $\psi_Y : \mathscr{E}_Y \to \mathscr{F}_Y$ a homomorphism. Then there is a unique closed subscheme $\mathfrak{Y} \subset Y$ with the universal property, that a morphism $f : T \to Y$ from an arbitrary scheme T of finite type over $\mathbb{C}$ factorizes over $\mathfrak{Y}$ if and only if $(f\times \mathrm{id}_X)^*(\psi_Y)\equiv 0$.*

1.15. For future use we collect some properties of pullback and direct image sheaves. First recall that for a morphism of schemes $f : Y \to T$ and a locally free sheaf E on T there are morphisms $E \to f_*f^*E$ and $f^*f_*E \to E$. Moreover direct images of isomorphisms/monomorphisms are again isomorphisms/monomorphisms. Note as well that for a commutative diagram of morphisms

$$\begin{tikzcd} Y\times X \arrow[r, "f\times\mathrm{id}_X"] \arrow[d, "\pi_Y"] & T\times X \arrow[d, "\pi_T"] \\ Y \arrow[r, "f"'] & T \end{tikzcd}$$

we have by construction $f_*\pi_{Y,*} = \pi_{T,*}(f \times \mathrm{id}_X)_*$. Furthermore for locally free sheaves F on T, H on $T\times X$, G on Y and $f_X := f\times \mathrm{id}_X$ we get using $f_{X,*}\pi_Y^* \simeq \pi_T^* f_*$ ([Ha77], III.9.3):

$$\begin{aligned}&\mathrm{Hom}((\pi_T f_X)^*(F), \pi_Y^*(G)\otimes f_X^*(H))\\ &\ \simeq \mathrm{Hom}\left(\mathscr{H}om(\pi_Y^* f^*(F), \pi_Y^*(G)))^\vee, f_X^*(H)\right)\\ &\ \simeq \mathrm{Hom}\left(\pi_T^* f_*\mathscr{H}om(f^*(F),G))^\vee, H\right)\\ &\ \simeq \mathrm{Hom}\left(\mathscr{H}om(F,\pi_{T,*}H))^\vee, f_*G\right)\\ &\ \simeq \mathrm{Hom}\left(f^*\mathscr{H}om(F,\pi_{T,*}H)^\vee, G\right).\end{aligned}$$

1.16. Since our family of semistable parabolic Higgs tuples of given type is bounded and all line bundles on X are semistable, i. e. the corresponding family is bounded as well, there is a $n_1 \in \mathbb{N}$ s. t. $\forall\ n \geq n_1$ $\forall a \in A$ and for all $L,\ L_a,\ [L] \in \mathrm{Jac}^l,\ [L_a] \in \mathrm{Jac}^{d_a},\ M := \bigotimes_{a\in A} L_a^{\otimes\kappa_a}$:

- $E_a(n),\ M^{\otimes w}\otimes L\otimes \mathcal{O}_X(un),\ L_a(n)$ are globally generated,
- $H^1(E_a(n)) = H^1(M^{\otimes w}\otimes L\otimes\mathcal{O}_X(un)) = H^1(L_a(n)) = \{0\}$.

Fix n big enough and p_a the Hilbert polynomial at n, i. e. $p_a = d_a + r_a(n+1-g)$, $a \in A$, $p = \sum_{a\in A}\kappa_a p_a$. Let $\mathfrak{Q}_a \subset \overline{\mathfrak{Q}}_a$ be the quasi-projective Quot scheme parametrizing quotients $q_a : V_a\otimes\mathcal{O}_X(-n) \to E_a$, V_a vector space of dimension p_a, E_a vector bundles of rank r_a and degree d_a, s. t. $H^0(q_a(n)) : V_a \to H^0(E_a(n))$ is an isomorphism. $\mathfrak{Q}_a$ comes with a universal quotient

$$q_{\mathfrak{Q}_a} : \pi_X^*(V_a\otimes\mathcal{O}_X(-n)) \simeq V_a\otimes\pi_X^*(\mathcal{O}_X(-n)) \to \mathscr{E}_{\mathfrak{Q}_a}$$

and a universal family $\mathscr{E}_{\mathfrak{Q}_a}$. Putting the several quotients together leads to $\mathscr{E}_{\mathfrak{Q}} := \bigoplus_{a\in A}\pi^*_{\mathfrak{Q}_a\times X}(\mathscr{E}_{\mathfrak{Q}_a}^{\kappa_a})$ on $\mathfrak{Q}\times X$, $\mathfrak{Q} := (\bigtimes_{a\in A}\mathfrak{Q}_a)\times\mathrm{Jac}^l$. Now take $\mathscr{P}^l$ a Poincaré line bundle on $\mathrm{Jac}^l\times X$ and let $\mathscr{P}_{\mathfrak{Q}} := \pi^*_{\mathrm{Jac}^l\times X}(\mathscr{P}^l)$ be the corresponding bundle on $\mathfrak{Q}\times X$. Let $V = \bigoplus_{a\in A} V_a^{\oplus\kappa_a}$ and $V_{u,v} := (V^{\otimes u})^{\oplus v}$. Define

$$\begin{aligned}\mathscr{F}_k &:= V_{u,v}\otimes\pi_X^*(\mathcal{O}_X(u(k-n))),\\ \mathscr{K}_k &:= \det(\mathscr{E}_{\mathfrak{Q}})^{\otimes w}\otimes\mathscr{P}_{\mathfrak{Q}}\otimes\pi_X^*(\mathcal{O}_X(uk)).\end{aligned}$$

Note that the general base change properties stated e. g. in [Ha77], III.12.11, guarantee that $\pi_{\mathfrak{Q},*}(F(k))$ is locally free for a coherent sheaf F on $\mathfrak{Q}\times X$ and k big enough. Therefore $\mathscr{F}_k$, $\mathscr{K}_k$ are locally free for k big enough, so is

$$\mathscr{H}om(\pi_{\mathfrak{Q},*}(\mathscr{F}_k), \pi_{\mathfrak{Q},*}(\mathscr{K}_k)).$$

Next define $\mathfrak{X} := \mathbb{P}(\mathscr{H}om(\pi_{\mathfrak{Q},*}(\mathscr{F}_k), \pi_{\mathfrak{Q},*}(\mathscr{K}_k))^{\vee}) \xrightarrow{\pi} \mathfrak{Q}$[5] and $\mathscr{P}_{\mathfrak{X}} = (\pi \times \mathrm{id}_X)^*(\mathscr{P}_{\mathfrak{Q}})$ as well as

$$q_{\mathfrak{X}} := (\pi \times \mathrm{id}_X)^*(q_{\mathfrak{Q}}) : V \otimes \pi_X^*(\mathcal{O}_X(-n)) \to \mathscr{E}_{\mathfrak{X}} := (\pi \times \mathrm{id}_X)^*(\mathscr{E}_{\mathfrak{Q}})$$

the corresponding pullbacks. Then [Ha77], II 7.12 implies that we find a surjective morphism $\pi^*\mathscr{H}om(\pi_{\mathfrak{Q},*}(\mathscr{F}_k), \pi_{\mathfrak{Q},*}(\mathscr{K}_k))^{\vee} \to \mathcal{O}_{\mathfrak{X}}(1)$. Note that $\pi_{\mathfrak{Q}}^*\pi_{\mathfrak{Q},*}\mathscr{F}_k \to \mathscr{F}_k$, $\pi_{\mathfrak{Q}}^*\pi_{\mathfrak{Q},*}\mathscr{K}_k \to \mathscr{K}_k$ are surjektive for our large k ([Ha77], III, Theorem 8.8) and hence 1.15 gives us a resulting morphism $(\pi \times \mathrm{id}_X)^*(\pi_{\mathfrak{Q}}^*\pi_{\mathfrak{Q},*}\mathscr{F}_k) \to (\pi \times \mathrm{id}_X)^*(\mathscr{K}_k) \otimes \pi_{\mathfrak{X}}^*(\mathcal{O}_{\mathfrak{X}}(1))$ which descends to $\psi_{\mathfrak{X}} : (\pi \times \mathrm{id}_X)^*(\mathscr{F}_k) \to (\pi \times \mathrm{id}_X)^*(\mathscr{K}_k) \otimes \pi_{\mathfrak{X}}^*(\mathcal{O}_{\mathfrak{X}}(1))$ on some closed subscheme $\mathfrak{X}'$.
Lemma 1.14 further provides us with the closed subscheme $\mathfrak{T} \subset \mathfrak{X}' \subset \mathfrak{X}$ s. t. $\psi_{\mathfrak{T}} := \psi_{\mathfrak{X}} \otimes \mathrm{id}_{\pi_X^*(\mathcal{O}_X(-uk))}|_{\mathfrak{T}\times X}$ vanishes on $\ker(q_{\mathfrak{T},u,v})$. The fundamental theorem on homomorphisms tells us now that $\psi_{\mathfrak{T}}$ factorizes over $\mathscr{E}_{\mathfrak{T},u,v}$:

$$\begin{tikzcd}
V_{u,v} \otimes \pi_X^*(\mathcal{O}_X(-un)) \arrow[rr, "\psi_{\mathfrak{T}}"] \arrow[dr, "q_{\mathfrak{T},u,v}"'] & & \det(\mathscr{E}_{\mathfrak{T}})^{\otimes w} \otimes \mathscr{P}_{\mathfrak{T}} \otimes \pi_{\mathfrak{X}}^*(\mathcal{O}_{\mathfrak{X}}(1))|_{\mathfrak{T}} \\
& \mathscr{E}_{\mathfrak{T},u,v}, \arrow[ur, "\varphi_{\mathfrak{T}}"'] & \ker(q_{\mathfrak{T},u,v}) \subset \ker(\psi_{\mathfrak{T}}).
\end{tikzcd}$$

Remark. Since all restrictions result from properties shared by at least all semistable Higgs tuples, $\mathfrak{T}$ still parametrizes (at least) all non-parabolic Higgs tuples underlying a semistable parabolic Higgs tuple.

1.17. Our parameter space $\mathfrak{T}$ so far does not account for the parabolic structure. Let $\mathfrak{G}_a^{ij}$ be the Graßmann variety parametrizing $r_a^{ij}-$dimensional subspaces of V_a, $1 \le i \le s_a^j$, $1 \le j \le |S|$, $a \in A$. Set $\mathfrak{G} := \times_{a\in A} \mathfrak{G}_a$, $\mathfrak{G}_a := \times_{j=1}^{|S|} \times_{i=1}^{s_a^j} \mathfrak{G}_a^{ij}$. We get a tautological quotient $q_{\mathfrak{G}_a^{ij}}^{ij} : V_a \otimes \mathcal{O}_{\mathfrak{G}_a^{ij}} \to \mathscr{V}_a^{ij}$ of vector bundles on $\mathfrak{G}_a^{ij}$. Define $q_{\mathfrak{T}\times\mathfrak{G},V_a}^{ij} = \pi_{\mathfrak{G}_a^{ij}}^*(q_{\mathfrak{G}_a^{ij}}^{ij})$ on $\mathfrak{T} \times \mathfrak{G} \times X$. In order for the parabolic quotients to factorize in the fibers over S we need to restrict again to a subscheme. Let $\mathfrak{G}_{\mathrm{par}} \subset \mathfrak{G}$ be the subscheme where $q_{\mathfrak{T}\times\mathfrak{G},V_a}^{ij}$ vanishes on $\ker(\pi_{\mathfrak{T}\times X}^*(q_{\mathfrak{T}})|_{\mathfrak{T}\times\mathfrak{G}\times\{x^j\}})$ for every $1 \le i \le s_a^j, 1 \le j \le |S|, a \in A$. Then we find quotients $q_{\mathfrak{T}\times\mathfrak{G},a}^{ij} : \pi_{\mathfrak{T}\times X}^*(\mathscr{E}_{\mathfrak{T},a})|_{\mathfrak{T}\times\mathfrak{G}_{\mathrm{par}}\times\{x^j\}} \to \pi_{\mathfrak{G}_a^{ij}}^*(\mathscr{V}_a^{ij})|_{\mathfrak{T}\times\mathfrak{G}_{\mathrm{par}}\times\{x^j\}}$ for every $1 \le i \le s_a^j, 1 \le j \le |S|, a \in A$. In order to get filtrations rather than only a collection of subspaces let $\mathfrak{F} \subset \mathfrak{G}$ be the closed subscheme such that for all $(q_a^{ij}) \in \mathfrak{F}$: $\ker(q_a^{ij}) \subset \ker(q_a^{i+1,j})$. Finally define $\mathfrak{T}_{\mathrm{par}} := \mathfrak{T} \times (\mathfrak{G}_{\mathrm{par}} \cap \mathfrak{F})$ as our parameter space.

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[5] Recall that φ is non-trivial.

Remark to 1.16 and 1.17. Replace $\mathfrak{Q}$ by an arbitrary scheme Y which parametrizes (surjective) quotients $q_{Y,a} : V_a \otimes \pi_X^*(\mathcal{O}_X(-n)) \to \mathscr{E}_Y$ with $\mathscr{E}_Y$ a family of coherent sheaves on $Y \times X$ and no restrictions imposed on the 1^{st} cohomology of any of the appearing sheaves. Since X is proper over $\mathbb{C}$, $Y \times X \to Y$ is proper. Now the generalization of [Ha77], III 12.11 to proper maps follows from [EGA] III, 3.2.1.[6] and 1.14 still holds in this more general case. The previous construction yields a closed subscheme $\mathfrak{T}_Y$ that parametrizes Higgs fields over Y.

1.3. Moduli Functor and Universal Properties

1.18. Definition. Let $\mathscr{P}^l$ be a Poincaré line bundle on $\mathrm{Jac}^l \times X$, Y a scheme of finite type over $\mathbb{C}$ and $\upsilon_Y : Y \to \mathrm{Jac}^l$ a morphism. A family of parabolic Y-Higgs tuples is a tuple $((\mathscr{E}_{Y,a},\ (q^{ij}_{Y,a},\ \mathscr{H}^{ij}_{Y,a})_{i[s^j_a]j[|S|]})_{a[|A|]},\ \upsilon_Y,\ \mathscr{H}_Y,\ \varphi_Y)$ s. t.

(i) $\mathscr{E}_{Y,a}$ is a vector bundle of rank r_a on $Y \times X$ with degree d_a on each fiber over $\{y\} \times X$,

(ii) $\upsilon_Y : Y \to \mathrm{Jac}^l$ is a morphism,

(iii) $\mathscr{H}_Y \to Y$ a line bundle, $\mathscr{H}^{ij}_{Y,a} \to Y \times \{x^j\}$ vector bundles of rank r^{ij}_a,

(iv) $\varphi_Y : (\mathscr{E}_Y^{\otimes u})^{\oplus v} \to \det(\mathscr{E})^w \otimes \mathscr{P}_{\upsilon_Y} \otimes \pi_Y^*(\mathscr{H}_Y)$, $\mathscr{P}_{\upsilon_Y} := (\upsilon_Y \times \mathrm{id}_X)^*(\mathscr{P}^l)$, a homomorphism non-trivial on $\{y\} \times X$ for all $y \in Y$ closed,

(v) $q^{ij}_{Y,a} : \mathscr{E}_{Y,a}|_{Y\times\{x^j\}} \to \mathscr{H}^{ij}_{Y,a}$ surjective morphisms on $Y \times \{x^j\}$ such that $\ker(q^{ij}_{Y,a}) \subset \ker(q^{i+1,j}_{Y,a})$, $\forall a \in A,\ 1 \leq j \leq |S|,\ 1 \leq i \leq s^j_a$.

1.19. Equivalence of $Y-$Families. Two Y-families $((\mathscr{E}^1_{Y,a},\ (q^{ij,1}_{Y,a},\ \mathscr{H}^{ij,1}_{Y,a})_{i[s^j_a]j[|S|]})_{a[|A|]},\ \upsilon^1_Y,\ \mathscr{H}^1_Y,\ \varphi^1_Y)$ and $((\mathscr{E}^2_{Y,a},\ (q^{ij,2}_{Y,a},\ \mathscr{H}^{ij,2}_{Y,a})_{i[s^j_a]j[|S|]})_{a[|A|]},\ \upsilon^2_Y,\ \mathscr{H}^2_Y,\ \varphi^2_Y)$ are isomorphic if $\upsilon^1_Y = \upsilon^2_Y = \upsilon_Y$ and there are isomorphisms $\psi_{Y,a} : \mathscr{E}^1_{Y,a} \to \mathscr{E}^2_{Y,a}$, $\gamma_Y : \mathscr{H}^1_Y \to \mathscr{H}^2_Y$, $\gamma^{ij}_{Y,a} : \mathscr{H}^{ij,1}_{Y,a} \to \mathscr{H}^{ij,2}_{Y,a}$ s. t.

$$\varphi^1_Y = \big(\det(\psi_Y)^{\otimes w} \otimes \mathrm{id}_{\mathscr{P}_{\upsilon_Y}} \otimes \pi_Y^*(\gamma_Y)\big)^{-1} \circ \varphi^2_Y \circ \psi_{Y,u,v}, \quad q^{ij,2}_{Y,a} \circ \psi_{Y,a}|_{Y\times\{x^j\}} = \gamma^{ij}_{Y,a} \circ q^{ij,1}_{Y,a}.$$

Remark. If there is no chance of confusion we will call a family of parabolic Y-Higgs tuples just Y-family.
The non-parabolic version of a family of Y-Higgs tuples admits no vector bundles $\mathscr{H}^{ij}_{Y,a}$ and no quotients $q^{ij}_{Y,a}$. The definition of equivalence is changed accordingly.

1.20. Moduli Functor. For stability parameters $(\xi_a,\ \delta,\ \beta^{ij}_a)$ we get the functor $\mathbf{M}^{(\xi_a,\delta,\beta^{ij}_a)-(s)s}(\kappa_a, u, v, w, r_a, d_a, l, S, r^{ij}_a)$:

---

[6] [Ha77], II 8.8.

$$\begin{array}{ccc} \mathbf{Sch}_{\mathbb{C}} & \longrightarrow & \mathbf{Set} \\ Y & \longmapsto & \left\{ \begin{array}{c} \text{Isomorphism classes of families of} \\ (\xi_a,\ \delta,\ \beta_a^{ij}) - \text{(semi)stable} \\ \text{parabolic } Y\text{-Higgs tuples} \end{array} \right\} \end{array}$$

Remark. Note that this functor still depends on the choice of a Poincaré line bundle. However since two Poincaré line bundles are isomorphic up to the pullback of a line bundle on the Jacobian, we may identify the corresponding moduli functors.

1.21. Definition. A quotient family of Y-Higgs tuples is a family of Y-Higgs tuples $((\mathscr{E}_{Y,a},\ (q^{ij}_{Y,a},\ \mathscr{H}^{ij}_{Y,a})_{i[s^j_a]j[|S|]})_{a[|A|]},\ \upsilon_Y,\ \mathscr{H}_Y,\ \varphi_Y)$ together with surjective morphisms

$$q_{Y,a} : V_a \otimes \pi_X^*(\mathcal{O}_X(-n)) \to \mathscr{E}_{Y,a}, \quad a \in A$$

s. t. $\pi_{Y,*}(q_{Y,a} \otimes \mathrm{id}_{\mathcal{O}_X(n)}) : V_a \otimes \mathcal{O}_Y \to \pi_{Y,*}(\mathscr{E}_{Y,a} \otimes \pi_X^*(\mathcal{O}_X(n)))$ are isomorphisms for all $a \in A$. Two quotient families are equivalent if there is an isomorphism $\psi_{Y,a}$ of the corresponding Y-Higgs tuples such that $q^2_{Y,a} = \psi_{Y,a} \circ q^1_{Y,a}$.

1.22. Proposition. *Every quotient family of Y-Higgs tuples is the pullback of the universal quotient family $((q_{\mathfrak{T}_{\mathrm{par}},a},\ \mathscr{E}_{\mathfrak{T}_{\mathrm{par}},a},\ (q^{ij}_{\mathfrak{T}_{\mathrm{par}},a},\ \mathscr{H}^{ij}_{\mathfrak{T}_{\mathrm{par}},a})_{i[s^j_a]j[|S|]})_{a[|A|]},\ \nu_{\mathfrak{T}_{\mathrm{par}}},$ $\mathcal{O}_{\mathfrak{X}\times\mathfrak{G}}(1)|_{\mathfrak{T}_{\mathrm{par}}},\ \varphi_{\mathfrak{T}_{\mathrm{par}}})$ by a unique morphism $h \times \mathrm{id}_X : Y \times X \to \mathfrak{T}_{\mathrm{par}} \times X$.*

Remark. Recall $\nu_{\mathfrak{T}_{\mathrm{par}}} : \mathfrak{T}_{\mathrm{par}} \to \mathfrak{Q} \xrightarrow{\pi_{\mathrm{Jac}^l}} \mathrm{Jac}^l$ and $\mathscr{P}_{\mathfrak{T}_{\mathrm{par}}} = \mathscr{P}_{\nu_{\mathfrak{T}_{\mathrm{par}}}}$ as well as $\mathscr{H}^{ij}_{\mathfrak{T}_{\mathrm{par}},a} = \pi^*_{\mathfrak{G}^{ij}_a}(\mathscr{V}^{ij}_a|_{\mathfrak{T}_{\mathrm{par}}\times\{x^j\}})$.

Proof. Let $((q_{Y,a}, \mathscr{E}_{Y,a},\ (q^{ij}_{Y,a},\ \mathscr{H}^{ij}_{Y,a})_{i[s^j_a]j[|S|]})_{a[|A|]},\ \upsilon_Y,\ \mathscr{H}_Y,\ \varphi_Y)$ be a quotient family. The universal property of the Quot schemes $\overline{\mathfrak{Q}}_a$ implies the existence of morphisms $f_{\mathfrak{Q}_a} : Y \to \overline{\mathfrak{Q}}_a$ s. t. $q_{Y,a} \simeq (f_{\mathfrak{Q}_a} \times \mathrm{id}_X)^*(q_{\overline{\mathfrak{Q}}_a})$. Using that $\pi_{Y,*}(q_{Y,a} \otimes \mathrm{id}_{\mathcal{O}_X(-n)})$ is an isomorphism, we see that $f_{\mathfrak{Q}_a,*}\pi_{Y,*}(q_{Y,a} \otimes \mathrm{id}_{\mathcal{O}_X(n)})$ is an isomorphism, too. Hence $f_{\mathfrak{Q}_a} : Y \to \mathfrak{Q}_a$. Together with υ_Y we get a morphism $f_{\mathfrak{Q}} : Y \to \mathfrak{Q}$. The morphism $f_{\mathfrak{X}}$ is the morphism induced by $\pi_{Y,*}(\psi_Y)$ considered as a morphism

$$\pi_{Y,*}(V_{u,v} \otimes \pi_X^*(\mathcal{O}_X(u(k-n)))) \otimes \left(\pi_{Y,*}\left(\det(\mathscr{E}_Y)^{\otimes w} \otimes \mathscr{P}_{\upsilon_Y} \otimes \pi_X^*(\mathcal{O}_X(uk))\right)\right)^{\vee} \to \mathscr{H}_Y.$$

using [Ha77], II.7.12. By definition of $q_{\mathfrak{T}} = \pi^*(q_{\mathfrak{Q}})|_{\mathfrak{T}}$ we get $(f_{\mathfrak{X}} \times \mathrm{id}_X)^*(q_{\mathfrak{T}}) = q_Y$. Furthermore by construction $f^*_{\mathfrak{X}}(\mathcal{O}_{\mathfrak{X}}(1)) = \mathscr{H}_Y$ and thus $(f_{\mathfrak{X}} \times \mathrm{id}_X)^*(\psi_{\mathfrak{X}}) = \psi_Y$ resp. $(f_{\mathfrak{X}} \times \mathrm{id}_X)^*(\varphi_{\mathfrak{X}}) = \varphi_Y$.

Now $f_{\mathfrak{X}}$ factorizes over $\mathfrak{T}$: By 1.14 it is enough to show that $(f_{\mathfrak{X}} \times \mathrm{id}_X)^*(\psi_{\mathfrak{X}} \otimes \mathrm{id}_{\pi_X^*(\mathcal{O}_X(-uk))}|_{\ker q_{\mathfrak{X},u,v}}) \equiv 0$. This on the other hand follows directly from general properties of the pullback

$$\begin{aligned} &(f_{\mathfrak{X}} \times \mathrm{id}_X)^*(\psi_{\mathfrak{X}} \otimes \mathrm{id}_{\pi_X^*(\mathcal{O}_X(-uk))}|_{\ker q_{\mathfrak{X},u,v}}) \\ &\quad = (f_{\mathfrak{X}} \times \mathrm{id}_X)^*(\psi_{\mathfrak{X}}) \otimes (\pi_X(f_{\mathfrak{X}} \times \mathrm{id}_X))^*(\mathrm{id}_{\mathcal{O}_X(-uk)})|_{\ker((f_{\mathfrak{X}}\times\mathrm{id}_X)^*(q_{\mathfrak{X},u,v}))} \end{aligned}$$

$$
\begin{aligned}
&= \psi_Y \otimes \mathrm{id}_{\pi_X^*(\mathcal{O}_X(-uk))} \,|_{\ker(q_{Y,u,v})} \\
&= (\varphi_Y \circ q_{Y,u,v})|_{\ker(q_{Y,u,v})} \equiv 0.
\end{aligned}
$$

The considerations above imply now that $f_{\mathfrak{T}} : Y \to \mathfrak{T}$ defined by $f_{\mathfrak{x}}$ is unique. Similarly the universality of the Graßmann variety provides us with morphisms $g_a^{ij} : Y \to \mathfrak{G}_a^{ij}$ s. t. $((f_{\mathfrak{T}} \times g_a^{ij} \times \mathrm{id}_X)^*(q^{ij}_{\mathfrak{T}\times\mathfrak{G},V_a}))|_{Y\times\{x^j\}} = q^{ij}_{Y,a} \circ q_{Y,a}$. Let $g : Y \to \mathfrak{G}$ be the resulting map on the product space. Obviously $g : Y \to \mathfrak{F}$. Now $h := f_{\mathfrak{T}} \times g : Y \to \mathfrak{T}_{\mathrm{par}}$ if $((f_{\mathfrak{T}} \times g \times \mathrm{id}_X)^*(q^{ij}_{\mathfrak{T}\times\mathfrak{G},V_a}|_{\ker(\pi^*_{\mathfrak{T}\times X}(q_{\mathfrak{T},a}))}))|_{Y\times\{x^j\}} \equiv 0$. But

$$
\begin{aligned}
&((f_{\mathfrak{T}} \times g \times \mathrm{id}_X)^*(q^{ij}_{\mathfrak{T}\times\mathfrak{G},V_a})|_{\ker(\pi^*_{\mathfrak{T}\times X}(q_{\mathfrak{T},a}))})|_{Y\times\{x^j\}} \\
&= ((f_{\mathfrak{T}} \times g \times \mathrm{id}_X)^*(q^{ij}_{\mathfrak{T}\times\mathfrak{G},V_a})|_{\ker((f_{\mathfrak{T}}\times g\times\mathrm{id}_X)^*\pi^*_{\mathfrak{T}\times X}(q_{\mathfrak{T},a}))})|_{Y\times\{x^j\}} \\
&= q^{ij}_{Y,a} \circ q_{Y,a}|_{\ker((f_{\mathfrak{T}}\times\mathrm{id}_X)^*(q_{\mathfrak{T},a}))} = q^{ij}_{Y,a} \circ q_{Y,a}|_{\ker(q_{Y,a})} \equiv 0.
\end{aligned}
$$

This proves the claim. □

1.23. Proposition. *Let Y be a scheme of finite type over $\mathbb{C}$ and $((\mathscr{E}_{Y,a}, (q^{ij}_{Y,a}, \mathscr{H}^{ij}_{Y,a})_{i[s_a^j]j[|S|]})_{a[|A|]}, \upsilon_Y, \mathscr{H}_Y, \varphi_Y)$ a semistable Y-Higgs tuple. There is an open covering $(Y_k)_{k\in I}$ of Y and morphisms $h_k : Y_k \to \mathfrak{T}_{\mathrm{par}}$, $k \in I$ index set, s. t. $((\mathscr{E}_{Y,a}, (q^{ij}_{Y,a}, \mathscr{H}^{ij}_{Y,a})_{i[s_a^j]j[|S|]})_{a[|A|]}, \upsilon_Y, \mathscr{H}_Y, \varphi_Y)|_{Y_k\times X} \simeq (h_k \times \mathrm{id}_X)^*((\mathscr{E}_{\mathfrak{T}_{\mathrm{par}},a}, (q^{ij}_{\mathfrak{T}_{\mathrm{par}},a}, \mathscr{H}^{ij}_{\mathfrak{T}_{\mathrm{par}},a})_{i[s_a^j]j[|S|]})_{a[|A|]}, \nu_{\mathfrak{T}_{\mathrm{par}}}, \mathcal{O}_{\mathfrak{X}\times\mathfrak{G}}(1)|_{\mathfrak{T}_{\mathrm{par}}}, \varphi_{\mathfrak{T}_{\mathrm{par}}})$ on $Y_k \times X$.*

Proof. $\mathscr{E}_{Y,a}$ is locally trivial in the product of the étale topology on Y and the Zariski topology on X, of rank r_a over $Y \times X$ and degree d_a on $\{y\} \times X$. Let $(Y_k)_k$ be a common refinement of the locally trivial coverings for $a \in A$, s. t. all $\mathscr{E}_{Y_k,a}$ are locally trivial on $Y_k \times X_l$ for a covering $(X_l)_l$ of X.[7] Then there exists a quotient $q_{Y_k,a} : V_a \otimes \mathcal{O}_{Y_k\times X}(-n) \to \mathscr{E}_{Y_k,a}$ s. t. $\pi_{Y_k,a,*}(q_{Y_k,a} \otimes \mathrm{id}_{\mathcal{O}_X(n)})$ is an isomorphism since $\pi_{Y_k,a,*}(q_{Y_k,a} \otimes \mathrm{id}_{\mathcal{O}_X(n)})|_{\{y\}} \simeq H^0(q_{y,a}(n)) : V_a \to H^0(\mathscr{E}_{Y_k,a}(n)|_{\{y\}})$ is an isomorphism on every fiber $y \in Y_k$. Hence $((q_{Y_k,a}, \mathscr{E}_{Y,a}, (q^{ij}_{Y,a}, \mathscr{H}^{ij}_{Y,a})_{i[s_a^j]j[|S|]})_{a[|A|]}, \upsilon_Y, \mathscr{H}_Y, \varphi_Y)|_{Y_k\times X}$ is a quotient family on $Y_k \times X$. The previous proposition together with the definition of equivalence of quotient families implies the claim. □

Remark. To construct a suitable quotient family it is in fact enough to show that $\pi_{Y,*}(E_{Y,a})(n)$ is locally trivial.[8]
Analogous results to 1.22 and 1.23 hold in the non-parabolic situation. The proofs are (almost) identical; a non-parabolic quotient family is defined as the extension of a non-parabolic Y-family.

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[7] cf. I.2 resp. [DS95].

[8] cf. for example [Be14], Lemma 4.10.

1.4. GROUP ACTION

Let $\mathfrak{T}_{\mathrm{par}}^{ss} \subset \mathfrak{T}_{\mathrm{par}}$ denote the subset of semistable Higgs tuples. We aim to define a group action of some group $\mathcal{G}_A$ on $\mathfrak{T}_{\mathrm{par}}$ that leaves $\mathfrak{T}_{\mathrm{par}}^{ss}$-invariant such that two Higgs tuples are isomorphic if and only if they lie in the same orbit. If we are able to show in a future step that the semistability condition defined before is in fact GIT-semistability (w. r. t. a linearization of the group action in some line bundle on $\mathfrak{T}_{\mathrm{par}}^{ss}$) then the good GIT-Quotient $\mathfrak{T}_{\mathrm{par}}^{ss} /\!\!/ \mathcal{G}_A$ will exist and the previously stated universal properties imply the existence of a coarse moduli space.
Let $\mathcal{G}_A := \bigtimes_{a \in A} \mathrm{Gl}(V_a)$ and

$$\begin{gathered} m : V \otimes \mathcal{O}_{\mathcal{G}_A} \to V \otimes \mathcal{O}_{\mathcal{G}_A}; \\ V \otimes \mathcal{O}_{\mathcal{G}_A}|_g \ni (v, s) \mapsto (g \cdot v, s) \end{gathered}$$

m is an automorphism.
Let $((\mathscr{E}_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}},a},\ (q^{ij}_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}},a},\ \mathscr{H}^{ij}_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}},a})_{i[s_a^j]j[|S|]})_{a[|A|]},\ \nu_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}}},\ \mathscr{H}_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}}},\ \varphi_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}}})$ be the pullback of the universal family to $\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}} \times X$ and

$$\begin{aligned} q_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}}} : V \otimes \pi_X^*(\mathcal{O}_X(-n)) &\xrightarrow{\pi^*_{\mathcal{G}_A}(m^{-1}) \otimes \mathrm{id}_{\pi_X^*(\mathcal{O}_X(-n))}} V \otimes \pi_X^*(\mathcal{O}_X(-n)) \longrightarrow \\ &- - \xrightarrow{\pi^*_{X \times \mathfrak{T}_{\mathrm{par}}}(q_{\mathfrak{T}_{\mathrm{par}}})} \mathscr{E}_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}}}. \end{aligned}$$

$q_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}}}$ is surjective, since $q_{\mathfrak{T}_{\mathrm{par}}}$ is surjective and m bijective. Furthermore $\pi_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}},*}(q_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}}} \otimes \mathrm{id}_{\mathcal{O}_X(n)})$ is an isomorphism, since m and $\pi_{\mathfrak{T}_{\mathrm{par}},*}(q_{\mathfrak{T}_{\mathrm{par}}} \otimes \mathrm{id}_{\mathcal{O}_X(n)})$ are isomorphisms.
Hence $(q_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}}},\ (\mathscr{E}_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}},a},\ (q^{ij}_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}},a}, \mathscr{H}^{ij}_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}},a})_{i[s_a^j]j[|S|]})_{a[|A|]},\ \nu_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}}},\ \mathscr{H}_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}}},\ \varphi_{\mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}}})$ is a quotient family. Proposition 1.22 provides a unique morphism

$$\alpha : \mathcal{G}_A \times \mathfrak{T}_{\mathrm{par}} \to \mathfrak{T}_{\mathrm{par}}.$$

α is a group action: α maps a quotient $q : V \otimes \mathcal{O}_X(-n) \to E$ to the quotient

$$V \otimes \mathcal{O}_X(-n) \xrightarrow{g^{-1} \otimes \mathrm{id}_{\mathcal{O}_X(-n)} =: \tilde{\alpha}(g)} V \otimes \mathcal{O}_X(-n) \xrightarrow{q} E.^9$$

Therefore

$$\alpha(e) \simeq \mathrm{id}_{\mathfrak{T}_{\mathrm{par}}},$$

$$\alpha(gh, q) = q \circ \tilde{\alpha}(gh) = q \circ \tilde{\alpha}(h) \circ \tilde{\alpha}(g) = \alpha(g, \alpha(h, q)),$$

for all $g,\ h \in \mathcal{G}_A$ and all quotients q.
The center $\mathbb{C}^* \cdot \mathrm{id}_V$ acts trivially: Let $m_c = c \cdot m$ for $c \in \mathbb{C}^*$. Then the induced

---

[9] More precisely: A class represented by a tuple $((q_a)_{a[|A|]},\ (E_a,\ (E_a^{ij})_{i[s_a^j]j[|S|]})_{a[|A|]},\ \varphi,\ L)$ is mapped to the class of $((q_a)_{a[|A|]}(g^{-1} \cdot *),\ (E_a,\ (E_a^{ij})_{i[s_a^j]j[|S|]})_{a[|A|]},\ \varphi,\ L)$.

quotient families are trivially equivalent as families on $\{\mathrm{pt}\} \times \mathfrak{T}_{\mathrm{par}} \times X$, i. e. Proposition 1.22 implies that the induced actions are the same.

1.24. Conclusion. *W. l. o. g. we may replace the* $\mathcal{G}_A$*-action by an action of* $\mathcal{S}_A^{\kappa_a} := \mathrm{Sl}(V) \cap \mathcal{G}_A = \{(g_a)_a \in \bigtimes_{a\in A} \mathrm{Gl}(V_a) | \prod_{a\in A} \det(g_a)^{\kappa_a} = 1\}$*. Note that we have in fact a* $\mathcal{PG}_A$*-action,* $\mathcal{PG}_A := \mathcal{G}_A/\mathbb{C}^*$*. Furthermore observe, that* $\mathcal{S}_A^{\kappa_a} \to \mathcal{PG}_A$ *has finite kernel, in particular the parabolic subgroups of* $\mathcal{S}_A^{\kappa_a}$ *and* $\mathcal{PG}_A$ *may be identified.*

1.25. Proposition. *Let* Y *be a scheme of finite type over* $\mathbb{C}$, $h_k : Y \to \mathfrak{T}_{\mathrm{par}}$, $k = 1, 2$ *two morphisms s. t.* $(h_1 \times \mathrm{id}_X)^*((\mathscr{E}_{\mathfrak{T}_{\mathrm{par}},a},\ (q^{ij}_{\mathfrak{T}_{\mathrm{par}},a},\ \mathscr{H}^{ij}_{\mathfrak{T}_{\mathrm{par}},a})_{i[s^j_a]j[|S|]})_{a[|A|]},\ \nu_{\mathfrak{T}_{\mathrm{par}}},\ \mathscr{H}_{\mathfrak{T}_{\mathrm{par}}},\ \varphi_{\mathfrak{T}_{\mathrm{par}}}) \simeq (h_2 \times \mathrm{id}_X)^*((\mathscr{E}_{\mathfrak{T}_{\mathrm{par}},a},\ (q^{ij}_{\mathfrak{T}_{\mathrm{par}},a},\ \mathscr{H}^{ij}_{\mathfrak{T}_{\mathrm{par}},a})_{i[s^j_a]j[|S|]})_{a[|A|]},\ \nu_{\mathfrak{T}_{\mathrm{par}}},\ \mathscr{H}_{\mathfrak{T}_{\mathrm{par}}},\ \varphi_{\mathfrak{T}_{\mathrm{par}}})$*. There is a morphism* $\Phi : Y \to \mathcal{G}_A$ *s. t.* $h_2 = \alpha(\Phi \times h_1)$*.*

Proof. Construct quotients $q^1_{Y,a}$, $q^2_{Y,a}$ like in the proof of 1.23 s. t. $((q^k_{Y,a},\ \mathscr{E}^k_{Y,a},\ (q^{ij,k}_{Y,a},\ \mathscr{H}^{ij,k}_{\mathfrak{T}_{\mathrm{par}},a})_{i[s^j_a]j[|S|]})_{a[|A|]},\ \upsilon^k_Y,\ \mathscr{H}^k_Y,\ \varphi_Y)$, $k = 1, 2$ are the pullbacks of the universal family by $(h_k \times \mathrm{id}_X)$. By assumption $\upsilon^1_Y = \upsilon^2_Y =: \upsilon_Y$ and there are morphisms

$$\psi_{Y,a} : \mathscr{E}^1_{Y,a} \to \mathscr{E}^2_{Y,a},\ \gamma_Y : \mathscr{H}^1_Y \to \mathscr{H}^2_Y,\ \gamma^{ij}_{Y,a} : \mathscr{H}^{ij,1}_{Y,a} \to \mathscr{H}^{ij,2}_{Y,a}$$

such that

$$\varphi^1_Y = \left(\det(\psi_Y)^{\otimes w} \otimes \mathrm{id}_{\mathscr{P}_{\upsilon_Y}} \otimes \pi^*_Y(\gamma_Y)\right)^{-1} \circ \varphi^2_Y \circ \psi_{Y,u,v}, \qquad q^{ij,2}_{Y,a} \circ \psi_{Y,a} = \gamma^{ij}_{Y,a} \circ q^{ij,1}_{Y,a}.$$

Consider next the isomorphism

$$\begin{aligned} V_a \otimes \mathcal{O}_Y &\xrightarrow[\simeq]{\pi_{Y,*}\left(q^1_{Y,a} \otimes \mathrm{id}_{\pi^*_X(\mathcal{O}_X(n))}\right)} \pi_{Y,*}(\mathscr{E}^1_{Y,a} \otimes \pi^*_X(\mathcal{O}_X(n))) \longrightarrow \\ -- &\xrightarrow[\simeq]{\pi_{Y,*}\left(\psi_{Y,a} \otimes \mathrm{id}_{\pi^*_X(\mathcal{O}_X(n))}\right)} \pi_{Y,*}(\mathscr{E}^2_{Y,a} \otimes \pi^*_X(\mathcal{O}_X(n))) \longrightarrow \\ -- &\xrightarrow[\simeq]{\pi_{Y,*}\left(q^2_{Y,a} \otimes \mathrm{id}_{\pi^*_X(\mathcal{O}_X(n))}\right)^{-1}} V_a \otimes \mathcal{O}_Y \end{aligned}$$

and the induced morphism $\Phi : Y \to \mathcal{G}_A$. By the uniqueness property of 1.22, it will be enough to show that h_2 and $\alpha \circ (\Phi \times h_1)$ induce isomorphic quotient families. But the quotient family to $\alpha \circ (\Phi \times h_1)$ is the tuple $((q^3_{Y,a},\ \mathscr{E}^1_{Y,a},\ (q^{ij,1}_{Y,a},\ \mathscr{H}^{ij,1}_{\mathfrak{T}_{\mathrm{par}},a})_{i[s^j_a]j[|S|]})_{a[|A|]},\ \upsilon^1_Y,\ \mathscr{H}^1_Y,\ \varphi_Y)$ with

$$q^3_{Y,a} : V_a \otimes \pi^*_X(\mathcal{O}_X(-n)) \xrightarrow{\pi^*_Y \Phi^*(m^{-1}) \otimes \mathrm{id}_{\pi^*_X(\mathcal{O}_X(-n))}} V_a \otimes \pi^*_X(\mathcal{O}_X(-n)) \xrightarrow{q^1_{Y,a}} \mathscr{E}^1_{Y,a}$$

since $(\alpha\circ(\Phi\times h_1)\times \mathrm{id}_X)^* = (\Phi\times h_1\times \mathrm{id}_X)^*(\alpha\times \mathrm{id}_X)^*$. Furthermore by construction using the natural map $\pi_Y^*\pi_{Y,*}(F) \to F$ which exists for every sheaf F, we get the commuting diagram

$$\begin{tikzcd}
V_a \otimes \pi_X^*(\mathcal{O}_X(-n)) \arrow[r, "\pi_Y^*\Phi^*(m^{-1})\otimes \mathrm{id}_{\pi_X^*(\mathcal{O}_X(-n))}"] \arrow[d, "q^2_{Y,a}"'] & V_a \otimes \pi_X^*(\mathcal{O}_X(-n)) \arrow[d, "q^1_{Y,a}"] \\
\mathscr{E}^2_{Y,a} \arrow[r, "\psi^{-1}_{Y,a}"'] & \mathscr{E}^1_{Y,a}.
\end{tikzcd}$$

Hence $q^2_{Y,a} = \psi_{Y,a}\circ q^3_{Y,a}$ and the two families are isomorphic. ☐

Remark. (i) By construction of α

$$\begin{aligned}
&(\alpha\times \mathrm{id}_X)^*((\mathscr{E}_{\mathfrak{T}_{\mathrm{par}},a},\ (q^{ij}_{\mathfrak{T}_{\mathrm{par}},a},\ \mathscr{H}^{ij}_{\mathfrak{T}_{\mathrm{par}},a})_{i[s_a^j]j[|S|]})_{a[|A|]},\ \nu_{\mathfrak{T}_{\mathrm{par}}},\ \mathscr{H}_{\mathfrak{T}_{\mathrm{par}}},\ \varphi_{\mathfrak{T}_{\mathrm{par}}})\\
&\quad = (\pi_{\mathfrak{T}_{\mathrm{par}}}\times \mathrm{id}_X)^*((\mathscr{E}_{\mathfrak{T}_{\mathrm{par}},a},\ (q^{ij}_{\mathfrak{T}_{\mathrm{par}},a},\ \mathscr{H}^{ij}_{\mathfrak{T}_{\mathrm{par}},a})_{i[s_a^j]j[|S|]})_{a[|A|]},\ \nu_{\mathfrak{T}_{\mathrm{par}}},\ \mathscr{H}_{\mathfrak{T}_{\mathrm{par}}},\ \varphi_{\mathfrak{T}_{\mathrm{par}}})
\end{aligned}$$

holds.

(ii) For $Y = \{\mathrm{pt}\}$ we see that two tuples are isomorphic if and only if they are in the same $\mathcal{G}_A$−orbit. The direction "$\Leftarrow$" is obvious from the definition of α.

(iii) All results of this section may be transferred to the non-parabolic setting, in particular 1.25 works w. r. t. the non-parabolic version of our group action.

1.5. GIESEKER SPACE AND GIESEKER MAP

Now that we have defined a group action we are left with the task to prove that semistability, as defined before, is in fact the notion of semistability that we would expect from Geometric Invariant Theory. This will be done in two steps: First we are going to construct a closed equivariant embedding **Gies** of our parameter scheme $\mathfrak{T}_{\mathrm{par}}$ into some projective space $\mathbb{P}$ following a well-known construction principle introduced by D. Gieseker in [Gi77]. Since GIT-semistability of points in a projective space is relatively easy characterized numerically, we are only left with the task to show that **Gies** maps semistable points to GIT-semistable points. As GIT-semistability is preserved under closed embeddings, the existence of the GIT-Quotient $\mathbb{P}^{ss} /\!\!/ \mathcal{G}_A$ guarantees the existence of $\mathfrak{T}^{ss}_{\mathrm{par}} /\!\!/ \mathcal{G}_A$.

1.26. Let $\mathscr{P}^{d_a}$ be a Poincaré line bundle on Jac^{d_a}. We get[10] a locally free sheaf

$$\mathscr{G}_a^1 := \mathscr{H}om\left(\bigwedge^{r_a} V_a \otimes \mathcal{O}_{\mathrm{Jac}^{d_a}}, \pi_{\mathrm{Jac}^{d_a},*}(\mathscr{P}^{d_a} \otimes \pi_X^*(\mathcal{O}_X(r_a n)))\right), \quad a \in A.$$

We can modify $\mathscr{P}^{d_a}$ by a line bundle $\mathscr{L}$ on Jac^0 such that $\mathscr{P}' := \mathscr{P}^{d_a} \otimes \pi_{\mathrm{Jac}^0}^*(\mathscr{L})$ is another Poincaré line bundle. The space $\mathscr{G}_a^1$ transforms into $\mathscr{G}_a^1 \otimes \mathscr{L}$ ([Ha77], Ex. II.5.1 (b), (d)). Furthermore $\mathcal{O}_{\mathbb{P}((\mathscr{G}_a^1)^\vee)}(1)$ transforms into $\mathcal{O}_{\mathbb{P}((\mathscr{G}_a^1)^\vee)}(1) \otimes \pi_a^*(\mathscr{L}^\vee)$ for the bundle projection $\pi_a : \mathbb{P}((\mathscr{G}_a^1))^\vee) \to \mathrm{Jac}^{d_a}$ ([Ha77], Lemma II.7.9, Proposition II.7.10). Therefore by choosing $\mathscr{L}$ suitably $\mathcal{O}_{\mathbb{P}((\mathscr{G}_a^1)^\vee)}(1) \otimes \pi_a^*(\mathscr{L}^\vee)$ is very ample ([Ha77], II.7.10 (b)). Thus w. l. o. g. we may assume that $\mathcal{O}_{\mathbb{P}((\mathscr{G}_a^1)^\vee)}(1)$ is very ample. Let $\mathbb{P}^1 = \bigtimes_{a \in A} \mathbb{P}((\mathscr{G}_a^1)^\vee)$.
Next define the locally free sheaf

$$\begin{aligned} \mathscr{G}^2 := \mathscr{H}om\big(&V_{u,v} \otimes \mathcal{O}_{\mathrm{Jac}^l \times \mathrm{Jac}_A}, \\ &\pi_{\mathrm{Jac}^l \times \mathrm{Jac}_A,*}\left(\pi_{\mathrm{Jac}_A \times X}^*\left(\mathscr{P}_A\right)^{\otimes w} \otimes \pi_{\mathrm{Jac}^l \times X}^*(\mathscr{P}^l) \otimes \pi_X^*(\mathcal{O}_X(un))\right)\big) \end{aligned}$$

for $\mathscr{P}_A := \bigotimes_{a \in A} \pi_{\mathrm{Jac}^{d_a} \times X}^*(\mathscr{P}^{d_a, \otimes \kappa_a})$, $\mathrm{Jac}_A := \bigtimes_{a \in A} \mathrm{Jac}^{d_a}$. For a suitable choice of $\mathscr{P}^l$, $\mathcal{O}_{\mathbb{P}((\mathscr{G}^2)^\vee)}(1)$ is very ample. Define $\mathbb{P} := \mathbb{P}^1 \times \mathbb{P}((\mathscr{G}^2)^\vee) \times \mathbb{P}_{\mathfrak{G}}$ the Gieseker space, where $\mathbb{P}_{\mathfrak{G}} = \bigtimes_{a \in A} \bigtimes_{j=1}^{|S|} \bigtimes_{i=1}^{s_a^j} \mathbb{P}\left(\bigwedge^{r_a^{ij}} V_a\right)$.

1.27. Define $\det_a : \mathfrak{T}_{\mathrm{par}} \to \mathrm{Jac}^{d_a}$, $t \mapsto [\det \mathscr{E}_{a,t}]$. By the universal property of the Poincaré line bundle $\mathscr{P}^{d_a}$

$$\det(\mathscr{E}_{a,t}) \simeq \mathscr{P}^{d_a}\big|_{\det_a(t)}.$$

For the varieties Jac^{d_a}, X, [Ha77], III.Ex.12.4 implies the existence of a line bundle $\mathscr{L}_{\mathfrak{T}_{\mathrm{par}}}$ on $\mathfrak{T}_{\mathrm{par}}$ s. t.

$$\det(\mathscr{E}_{\mathfrak{T}_{\mathrm{par}},a}) \simeq (\det_a \times \mathrm{id}_X)^*(\mathscr{P}^{d_a}) \otimes \pi_{\mathfrak{T}_{\mathrm{par}}}^*(\mathscr{L}_{\mathfrak{T}_{\mathrm{par}},a}).$$

In other words we use the universal property of the Jacobian variety ([Ha77], IV.4.10).
We want to construct a morphism $\mathrm{Gies}_a^1 : \mathfrak{T}_{\mathrm{par}} \to \mathbb{P}((\mathscr{G}_a^1)^\vee)$ s. t. $(\mathrm{Gies}_a^1)^*(\mathcal{O}_{\mathbb{P}((\mathscr{G}_a^1)^\vee)}(1)) \simeq \mathscr{L}_{\mathfrak{T}_{\mathrm{par}},a}$. For any morphism $g : \mathfrak{T}_{\mathrm{par}} \to \mathrm{Jac}^{d_a}$ it is known that to give a morphism $\mathfrak{T}_{\mathrm{par}} \to \mathbb{P}((\mathscr{G}_a^1)^\vee)$ is equivalent to give an invertible sheaf $\mathscr{L}$ on $\mathfrak{T}_{\mathrm{par}}$ and a surjective map of sheaves on $\mathfrak{T}_{\mathrm{par}}$, $g^*((\mathscr{G}_a^1)^\vee) \to \mathscr{L}$ ([Ha77], II.7.12). In particular the morphism can be chosen to satisfy $g^*(\mathcal{O}_{\mathbb{P}((\mathscr{G}_a^1)^\vee)}(1)) \simeq$

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[10] compare with the construction of $\mathfrak{X}$ in 1.16.

$\mathscr{L}$.[11] We choose $g = \det_a$ and $\mathscr{L} := \mathscr{L}_{\mathfrak{T}_{\text{par}},a}$. The surjective morphism $\tilde{q}_a := \bigwedge^{r_a}\left(q_{\mathfrak{T}_{\text{par}},a} \otimes \text{id}_{\pi_X^*(\mathcal{O}_X(n))}\right)$ induces by 1.15 a surjective morphism

$$\tilde{q}_a \in \text{Hom}\left(\bigwedge^{r_a} V_a \otimes \mathcal{O}_{\mathfrak{T}_{\text{par}}\times X}, (\det_a \times \text{id}_X)^*(\mathscr{P}^{d_a} \otimes \pi_X^*(\mathcal{O}_X(r_a n))) \otimes \pi_{\mathfrak{T}_{\text{par}}}^*(\mathscr{L}_{\mathfrak{T}_{\text{par}},a})\right)$$
$$\simeq \text{Hom}\left((\det_a)^*\left(\mathscr{G}_a^1\right)^{\vee}, \mathscr{L}_{\mathfrak{T}_{\text{par}},a}\right)$$

Define now $\text{Gies}^1 := \times_{a\in A} \text{Gies}_a^1$.

1.28. The process can be transferred to $\mathscr{G}^2$. Note that for $\mathscr{L}_{\mathfrak{T}_{\text{par}},A} := \bigotimes_{a\in A} \mathscr{L}_{\mathfrak{T}_{\text{par}},a}^{\otimes \kappa_a}$

$$\det(\mathscr{E}_{\mathfrak{T}_{\text{par}}}) \simeq \left(\times_{a\in A} \det_a \times \nu_{\mathfrak{T}_{\text{par}}} \times \text{id}_X\right)^* (\pi_{\text{Jac}_A \times X})^*(\mathscr{P}_A) \otimes \pi_{\mathfrak{T}_{\text{par}}}^*(\mathscr{L}_{\mathfrak{T}_{\text{par}},A})$$

Consider the map $\hat{\psi} := (\varphi_{\mathfrak{T}_{\text{par}}} \circ q_{\mathfrak{T}_{\text{par}},u,v}) \otimes \text{id}_{\pi_X^*(\mathcal{O}_X(un))}$. Again using 1.15

$$\hat{\psi} \in \text{Hom}\left(V_{u,v} \otimes \mathcal{O}_{\mathfrak{T}_{\text{par}}\times X}, \left(\bigtimes_{a\in A} \det_a \times \nu_{\mathfrak{T}_{\text{par}}} \times \text{id}_X\right)^* (\pi_{\text{Jac}_A \times X})^*(\mathscr{P}_A^{\otimes w})\right.$$
$$\left. \otimes \pi_{\mathfrak{T}_{\text{par}}}^*(\mathscr{L}_{\mathfrak{T}_{\text{par}},A}^{\otimes w} \otimes \mathscr{H}_{\mathfrak{T}_{\text{par}}}) \otimes \mathscr{P}_{\mathfrak{T}_{\text{par}}} \otimes \pi_X^*(\mathcal{O}_X(un))\right)$$
$$\simeq \text{Hom}\left(\left(\times_{a\in A} \det_a \times \nu_{\mathfrak{T}_{\text{par}}}\right)^* (\mathscr{G}^2)^{\vee}, \mathscr{L}_{\mathfrak{T}_{\text{par}},A}^{\otimes w} \otimes \mathscr{H}_{\mathfrak{T}_{\text{par}}}\right)$$

Hence we find a morphism $\text{Gies}^2 : \mathfrak{T}_{\text{par}} \to \mathbb{P}((\mathscr{G}^2)^{\vee})$ ([Ha77], II.7.12).

1.29. For the final component we may use the Plücker embedding $\text{Gies}^3 := \times_{a\in A} \times_{j=1}^{|S|} \times_{i=1}^{s_a^j} \text{Gies}_a^{ij}|_{\mathfrak{F}\cap\mathfrak{G}_{\text{par}}}$ and $\text{Gies}_a^{ij} : \mathfrak{G}_a^{ij} \hookrightarrow \mathbb{P}\left(\bigwedge^{r_a^{ij}} V_a\right)$.

1.30. Define the Gieseker morphism $\text{Gies} = (\text{Gies}^1 \times \text{Gies}^2 \times \text{Gies}^3) : \mathfrak{T}_{\text{par}} \to \mathbb{P}$. Note that by the definition of Gies, Gies_a uniquely defines $\tilde{q}_a$. Now $\tilde{q}_a$ is the sheaf morphism that induces the Plücker embedding and therefore $\tilde{q}_a$ defines q_a uniquely. Furthermore note, that once $\text{Gies}(t)$ is fixed, $q_{t,a}$ is uniquely defined and thus by construction φ_t is uniquely defined. Finally the Plücker embedding induces a unique $q_{t,a}^{ij} \in \mathfrak{G}_a^{ij}$ mapped to $\text{Gies}_a^{ij}(t)$, i. e. $\text{Gies}(t)$ is one-to-one.
Furthermore Gies^1 commutes with the $\mathcal{G}_A$-action, where $\mathcal{G}_A$ acts on $\mathbb{P}$ in the natural way. Moreover we have already seen that the action descends to an action of $\mathcal{PG}_A$, i. e. it is in particular well-defined on our projective space $\mathbb{P}$.

Remark to 1.30. It is sometimes possible to repeat the construction of a morphism to Giesecker space $\mathbb{P}$ for Y-families even if a given morphism q_Y is not everywhere surjective. For example if $q_Y|_{\{y\}\times X}$ is surjective for every $y \notin T$ with T a

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[11] There is a unique morphism satisfying this additional property.

closed codimension 2 subset of a regular (or normal) scheme Y, then q_Y induces a morphism $Y \setminus T \to \mathbb{P}$, which extends uniquely to a morphism $\mathrm{Gies}_Y : Y \to \mathbb{P}$. Observe that for a Y-family $\mathscr{E}_Y$, a Y'-family $\mathscr{E}_{Y'}$ and a morphism $f : Y \to Y'$ such that $\mathscr{E}_Y = f^*\mathscr{E}_{Y'}$ and $\mathscr{L}_Y \simeq f^*\mathscr{L}_{Y'}$ the functorial properties of the pullback imply that $\mathrm{Gies}_Y = \mathrm{Gies}_{Y'} \circ f$ whenever $\mathrm{Gies}_Y, \mathrm{Gies}_{Y'}$ exist.

1.6. GIT-Semistability and Linearizations

In the following section we will give a brief account of the definitions and theorems from Geometric Invariant Theory that will be applied later on. Subsequently we will use these criteria to derive GIT-semistability conditions for certain (model) types of linear actions of $\mathcal{S}_A^{\kappa_a}$.

1.31. Theorem. (Hilbert-Mumford-Criterion) *Let Y be proper over k, k an arbitrary field of characteristic* 0*, L an ample line bundle on Y with a linearization of a group action by a reductive linear algebraic group G on Y. Let λ be a one-parameter subgroup of G, then for any $y \in Y$, consider the morphism $z \mapsto \lambda(z)^{-1}y$. Since $\mathbb{G}_m$ identifies with* $\mathrm{Spec}(k)[\alpha, \alpha^{-1}]$ *we may embed $\mathbb{G}_m$ into $\mathbb{A}^1 = \mathrm{Spec}(k)[\alpha]$. We find a unique extension $f_y : \mathbb{A}^1 \to Y$.*[12] *Now the action of $\mathbb{G}_m$ on L_{y_∞}, $y_\infty := f_y(0)$ is given by a character $\chi(z) = z^r$ for $z \in \mathbb{G}_m$. Define $\mu(y, \lambda) = -r$.*
A rational point y in Y is semistable if and only if $\mu(y, \lambda) \geq 0$ holds for every one-parameter subgroup λ of G.
y is stable if and only if $\mu(y, \lambda) > 0$ holds for every (non-trivial) one-parameter subgroup λ of G.

Proof. [MFK], Theorem 2.1, Proposition 2.2 and 1.§3. The case of a not necessarily algebraically closed ground field k is treated e.g. in Théorème 5.2 of [Rou78] and [RR84]. □

Remark.

1. For Y projective over $\mathbb{C}$, define $y_\infty = \lim_{z\to\infty} \lambda(z)y$ and $\mathbb{C}^*$ acts on L_{y_∞} by $l \mapsto z^r \cdot l$.

2. For every $g \in G$ and every one-parameter subgroup: $\mu(gy, \lambda) = \mu(y, g^{-1}\lambda g)$.

3. Given a closed G-invariant subscheme $Z \subset Y$, $z \in Z$ already implies $z_\infty \in Z$. Thus given the induced linearization on $L|_Z$ a point $z \in Z$ is (semi)stable w. r. t. this linearization in L_Z if and only if $z \in Y$ is (semi)stable w. r. t. the linearization in L.[13]

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[12] $\mathbb{A}^1_{(0)}$ is a valuation ring, X proper over k.

[13] This statement can be proved without using the Hilbert-Mumford-Criterion, cf. [Sch08], 1.4.3, The General Theory.

1.32. One-Parameter Subgroups and Filtrations. Let $\nu : G \to \mathrm{Gl}(W)$ be the action of a linear algebraic group G on some vector space W. Let λ be a one-parameter subgroup of G, then $\nu \circ \lambda$ is a one-parameter subgroup of $\mathrm{Gl}(W)$. There is a basis $(w^i)_{i[\dim(W)]}$ and integer weights $\gamma^i \in \mathbb{Z}$, $\forall 1 \leq i \leq \dim(W)$ such that $\nu \circ \lambda(z) w^i = z^{\gamma^i} w^i$ and $\gamma^i \leq \gamma^{i+1}$, $\forall 1 \leq i \leq \dim(W) - 1$. If ν maps to $\mathrm{Sl}(W)$, then $\sum_{i=1}^{\dim(W)} \gamma^i = 0$. On the other hand given a basis $(w^i)_{i[\dim(W)]}$ and ascending integer weights $\gamma^i \in \mathbb{Z}$, $\forall 1 \leq i \leq \dim(W)$ we receive a one-parameter subgroup of $\mathrm{Gl}(W)$. If $\sum_{i=1}^{\dim(W)} \gamma^i = 0$ the image of this one-parameter subgroup lies in $\mathrm{Sl}(W)$ (cf. [Sch08], Example 1.1.2.3 and 1.5.1.11).
Every basis $(w^i)_{i[\dim(W)]}$ as above defines subspaces $\langle w^j : \gamma^j \leq \gamma^i \rangle$ of W and hence $(w^i, \gamma^i)_{i[\dim(W)]}$ defines a proper weighted filtration $(W^i, \gamma^i)_{i[m]}$ of W. The filtration $(W^i)_{i[m]}$ is uniquely defined by λ. The converse is obviously not true. However, we will see below that the value of our weight functions $\mu(p, \lambda)$, $p \in \mathbb{P}(W)$ solely depends on the weighted filtration $(W^i, \gamma^i)_{i[m]}$ induced by λ. To check (semi)stability of a point $p \in \mathbb{P}(W)$ it is therefore enough to choose for every proper (non-trivial) filtration $(W^i)_{i[m]}$ with strictly ascending integer weights $(\gamma^i)_{i[m]}$ a single one-parameter subgroup λ that induces $(W^i)_{i[m]}$ and to verify that $\mu(p, \lambda)$ $(\geq)$ 0.

1.33. Semistability in Projective Space. Let $L \xrightarrow{\pi} P$ be a very ample line bundle on a projective variety P with a linearization $\sigma : G \times L \to L$ of a G-action υ on P. Recall that $\pi \circ \sigma = \upsilon \circ \pi$ and $L_p \to L_{\upsilon(g,p)}$, $l \mapsto \sigma(g, l)$ is linear for every $g \in G$, $\forall p \in P$.
Then there exists an immersion $\iota : P \to \mathbb{P}_n$, an action $\upsilon_{\mathbb{P}_n}$ of G on $\mathbb{P}_n$, and a G-linearization $\sigma_{\mathbb{P}_n}$ in $\mathcal{O}_{\mathbb{P}_n}(1)$ such that ι is G-linear and such that L together with its G-linearization σ is induced via ι from $\mathcal{O}_{\mathbb{P}_n}(1)$ and its G-linearization $\sigma_{\mathbb{P}_n}$ ([MFK], Proposition 1.7).

Let $W := \mathbb{C}^{n+1}$. Now since $H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(1)) \simeq W$, the linear action $\sigma_{\mathbb{P}(W)}$ on $\mathcal{O}_{\mathbb{P}(W)}(1)$ to $\upsilon_{\mathbb{P}(W)}$ on $\mathbb{P}(W)$ induces a linear action

$$\begin{aligned} \upsilon_W : \ & G \times W \to W \\ & (g, w) \mapsto \{p \mapsto \sigma_{\mathbb{P}(W)}(g, w(\upsilon_{\mathbb{P}(W)}(g^{-1}, p)))\}. \end{aligned}$$

Then the dual linear action

$$\begin{aligned} \upsilon_{W^\vee}: \quad G \times W^\vee & \to W^\vee \\ (g, w^\vee) & \mapsto \{u \mapsto w^\vee(\upsilon_W(g^{-1}, u))\}, \end{aligned}$$

descends via the natural projection $\pi^\vee : W^\vee \setminus \{0\} \to \mathbb{P}(W)$ onto $\mathbb{P}(W)$ s. t. $\pi^\vee(\upsilon_{W^\vee}(g, w^\vee)) = \upsilon_{\mathbb{P}(W)}(g, \pi^\vee(w^\vee))$. On the other hand we will soon see, that if $\pi^\vee$ is equivariant with respect to two actions then there is a linearization of the action on $\mathbb{P}(W)$ in $\mathcal{O}_{\mathbb{P}(W)}(1)$.
Hence we can lift the action $\upsilon_{\mathbb{P}(W)}$ to a vector space action υ_W on W. Let λ be a one-parameter subgroup of G acting diagonally on a basis $(w^i)_{i[n+1]}$ of W with weights $\gamma^1_{\upsilon_W} \leq \cdots \leq \gamma^{\dim(W)}_{\upsilon_W}$. Define $\lambda^\vee := \upsilon_{W^\vee}(\lambda, \,\cdot\,)$. Now $\lambda^\vee$ acts diagonal w. r. t. the dual basis $(w^{i,\vee})_{i[\dim(W)]}$ of $W^\vee$ with weights $-\gamma^1_{\upsilon_W}, \ldots, -\gamma^{\dim(W)}_{\upsilon_W}$. If $p = \sum_{i=1}^{\dim(W)} p^i w^{i,\vee}$ and $\gamma^{l-1}_{\upsilon_W} < \gamma^l_{\upsilon_W} = \cdots = \gamma^{l+k}_{\upsilon_W} = \min\{\gamma^i_{\upsilon_W} : p^i \neq 0\} < \gamma^{l+k+1}_{\upsilon_W}$ then $p_\infty := \lim_{z \to \infty} \lambda(z)[p] = [\ldots, 0, p^l, \ldots, p^{l+k}, 0, \ldots]$. In particular we get $\mu([p], \lambda) = -\min\{\gamma^i_{\upsilon_W} : p^i \neq 0\}$.

1.34. If $L \to P$, $L' \to P'$ are line bundles with a linearization σ of a G-action on P and a linearization σ' of a G'-action on P', then $\pi^*_P(L) \otimes \pi^*_{P'}(L') \to P \times P'$ has an induced linearization

$$\sigma \otimes \sigma'((g, g'), (l_p \otimes l'_{p'})) := \sigma(g, l_p) \otimes \sigma'(g', l'_{p'}) \in L \otimes L'|_{(gp, g'p')}$$

$\forall\, l_p \in L|_p$, $\forall\, l'_{p'} \in L'|_{p'}$, $\forall g \in G$, $\forall g' \in G'$ s. t. $\mu_{\sigma\otimes\sigma'}((p, p'), \lambda \times \lambda') = \mu_\sigma(p, \lambda) + \mu_\sigma(p', \lambda')$, $\forall p \in P$, $\forall p' \in P'$ and one-parameter subgroups $\lambda : \mathbb{C}^* \to G$, $\lambda' : \mathbb{C}^* \to G'$. For $P = P'$ we use the pullback of $\sigma \otimes \sigma'$ by the diagonal embedding $P \hookrightarrow P \times P$ instead. The linearization in the tensor product $\mathcal{O}_{\mathbb{P}(W)}(k)$ from 1.33 leads us consequentially to $\mu_k([l], \lambda) = k \cdot \mu_1([l], \lambda)$ for the μ-functions μ_1 and μ_k w. r. t. the natural linearizations in $\mathcal{O}_{\mathbb{P}(W)}(1)$ and $\mathcal{O}_{\mathbb{P}(W)}(k)$.
Let $\chi : G \to \mathbb{C}^*$ be a character of G. Then $\sigma_\chi := \chi \cdot \sigma$ is another linearization, since σ is linear. Further $\sigma_\chi(\lambda(z), \,\cdot\,) = \chi(\lambda(z)) \cdot \sigma(\lambda(z), \,\cdot\,)$ implies

$$\mu_\chi(\,\cdot\,, \lambda) = \mu(\,\cdot\,, \lambda) + \langle \lambda, \chi \rangle$$

for μ_χ the μ-function to the linearization σ_χ and $\langle \lambda, \chi \rangle$ the dual pairing, i. e. $\langle \lambda, \chi \rangle = \gamma$ unique with $\chi(\lambda(z)) = z^\gamma$, $z \in \mathbb{C}^*$ ([Bo91], 8.6).
Fix a basis $(w^i)_{i[n+1]}$ of W and consider all one-parameter subgroups with respect to this basis. Assume that there is a one-parameter subgroup $\lambda(\gamma^i_{\upsilon_W})$ of G that corresponds to an integer tuple $(\gamma^i_{\upsilon_W})_{i[n+1]}$, i. e. $\nu_W(\lambda, w^i)(z) = z^{\gamma^i_{\upsilon_W}} w^i$, $\forall 1 \leq i \leq n+1$, $z \in \mathbb{C}^*$, as in the previous paragraph 1.33. Then $\mu([l], \lambda(c \cdot \gamma^i_{\upsilon_W})) = c \cdot \mu([l], \lambda(\gamma^i_{\upsilon_W}))$, $\forall [l] \in \mathbb{P}(W)$, $c \in \mathbb{Q}_+$ by definition of the $\mu-$function. If $(\gamma^i_{\upsilon_W})_{i[n+1]}$, $(\delta^i_{\upsilon_W})_{i[n+1]}$ are ordered tuples, i. e. $\gamma^{i+1}_{\upsilon_W} \geq \gamma^i_{\upsilon_W}$ and $\delta^{i+1}_{\upsilon_W} \geq \delta^i_{\upsilon_W}$ for all $1 \leq i \leq n$, then $\mu([l], \lambda(\gamma^i_{\upsilon_W} + \delta^i_{\upsilon_W})) = \mu([l], \lambda(\gamma^i_{\upsilon_W})) + \mu([l], \lambda(\delta^i_{\upsilon_W}))$, $\forall [l] \in \mathbb{P}(W)$. The same result holds for $\geq$ replaced by $\leq$, i. e. $\gamma^{i+1}_{\upsilon_W} \leq \gamma^i_{\upsilon_W}$ and $\delta^{i+1}_{\upsilon_W} \leq \delta^i_{\upsilon_W}$ for all $1 \leq i \leq n$. For the special case

when $\gamma^i_{v_W}$ is chosen arbitrary and $(\delta^i_{v_W})_{i[n+1]} = (1, \dots, 1)$, $c \in \mathbb{Q}$ we receive $\mu([l], \lambda(\gamma^i_{v_W} + c\delta^i_{v_W})) = \mu([l], \lambda(\gamma^i_{v_W})) + c \cdot \mu([l], \lambda(\delta^i_{v_W}))$, $\forall [l] \in \mathbb{P}(W)$. Note in particular $\mu([l], \lambda(\gamma^i_{v_W})) = \sum_{i=1}^{n+1} \alpha^i \mu([l], \lambda(\gamma^i_{n+1}))$ in the notation of 1.3.

1.35. Semistability in Quotients and Tensor Products. Let V be a vector space of dimension p, $(v^i)_{i[p]}$ a basis of V and $\left(\bigwedge_{j=1}^r v^{\theta(j)}\right)_{\theta \in \mathrm{Mon}^p_r}$, $\mathrm{Mon}^p_r := \{\theta : \{1, \dots, r\} \to \{1, \dots, p\} \text{ monotone increasing}\} \subset \mathrm{Map}^p_r := \{1, \dots, r\}^p$ a corresponding basis of $\bigwedge^r V$. Consider the usual action of $\mathrm{Gl}(V)$ on $\bigwedge^r V$ with linearization in $\mathcal{O}_{\mathbb{P}(\bigwedge^r V)}(1)$. Now let λ be a one-parameter subgroup of $\mathrm{Gl}(V)$ with associated ascending weights γ^i w. r. t. a basis of eigenvectors $(v^i)_{i[p]}$. Note, that for quotients $q : V \to W$ with the rules derived in 1.34 and the linearization induced by the Plücker embedding we have $\mu([q], \lambda(\gamma^i_p)) = p \dim(\mathrm{im}(q(\langle v^1, \dots, v^i \rangle)))$ and $\mu([q], \lambda(\gamma^i)) = \sum_{i=1}^p \alpha^i \mu([q], \lambda(\gamma^i_p))$.
More precisely: If $q : V \to W$ is a quotient, then $\bigwedge^r q$, $r = \dim W$ induces a morphism to projective space ([Ha77], II.7.12).[14] This is the Plücker embedding of the Graßmannian into $\mathbb{P}(\bigwedge^r V)$. Then 1.34 implies that the natural weight function is defined as $-\min\{\sum_{j=1}^r \gamma^{\theta(j)} |\ q(v^\theta) \neq 0,\ \theta \in \mathrm{Mon}^p_r\}$.
Denote by $\theta_q \in \mathrm{Map}^p_r :\ \theta_q(j) := \min\{k : \dim(q(v^1, \dots, v^k)) = j\}$. Then $\sum_{j=1}^r \gamma^{\theta_q(j)}$ is minimal for any choice of a (ascending) weight vector $(\gamma^j)_{j[p]}$. Assume there was a $\theta' \in \mathrm{Map}^p_r$ such that $q(v^{\theta'}) \neq 0$, then the $q(v^{\theta'(j)})$ are linearly independent. Hence by the definition of θ_q we get $\theta_q(j) \leq \theta'(j)$ for all $1 \leq j \leq r$. In particular for monotone increasing weights $\sum_{j=1}^r \gamma^{\theta_q(j)} \leq \sum_{j=1}^r \gamma^{\theta'(j)}$.
Thus the weight function becomes with 1.34

$$\mu([q], \lambda) = -\min\left\{\sum_{j=1}^r \gamma^{\theta(j)} |\ q(v^\theta) \neq 0,\ \theta \in \mathrm{Mon}^p_r\right\} = \sum_{k=1}^p \alpha^k p v\,(k, \theta_q)$$

$$= \sum_{k=1}^p \alpha^k p \dim(\mathrm{im}(q(\langle v^1, \dots, v^k \rangle))), \quad v\,(k, \theta_q) := \#\{j : \theta_q(j) \leq k\}.$$

Finally note that for a tensor product $(V^{\otimes u})^{\oplus v}$ and a basis $(v^i)_{i[p]}$ of V:

$$v^\theta_s := \left(\delta_{st} \bigotimes_{j=1}^u v^{\theta(j)}\right)_{t[v]}$$

is a basis of $(V^{\otimes u})^{\oplus v}$ if s varies over $1 \leq s \leq v$ and $\theta \in \mathrm{Map}^p_u$. Then for $q = \sum_{s=1}^v \sum_{\theta \in \mathrm{Map}^p_u} a^\theta_s v^{\theta,\vee}_s$

$$\mu([q], \lambda) = -\min\left\{-\sum_{i=1}^p \alpha^i p v\,(i, \theta) \mid \exists 1 \leq s \leq v,\ \theta \in \mathrm{Map}^p_u :\ q(v^\theta_s) \neq 0\right\}.$$

---

[14] Take the universal quotient over the Graßmannian of r-dimensional quotients of p-dimensional space and the determinant thereof.

Remark. Observe that the weight functions in 1.35 depend only on the proper weighted filtration induced by λ (see 1.32). This is a general property of weight functions (see [Sch08], Proposition 1.5.1.35.).

Example. Let $\lambda(z) = \operatorname{diag}(z^{-1}, 1, z) = \operatorname{diag}(z^{\gamma^1}, z^{\gamma^2}, z^{\gamma^3})$ be a one-parameter subgroup of $\mathrm{Sl}(V)$, $V = \mathbb{C}^3$ w. r. t. a basis (v^1, v^2, v^3) and $\nu_{V^{\otimes 2}}$ the natural action of $\mathrm{Sl}(V)$ on $W = V^{\otimes 2}$. By 1.3 we get $\alpha^1 = \alpha^2 = \frac{1}{3} = -\alpha^3$. The weights of $\nu_{V^{\otimes 2}} \circ \lambda$ are $\gamma^{jk} = \gamma^j + \gamma^k = j + k - 4$ w. r. t. the basis $(v^j \otimes v^k)_{j[3]k[3]}$. Now consider $q = (v^1 \otimes v^3)^\curlyvee + (v^2 \otimes v^2)^\curlyvee \in (V^{\otimes 2})^\curlyvee$. We get

$$\begin{aligned}&-\sum_{i=1}^{3} 3 \cdot \alpha^i \min\left\{-v(i,\theta) |\ \exists \theta \in \mathrm{Map}_2^3 : q(v^\theta) \neq 0\right\}\\ &= -1 \cdot (-v(1,(1,3)) - 1 \cdot (-v(2,(2,2)) + 1 \cdot (-v(3,(1,3)) = 1 + 2 - 2 = 1\end{aligned}$$

but

$$\begin{aligned}\mu([q],\lambda) &= -\min\left\{-\sum_{i=1}^{3} 3 \cdot \alpha^i v\,(i,\theta)\ |\ \exists \theta \in \mathrm{Map}_2^3 :\ q(v^\theta) \neq 0\right\}\\ &= v\,(1,(1,3)) + v\,(2,(1,3)) - v\,(3,(1,3)) = 1 + 1 - 2 = 0.\end{aligned}$$

We see that we cannot sum over the minimized $v(i,\theta)$ as in the case of an action on an exterior product. However the additivity in 1.34 still holds: Order the γ^{ij} as $\gamma^k_{\nu_{V^{\otimes 2}}} \in \{\gamma^{ij} : i + j = k + 1\}$. Then $\alpha^k_{\nu_{V^{\otimes 2}}} = \frac{1}{9}$ for $1 \leq k \leq 4$ and $\alpha^5_{\nu_{V^{\otimes 2}}} = -\frac{2}{9}$ and

$$r^1_{\nu_{V^{\otimes 2}}} = 1,\ r^2_{\nu_{V^{\otimes 2}}} = 3,\ r^3_{\nu_{V^{\otimes 2}}} = 6,\ r^4_{\nu_{V^{\otimes 2}}} = 8,\ r^5_{\nu_{V^{\otimes 2}}} = 9.$$

We get

$$\mu([q],\lambda) = -\min\{\gamma^{ij} : q(v^{ij}) \neq 0\} = -\min\{\gamma^i + \gamma^j : (i,j) \in \{(1,3),(2,2)\}\} = 0$$

as well as

$$\begin{aligned}&\mu([q],\lambda)\\ &= -\sum_{k=1}^{5} \alpha^k_{\nu_{V^{\otimes 2}}} \min\left\{(\gamma_9^{r^k_{\nu_{V^{\otimes 2}}}})_{r^l_{\nu_{V^{\otimes 2}}}} = (\gamma^k_{5,9})_l : q(v^{ij}) \neq 0 \text{ for some } i + j = l + 1\right\}\\ &= \left(-\frac{1}{9}\right) \cdot 0 + \left(-\frac{1}{9}\right) \cdot 0 + \left(-\frac{1}{9}\right) \cdot (-9) + \left(-\frac{1}{9}\right) \cdot (-9) + \left(\frac{2}{9}\right) \cdot (-9)\\ &= 0.\end{aligned}$$

1.36. Let G act on a vector space V and let W be another vector space. Then G acts on W trivially and we get an action on the tensor product $g(v \otimes w) = gv \otimes w$

for all $g \in G,\ v \in V,\ w \in W$. Let $\lambda : \mathbb{C}^* \to G$ be a one-parameter subgroup and $(v^i)_{i[\dim V]}$ a basis of V such that λ is diagonal w. r. t. this basis with weights γ^i. Choose any basis $(w^j)_{j[\dim W]}$ of W, then $\mu(L, \lambda) = -\min\{\gamma^i : L(v^i \otimes w^j) \neq 0 \text{ for some } w^j\}$ and $L = \sum_{ij} L^{ij}(v^i \otimes w^j)^\vee$ a non-trivial linear form. The isomorphism $\Psi : V^\vee \otimes W^\vee \to (V \otimes W)^\vee,\ l \otimes k \mapsto \{\sum_{ij} a^{ij} v^i \otimes w^j \mapsto \sum_{ij} a^{ij} l(v^i) k(w^j)\}$ shows that for $L = \Psi(l, k) : \mu(L, \lambda) = -\min\{\gamma^i : l(v^i) \neq 0\} = \mu(l, \lambda)$.

1.37. As we have seen above GIT-semistability depends on the choice of a linearization in some suitable (ample) line bundle. We make the following choices

$$\begin{aligned}\tilde{\nu}_a &:= \frac{\kappa_a \left(p - u\delta - \sum_{j=1}^{|S|} \sum_{i=1}^{s^j} \delta^{ij}(r - r^{ij})\right) - \xi_a \cdot r}{p},\\ \tilde{\nu} &:= \frac{r \cdot \delta}{p},\\ \tilde{\nu}_a^{ij} &:= \frac{r \cdot \kappa_a \cdot \delta_a^{ij}}{p}\end{aligned}$$

and $l \in \mathbb{N}$ minimal s. t. $\nu_a := l\tilde{\nu}_a,\ \nu := l\tilde{\nu},\ \nu^{ij} := l\tilde{\nu}^{ij} \in \mathbb{Z}$.

For line bundles $L_a,\ L$ on X, the fiber over $(\bigtimes_{a\in A}[L_a],\ [L])$ in $\mathbb{P}$ is $\bigtimes_{a\in A} \mathbb{P}(\mathbb{L}_a) \times \mathbb{P}(\mathbb{L}) \times \mathbb{P}_{\mathfrak{G}}$ with

$$\mathbb{L}_a := \operatorname{Hom}\left(\bigwedge^{r_a} V_a, H^0(L_a(r_a n))\right)^\vee \simeq \bigwedge^{r_a} V_a \otimes H^0(L_a(r_a n))^\vee$$

$$\mathbb{L} := \operatorname{Hom}\left(V_{u,v}, H^0\left(\bigotimes_a L_a^{\otimes \kappa_a w} \otimes L(un)\right)\right)^\vee \simeq V_{u,v} \otimes H^0\left(\bigotimes_a L_a^{\otimes \kappa_a w} \otimes L(un)\right)^\vee.$$

Remark. Obviously every point y in $\mathbb{P}$ belongs to one of these closed fibers. By definition of (semi)stability (resp. the Hilbert-Mumford criterion 1.31 and its following remark) we are allowed to check the (semi)stability of y considered as an element of some fiber $\bigtimes_{a\in A} \mathbb{P}(\mathbb{L}_a) \times \mathbb{P}(\mathbb{L}) \times \mathbb{P}_{\mathfrak{G}}$. The fibers are tuples of projective spaces (or closed subschemes thereof) and the GIT-weight functions μ are well-known in this situation (cf. 1.35 and 1.36).

Assume there is an action σ of $\mathrm{Gl}(W)$ on $\mathbb{P}(W^\vee)$, W some vector space. Then σ lifts to an action on W and by definition of $\mathcal{O}_{\mathbb{P}(W^\vee)}(-1) \subset \mathbb{P} \times W^\vee$, we receive a linearization in $\mathcal{O}_{\mathbb{P}(W^\vee)}(-1)$ and hence a linearization in $\mathcal{O}_{\mathbb{P}(W^\vee)}(1)$ resp. $\mathcal{O}_{\mathbb{P}(W^\vee)}(m),\ m \in \mathbb{Z}$. By construction of the $\mathcal{G}_A$-action on the components of $\mathbb{P}$, this action lifts to $\mathscr{G}_a^1,\ a \in A,\ \mathscr{G}^2$ and we get a linearization of the $\mathcal{G}_A$-action in

$$\begin{aligned}&\mathcal{O}_{\mathbb{P}}(\nu_a, \nu, \nu^{ij})\\ &:= \bigotimes_{a\in A}\left(\left(\pi^*_{\mathbb{P}(\mathbb{L}_a)}(\mathcal{O}_{\mathbb{P}(\mathbb{L}_a)}(\nu_a))\right) \otimes \bigotimes_{j=1}^{|S|} \bigotimes_{i=1}^{s_a^j} \left(\pi^*_{\mathfrak{G}_a^{ij}}(\mathcal{O}_{\mathfrak{G}_a^{ij}}(\nu_a^{ij}))\right)\right) \otimes \left(\pi^*_{\mathbb{P}(\mathbb{L})}(\mathcal{O}_{\mathbb{P}(\mathbb{L})}(\nu))\right).\end{aligned}$$

1.38. Modification by a Character. Let λ be a one-parameter subgroup of $\mathcal{S}_A^{\kappa_a}$ with associated filtration $(V^i)_{i[p]}$ and weights γ^i. Using 1.34 for the character $\chi : \mathcal{S}_A^{\kappa_a} \to \mathbb{C}^*$, $(M_a)_{a\in A} \mapsto \prod_{a\in A} \det(M_a)^{\chi_a}$ with $\sum_{a\in A} \chi_a \dim(V_a) = 0$ and $\sigma : \mathcal{S}_A^{\kappa_a} \times \mathcal{O}_{\mathbb{P}}(\nu_a, \nu, \nu^{ij}) \to \mathcal{O}_{\mathbb{P}}(\nu_a, \nu, \nu^{ij})$ the linearization of our $\mathcal{S}_A^{\kappa_a}$-action, we receive

$$\langle \lambda, \chi \rangle = \sum_{i=1}^{p} \gamma^i \sum_{a\in A} \chi_a \dim(V_a^i / V_a^{i-1}),$$

for $V_a^i := V_a \cap V^i$. By using 1.3 and some index shifting we may rewrite this expression as

$$\langle \lambda, \chi \rangle = \sum_{i=1}^{p-1} \alpha^i \dim(V) \cdot \sum_{a\in A} \chi_a (\dim(V_a) - \dim(V_a^i)).$$

Denote by σ_χ the group action altered by χ.

1.39. Definition. A point $p \in \mathbb{P}$ is χ-(semi)stable if $\mu_\chi(p, \lambda)$ $(\geq)$ 0 holds for every one-parameter subgroup $\lambda : \mathbb{C}^* \to \mathcal{S}_A^{\kappa_a}$, where μ_χ is the μ-function w. r. t. the linearization σ_χ.
Fix $\chi : \mathcal{S}_A^{\kappa_a} \to \mathbb{C}^*$, $(M_a)_{a[|A|]} \mapsto \prod_{a\in A} \det(M_a)^{\chi_a}$ with $\chi_a := \chi_a^1 + \chi_a^2$

$$\chi_a^1 := \left(\sum_{j=1}^{|S|} \sum_{i=1}^{s^j} \delta_a^{ij} \kappa_a \left(\frac{r_a^{ij} r - r_a r^{ij}}{p p_a} \right) \right),$$
$$\chi_a^2 := \left(\left(1 - \frac{u\delta}{p} \right) \cdot \kappa_a \left(\frac{r}{p} - \frac{r_a}{p_a} \right) + \left(\frac{\xi_a r}{p} \cdot \frac{r_a}{p_a} \right) \right).$$

1.7. Main Calculations

From now on we will use the following notational conventions.

1.40. Notation. Let λ be a one one-parameter subgroup of $\mathcal{S}_A^{\kappa_a}$ defined by a basis $(v^k)_{k[p]}$ of V and ascending weights $(\gamma^k)_{k[p]}$ and let $(V^k)_{k[p]}$ be the corresponding complete filtration. As usual we will receive filtrations $(V_a^k)_{k[p]}$ and weights $(\gamma_a^k)_{k[p]}$ for every $a \in A$. Given a filtration $(V^k)_{k[p]}$ of V we choose a suitable basis $(v^k)_{k[p]}$ and proceed as before. We will sometimes write $\lambda((\gamma^k)_{k[p]})$ if we want to lay special emphasis on the weights of the one-parameter subgroup.
Vektor subspace filtrations $(V_a^k)_{k[p]}$ of V_a generate filtrations of E_a by coherent subsheaves $F_a^{k,coh} = q_a(V_a^k \otimes \mathcal{O}_X(-n))$. These add up to a filtration of coherent subsheaves $F^{k,coh}$ of E in the usual way. Let $F_a^k = \overline{F_a^{k,coh}}$ (resp. $F^k = \overline{F^{k,coh}}$) be the vector subbundles of E_a (resp. E) generated by V_a^k (resp. V^k). Furthermore

there is an induced filtration of each $\ker(q_a^{ij}) = E_a^{ij} \subset E_a|_{\{x^j\}}$ over the puncture x^j by vector subspaces $F_{a,coh}^{ij,k} = E_a^{ij} \cap F_{a,coh}^k|_{\{x^j\}}$, $1 \leq i \leq s^j$.[15] We receive another filtration of E_a^{ij} by $F_a^{ij,k} := E_a^{ij} \cap F_a^k|_{\{x^j\}}$. Denote by $F_{coh}^{ij,k}$ and $F^{ij,k}$ the corresponding subspaces of $E^{ij} \subset E|_{\{x^j\}}$.

The dimensions and ranks of vector (sub)spaces and coherent (sub)sheaves are denoted in the usual way, i. e. $r_\star^* = \mathrm{rk}(F_\star^*)$ and $r_\star = \mathrm{rk}(E_\star)$ as well as $p_\star^* = \dim(V_\star^*)$. Note that for a complete filtration $(V_\star^k)_{k[p]}$ we get $p_\star^k = k$.

Finally define $f_{a,coh}^{ij,k} = \dim(\mathrm{im}\left(q_a^{ij}\left(F_a^{k,coh}|_{\{x^j\}}\right)\right))$, $f_a^{ij,k} = \dim\left(\mathrm{im}\left(q_a^{ij}(F_a^k|_{\{x^j\}})\right)\right)$ and $f_{coh}^{ij,k}$, $f^{ij,k}$, f_a^{ij}, f^{ij} accordingly. Note that $f_{a,coh}^{ij,k} = r_a^k - r_{a,coh}^{ij,k}$, $f_a^{ij,k} = r_a^k - r_a^{ij,k}$. Let the weights δ_a^{ij} be defined as in 1.8 from parabolic weights $\beta_a^{ij} \in \mathbb{Q}$. Furthermore let ν_a, ν, ν_a^{ij} and χ_a^1, χ_a^2 be defined by the expressions in 1.37 and 1.39 as functions of the already defined parameters r, r_a, etc.

For a function $\theta \in \mathrm{Map}_{m^1}^{m^2}$ denote $v(k,\theta) = \#\{j \in \{1,\dots,m^1\} : \theta(j) \leq k\}$ (cf. 1.35). We identify quotients with the corresponding elements of an exterior power. Further if $q = \sum_{\theta \in \mathfrak{l}} q^\theta v^{\theta(1)} \star \cdots \star v^{\theta(m^1)}$, $\mathfrak{l} = \mathrm{Mon}_{m^1}^{m^2} \vee \mathrm{Map}_{m^1}^{m^2}$ for $(v^i)_{i[m^2]}$ basis of $\mathbb{C}^{m^2}$ and $\star = \otimes, \wedge$ then θ_q^λ denotes the element of $\mathfrak{l}$ such that $-\sum_{k=1}^{m^1} \alpha^k v(k,\cdot)$ is minimal within the set $\{\theta \in \mathfrak{l} : q^\theta \neq 0\}$.

1.41. Semistability for points in $\mathrm{Gies}(\mathfrak{T}_{\mathrm{par}})$**.** We use the notation from 1.40. Let $t \in \mathfrak{T}_{\mathrm{par}}$ resp. $q := q_t$ correspond to E. $\mathrm{Gies}(t)$ is represented by a tuple $(q_a,\ q_{\varphi\circ q_{u,v}},\ q_a^{ij}) \in \bigtimes_{a\in A} \bigwedge^{r_a} V_a \otimes H^0(L_a(r_a n))^\vee \times V_{u,v} \otimes H^0\left(\bigotimes L_a^{\otimes \kappa_a w} \otimes L(un)\right)^\vee \times \bigtimes_{a\in A} \bigtimes_{j=1}^{|S|} \bigtimes_{i=1}^{s_a^j} \bigwedge^{r_a^i} V_a$. Using 1.36 we can apply the general calculations in 1.35 to get with resp. to the standard linearization in the corresponding $\mathcal{O}_\star(1)$:

$$\begin{aligned}
\mu(\mathrm{Gies}_a^1(t), \lambda(\gamma_{p_a}^k)) &= p_a v(k, \theta_{q_a}^{\lambda(\gamma_{p_a}^k)}), = p_a r_a^k \\
\mu(\mathrm{Gies}^2(t), \lambda(\gamma_p^k)) &= p v(k, \theta_{\varphi\circ q_{u,v}}^{\lambda(\gamma_p^k)}), \\
\mu(\mathrm{Gies}_a^{ij}(t), \lambda(\gamma_{p_a}^k)) &= p_a v(k, \theta_{q_a^{ij}\circ q_a}^{\lambda(\gamma_{p_a}^k)}) = p_a f_{a,coh}^{ij,k} = p_a(r_a^k - r_{a,coh}^{ij,k}).
\end{aligned}$$

Recall that γ_p^k, $\gamma_{p_a}^k$ were defined in 1.3. Observe that $\theta_{\varphi\circ q_{u,v}}^{\lambda(\gamma_p^k)}$ might change if we consider $\lambda = \lambda((\gamma^k)_{k[p]})$ instead of $\lambda(\gamma_p^k)$, i. e. $\theta_{\varphi\circ q_{u,v}}^{\lambda} \neq \theta_{\varphi\circ q_{u,v}}^{\lambda(\gamma_p^k)}$ in general. We are mainly interested in $\mu(\mathrm{Gies}^2(t), \lambda)$, thus we usually consider the minimizing element $\theta_{\varphi\circ q_{u,v}}^{\lambda}$ for λ.

1.42. We use the notation from 1.40 and $q = q_t$ for the quotient corresponding to E. For the linearization 1.37 we deduce from 1.34 that

$$\begin{aligned}
\mu_\chi(\mathrm{Gies}(t), \lambda) = \sum_{a\in A} \nu_a \mu(\mathrm{Gies}_a^1(t), \lambda) + \left(\sum_{j=1}^{|S|} \sum_{i=1}^{s_a^j} \nu_a^{ij} \mu(\mathrm{Gies}_a^{ij}(t), \lambda) \right) \\
+ \nu \cdot \mu(\mathrm{Gies}^2(t), \lambda) + \langle \lambda, \chi \rangle.
\end{aligned}$$

---

[15] See 1.4 for the transition from a filtration of length s_a^j to a filtration of length s^j.

From 1.35 we know that $\mu(\mathrm{Gies}^1_a(t), \lambda) = \sum_{k=1}^p \alpha_a^k \mu(\mathrm{Gies}^1_a(t), \lambda(\gamma^k_{p_a}))$ and analogously for the parabolic components.
In a first step we will bring our weight function above into the more classical form[16]

$$\begin{aligned}\mu_\chi(\mathrm{Gies}(t), \lambda) = {} & \underbrace{\sum_{a\in A} \nu_a \sum_{k=1}^p \alpha_a^k (p_a r_a^k - p_a^k r_a)}_{\mathrm{Comp}^1} + \underbrace{\nu \cdot \sum_{k=1}^p \alpha^k \left(pv(p^k, \theta^\lambda_{\varphi\circ q_{u,v}}) - p^k u \right)}_{\mathrm{Comp}^2} \\ & + \underbrace{\sum_{a\in A} \sum_{j=1}^{|S|} \sum_{i=1}^{s^j} \nu_a^{ij} \sum_{k=1}^p \alpha_a^k \left(p_a f^{ij,k}_{a,coh} - p_a^k f_a^{ij} \right)}_{\mathrm{Comp}^3} + \langle \lambda, \chi \rangle. \qquad (\star)\end{aligned}$$

Note that in the case $|A| = 1$ this equality obiously holds as λ is a one-parameter subgroup of $\mathcal{S}_A^{\kappa_a}$ and hence in particular $\sum_{k=1}^p \alpha^k p^k = 0 \Rightarrow \alpha^p = -p^{-1} \sum_{k=1}^{p-1} \alpha^k p^k$. However in the tuple case we cannot replace the α_a^p-term that easily, since in general $\alpha_a^p \neq -p_a^{-1} \sum_{k=1}^{p-1} \alpha^k p_a^k$. It is fortunately not very difficult to show

$$\begin{aligned} & \sum_{a\in A} \sum_{k=1}^p \alpha_a^k \left(\nu_a(-p_a^k r_a) + \sum_{j=1}^{|S|} \sum_{i=1}^{s^j} \nu_a^{ij} \left(-p_a^k f_a^{ij} \right) \right) + \nu \cdot \sum_{k=1}^p \alpha^k \left(-p^k u \right) \\ & = \sum_{k=1}^p \alpha^k \left(-p^k r \right) = 0 \qquad (\circledast)\end{aligned}$$

and therewith $(\star)$. Here the final step follows from $\sum_{k=1}^p \alpha^k p^k = 0$ for the $\mathcal{S}_A^{\kappa_a}$-one-parameter subgroup λ. The first equation will result from the upcoming more general calculations that lead to (M1) (see remark to 1.47).

The following numerical calculations are the main technical difficulty in proving the existence of our moduli space with Geometric Invariant Theory. We will first prove two technical lemmas.

1.43. Lemma. *Let $\eta_a := \xi_a \cdot c$, $c \in \mathbb{R}$. Then the following formula holds:*

$$\sum_{a\in A} \sum_{k=1}^p \alpha_a^k (-\eta_a (p_a r_a^k - p_a^k r_a) + \eta_a r_a (p_a - p_a^k)) = \sum_{a\in A} \sum_{k=1}^p \alpha^k p \frac{\eta_a}{r} (r_a r^k - r_a^k r).$$

Proof. We have

$$\begin{aligned} & - \eta_a (p_a r_a^k - p_a^k r_a) + \eta_a r_a (p_a - p_a^k) \\ & = -\eta_a p_a r_a^k + \eta_a r_a p_a = \eta_a p_a (r_a - r_a^k) \end{aligned}$$

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[16] As usual we replace every length s_a^j weighted filtration by a length s^j filtration (cf. 1.4).

and therefore using 1.5[17]

$$\sum_{a\in A}\sum_{k=1}^{p}\alpha_a^k\eta_a p_a(r_a-r_a^k)=\sum_{k=1}^{p}\alpha^k p\sum_{a\in A}\eta_a(r_a-r_a^k).$$

By definition (cf. 1.7)

$$\sum_{a\in A}\sum_{k=1}^{p}\alpha_a^k(-\eta_a(p_a r_a^k-p_a^k r_a)+\eta_a r_a(p_a-p_a^k))=\sum_{k=1}^{p}\sum_{a\in A}\alpha^k p\frac{\eta_a}{r}(r_a r^k-r_a^k r).$$

□

1.44. Lemma. *Under the same hypotheses as before and additionally assuming that* $(r_a^k)_{a[|A|]k[p]}$ *and* $(p_a^k)_{a[|A|]k[p]}$ *induce the same* $(\gamma_a^k)_{a[|A|]k[p]}$ *from given* $(\gamma^k)_{k[p]}$*, we receive*

$$\begin{aligned}&\sum_{k=1}^{p}\alpha^k(pr^k-p^k r)\\&=\sum_{a\in A}\sum_{k=1}^{p}\kappa_a\alpha_a^k(p_a r_a^k-p_a^k r_a)-\sum_{a\in A}\sum_{k=1}^{p}\alpha_a^k p_a\kappa_a\left(\frac{r_a}{p_a}-\frac{r}{p}\right)(p_a-p_a^k).\end{aligned}$$

Proof. By using 1.5 again, we already see that

$$\sum_{a\in A}\sum_{k=1}^{p}\alpha_a^k p_a\kappa_a\left(\frac{r}{p}\right)(p_a-p_a^k)=\sum_{k=1}^{p}\alpha^k r(p-p^k).$$

Furthermore

$$\begin{aligned}&\sum_{a\in A}\sum_{k=1}^{p}\kappa_a\alpha_a^k(p_a r_a^k-p_a^k r_a)-\sum_{a\in A}\sum_{k=1}^{p}\alpha_a^k p_a\kappa_a\left(\frac{r_a}{p_a}\right)(p_a-p_a^k)\\&=\sum_{a\in A}\sum_{k=1}^{p}\kappa_a\alpha_a^k(p_a r_a^k-p_a^k r_a-r_a(p_a-p_a^k))=\sum_{k=1}^{p}\alpha^k p(r^k-r)\end{aligned}$$

implies the claim. □

Remark. It is in fact enough to assume that $(r_a^k)_{a[|A|]k[p]}$ induces a sub-weight of the weight $(\gamma_a^k)_{a[|A|]k[p]}$ induced by $(p_a^k)_{a[|A|]k[p]}$ from a given $(\gamma^k)_{k[p]}$.

These preparatory results will help us in the next step to split up the (semi)stability concept into parts:

---

[17] Apply 1.5 for r_a^k replaced by $\frac{\eta_a}{\kappa_a}(r_a-r_a^k)$.

1.45. The Parabolic Contribution. Recall 1.4. Observe that for all $1 \leq k \leq p$ the weight vector $(\gamma^k_a)_{a[|A|]k[p]}$ induced by the (parabolic) data $(f^{ij,k}_{a,coh})_{a[|A|]k[p]}$ from $(\gamma^k)_{k[p]}$ is coarser than the one induced by $(r^k_a)_{a[|A|]k[p]}$ resp. $(p^k_a)_{a[|A|]k[p]}$, i. e. we can apply lemma 1.44. Analogously the $(\beta^{ij}_a)_{a[|A|]i[s^j]}$ induced by the (parabolic) data $(f^{ij,k}_{a,coh})_{a[|A|]i[s^j]}$ form only part of the weights of the $(\beta^{ij}_a)_{a[|A|]i[s^j]}$ induced by $(f^{ij}_a)_{a[|A|]i[s^j]}$ from $(\beta^{ij})_{i[s^j]}$, $\forall 1 \leq j \leq |S|$. Hence we can apply 1.5 in this situation.[18]
The following calculation will be used to simplify the parabolic part:

$$
\begin{aligned}
&\mathrm{Comp}^3\\
&= \sum_{a\in A}\sum_{k=1}^{p}\sum_{j=1}^{|S|}\sum_{i=1}^{s^j} \frac{\kappa_a \alpha^k_a r \delta^{ij}_a}{p}\left(p_a f^{ij,k}_{a,coh} - p^k_a f^{ij}_a\right)\\
&= \sum_{j=1}^{|S|}\sum_{i=1}^{s^j} \frac{r\delta^{ij}}{p} \sum_{a\in A}\sum_{k=1}^{p} \kappa_a \alpha^k_a \left(p_a f^{ij,k}_{a,coh} - p^k_a f^{ij}_a\right)\\
&= \sum_{j=1}^{|S|}\sum_{i=1}^{s^j} \frac{r\delta^{ij}}{p} \left(\sum_{k=1}^{p} \alpha^k (p f^{ij,k}_{coh} - p^k f^{ij}) + \sum_{a\in A}\sum_{k=1}^{p} \alpha^k_a \kappa_a p_a \left(\frac{f^{ij}_a}{p_a} - \frac{f^{ij}}{p}\right)(p_a - p^k_a)\right).
\end{aligned}
$$

The parabolic part of the first component Comp^1 is

$$
\begin{aligned}
&-\sum_{a\in A}\sum_{k=1}^{p}\sum_{j=1}^{|S|}\sum_{i=1}^{s^j} \frac{\kappa_a \alpha^k_a f^{ij}\delta^{ij}}{p}(p_a r^k_a - p^k_a r_a)\\
&= -\sum_{j=1}^{|S|}\sum_{i=1}^{s^j} \frac{f^{ij}\delta^{ij}}{p} \sum_{a\in A}\sum_{k=1}^{p} \kappa_a \alpha^k_a \left(p_a r^k_a - p^k_a r_a\right)\\
&= -\sum_{j=1}^{|S|}\sum_{i=1}^{s^j} \frac{f^{ij}\delta^{ij}}{p} \left(\sum_{k=1}^{p} \alpha^k (p r^k - p^k r) + \sum_{a\in A}\sum_{k=1}^{p} \alpha^k_a \kappa_a p_a \left(\frac{r_a}{p_a} - \frac{r}{p}\right)(p_a - p^k_a)\right).
\end{aligned}
$$

Now adding the last summands of each term we have

$$
\begin{aligned}
&\sum_{j=1}^{|S|}\sum_{i=1}^{s^j} \frac{\delta^{ij}}{p} \sum_{a\in A}\sum_{k=1}^{p} \alpha^k_a \kappa_a p_a \left(\left(\frac{f^{ij}_a r}{p_a} - \frac{f^{ij} r}{p}\right) - \left(\frac{r_a f^{ij}}{p_a} - \frac{r f^{ij}}{p}\right)\right)(p_a - p^k_a)\\
&= \sum_{k=1}^{p} \alpha^k p \sum_{a\in A} \underbrace{\left(\sum_{j=1}^{|S|}\sum_{i=1}^{s^j} \delta^{ij}_a \kappa_a \left(\frac{f^{ij}_a r - r_a f^{ij}}{p p_a}\right)\right)}_{=-\chi^1_a} (p_a - p^k_a)\\
&= -\langle \lambda, \chi^1 \rangle.
\end{aligned}
$$

---

[18] Recall that $(\delta^{ij})_{i[s^j]}$ is induced by $(\beta^{ij})_{i[s^j]}$ as $(\alpha^i)_{i[p]}$ is from $(\gamma^i)_{i[p]}$ (up to a scalar multiplication).

for $r_a^{ij} = r - f_a^{ij}$.
Furthermore the first summands of each term add up to

$$\sum_{k=1}^{p} \alpha^k \left(\sum_{j=1}^{|S|} \sum_{i=1}^{s^j} \delta^{ij} \left(f_{coh}^{ij,k} r - f^{ij} r^k \right) \right).$$

1.46. Non-Parabolic Contribution. Next consider the non-parabolic part of the first Gieseker component Comp^1 again using 1.43 and 1.44:

$$\begin{aligned}
&\sum_{a \in A} \sum_{k=1}^{p} \alpha_a^k \left(\left(1 - \frac{u\delta}{p} \right) \cdot \kappa_a - \frac{\xi_a r}{p} \right) (p_a r_a^k - p_a^k r_a) \\
&= \left(1 - \frac{u\delta}{p} \right) \cdot \left(\sum_{k=1}^{p} \alpha^k (p r^k - p^k r) + \sum_{a \in A} \sum_{k=1}^{p} \alpha_a^k p_a \kappa_a \left(\frac{r_a}{p_a} - \frac{r}{p} \right) (p_a - p_a^k) \right) - \dots \\
&\qquad \dots - \sum_{a \in A} \sum_{k=1}^{p} \alpha_a^k \left(\frac{\xi_a}{p} r r_a \right) (p_a - p_a^k) + \sum_{k=1}^{p} \sum_{a \in A} \alpha^k \xi_a (r_a r^k - r_a^k r) \\
&= \sum_{k=1}^{p} \alpha^k \left(\left(1 - \frac{u\delta}{p} \right) \cdot \left(p r^k - p^k r \right) + \sum_{a \in A} \xi_a (r_a r^k - r_a^k r) \right) + \dots \\
&\qquad \dots + \sum_{k=1}^{p} \alpha^k p \sum_{a \in A} \underbrace{\left(\left(1 - \frac{u\delta}{p} \right) \cdot \kappa_a \left(\frac{r_a}{p_a} - \frac{r}{p} \right) - \left(\frac{\xi_a r}{p} \cdot \frac{r_a}{p_a} \right) \right)}_{= -\chi_a^2} (p_a - p_a^k) \\
&= \sum_{k=1}^{p} \alpha^k \left(\left(1 - \frac{u\delta}{p} \right) \cdot \left(p r^k - p^k r \right) + \sum_{a \in A} \xi_a (r_a r^k - r_a^k r) \right) - \langle \lambda, \chi^2 \rangle \\
&= \sum_{k=1}^{p} \alpha^k \left(p r^k - p^k r + \sum_{a \in A} \xi_a (r_a r^k - r_a^k r) \right) + \sum_{k=1}^{p} \alpha^k \left(\frac{r}{p} \delta p^k u - u \delta r^k \right) - \langle \lambda, \chi^2 \rangle.
\end{aligned}$$

1.47. The Higgs Field Contribution. Finally we consider the second component Comp^2 of $\mu(\mathrm{Gies}(t), \lambda)$:

$$\mathrm{Comp}^2 = \frac{r \cdot \delta}{p} \sum_{k=1}^{p} \alpha^k \left(p v(p^k, \theta^{\lambda}_{\varphi \circ q_{u,v}}) - p^k u \right) = \sum_{k=1}^{p} \alpha^k \left(r \delta v(p^k, \theta^{\lambda}_{\varphi \circ q_{u,v}}) - \frac{r}{p} \delta p^k u \right).$$

Putting all components (apart from $-\langle \lambda, \chi^1 \rangle$ and $-\langle \lambda, \chi^2 \rangle$) together we receive $\mu_\chi(\mathrm{Gies}(t), \lambda)$, i. e.

$$\begin{aligned}
&\sum_{k=1}^{p} \alpha^k \left(p r^k - p^k r + \sum_{a \in A} \xi_a (r_a r^k - r_a^k r) - \sum_{j=1}^{|S|} \sum_{i=1}^{s^j} \delta^{ij} \left(f^{ij} r^k - f_{coh}^{ij,k} r \right) \right) \\
&\qquad + \delta \sum_{k=1}^{p} \alpha^k \left(r v(p^k, \theta^{\lambda}_{\varphi \circ q_{u,v}}) - u r^k \right). \qquad \text{(M1)}
\end{aligned}$$

Remark. The equation ($\circledast$) is the special case where we remove r_a^k, r^k, $f_{a,coh}^{ij,k}$, $f_{coh}^{ij,k}$, $v(p^k, \theta^\lambda_{\varphi\circ q_{u,v}})$ from the calculations in 1.45 through 1.47.

1.8. Simplifications of the Semistability Concept

Before we can apply the previous numerical calculations we still need to show that all assumptions made above are satisfied. In this section we will therefore show that the semistability concept of tuples has to be checked only against a bounded family of filtrations. This implies for a suitable natural number n that $F^k(n)$ is globally generated, $H^1(F^k(n)) = 0$ and in particular that $p^k = d^k + r^k(n+1-g)$. On the other hand one-parameter subgroups of $\mathcal{S}_A^{\kappa_a}$ may come with far more weights γ^k than weights accessible in a weighted bundle filtration. Hence we will need to find a way to produce suitable weights for the subbundles F^k (resp. subsheaves $F^{k,coh}$) induced by V^k.
In order to find a natural number n as stated above, we start with another boundedness result:

1.48. Lemma. *The family of rank r and degree d vector bundles E with $E \simeq E_t$, $t \in \mathfrak{T}_{\text{par}}$ s. t.* Gies(t) *is χ-semistable for some n big enough, is bounded.*

Proof. Let $q := q_t : V \otimes \mathcal{O}_X(-n) \to E$ and use the notation from 1.40. In order to apply 1.9 we start with a subbundle $F \subset E$ and $Q = E/F$ the quotient bundle, i. e. the long exact sequence corresponding to $0 \to F \to E \to Q$ implies $H^0(F(n)) \hookrightarrow H^0(E(n))$ and we may define $W := H^0(q(n))^{-1}(H^0(F(n))) \subset V$. Denote $W \cap V_a = W_a$ and $w_a := \dim(W_a)$, $w =: \dim(W)$. Let $(v_a^i)_{i[p_a]}$ be a basis of V_a such that $(v^i)_{i[w_a]}$ is a basis of W_a. Now consider the one-parameter subgroup $\lambda = (\lambda_a)_{a[|A|]} : \mathbb{C}^* \to \mathcal{S}_A^{\kappa_a}$ with weights $\gamma_a^j = \begin{cases} w_a - p_a & \text{for } j \leq w_a \\ w_a & \text{for } j > w_a \end{cases}$.
Using 1.3 as well as 1.34, 1.41 and 1.42 we find

$$p_a \operatorname{rk}(F_a) - w_a r_a \leq r_a(p_a - w_a),$$

$$pv(w, \theta^\lambda_{\varphi\circ q_{u,v}}) - wu \leq u(p-w),$$

$$p_a(r_a^k - r_{a,coh}^{ij}) - w_a(r_a - r_a^{ij}) \leq r_a(p_a - w_a).$$

Recalling the ν_a, ν and ν_a^{ij} w. r. t. which we chose our linearization, we see that all but the first component are at most constant in p[19], i. e. one finds $c_a > 0$ s. t. $0 \leq \mu_\chi(\text{Gies}(t), \lambda)$ implies

$$0 \leq p_a \operatorname{rk}(F_a) - w_a r_a + c_a, \quad \forall a \in A.$$

---

[19] $w \leq p$, $w_a \leq p_a \leq p$ and all factors are positive for big n (independent of the data of F). For the character part χ_2 this follows from $r_a p - p_a r = r_a d - d_a r$ for n big.

Furthermore for all $b \in A$ we have

$$\frac{r_a}{r_b}\left(p_b \operatorname{rk}(F_a) - w_a r_b\right) - \left(p_a \operatorname{rk}(F_a) - w_a r_a\right) = \left(\frac{r_a}{r_b} p_b - p_a\right) \operatorname{rk}(F_a).$$

Now the right hand side is bounded for n big enough since $\operatorname{rk}(F_b) \leq r_b$. Observe, that this bound depends only on the data of E. Hence we find another constant c such that

$$p \operatorname{rk}(F) - wr + c = \sum_{a,b \in A} \kappa_a \kappa_b (p_b \operatorname{rk}(F_a) - w_a r_b) + c \geq 0.$$

We are in a similar situation as considered in [Sch08], 2.3.5.12. For completion we repeat the argument here: Since by exactness $w \geq p - h^0(Q(n))$ our inequality implies

$$p(\operatorname{rk}(F) - r) + h^0(Q(n))r \geq -c \Rightarrow \frac{h^0(Q(n))}{r} \geq \frac{p}{r} - \frac{c}{r}.$$

Choose Q the minimal destabilizing quotient of E coming from the Harder-Narasimhan filtration (1.10). In particular Q is semistable with minimal $\mu(Q)$-value. Lemma 7.1.2 in [LP97] shows for the semistable bundle Q[20] that

$$\frac{h^0(Q)}{\operatorname{rk}(Q)} \leq \max\{0,\ \mu(Q) + 1\}. \tag{B1}$$

$$\mu_{\min}(E) + n + 1 \geq \mu(Q(n)) + 1 \geq \frac{p}{r} - \frac{c}{r} = \frac{d}{r} + n + 1 - g - \frac{c}{r}$$
$$\Rightarrow \mu_{\min}(E) \geq \mu(E) - g - \frac{c}{r}.$$

Therefore 1.9 implies the claim. □

Remark to 1.48. If we replace $\mathfrak{T}_{\text{par}}$ by any scheme that ensures that we still have a Gieseker morphism (cf. remark 1.30) and such that $H^0(q(n))$ is one-to-one, the claim still holds. Observe that $w \leq \dim(H^0(F(n))$ and $w_a \leq \dim(H^0(F_a(n))$.

1.49. Stability Simplification I. Let $\mathfrak{F}^c$ be the bounded family of vector bundles E for which $\mu(F) \leq \mu(E)+c$ holds for every subbundle $F \subset E$ and a fixed constant c (cf. 1.9). In this paragraph we will assume that E belongs to $\mathfrak{F}^c$. In particular the degree of a subbundle is bounded from above.

Claim. There is a finite set $\Xi^{(s)s}$ such that (semi)stability of a tuple has to be checked only for filtrations $(F_a^k)_{a[|A|]k[r]}$ with $(\operatorname{rk}(F^k), \alpha^k) \in \{(r^k)_{k[r]} : 0 \leq r_1 \leq \ldots \leq r\} \times \Xi^{(s)s}$.

---

[20] For $\mu(Q) < 0$ we have $h^0(Q) = 0$.

Proof. For now call $S_{[\mathrm{par}]}(F^k,\alpha^k) := M^{\kappa,\xi}_{[\mathrm{par}]}(F^k,\alpha^k) + \delta\cdot\mu(F^k,\alpha^k,\varphi)$. For $(\alpha^k)_k \to (z\alpha^k)_{k[r]},\ z>0$ we have $S_{[\mathrm{par}]}(F^k,\alpha^k) \to z\cdot S_{[\mathrm{par}]}(F^k,\alpha^k)$, i. e. we can restrict to $(\alpha^k)_{k[r]} \in ([0,1]\cap\mathbb{Q}_{\geq 0})^r$ in the (semi)stability criterion 1.6. We may extend $S_{[\mathrm{par}]}$ to a continuous function $[0,1]^r \to \mathbb{R},\ (\alpha^k)_{k[r]} \mapsto S_{[\mathrm{par}]}(\alpha^k)$, where $S_{[\mathrm{par}]}(\ \cdot\ )$ is the extension of $S_{[\mathrm{par}]}(F^k,\ \cdot\ )$ to arbitrary degrees d^k and ranks r^k (independent of the existence of a suitable filtration). Of course we get different functions $S_{[\mathrm{par}]}$ for a different choice of the data $r^k,\ d^k,\ r^{ij,k}$ and so on.
Note however that apart from the degrees of the F^k we only have finitely many choices for $0 \leq r^k \leq r$ as well as $r^{ij,k}$, etc. In particular, since $0 \leq v(i,\theta^*_\star) \leq u$ for fixed $(\alpha^k)_{k[r]}$, $\mu(\ \cdot\ ,\ \alpha^k,\ \cdot\ )$ has only finitely many possible values. Since our family is bounded by assumption, the degree is bounded from above. Thus we only have to care about small degrees and of course about the $(\alpha^k)_{k[r]}$. For all data (apart from the $(\alpha^k)_{k[r]}$) fixed, the continuous function $S_{[\mathrm{par}]}(\alpha^k)$ on the compact set $[0,1]^r$ has a global minimum m for some $(\alpha^k_{\mathrm{min}})_{k[r]}$.
Assume that $d^i < -|d| - u\delta$, then

$$\begin{aligned}\alpha^i(dr^i - rd^i) - \delta\cdot\max\{0,\alpha^i(ur^i - v(i,\theta)r)\} &\geq \alpha^i(dr^i - rd^i - \delta ru)\\ &\geq \alpha^i r(-|d| - \delta u - d^i) \geq 0\end{aligned}$$

and hence the function $S_{[\mathrm{par}]}$ can be minimal only if $\alpha^i = 0$ in the tuple $(\alpha^k_{\mathrm{min}})_{k[r]}$. We repeat the argument for all other d^j with $d^j < -|d| - u\delta$. Of course we find two disjoint subsets $I,\ J \subset \{1,\dots r\}$ such that $d^i < -|d| - u\delta$ if and only if $i \in I$ and $d^j \geq -|d| - u\delta$ if and only if $j \in J$. Therefore all tuples $(d^i)_{i[r]}$ that share the same I, the same J and that are equal on J, share the same $(\alpha^k_{\mathrm{min}})_{k[r]}$ that minimizes $S_{[\mathrm{par}]}(\alpha^k)$. Note that $\alpha^i_{\mathrm{min}} = 0,\ i \in I$. Since there is only a finite choice of sets $I,\ J$ and the $d^j, j \in J$ are additionally bound from above, we find a finite set Ξ of $(\alpha^k_{\mathrm{min}})_{k[r]}$ that contains the minimizer $S_{[\mathrm{par}]}(\alpha^k)$ for any choice of data.

If the actual minimum is smaller than 0, by density of $\mathbb{Q}$ in $\mathbb{R}$ we find a rational $(\hat{\alpha}^k_{\mathrm{min}})_{k[r]} \in (\mathbb{Q}_{\geq 0})^r$ such that $S_{[\mathrm{par}]}(\hat{\alpha}^k_{\mathrm{min}}) < 0$ holds. Index these $(\hat{\alpha}^k_{\mathrm{min}})_{k[r]}$ by the finite set Ξ^{ss}. If the minimum is exactly 0 set $\Xi' = \{(\alpha^k_{\mathrm{min}}) \in S^{-1}_{[\mathrm{par}]}(0) \cap (\mathbb{Q}_{\geq 0})^r \cap \Xi\}$ and $\Xi^s = \Xi^{ss} \cup \Xi'$. Finally we get

$$\exists (F^k,\alpha^k)_{k[r]} : S_{[\mathrm{par}]}(F^k,\alpha^k)\ (\leq)\ 0 \Leftrightarrow \exists (\hat{\alpha}^k)_{k[r]} \in \Xi^{(s)s} : S_{[\mathrm{par}]}(F^k,\hat{\alpha}^k)\ (\leq)\ 0.$$

□

Remark. Observe that by definition $\mu(F^k,\alpha^k,\varphi) := \frac{1}{q}\cdot\mu(F^k,q\cdot\alpha^k,\varphi)$ for $q\cdot\alpha^k \in \mathbb{Z}_{\geq 0}[1/r],\ q \in \mathbb{Q}^*$.
Once we established that there is only a finite number of $(\hat{\alpha}^k)_{k[r]}$ to be checked, we find an integer z as above such that we may check (semi)stability against a finite subset of $(\mathbb{Z}_{\geq 0}[1/r])^r$.

Using $(\alpha^i)_{i[r]} \in \Xi^{(s)s}$ finite and the proof of lemma 1.13, we find a constant c_1^j with

$$\begin{aligned} M^{\kappa,\xi}_{[\mathrm{par}]}(F^i,\alpha^i) \geq & -\sum_{i=1}^{m} \alpha^i \Bigg(\text{par-deg}(F^i)\operatorname{rk}(E) - \text{par-deg}(E)\operatorname{rk}(F^i) \\ & \qquad\qquad + \sum_{a\in A} \xi_a \operatorname{rk}(F^i_a)\operatorname{rk}(E)) \Bigg) \\ \geq & \; c_1^j - \alpha^j \left(\text{par-deg}(F^j)\operatorname{rk}(E)\right). \end{aligned}$$

Therefore we find for any constant c_3 a constant c_2 such that the existence of a j with $\mu(F^j) \leq c_2$ and $\alpha^j \neq 0$ implies $M^{\kappa,\xi}_{[\mathrm{par}]}(F^k,\alpha^k) \geq c_3$. Furthermore we saw in the proof of lemma 1.13 that $\mu(F^k,\alpha^k,\varphi) \geq -ur\sum_{k=1}^{r-1}\alpha^k$, i. e. $\mu(F^k,\alpha^k,\varphi) \geq c_4$ on $\Xi^{(s)s}$ finite. Choosing $c_3 > -\delta c_4$ implies that our tuple is already (semi)stable, i. e. it is enough to check (semi)stability for filtrations $(F^k_a)_{a[|A|]k[r]}$ with $\mu(F^k_a) > c_2, \forall k$ with $\alpha^k \neq 0$. By the following lemma 1.50 the $F^k(n)$ for some n big enough are now globally generated with vanishing first cohomology.

1.50. Lemma. *Let $\mathfrak{Fil}$ be the family of subbundles F of a rank r and degree d vector bundle E in a bounded family $\mathfrak{F}^c$, such that $\mu(F) \geq c_2$ given a fixed constant c_2. Then there is a $n_0 \in \mathbb{N}$ s. t. for all $n \geq n_0$ one has $h^1(F(n)) = 0$ and $F(n)$ globally generated.*

Proof. For a n with $H^1(F(n)) \simeq \operatorname{Hom}(F(n),\omega_X) \neq 0$ (Serre duality) we find a homomorphism $\varphi : F(n) \to \omega_X$, $\varphi \not\equiv 0$ and a short exact sequence

$$0 \longrightarrow \ker(\varphi) \longrightarrow F(n) \longrightarrow \varphi(F(n)) \longrightarrow 0.$$

Therefore

$$\begin{aligned} \operatorname{rk}(F)\cdot n + \operatorname{rk}(F)\cdot c_2 &\leq \deg(F(n)) = \deg(\ker(\varphi)) + \deg(\varphi(F(n))) \\ &\leq (\operatorname{rk}(F)-1)\cdot\mu(\ker(\varphi)) + 2g-2 \\ &\leq (\operatorname{rk}(F)-1)\left(\frac{d}{r}+n+c\right) + 2g-2 \\ \Rightarrow n &\leq \frac{d(r-1)}{r} + rc + 2g - 2 + |r\cdot c_2| =: n_0 - 1 \end{aligned}$$

Thus for $n > n_0 - 1$ big enough $H^1(F(n)) = 0$, $\forall j$. Analogously we can show that $H^1(F(n)(-x)) = 0$ for every $x \in X$ if $n \geq n_0$, i. e. $F(n)$ is then globally generated. □

Remark. This proof is an adaption of the proof of proposition 2.2.3.7 in [Sch08].

1.51. Stability Simplification II. Let $E \in \mathfrak{F}^c$ be a vector bundle and $(F^k)_{k[r]}$ a filtration of type $(r^k, d^k)_{k[r]}$ of E such that $F^j \in \mathfrak{Fil}$ as before. From the previous lemma we know that $H^1(F^k(n)) = 0$. We get a filtration $(V^k)_{k[r]}$ of V under the bijection $H^0(q(n))$ such that $p^k = \dim V^k = d^k + r^k(n+1-g)$ together with weights $(\alpha^k)_{k[r]}$. As usual we will extend the weights and the filtration trivially to $(V^k, \alpha^k)_{k[p]}$. By 1.40 we receive an associated one-parameter subgroup λ of $\mathcal{S}_A^{\kappa_a}$.

We extend a result from [Sch08], 2.3.5.15 to the tuple case:

Claim. For every (semi)stable tuple $((E_a,\ (E_a^{ij})_{i[s_a^j]j[|S|]})_{a[|A|]},\ \varphi,\ L)$ and every weighted filtration $(F^k)_{k[p]}$ with global weights α^k, there is a possibly different weight vector $(\alpha_J^k)_{k[p]}$ such that

$$\begin{aligned}
&\sum_{k=1}^{p} \alpha^k \left(pr^k - p^k r + \sum_{a\in A} \xi_a (r_a r^k - r_a^k r) + \sum_{j=1}^{|S|} \sum_{i=1}^{s^j} \delta^{ij} \left(r^{ij} r^k - r^{ij,k} r \right) \right) \\
&+ \delta \sum_{k=1}^{p} \alpha^k \left(rv(p^k, \theta_{\varphi\circ q_{u,v}}^{\lambda}) - ur^k \right) \\
&\geq M_{\mathrm{par}}^{\kappa,\xi}(F^k, \alpha_J^k) + \delta\mu(F^k, \alpha_J^k, \varphi),
\end{aligned}$$

where we used the notation 1.40.

Proof. By 1.13 we find a constant c_1 such that $\mu(F^k) \leq c_1$. Let $I(c_2) := \{k \in \{1, \ldots, r\} : \mu(F^k) < c_2\}$ and $J(c_2) = \{1, \ldots, r\} \setminus I(c_2)$. Again by (B1) and the Harder-Narasimhan filtration $0 = F^{k,0} \subset F^{k,1} \subset \ldots \subset F^{k,m} = F^k$ we get

$$h^0(F^k) \leq \sum_{i=0}^{m-1} h^0(F^{k,i+1}/F^{k,i}) \leq (\mathrm{rk}(F^k) - 1)(c_1 + 1) + \mu(F^k) + 1^{21}$$

$$\Rightarrow p^k = h^0(F^k(n)) < (\mathrm{rk}(F^k) - 1)(c_1 + 1) + c_2 + \mathrm{rk}(F^k) \cdot n + 1, \quad \forall k \in I(c_2).$$

Choose c_2 small enough that

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[21] The expression holds for $h^0(F^k) \geq 0$; else $h^0(F^k) = 0$. Observe that $\mu(F^{k,m}/F^{k,m-1}) \leq \mu(F^k)$.

$$\Big(pr^k - p^k r + \sum_{a\in A}\xi_a(r_a r^k - r^k_a r) + \sum_{j=1}^{|S|}\sum_{i=1}^{s^j}\delta^{ij}\left(r^{ij}r^k - r^{ij,k}r\right)\Big) + \delta\cdot\Big(rv(p^k,\theta^{\lambda}_{\varphi\circ q_{u,v}}) - ur^k\Big) \geq 0.\text{[22]} \tag{S2}$$

Next consider the filtrations $(F^k_I)_{k\in I(c_2)} = (F^k)_{k\in I(c_2)}$ and $(F^k_J)_{k\in J(c_2)} = (F^k)_{k\in J(c_2)}$ and p^k_I, p^k_J, a. s. o. the corresponding dimensions. Then

$$\begin{aligned}
&\sum_{k=1}^{p}\alpha^k\left(pr^k - p^k r + \sum_{a\in A}\xi_a(r_a r^k - r^k_a r) + \sum_{j=1}^{|S|}\sum_{i=1}^{s^j}\delta^{ij}\left(r^{ij}r^k - r^{ij,k}r\right)\right)\\
&\quad + \delta\sum_{k=1}^{p}\alpha^k\Big(rv(p^k,\theta^{\lambda}_{\varphi\circ q_{u,v}}) - ur^k\Big)\\
&\geq \sum_{k=1}^{p}\alpha^k_J\left(pr^k_J - p^k_J r + \sum_{a\in A}\xi_a(r_a r^k_J - r^k_{J,a} r) + \sum_{j=1}^{|S|}\sum_{i=1}^{s^j}\delta^{ij}\left(r^{ij}r^k_J - r^{ij,k}_J r\right)\right)\\
&\quad + \delta\sum_{k=1}^{p}\alpha^k_J\Big(rv(p^k_J,\theta^{\lambda}_{\varphi\circ q_{u,v}}) - ur^k_J\Big)\\
&\quad + \sum_{k=1}^{p}\alpha^k_I\left(pr^k_I - p^k_I r + \sum_{a\in A}\xi_a(r_a r^k_I - r^k_{I,a} r) + \sum_{j=1}^{|S|}\sum_{i=1}^{s^j}\delta^{ij}\left(r^{ij}r^k_I - r^{ij,k}_I r\right)\right)\\
&\quad + \delta\sum_{k=1}^{p}\alpha^k_I\Big(rv(p^k_I,\theta^{\lambda}_{\varphi\circ q_{u,v}}) - ur^k_I\Big)\\
&\geq M^{\kappa,\xi}_{\mathrm{par}}(F^k,\alpha^k_J) + \delta\cdot\mu(F^k,\alpha^k_J,\varphi).
\end{aligned}$$

In the last step we used 1.50, i. e. $pr^k_J - p^k_J r = dr^k_J - d^k_J r$[23], as well as the estimate (S2). Also note the remark below. □

Remark. By construction of the parameter space $\varphi|_{\left(\bigotimes_{i_j} F^{i_j}\right)^{\oplus v}} \equiv 0 \Leftrightarrow \psi|_{\left(\bigotimes_{i_j} V^{i_j}\right)^{\oplus v}} \equiv 0$ and hence by definition of the Gieseker map $\mu(F^k,\alpha^k_*,\varphi) = \sum_{k=1}^{p}\alpha^k\Big(rv(p^k_*,\theta^{\lambda}_{\varphi\circ q_{u,v}}) - ur^k_*\Big)$.

Remark. We find a non-parabolic version of 1.51 if we drop the parabolic contribution everywhere, i. e. set $\delta^{ij} = 0$.

---

[22] Note that n drops out and that apart from $pr^k - p^k r$ all other terms are uniformly bounded, i. e. can be controlled in terms of c_2. In particular $0 \leq v(p^k,\theta^{\lambda}_{\varphi\circ q_{u,v}}) \leq u$.

[23] Apply 1.50 to the family of filtrations with $I = \emptyset$.

1.9. Main Technical Theorem

We are now able to prove that **Gies** respects (semi)stability.

1.52. Theorem. *Choose admissible weights* $(\delta^{ij})_{i[s^j]j[|S|}$, *i. e.* $\sum_{i=1}^{s^j} \delta^{ij} < 1$ *for every* $x^j \in S$. *There is a* $N \in \mathbb{Z}$ *such that for all* $n \geq N$ *and all* $t \in \mathfrak{T}_{\text{par}}$ *the following two properties are equivalent:*

(i) t *is a* $(\delta,\ \xi_a,\ \delta^{ij})$*-(semi)stable tuple,*

(ii) Gies(t) *is* χ*-(semi)stable.*

Proof. (ii) ⇒ (i): By lemma 1.48, we may apply the semistability simplification 1.49, i. e. it will be enough to check (semi)stability for filtrations $(F_a^k)_{a[|A|]k[r]}$ with $F_a^k(n)$ globally generated and $h^1(F_a^k(n)) = 0$. Let $(V_a^k)_{a[|A|]k[p]}$ be the resulting filtration of V under the bijection $H^0(q(n))$ with the weights $(\gamma^k)_{k[p]}$ induced in the usual way from given $(\alpha^k)_{k[p]}$ with $\alpha^p = -p^{-1} \sum_{k=1}^{p-1} \alpha^k p^k$.[24] Let $\lambda := \lambda(\gamma^k)$ be a corresponding one-parameter subgroup w. r. t. the filtration $(V_a^k)_{a[|A|]k[p_a]}$.[25] Now we are in the situation of section 1.7.
If n is big enough, $p = d + r(n+1-g)$ and $p^k = d^k + r^k(n+1-g)$ and thus $pr^k - p^k r = dr^k - d^k r$, i. e. (M1) becomes

$$\mu_\chi(\text{Gies}(t), \lambda) = M_{\text{par}}^{\kappa,\xi}(F^k, \alpha^k) + \delta \cdot \mu(F^k, \alpha^k, \varphi).$$

This proves the claim.
(i) ⇒ (ii): Start with an arbitrary one-parameter subgroup $\lambda : \mathbb{C}^* \to \mathcal{S}_A^{\kappa_a}$ and use the notation of 1.40. First note, that on the side of the Gieseker space there is in general a bigger choice of one-parameter subgroup weights $(\gamma^i)_{i[p]}$ and corresponding $(\alpha^i)_{i[p]}$ than there is in the semistability condition of Higgs tuples. Thus we have to produce from the given $(\alpha^i)_{i[p]}$ new weights $(\hat{\alpha}^i)_{i[p]}$, about which we can say anything. More precisely they should obey the semistability condition for Higgs tuples. Define $\hat{p}_a^k := h^0(F_a^k(n))$, $\hat{p}^k := h^0(F^k(n))$ and note that $\hat{p}_a^k \geq p_a^k$. Now take $J_a^k = \{l : F_a^k = F_a^l\}$ and $\hat{\alpha}_a^k = \sum_{l \in J_a^k} \alpha_a^l$. Let $\hat{\lambda}$ be the one-parameter subgroup $\hat{\lambda} : \mathbb{C}^* \to \mathcal{S}_A^{\kappa_a}$ corresponding to $(\hat{\alpha}^k)_{k[p]}$ and $H^0(F^k(n))$. Furthermore note that the torsion decomposes $\hat{p}^k - p^k \geq \sum_{j \in \text{supp}(F^k/F^{k,coh}) \cap S} t^{j,k}$[26] with $t^{j,k} \geq f^{ij,k} - f_{coh}^{ij,k}$.
Application of the main calculations 1.7 shows, that

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[24] Extend the $(V^k, \alpha^k)_{k[r]}$ as in 1.4 to $(V^k, \alpha^k)_{k[p]}$.

[25] As mentioned before λ depends on the choice of a suitable basis (see 1.32).

[26] The part of the torsion that hits the punctures and therefore contributes to the (semi)stability calculations.

$$\begin{aligned}
&\mu_\chi(\mathrm{Gies}(t),\lambda)\\
&\overset{(\mathrm{M1})}{=} \sum_{k=1}^{p}\alpha^k\left(pr^k - p^k r + \sum_{a\in A}\xi_a(r_a r^k - r^k_a r) - \sum_{j=1}^{|S|}\sum_{i=1}^{s^j}\delta^{ij}\left(f^{ij}r^k - f^{ij,k}_{coh}r\right)\right)\\
&\quad + \delta\sum_{k=1}^{p}\alpha^k\left(rv(p^k,\theta^\lambda_{\varphi\circ q_{u,v}}) - ur^k\right)\\
&= \mu_\chi(\mathrm{Gies}(t),\hat\lambda) + \sum_{k=1}^{p}\alpha^k\left((\hat p^k - p^k)r - \sum_{j=1}^{|S|}\sum_{i=1}^{s^j}\delta^{ij}\left((f^{ij,k} - f^{ij,k}_{coh})r\right)\right)\\
&\geq \mu_\chi(\mathrm{Gies}(t),\hat\lambda) + \sum_{k=1}^{p}\alpha^k\sum_{j=1}^{|S|}t^{k,j}r\left(1 - \sum_{i=1}^{s^j}\delta^{ij}\right) \qquad (\mathrm{Tor}\ 1)\\
&\geq \mu_\chi(\mathrm{Gies}(t),\hat\lambda).
\end{aligned}$$

For the second equality we used 1.35 and the remark above, i. e. $v(\hat p^k,\theta^\lambda_\varphi) = v(p^k,\theta^\lambda_{\varphi\circ q_{u,v}})$. The last two inequalities are a consequence of $\hat p^k - p^k \geq \sum_{j\in\mathrm{supp}(F^k/F^{k,coh})\cap S} t^{j,k}$ and admissibility of the weights δ^{ij}. Note that by choosing subsheaves with suitable associated torsion sheaves F^k/F^k_{coh} supported exactly on the punctures such that $f^{ij,k} - f^{ij,k}_{coh} = t^{j,k}$, admissibility becomes a necessary condition for our construction to work.

Now in order for the claim to hold, we only need to show that we find for each tuple $(\hat\alpha^k)_{k[p]}$ a tuple $(\tilde\alpha^k)_{k[p]}$ such that

$$\begin{aligned}
&\sum_{k=1}^{p}\hat\alpha^k\left(pr^k - \hat p^k r + \sum_{a\in A}\xi_a(r_a r^k - r^k_a r) + \sum_{j=1}^{|S|}\sum_{i=1}^{s^j}\delta^{ij}\left(r^{ij}r^k - r^{ij,k}r\right)\right)\\
&\quad + \delta\sum_{k=1}^{p}\hat\alpha^k\left(rv(\hat p^k,\theta^\lambda_{\varphi\circ q_{u,v}}) - ur^k\right)\\
&\quad \geq M^{\kappa,\xi}_{\mathrm{par}}(F^k,\tilde\alpha^k) + \delta\cdot\mu(F^k,\tilde\alpha^k,\varphi).
\end{aligned}$$

But this is the statement of 1.51. □

1.10. GEOMETRIC INVARIANT THEORY

We will repeat some of the main results from Geometric Invariant Theory. Moreover it will be shown that **Gies** is a closed embedding.

1.53. Definition. Let G be an algebraic group acting by σ on the algebraic prescheme X (both over $\mathbb{C}$). A pair $(X /\!\!/ G, \pi_X)$ for a prescheme $X /\!\!/ G$ and a morphism $\pi_X : X \to X /\!\!/ G$ is called a categorical quotient (of X by G) if

(Cat^1) the diagram

$$\begin{tikzcd}
G \times X \arrow[r, "\sigma"] \arrow[d, "\mathrm{pr}_2"'] & X \arrow[d, "\pi_X"] \\
X \arrow[r] & X /\!\!/ G
\end{tikzcd}$$

commutes.

(Cat^2) Given any pair $(Z,\ \pi_Z)$ with a prescheme Z and a morphism $\pi_Z : X \to Z$ such that $\pi_Z \circ \sigma = \pi_Z \circ \mathrm{pr}_2$, there is a unique morphism $f : X /\!\!/ G \to Z$ such that $\pi_Z = f \circ \pi_X$.

$(X /\!\!/ G, \pi_X)$ is a good quotient of X by G if π_X is a surjective affine morphism that satisfies (Cat^1) and

(G^1) $\pi_{X/\!\!/G,*}(\mathcal{O}_X)^G = \mathcal{O}_{X/\!\!/G}$ where $\pi_{X/\!\!/G,*}(\mathcal{O}_X)^G(U)$ is the set of G-invariant functions on $\pi^{-1}_{X/\!\!/G}(U)$ for $U \subset X /\!\!/ G$ offen.

(G^2) closed G-invariant subsets are mapped to closed subsets.

(G^3) disjoint G-invariant closed subsets stay disjoint under π_X.

$(X /\!\!/ G, \pi_X)$ is a geometric quotient of S by G if $X /\!\!/ G$ is a good quotient such that

(Geo^1) for every $x \in X /\!\!/ G$, $\pi_X^{-1}(\{x\})$ contains at most one orbit.

$X /\!\!/ G$ is a universal categorical (resp. universal good, resp. universal geometric) quotient if it is a categorical (resp. good, resp. geometric) quotient under every base change, i. e. $X \times_{X/\!\!/G} Z \to Z$ is a categorical (resp. good, resp. geometric) quotient for every scheme $Z \to X /\!\!/ G$.

Remark.

1. A (universal) good quotient is a (universal) categorical quotient ([MFK].0.§2 Proposition 0.1). (G^3) guarantees that φ (in the definition of a categorical quotient) exists as a map of the underlying sets, (G^2) shows that φ is continuous and (G^1) makes φ algebraic.
2. If $X /\!\!/ G$ is a good quotient of X by G, then $\pi_{X/\!\!/G}(x) = \pi_{X/\!\!/G}(y)$ if and only if $\overline{Gx} \cap \overline{Gy} \neq \emptyset$ (Seshadri [Ses77], remark 8 to theorem 4). Furthermore every fiber $\pi^{-1}_{X/\!\!/G}(x)$ of every closed point $x \in X /\!\!/ G$ contains exactly one closed orbit.[27] Note that for $x, y \in X$ with $\overline{Gx} \cap \overline{Gy} \neq \emptyset$ and $Gy = \overline{Gy}$ there is a one-parameter subgroup λ of G such that $\lim_{z\to\infty} \lambda(z) \cdot x \in Gy$.[28]

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[27] e. g. [LP97], Proposition 6.1.7.

[28] see [Bir71], theorem 4.2 by R. Richardson.

1.54. Theorem ([MFK], Theorem 1.1). *Let Y be an affine scheme of finite type over $\mathbb{C}$, let G be a reductive linear algebraic group acting on Y. Then the $Y /\!\!/ G$ exists as a universal good quotient.*

1.55. Theorem. *Let X, G be as in 1.53 and L a G-linearized line bundle on X. Then a universal categorical quotient $(X^{ss} /\!\!/ G, \pi_{X^{ss}})$ exists for X^{ss} the open set of semistable points w. r. t. the G-linearization on L. Moreover*

(i) *$\pi_{X^{ss}}$ is affine and universally submersive;*

(ii) *there is an ample invertible sheaf M on $X^{ss} /\!\!/ G$ such that $\pi^*_{X^{ss}}(M) \simeq L^k$ for some k;*

(iii) *$X^{ss} /\!\!/ G$ is a quasi-projective algebraic scheme;*

(iv) *there is an open subset $Y \subset X^{ss} /\!\!/ G$ such that $X^s = \pi^{-1}_{X^{ss}}(Y)$ and X^s the set of stable points w. r. t. the G-linearization on L. Then $(Y, \pi_{X^{ss}}|_{X^s})$ is a universal geometric quotient, i. e. $Y = X^s/G$.*

Proof. [MFK], Ch. 1, §4.1.10. □

Remark 1.55. If X is proper over $\mathbb{C}$ and L is ample, $X^{ss} /\!\!/ G$ is projective ([MFK], Amplification 1.11 in §1).

1.56. Theorem. *Let $f : X \to Y$ be a finite G-linear morphism between algebraic preschemes. Let σ be a G-linearization on an ample line bundle L and σ_X the induced linearization on f^*L, then*

$$X^s = f^{-1}(Y^s), \quad X^{ss} = f^{-1}(Y^{ss}).$$

Proof. [MFK], Ch.1, §5.1.19. and the comment after Corollary Ch.1, §5.1.20.. □

1.57. Proposition ([Ram96ii], 5.1). *Let G be a linear algebraic group acting on two schemes X, Y and $f : X \to Y$ an affine G-equivariant morphism, then if $Y /\!\!/ G$ exists as a good quotient, so does $X /\!\!/ G$ and the induced morphism $\overline{f} : X /\!\!/ G \to Y /\!\!/ G$ is also affine. Furthermore if f is proper (i. e. finite), then $\overline{f}$ is finite; if additionally $Y /\!\!/ G$ is a geometric quotient, so is $X /\!\!/ G$.*

1.58. Definition. Let G be an action of a linear algebraic group on a scheme X over $\mathbb{C}$ and let X^{ss} be the open subset of semistable objects with respect to a linearization of the G-action in some ample line bundle. We call two points x, $y \in X^{ss}$ strongly equivalent, or short S-equivalent, if $\overline{G}x \cap \overline{G}y \neq \emptyset$. We have seen above that S-equivalent points are mapped to the same point in a good quotient.

In view of 1.57 we still need to show that our Gieseker map is finite to pull back the GIT-quotient.

1.59. Lemma. *The Gieseker map is a closed immersion* $\mathfrak{T}^{ss}_{\mathrm{par}} \hookrightarrow \mathbb{P}^{ss}$. *In particular* Gies *finite.*[29]

Proof. ([Sch08], 2.3.5.17) Since Gies is one-to-one, it is enough to show the properness[30] using the properness valuation criterion. Let R be a discrete valuation ring R with a fraction field K. Start with maps $\hat{f} : \mathrm{Spec}(R) \to \mathbb{P}^{ss}$ and $f : \mathrm{Spec}(K) \to \mathfrak{T}^{ss}_{\mathrm{par}}$ such that

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "f"] \arrow[d, "j"'] & \mathfrak{T}^{ss}_{\mathrm{par}} \arrow[d, "\mathrm{Gies}"] \\
\mathrm{Spec}(R) \arrow[r, "\hat{f}"'] & \mathbb{P}^{ss}
\end{tikzcd}$$

commutes. We need to find a map $\overline{f} : \mathrm{Spec}(R) \to \mathfrak{T}^{ss}_{\mathrm{par}}$ for every discrete valuation ring R such that the extended diagram commutes.[31] Using 1.22 we find a quotient family $((q_{\mathrm{Spec}(K),a}, \mathscr{E}_{\mathrm{Spec}(K),a}, (q^{ij}_{\mathrm{Spec}(K),a}, \mathscr{H}^{ij}_{\mathrm{Spec}(K),a})_{i[s^j_a]j[|S|]})_{a[|A|]}, \upsilon_{\mathrm{Spec}(K)}, \mathscr{H}_{\mathrm{Spec}(K)}, \varphi_{\mathrm{Spec}(K)})$ on $\mathrm{Spec}(K)$. Let $\pi \circ f : \mathrm{Spec}(K) \to \overline{\mathfrak{Q}}$ with $\overline{\mathfrak{Q}}$ projective. Then $q_{\mathrm{Spec}(K),a}$ extends to $q_{\mathrm{Spec}(R),a}$ but the special fiber over the special point $\mathfrak{p}$ will in general be only a coherent sheaf on X. Analogously we define the extension $\upsilon_{\mathrm{Spec}(R)}$ by the projectivity of Jac^l and $q^{ij}_{\mathrm{Spec}(R),a}$ by the projectivity of the Graßmann variety.

We have to deal with the possible torsion of E. For $T \subset \{\mathfrak{p}\} \times X$ the support of the torsion on the special fiber, we may extend $\det(\mathscr{E}_{\mathrm{Spec}(R)}|_{(\mathrm{Spec}(R)\times X)\setminus T})$ uniquely over the codimension 2 torsion T to the regular two dimensional scheme $\mathrm{Spec}(R) \times X$. Name the corresponding line bundles on $\mathrm{Spec}(R) \times X$ resp. $\mathrm{Spec}(R) \times \{x^j\}$: $\mathscr{H}_{\mathrm{Spec}(R)}$, $\det(\mathscr{E}_{\mathrm{Spec}(R)})$ and $\mathscr{H}^{ij}_{\mathrm{Spec}(R)}$.

Next the repetition of the construction of the parameter space with $\mathfrak{Q}$ replaced by $\mathrm{Spec}(R)$, will lead us to a closed subscheme $\mathfrak{R}$ and a projective morphism $\pi_{\mathfrak{R}} : \mathfrak{R} \to \mathrm{Spec}(R)$.[32] Since we already know that some morphism, namely $\psi_{\mathrm{Spec}(K)} = \varphi_{\mathrm{Spec}(K)} \circ q_{\mathrm{Spec}(K),u,v}$ splits, $\mathfrak{R}$ is not empty and we get a morphism $\mathrm{Spec}(K) \to \mathfrak{R}$

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[29] Recall that finite maps are proper and affine.

[30] [EGA] IV, Corollaire 18.12.4 shows that proper injections are proper and quasi-finite, and therefore finite.

[31] [Ha77], II.4.7.

[32] cf. remark 1.16 and 1.17.

over $\operatorname{Spec}(R)$. Now the valuation criterion applied to the projective morphism $\pi_{\mathfrak{R}}$ shows the existence of $\varphi_{\operatorname{Spec}(R)} : \mathscr{E}_{\operatorname{Spec}(R),u,v} \to \det(\mathscr{E}_{\operatorname{Spec}(R)})^{\otimes w} \otimes \mathscr{P}_{\upsilon_{\operatorname{Spec}(R)}}$ on $\operatorname{Spec}(R) \times X$ which extends $\varphi_{\operatorname{Spec}(K)}$ ([Ha77], II.4.7).
Let $\iota : \mathscr{E}_{\operatorname{Spec}(R)} \to \hat{\mathscr{E}}_{\operatorname{Spec}(R)} := \mathscr{E}^{\vee\vee}_{\operatorname{Spec}(R)}$. By [Ha80], Corollary 1.2 $\mathscr{E}^{\vee\vee}_{\operatorname{Spec}(R)}$ is reflexive and thus by [Ha80], Corollary 1.4 it is already flat on $\operatorname{Spec}(R) \times X$. Observe that the torsion of $\mathscr{E}_{\operatorname{Spec}(R)}|_{\{\mathfrak{p}\}\times X}$ is just $\ker(\iota|_{\{\mathfrak{p}\}\times X})$. In particular $\hat{q}|_{\{\mathfrak{p}\}\times X}$ is in general not surjective.[33] Hence we still need to define $\hat{\varphi}_{\operatorname{Spec}(R)}$. Given the embedding $i : (\operatorname{Spec}(R) \times X) \setminus T \hookrightarrow \operatorname{Spec}(R) \times X$, we then define

$$\hat{\varphi}_{\operatorname{Spec}(R)}(e) := i_*(\varphi_{\operatorname{Spec}(R)}|_{(\operatorname{Spec}(R)\times X)\setminus T})(i_*(e))$$

for all $e \in \hat{\mathscr{E}}_{\operatorname{Spec}(R),u,v}$, $i_*(e)$ the corresponding point in

$$i_*\left(\hat{\mathscr{E}}_{\operatorname{Spec}(R),u,v}|_{(\operatorname{Spec}(R)\times X)\setminus T}\right) = i_*\left(\mathscr{E}_{\operatorname{Spec}(R),u,v}|_{(\operatorname{Spec}(R)\times X)\setminus T}\right).$$

Note that we have again[34] used that in the image

$$\begin{aligned} i_*\big(\det(\mathscr{E}_{\operatorname{Spec}(R)}|_{(\operatorname{Spec}(R)\times X)\setminus T})^{\otimes w} \otimes \mathscr{P}_{\upsilon_{\operatorname{Spec}(R)}|_{(\operatorname{Spec}(R)\times X)\setminus T}}\big) \\ = \det(\mathscr{E}_{\operatorname{Spec}(R)})^{\otimes w} \otimes \mathscr{P}_{\upsilon_{\operatorname{Spec}(R)}} = \det(\hat{\mathscr{E}}_{\operatorname{Spec}(R)})^{\otimes w} \otimes \mathscr{P}_{\upsilon_{\operatorname{Spec}(R)}}. \end{aligned}$$

We proceed analogously for the parabolic quotients. By construction the family

$$\begin{aligned} ((\hat{q}_{\operatorname{Spec}(R),a},\ \hat{\mathscr{E}}_{\operatorname{Spec}(R),a},\ (\hat{q}^{ij}_{\operatorname{Spec}(R),a},\ \hat{\mathscr{H}}^{ij}_{\operatorname{Spec}(R),a})_{i[s^j_a]j[|S|]})_{a[|A|]},\ \hat{\upsilon}_{\operatorname{Spec}(R)}, \\ \hat{\mathscr{H}}_{\operatorname{Spec}(R)},\ \hat{\varphi}_{\operatorname{Spec}(R)}) \end{aligned}$$

defines a morphism to $\mathbb{P}$ which coincides with $\hat{f}$ first on $\operatorname{Spec}(K)$ and (since $\mathbb{P}$ is projective) then already on all of $\operatorname{Spec}(R)$. Now restricting our family to a family $r := ((\hat{q}_a,\ \hat{\mathscr{E}}_a,\ (\hat{q}^{ij}_a,\ \hat{\mathscr{H}}^{ij}_{\operatorname{Spec}(R),a})_{i[s^j_a]j[|S|]})_{a[|A|]},\ \hat{\upsilon},\ \hat{\mathscr{H}},\ \hat{\varphi})$ on $\{\mathfrak{p}\} \times X$, by definition of $\hat{f}$ the point $\hat{f}(r)$ is semistable. Observe that $H^0(\hat{q}(n))$ is one-to-one - for the kernel k of $H^0(\hat{q}(n))$ we get by (Tor 1) $\sum_{j=1}^{|S|}\sum_{i=1}^{s^j} \delta^{ij} f^{ij}_{coh} r - \dim(k) r \geq 0$ and $\sum_{j=1}^{|S|} f^{1j}_{coh} \leq \dim(k)$[35]$\overset{\sum_{i=1}^{s^j}\delta^{ij}<1}{\Longrightarrow} \dim(k) = 0$.
Now the remarks to 1.30 and 1.48 as well as the corresponding modification of the main calculation imply that r is δ-semistable. But the vector bundle associated to r lives in a bounded family and the monomorphisms $H^0(\hat{q}_a(n))$ into spaces of

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[33] Coherence implies that the corresponding modules are of finite length, i. e. $l(M) = l(M^\vee) = l(M^{\vee\vee})$ for the length. Then $\operatorname{coker}(\iota) \simeq M^{\vee\vee}/\iota(M) \simeq M^{\vee\vee}/(M/\ker(\iota))$ for $\iota : M \to M^{\vee\vee}$ has length $\dim(\operatorname{Tor}(M))$, i. e. $\hat{q} = q \circ \iota$ is surjective if the torsion vanishes.

[34] as line bundles on a regular two-dimensional scheme minus a codimension 2 torsion T extend uniquely to line bundles on $\operatorname{Spec}(R) \times X$.

[35] $f^{1j}_{coh} \geq f^{ij}_{coh}$ for all $i \in \{1, \ldots, s^j\}$.

the same dimensions are isomorphisms. Thus r is a quotient family. Now the universal property 1.22 provides us with a unique morphism $\overline{f} : \mathrm{Spec}(R) \to \mathfrak{T}^{ss}_{\mathrm{par}}$ s. t. $((\hat{q}_{\mathrm{Spec}(R),a},\ \hat{\mathscr{E}}_{\mathrm{Spec}(R),a},\ (\hat{q}^{ij}_{\mathrm{Spec}(R),a},\ \hat{\mathscr{H}}^{ij}_{\mathrm{Spec}(R),a})_{i[s^j_a]j[|S|]})_{a[|A|]},\ \hat{\upsilon}_{\mathrm{Spec}(R)},\ \hat{\mathscr{H}}_{\mathrm{Spec}(R)},$ $\hat{\varphi}_{\mathrm{Spec}(R)})$ is the pullback of the universal family. By remark 1.30 $\mathrm{Gies} \circ \overline{f} = \hat{f}$ shows that $\overline{f}$ lifts $\hat{f}$. On the other hand $\overline{f} \circ j = f$ by the uniqueness property in 1.22. Hence Gies is proper and therefore finite. □

1.11. Existence of the Moduli Space

We are now in the position to state our main result, namely the existence of the moduli space of Higgs tuples. Before we do so, we repeat the definition of fine and coarse moduli spaces and establish some conventions that we will use.
In Balaji [Bal10] a detailed discussion of the underlying categorical properties of moduli spaces is given. Classical references for the construction of moduli spaces are Newstead's book [New78] and of course Mumford, Fogarty, Kirwan [MFK].

1.60. Definition. Let $\mathbf{M} : \mathbf{Sch}_{\mathbb{C}} \to \mathbf{Sets}$ be a functor. A scheme $\mathscr{M}$ together with a natural transformation $\psi_{\mathscr{M}}$ from $\mathbf{M}$ to $\mathrm{Hom}(\,\cdot\,, \mathscr{M})$ is called a *coarse* moduli scheme for $\mathbf{M}$ if

1. ψ_M is a bijection over $\mathrm{Spec}(\mathbb{C})$ and,
2. for any scheme S and any natural transformation $\psi_S : \mathbf{M} \to \mathrm{Hom}(\,\cdot\,, S)$, there is a unique morphism of functors $\varphi : \mathrm{Hom}(\,\cdot\,, \mathscr{M}) \to \mathrm{Hom}(\,\cdot\,, S)$ such that $\psi_S = \varphi \circ \psi_{\mathscr{M}}$.

Remark. Condition (ii) is equivalent to saying that $\mathscr{M}$ corepresents $\mathbf{M}$, i. e., that $\psi_{\mathscr{M}}$ induces $\mathrm{Mor}(\mathbf{M}, \mathrm{Hom}(\,\cdot\,, S)) \simeq \mathrm{Mor}(\mathscr{M}, S)$ for every scheme S over $\mathbb{C}$. Sometimes authors (see [HL10], Def. 4.1.1) require only (ii) as the definition of a moduli space.

1.61. Definition. Let $\mathbf{M} : \mathbf{Sch}_{\mathbb{C}} \to \mathbf{Sets}$ be a functor, that associates to every scheme S the set of equivalence classes of S-families of objects. A scheme $\mathscr{M}$ is called a *fine* moduli space for the functor $\mathbf{M}$ if there is a universal family $\mathscr{U}$ on $\mathscr{M}$. A universal family $\mathscr{U}$ on $\mathscr{M}$ is a $\mathscr{M}$-family such that for every scheme S and every S-family $\mathscr{S}$ there is a unique morphism $\psi : S \to \mathscr{M}$ such that $\mathscr{S} \simeq \psi^*(\mathscr{U})$.

Remark. (i) In order for the definition to make sense, a specific definition of a S-family should allow pullbacks, i. e. given a morphism between two schemes $f : T \to S$ and a universal S-family $\mathscr{S}$ there is a T-family $f^*(\mathscr{S})$. The pullback should naturally have some functorial properties, i. e. for two morphisms $f,\ g$: $(f \circ g)^* = g^* \circ f^*$, $\mathrm{id}_S^* = \mathrm{id}_T$. In our applications we will have additional

equivalence relations on S-families, e. g. isomorphy of S-families. Naturally the pullback operation should respect a given equivalence relation. An equivalence relation on families on the other hand will have to be compatible with a previously fixed equivalence relation of the underlying objects, i. e. the equivalence relation of families restricts to the equivalence relation of objects for $S = \mathrm{Spec}(\mathbb{C})$.
Whenever we define S-families of objects and equivalence of S-families, these conditions will be satisfied.
The existence of a pullback may be formalized in the language of fibered categories (over $\mathbf{Sch}_{\mathbb{C}}$).[36]
(ii) Equivalently $\mathscr{M}$ together with a natural transformation $\psi_{\mathscr{M}}$ from $\mathbf{M}$ to $\mathrm{Hom}(\,\cdot\,, \mathscr{M})$ is called a *fine* moduli scheme for $\mathbf{M}$ if $\mathscr{M}$ represents $\mathbf{M}$, i. e. for any scheme S, $\psi_{\mathscr{M}}$ induces $\mathrm{Mor}(\mathrm{Hom}(\,\cdot\,, S), \mathbf{M}) \simeq \mathrm{Mor}(S, \mathscr{M})$.[37]

1.62. Proposition ([New78], 2.13). *For a scheme T and a T-family $\mathscr{T}$ which satisfy*

(U^1) *local universality, i. e. for any point s of a scheme S that admits an S-family $\mathscr{S}$, there is a neighborhood V of s in S and a morphism $\varphi : V \to T$ such that $\mathscr{S}|_V \simeq \varphi^*(\mathscr{T})$;*

(U^2) *and which admits an action of a linear algebraic group G such that for any two morphisms $h_1,\ h_2 : S \to T$ we get $h_1^*(\mathscr{T}) \simeq h_2^*(\mathscr{T})$ if and only if there is a morphism $\Phi : G \to T$ such that $\Phi \cdot h_1 = h_2$;*

the following two statements hold:

(M^1) *A coarse moduli space, if it exists is a categorical quotient of T by G,*

(M^2) *A categorical quotient T by G is a coarse moduli space exactly when T is an orbit space, i. e. if every fiber of $T \to T /\!\!/ G$ contains exactly one orbit.*

Remark. Given a fixed equivalence relation on objects, it can be shown that a coarse moduli space is independent of the chosen extension of that equivalence relation to families. The result does not hold for fine moduli spaces, whose existence usually depends on the chosen equivalence relation ([New78], Def. 1.6', Pro. 1.8 and Lemma 5.10.). For vector bundles for example, if we define equivalence of families by simply requiring that two bundles $\mathscr{E}_Y$, $\mathscr{F}_Y$ of rank r and fiberwise degree d on $Y \times X$ are isomorphic as vector bundles, we will only get a coarse moduli space of stable vector bundles. However, if we define $\mathscr{E}_Y \sim \mathscr{F}_Y \Leftrightarrow \exists \mathscr{L}_Y \to Y$ line bundle such that $\mathscr{E}_Y \simeq \mathscr{F}_Y \otimes \pi_Y^*(\mathscr{L}_Y)$ as vector bundles, then in

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[36] Classical references are [SGA] or [Gr66]. Nicolai Beck gives an excellent account thereof in [Be14].

[37] Newstead [New78], Def. 1.5'.

some cases the stable vector bundles even form a fine moduli space. More precisely if $\gcd(r,d) = 1$ the fine moduli space of stable vector bundles does exist ([New78] 5.12), for $\gcd(r,d) \neq 1$ not (Ramanan [Ra73], theorem 2).
The concept can be easily extended to tuples and vector bundles with more additional structure. For tuples for example $((\mathscr{E}_Y^2 \otimes \pi_Y^*(\mathscr{L}_Y))^{\otimes u})^{\oplus v} \simeq ((\mathscr{E}_Y^2)^{\otimes u})^{\oplus v} \otimes \pi_Y^*(\mathscr{L}_Y^{\otimes u})$ implies that in 1.19 we should require $\gamma_Y : \mathscr{H}_Y^1 \to \mathscr{H}_Y^2 \otimes \mathscr{L}_Y^{\otimes u}$ such that given $\psi_Y : \mathscr{E}_Y^1 \to \mathscr{E}_Y^2 \otimes \pi_Y^*(\mathscr{L}_Y)$ the formula $\varphi_Y^1 = \left(\det(\psi_Y)^{\otimes w} \otimes \mathrm{id}_{\mathscr{P}_{v_Y}} \otimes \pi_Y^*(\gamma_Y)\right)^{-1} \circ \varphi_Y^2 \circ \psi_{Y,u,v}$ still makes sense.
Since we are temporarily only interested in the construction of coarse moduli spaces of stable objects we will stick with the easier condition given at the beginning. However all proofs given should work for the second equivalence relation on families as well. Similar results to the ones by Newstead and Ramanan in the case of vector bundles with additional structure seem desirable.

Remark. Note that the existence and if so the structure of a moduli space depends on the choice of the equivalence relation on objects. For example if we consider S-equivalence classes instead of isomorphism classes of semistable Higgs tuples, we are able to construct a coarse moduli space for the resulting functor of S-equivalence classes.

1.63. Definition. We call a scheme $\mathscr{M}^{ss}$ and a natural transformations $\psi_{\mathscr{M}^{(s)s}} : \mathbf{M}^{\mathbf{(s)s}} \to \mathrm{Hom}(\,\cdot\,,\mathscr{M}^{(s)s})$ a **coarse moduli space** for the functors $\mathbf{M}^{\mathbf{(s)s}}$ that associate to a scheme S of finite type over $\mathbb{C}$ an isomorphism class of S-families of (semi)stable objects, if

(i) $(\mathscr{M}^{ss}, \psi_{\mathscr{M}^{ss}})$ corepresents (cf. 1.60 (ii)) $\mathbf{M}^{\mathbf{ss}}$.

(ii) $(\mathscr{M}^{s}, \psi_{\mathscr{M}^{s}})$ is a coarse moduli space for $\mathbf{M}^{\mathbf{s}}$.

(iii) $\psi_{\mathscr{M}^{ss}}$ is surjective and every fiber contains at most one S-equivalence class.

Remark. This abuse of notation[38] is justified: For vector bundles Seshadri shows that the map that associates to every T-family of S-equivalence classes of semistable vector bundles the associated graded $t \mapsto \mathrm{Gr}(E_t) \in \mathscr{M}^{ss}$ is a morphism of schemes.[39] By definition 1.6' in Newstead $\mathscr{M}^{ss}$ is therefore the coarse moduli space of S-equivalence classes of semistable vector bundles.

1.64. Theorem. (i) *The coarse moduli space* $\mathscr{M}^{ss} := \mathfrak{T}^{ss}_{\mathrm{par}} /\!\!/ \mathcal{G}_A$ *for the functors in 1.20 exists as a projective scheme.*

(ii) *The geometric quotient* $\mathfrak{T}^{s}_{\mathrm{par}} / \mathcal{G}_A =: \mathscr{M}^{s} \subset \mathscr{M}^{ss}$ *exists as an open subscheme.*

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[38] We stick with the notation in [Sch08], 2.2.

[39] Theorem 8.1 in [Ses67].

Proof. Observe that by [MFK], 1.7. $\mathbb{P}^{(s)s}$ is open and hence $\mathfrak{T}^{(s)s}_{\text{par}} = \text{Gies}^{-1}(\mathbb{P}^{(s)s})$ is open. By 1.55 $\mathbb{P}^{ss} /\!\!/ \mathcal{G}_A$ exists. Furthermore 1.56 implies that the preimage $\text{Gies}^{-1}(\mathbb{P}^{ss})$ is the set of (semi)stable points with respect to the by Gies pulled back linearization and 1.52 shows that $\text{Gies}^{-1}(\mathbb{P}^{ss}) = \mathfrak{T}^{ss}_{\text{par}}$. Thus the quotient $\mathfrak{T}^{ss}_{\text{par}} /\!\!/ \mathcal{G}_A$ exists again using 1.55. Furthermore $\mathbb{P}^{ss} /\!\!/ \mathcal{G}_A$ is projective by 1.10, since $\mathcal{O}_{\mathbb{P}}(1)$ is ample. As the pullback of an ample line bundle by a finite map is ample again, $\mathscr{M}^{ss}$ is projective.
The universal property 2. in 1.53 together with 1.23, 1.25, 1.62 imply that $\mathscr{M}^{ss}$ is a coarse moduli space.[40]
(ii) is proved the same way, i. e. $\mathfrak{T}^{s}_{\text{par}}/\mathcal{G}_A$ is a geometric quotient as pullback of the geometric quotient $\mathbb{P}^s/\mathcal{G}_A$ by a finite stability-preserving equivariant morphism; $\mathbb{P}^s/\mathcal{G}_A \subset \mathbb{P}^{ss} /\!\!/ \mathcal{G}_A$ is open, so is $\mathfrak{T}^{s}_{\text{par}}/\mathcal{G}_A$ in $\mathfrak{T}^{ss}_{\text{par}} /\!\!/ \mathcal{G}_A$. ☐

Remark. We omit the discussion of S-equivalence for now. It will be shown later on that the Gieseker morphism does respect a still to be given definition of S-equivalence of tuples.

1.12. Further Extension

In the next two sections we study objects closely related to the parabolic Higgs tuples considered before. The calculations and constructions will transfer easily to the new setting and provide moduli spaces in these cases as well.

1.65. Recall that a parabolic filtration of $(E_a)_{a[|A|]}$ of type $(r_a^{ij})_{i[s_a^j]}$ over the puncture x^j consists of vector space filtration $0 \subset E_a^{1j} \subset \ldots \subset E_a^{s_a^j j} \subset E_a|_{x^j}$, $\dim(E_a^{ij}) = r_a^{ij}$ and weights $(\beta_a^{ij})_{i[s_a^j]}$ for every $a \in A$. A first extension of the concept considers filtrations of the full bundle $E = \bigoplus_{a \in A} E_a^{\oplus \kappa_a}$ of type $(r^{ij})_{i[s^j]j[|S|]}$ rather than filtration of each E_a. The construction in this case stays (almost) the same, we only have to replace $\bigtimes_{a\in A} \bigtimes_{j:x^j\in S} \bigtimes_{i=1}^{s_a^j} \mathfrak{G}_a^{ij}$ by $\bigtimes_{j:x^j\in S} \bigtimes_{i=1}^{s^j} \mathfrak{G}^{ij}$ with $\mathfrak{G}^{ij}$ the Graßmannian variety parametrizing r^{ij}-dimensional subspace of V. Of course this space is larger than the one discussed before. However, the construction depends on less parameters and is therefore easier. In fact only the parabolic contribution calculated in section 1.7 changes. To get the expected results replace $\chi_a^1 = \sum_{j=1}^{|S|} \sum_{i=1}^{s^j} \frac{\kappa_a f^{ij} \delta^{ij}}{p} \cdot \left(\frac{r_a}{p_a} - \frac{r}{p} \right)$ and $\tilde{\nu}^{ij} = \frac{r\delta^{ij}}{p}$ and $\tilde{\nu}_a := \frac{\kappa_a(p-u\delta-\sum_{j=1}^{|S|}\sum_{i=1}^{s^j}\delta^{ij}f^{ij})-\xi_a r}{p}$ and keep $\tilde{\nu}$ and χ_a^2 the same. Now the given proofs transfer easily to the new situation and result in the existence of the corresponding moduli space.

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[40] Observe that the universal properties hold on the G-invariant open subset of semistable tuples.

1.66. In his dissertation [Be14] Nikolai Beck considers decorated tuples. These are non-parabolic Higgs tuples with a point in $\mathbb{P}(E_\sigma|_S)$ for one puncture $S \in X$ and a new homogeneous representation $\sigma : \mathrm{Gl}(\mathbb{C}^{r_a})_a := \bigtimes_{a\in A} \mathrm{Gl}(\mathbb{C}^{r_a}) \to \mathrm{Gl}(V_\sigma)$ on some vector space V_σ. We would like to compare the two concepts for one puncture S. Since σ decomposes into finite-dimensional irreducible representations, we restrict our attention to the case σ irreducible for now. The irreducible representations of $\bigtimes_{a\in A} \mathrm{Gl}(\mathbb{C}^{r_a})$ are tensor products of irreducible representations $\sigma_a : \mathrm{Gl}(\mathbb{C}^{r_a}) \to \mathrm{Gl}(V_{\sigma_a})$.[41] Furthermore the irreducible representations of $\mathrm{Gl}(\mathbb{C}^{r_a})$ are parametrized by tuples $(\sigma_a^1, \dots, \sigma_a^{r_a}) \in \mathbb{Z}^{r_a}$. We denote by $(\sigma^i)_{i[r]} \in \mathbb{Z}^r$ the resulting weight of σ. For irreducible polynomial representations the last entry is trivial.

Two irreducible representations are isomorphic if they possess the same Schur character χ and the Borel-Weil theorem tells us that $\mathrm{Gl}(\mathbb{C}^{r_a})_a/Q \hookrightarrow \mathbb{P}(V_\sigma)$ is a closed embedding, where the parabolic subgroup Q is the stabilizer of the orbit of the unique maximal weight vector corresponding to χ and the image of $\mathrm{Gl}(\mathbb{C}^{r_a})_a/Q$ is the orbit of the maximal weight vector. Thus the parameter space $\mathbb{P}(E_\sigma|_S)$ contains more points than some flag variety $\mathrm{Gl}(\mathbb{C}^{r_a})_a/Q$ that parametrizes parabolic filtrations.

Moreover note, that $Q = \bigtimes_{a\in A} Q_a$ for parabolic subgroups $Q_a \subset \mathrm{Gl}(\mathbb{C}^{r_a})$ and $\mathrm{Gl}(\mathbb{C}^{r_a})/Q_a \hookrightarrow \mathbb{P}(V_{\sigma_a})$. Using the Segre embedding the following diagram of monomorphisms commutes

$$\begin{tikzcd}
\bigtimes_{a\in A} (\mathrm{Gl}(\mathbb{C}^{r_a})/Q_a) \arrow[r] \arrow[d] & \bigtimes_{a\in A} \mathbb{P}(V_{\sigma_a}) \arrow[d] \\
\mathrm{Gl}(\mathbb{C}^{r_a})_a/Q \arrow[r] & \mathbb{P}(V_\sigma).
\end{tikzcd}$$

Nikolai Beck defines the μ-function for points in $\mathbb{P}(E_\sigma|_s)$ w. r. t. the standard linearization of the natural action of $\mathrm{Sl}(\mathbb{C}^{r_a})_a := \{(g_a)_{a[|A|]} \in \mathrm{Gl}(\mathbb{C}^{r_a})_a | \prod_{a\in A} \det(g_a)^{\kappa_a} = 1\}$ in $\mathcal{O}(\hat{\delta})$.[42] Using this definition the moduli space of tuples is constructed under the restriction $\hat{\delta} u^\sigma < 1$ for $u^\sigma = \sum_{j=1}^r \sigma^j$ the homogeneous degree of σ. Unfortunately this concept implies the existence of the moduli space of parabolic Higgs tuples only in the case $\sum_{i=1}^r \delta^{ij} r^{ij} < 1$.

Using the main calculations above, we can however strengthen the result if we use a stability concept closer related to the properties of σ. The semistability concept in [Be14] depends on the homogeneous degree u^σ of the representation σ rather than the classifying data $(\sigma^1, \dots, \sigma^r)$.

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[41] [KP00] 5.7.

[42] We allow $\hat{\delta} \in \mathbb{Q}_+$. This is a slight (but quite common) abuse of notation.

1.67. Semistability - An Intrinsic Definition. We will consider $G = \mathrm{Sl}(\mathbb{C}^{r_a})_{a[|A|]}$ in this paragraph. However the representation theory that follows works for arbitrary reductive groups G as well.
As usual denote $r = \sum_{a \in A} r_a$. Given a one-parameter subgroup $\lambda : \mathbb{C}^* \to G$ with weights $(\gamma^l)_{l[r]}$ and a representation $\sigma : G \to \mathrm{Gl}(V_\sigma)$ with corresponding character $\chi_\sigma = \prod_{a \in A} \chi_{\sigma_a}$, $\lambda(\mathbb{C}^*)$ is contained in a maximal torus $T_\lambda \subset G$.[43] Therefore we find a basis of weight vectors v^i_λ of V_σ with corresponding weights $(\sigma^{il}_\lambda)_{l[r]} \in \mathbb{Z}^r$ such that $\sigma(\lambda(z))v^i_\lambda = \prod_{l=1}^r z^{\gamma^l \sigma^{il}_\lambda} v^i_\lambda$. Now given an arbitrary point $v \in V^\vee_\sigma$ representing a $[v] \in \mathbb{P}(V_\sigma)$, $v = \sum_{i=1}^{\dim(V_\sigma)} a^i v^{i,\vee}_\lambda$ we define

$$\begin{aligned}\mu^\sigma(\lambda, v) &= -\min\left\{ (\gamma^l)^t_{l[r]} \cdot (\sigma^{im}_\lambda)_{m[r]} = \sum_{l=1}^r \gamma^l \sigma^{il} \,\middle|\, a^i \neq 0 \right\}\\ &= -\min\{\chi^i_\lambda(\lambda) |\, a^i \neq 0\},\end{aligned} \tag{WF}$$

where χ^i_λ denotes the character of T_λ associated to $(\sigma^{il}_\lambda)_{l[r]}$.

1.68. Note that this definition does not depend on any embedding into a tensor product. However it agrees with the definition given before. Since $V_\sigma \subset (\mathbb{C}^r)^{\otimes u}$ is a subrepresentation[44] the representation on the tensor product splits as $\sigma \oplus \upsilon$ for some representation υ. Hence $(\mathbb{C}^r)^{\otimes u}$ decomposes into T_λ-weight spaces $V^i_\sigma \subset V_\sigma$ and $V^l_\upsilon \subset V_\upsilon$. On the other hand given a basis $(w^i)_{i[r]}$ such that T_λ acts diagonal we get for $(t^j)_{j[r]} \in T_\lambda$ and $w^\theta = \bigotimes_{i=1}^u w^{\theta(i)}$, $\theta \in \mathrm{Map}^r_u$

$$(t^j)_j w^\theta = \bigotimes_{i=1}^u (t^j)_j w^{\theta(i)} = \prod_{i=1}^u t_i^{\chi^{\theta,i}_\lambda} w^\theta, \quad \chi^{\theta,i}_\lambda := \#\{j : \theta(j) = i\}.$$

Denote by $W^\theta = \langle w^\theta \rangle$ and by I the subset of $\{1, \dots, r\}^u$ such that $(w^\theta)_{\theta \in I}$ is a basis of V_σ. We have

$$\begin{aligned}\chi^\theta_\lambda(\lambda) &= \sum_{j=1}^r \gamma^j \chi^{\theta,j}_\lambda = \sum_{j=1}^r \sum_{k=1}^r \alpha^k (\gamma^k_r)_j \chi^{\theta,j}_\lambda\\ &= \sum_{k=1}^r \alpha^k (-r) \sum_{j=1}^k \#\{i : \theta(i) = j\} = -r \sum_{k=1}^r \alpha^k \#\{i : \theta(i) \leq k\}\\ &= -r \sum_{k=1}^r \alpha^k v(k, \theta) = \sum_{k=1}^{r-1} \alpha^k (k \cdot u - v(k, \theta) r),\end{aligned}$$

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[43] For an arbitrary group G, the representation $\varsigma|_{T_\lambda}$, T_λ abelian, decomposes into 1-dimensional irreducible representations, i. e. characters of T_λ.

[44] cf. 2.15.

where χ^θ_λ denotes the character of T_λ with weights $(\chi_\lambda^{\theta,j})_{j[r]}$.
Let $W_\theta = \bigotimes_{i=1}^u W_{\theta(i)}, W_{\theta(i)} = \langle w^1, \dots, w^{\theta(i)} \rangle$ and $v^\vee = \sum_{\theta \in I} a^\theta (w^\theta)^\vee$ be an arbitrary element in $V_\sigma^\vee$. Observe that

$$v^\vee|_{W_\theta} \neq 0 \Leftrightarrow a^{\tilde\theta} \neq 0 \text{ for some } \tilde\theta \in I : \ \tilde\theta(i) \leq \theta(i), \forall 1 \leq i \leq u.$$

Moreover in this case $v(k,\tilde\theta) \geq v(k,\theta)$, $\forall 1 \leq k \leq r$ and therefore $\chi_\lambda^{\tilde\theta}(\lambda) \leq \chi_\lambda^\theta(\lambda)$. We conclude if θ minimizes $\chi_\lambda^\theta(\lambda)$, $a^\theta \neq 0$ then $v^\vee|_{W_\theta} \neq 0$. On the other hand if θ minimizes $\chi_\lambda^\theta(\lambda)$, $v^\vee|_{W_\theta} \neq 0$, then there is a $\tilde\theta$ with $a^{\tilde\theta} \neq 0$ such that $\chi_\lambda^{\tilde\theta}(\lambda) \leq \chi_\lambda^\theta(\lambda)$. Hence

$$-\min\{\chi_\lambda^\theta(\lambda) | \ a^\theta \neq 0\} = -\min\left\{ \sum_{k=1}^{r-1} \alpha^k (k \cdot u - v(k,\theta) r) \middle| \ v^\vee|_{W_\theta} \neq 0 \right\}.$$

1.69. Parabolic Filtrations as Elements of a Representation Space. Given a reduction $\mathscr{R} : X \to P/Q_G(\lambda)$ and a point $s : \{S\} \to P_\sigma|_S$, consider the $Q_G(\lambda)$-bundle $\mathscr{R}^*(P)$ defined by $P \to P/Q_G(\lambda)$ and note that $\sigma|_{Q_G(\lambda)}(\mathscr{R}^*(P)) = P_\sigma$.[45] Now the transition functions of P_σ may be chosen, such that they split over $Q_G(\lambda)$, i. e. $Q_G(\lambda)$ induces a filtration of P_σ by subbundles. A (up to an element of $\sigma(Q_G(\lambda))$ uniquely) chosen trivialization ϕ, identifies s with a point $\phi(s(S)) \in V_\sigma$ and we can apply the semistability criterion defined before. Note that a different choice of $\mathscr{R}$ changes the weight function by a factor in $G/Q_G(\lambda)$.
More precisely: A quotient P/Q is a locally trivial bundle if and only if it possesses local sections.[46] For a Lie subgroup $Q \subset G$ this condition is satisfied.[47] If $(t^i)_i$ denote local sections of $\pi : P \to P/Q$ and $(\psi^i)_i$ the extensions to local trivializations with Q-valued transition functions $(g^{ij})_{ij}$, then $(t^i \circ \mathscr{R}|_{U_i})_i$ extend to local trivializations $(\varphi^i)_i$, $\varphi^i : P|_{U_i} \to U_i \times G$ of P with the same transition functions. Of course this construction depends on the choice of the $(t^i)_i$, i. e. on a map to Q. If we further denote by $[(p,v)] \in P_\sigma$ an element represented by $(p,v) \in P \times V_\sigma$, then $(\phi^i)_i$ for $\phi^i([p,v]) = (\pi(p), \sigma(\mathrm{pr}_2 \varphi^i(p), v))$ are trivializations of P_σ with transition functions $(\sigma(g^{ij}))_{ij}$. Note that by construction $\phi^i([t^i \circ \mathscr{R}|_{U_i}(x), v]) = (x, v)$. If ϕ denotes the trivialization at the puncture: $\phi(s(S)) \in V_\sigma$.

1.70. In the special case of a parabolic filtration, namely an element $s^j \in P|_{x^j}/Q_G(a^j)$ for a^j a one-parameter subgroup, we find the anti-dominant character $\chi_{a^j} : Q_G(a^j) \to \mathbb{C}^*$. Let $\sigma : G \to V_\sigma$ be the representation induced by χ_{a^j} and T_{a^j} the maximal torus to a^j. Given a one-parameter subgroup λ of G

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[45] The reduction of the structure group implies the existence of suitable transition functions $U_{ij} \to Q_G(\lambda)$ of P.

[46] [St51], §7.4, Theorem.

[47] Chevalley, [Ch46], Prop. 1, p. 110.

as well as a reduction $\mathscr{R} : X \to P/Q_G(\lambda)$, we find for every choice of two representatives of $s^j(x^j)Q_G(a^i)$ and $\mathscr{R}(x^j)Q_G(\lambda)$ a $g \in Q_G(\lambda)\backslash G/Q_G(a^i)$ that maps one to the other - $s^j(x^j)g = \mathscr{R}(x^j)$.[48] In 2.7 we will define the weight function $\mu(\lambda, s^j) = -\langle \lambda, g^{-1}a^j g\rangle$. Since the value of μ depends only on the class of $g \in Q_G(\lambda)\backslash G/Q_G(a^j)$ we may assume w. l. o. g. that $g\lambda(z)g^{-1} \in T_{a^j}$. Let $(v^i_{a^j})_i$ be the weight vectors to T_{a^j} with weights $(\chi^i_{a^j})_i$. Then

$$\begin{aligned} g\lambda(z)g^{-1}v^i_{a^j} &= \sigma(g\lambda(z)g^{-1}, v^i_{a^j}) = \chi^i_{a^j}(g\lambda(z)g^{-1})v^i_{a^j} \\ \Rightarrow \lambda(z)g^{-1}v^i_{a^j} &= \chi^i_{a^j}(g\lambda(z)g^{-1})g^{-1}v^i_{a^j}, \end{aligned}$$

i. e. $v^{ij}_\lambda := g^{-1}v^i_{a^j}$ is a weight vector of T_λ with weight $\chi^{ij}_\lambda = \chi^i_{a^j}(g \cdot \ldots \cdot g^{-1})$. In particular there is an i_0 such that $\chi^{i_0}_{a^j} = \chi_{a^j}$ and thus

$$\chi^{i_0 j}_\lambda(\lambda) = \chi_{a^j}(g\lambda g^{-1}) = \langle \lambda, g^{-1}a^j g\rangle.$$

We have seen above, that using a trivialization ϕ associated to $\mathscr{R}$, $s^j(x^j)$ maps to $g^{-1}v^j$ while $\mathscr{R}(x^j)$ maps to v^j - v^j weight vector to χ_{a^j}. Hence

$$\begin{aligned} -\min\left\{ \chi^{ij}_\lambda(\lambda) \,\middle|\, a^{ij} \neq 0, \phi(s^j(x^j)) = \sum_{k=1}^{\dim(V_\sigma)} a^{kj}v^{kj}_\lambda \right\} &= -\chi^{i_0 j}_\lambda(\lambda) \\ &= -\langle \lambda, g^{-1}a^j g\rangle. \end{aligned}$$

Remark. In 2.7 it can be seen, that the right-hand term is constant on the class $Q_G(\lambda)\backslash G/Q_G(a^j)$, i. e. independent of the chosen trivializations used to define $s^j(x^j)$ and $\mathscr{R}(x^j)$.

1.13. New Moduli Spaces

1.71. Let $G = \mathrm{Gl}(\mathbb{C}^{r_a})_a = \{(g_a)_{a\in A} \in \mathrm{Gl}(\mathbb{C}^{r_a})_a\}$ and choose $B_a \subset \mathrm{Gl}(\mathbb{C}^{r_a})$ the Borel subgroup of upper triangular matrices, $B = \{b \in \mathrm{Gl}(\mathbb{C}^{r_a})_a : b_a \in B_a\}$. Recall that a character on B takes the form $\prod_{a\in A}\prod_{i=1}^{r_a}(b^{ii}_a)^{c^i_a}$ for some $c^i_a \in \mathbb{Z}$. More generally let P_a be a parabolic subgroup containing B_a, then

$$P_a = \begin{pmatrix} P^{r^1_a} & & * \\ & \ddots & \\ 0 & & P^{r^{m(a)}_a} \end{pmatrix},\ P^{r^1_a} \in \mathrm{Gl}(\mathbb{C}^{r^1_a})$$

and a character on $\times_{a\in A} P_a$ takes the form $\prod_{a\in A}\prod_{i=1}^{m(a)} \det(P^{r^i_a})^{c^i_a}, c^i_a \in \mathbb{Z}$.
As we have seen above every representation $\sigma : \mathrm{Gl}(\mathbb{C}^{r_a})_a \to \mathrm{Gl}(V_\sigma)$ comes from

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[48] In abuse of notation denote by $s^j(x^j)$ resp. $\mathscr{R}(x^j)$ the image in P under suitable fixed trivializations.

a tuple of representations $\sigma_a : \mathrm{Gl}(\mathbb{C}^{r_a}) \hookrightarrow \mathrm{Gl}(V_{\sigma_a})$. The weights of σ_a and σ are connected as usual (cf. 1.5). We denote the weights as in the classical parabolic case by $(\beta^{ij})_{i[r]j[|S|]}$ and $(\beta^{ij}_a)_{a[|A|]i[r_a]j[|S|]}$.

1.72. Definition. Choose for every puncture $x^j \in S$ a tuple of representations $\sigma^j : \mathrm{Gl}(\mathbb{C}^{r_a})_a \to \mathrm{Gl}(V_\sigma)$ as above and denote by σ^j_a the resulting representations of $\mathrm{Gl}(\mathbb{C}^{r_a})$. Denote by $(\beta^{ij})_{i[r]}$, $\beta^{1j} \geq \ldots \geq \beta^{rj}$ the maximal weight of σ^j.
Consider a tuple $((E_a,\ (s^j_a)_{j[|S|]})_{a[|A|]},\ \varphi,\ L)$ for $s^j_a \in \bigtimes_{a\in A} \mathbb{P}(E_{\sigma_a})$ (cf. 1.2). Recall that every proper filtration $(F^k,\ \alpha^k)_{k[m]}$, $\alpha^k \in \mathbb{Q}_+$ as in 1.6 comes from a one-parameter subgroup $\lambda : \mathbb{C}^* \to \mathrm{Sl}(\mathbb{C}^{r_a})_a$ resp. one-parameter subgroups $\lambda_a : \mathbb{C}^* \to \mathrm{Gl}(\mathbb{C}^{r_a})$. We call a tuple $(\kappa_a, \xi_a, \delta, \varepsilon^j)-$(semi)stable if

$$M^{\kappa,\xi}(F^i,\alpha^i) + \delta\mu(F^i,\alpha^i,\varphi) + \varepsilon^j \sum_{j:x^j\in S} \mu^{\sigma^j}(\lambda, s^j)\ (\geq)\ 0$$

holds for all weighted filtrations $(F^i,\alpha^i)_{i[r]}$ (as in 1.6) and

$$\begin{aligned} M^{\kappa,\xi}_{\mathrm{par}}(F^i,\alpha^i) =& \sum_{i=1}^r \alpha^i \cdot (\deg(E)\,\mathrm{rk}(F^i) - \deg(F^i)\,\mathrm{rk}(E) \\ &+ \sum_{a\in A} \xi_a(\mathrm{rk}(E_a)\,\mathrm{rk}(F^i) - \mathrm{rk}(F^i_a)\,\mathrm{rk}(E))) \\ \mu(F^i,\alpha^i,\varphi) :=& -\min\left\{ \sum_{j=1}^u \gamma^{i_j} \,\middle|\, (i_j)_{j[u]} \in \{1,\ldots,r\}^u : \varphi|_{(\bigotimes_{j=1}^u F^{i_j})^{\oplus v}} \not\equiv 0 \right\}, \end{aligned}$$

where $\mu^{\sigma^j}(\lambda, s^j)$ is defined as in (WF) with a trivialization like the one of 1.69.

Remark. Using 1.66 the following construction extends as in 1.65 to tuples $((E_a)_{a[|A|]},\ \varphi,\ (s^j)_{j[|S|]})$ with $s^j \in \mathbb{P}(E_{\sigma_a})|_{x^j}$.

1.73. Recall that in 1.16 we constructed a parameter space $\mathfrak{X}$ and then added Graßmannian varieties $\mathfrak{G}^{ij}_a$ to parametrize parabolic filtrations. Now we know that there are $u^{\sigma^j_a}$, $v^{\sigma^j_a}$, $w^{\sigma^j_a} \in \mathbb{Z}$ such that $E^j_{\sigma_a} \subset (E_a^{\otimes u^{\sigma^j_a}})^{\oplus v^{\sigma^j_a}} \otimes \det(E_a)^{\otimes w^{\sigma^j_a},\vee}$ and as before we find sheaves $\mathscr{F}^{\sigma^j_a}_j = \left(V_a^{\otimes u^{\sigma^j_a}}\right)^{\oplus v^{\sigma^j_a}} \otimes \pi^*_X(\mathcal{O}_X(-u^{\sigma^j_a}\cdot n))|_{x^j}$, $\mathscr{K}^{\sigma^j_a}_j = \det(\mathscr{E}_{\mathfrak{Q}_a})^{\otimes w^{\sigma^j_a}}|_{x^j}$, $\mathfrak{X}^{\sigma^j_a} = \mathbb{P}(\mathscr{H}om(\pi_{\mathfrak{Q},*}(\mathscr{F}^{\sigma^j_a}_j), \pi_{\mathfrak{Q},*}(\mathscr{K}^{\sigma^j_a}_j))^\vee) \xrightarrow{\pi} \mathfrak{Q}$. We have again a tautological morphism

$$\psi_{\mathfrak{X}^{\sigma^j_a}} : (\pi\times \mathrm{id}_X)^*(\mathscr{F}^{\sigma^j_a}_j) \to (\pi\times\mathrm{id}_X)^*(\mathscr{K}^{\sigma^j_a}_j) \otimes \pi^*_{\mathfrak{X}^{\sigma^j_a}}(\mathcal{O}_{\mathfrak{X}^{\sigma^j_a}}(1)).$$

We may pull these morphisms back to $X \times \mathfrak{T} \times \bigtimes_{\substack{1 \leq j \leq |S| \\ a \in A}} \mathfrak{X}^{\sigma_a^j}$ and find a closed subscheme $\mathfrak{G}_{\mathrm{par}} \subset \bigtimes_{\substack{1 \leq j \leq |S| \\ a \in A}} \mathfrak{X}^{\sigma_a^j}$ such that $\bigtimes_{a \in A} \psi_{\mathfrak{X}^{\sigma_a^j}}$ splits over $\mathscr{E}_{\mathfrak{T} \times \mathfrak{G}_{\mathrm{par}}, \sigma}$. The universal properties are proved as in the classical case.
For the Gieseker morphism we replace the Plücker embedding in the last component by the identity. Then all proofs and calculations of the classical case apply in this situation as well, with one exception; namely the admissibility condition. It may happen that given a filtration $(V^k)_{k[p]}$ of V as before, $q_{u^{\sigma^j},\, v^{\sigma^j}}|_{(V_\theta)^{\oplus v^{\sigma^j}}} = 0$, $V_\theta = \bigotimes_{i=1}^{u^{\sigma^j}} V^{\theta(i)}$ even if the induced map on $E_\sigma|_{x^j}$ is non-zero on the subbundles generated by $(V_\theta)^{\oplus v^\sigma}$ resp. their intersection with $E_\sigma|_{x^j}$. Thus we have to bind $-(\chi_\lambda^{i_0 j}(\gamma_r^{r^{jk}}) - \chi_\lambda^{i_0 j}(\gamma_r^{r^{jk}_{coh}}))$ where $r^{jk} = \dim(F^k)|_{x^j}$ and $r^{jk}_{coh} = \dim(F^{k,coh})|_{x^j}$.[49] This term is bounded by $r \sum_{i=r^{jk}_{coh}}^{r^{jk}} \beta_{i_0}^{ij}$ and therefore $-(\chi_\lambda^{i_0 j}(\gamma_r^{r^{jk}}) - \chi_\lambda^{i_0 j}(\gamma_r^{r^{jk}_{coh}})) \leq \beta^{1j}(r^{jk} - r^{jk}_{coh})r$. Hence we call the stability parameters ε^j admissible if they are positive, decreasing and $\varepsilon^j \beta^{1j} < 1$ or equivalently if $\varepsilon^j \sum_{i=1}^{s^j} \delta^{ij} < 1$. If σ is not irreducible, i. e. decomposes $\sigma^j = \oplus_{t=1}^m \sigma_t^j$, the admissibility condition becomes $\varepsilon^j \max\{\beta_t^{1j} : 1 \leq t \leq m\} < 1$ for $(\beta_t^{ij})_{i[r]}$ the maximal weight of σ_t^j.

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[49] $\chi_\lambda^{i_0 j}$ is the character of T_λ to the weight vector $v^{i_0 j}$ with non-zero coefficient $a^{i_0 j}$ and minimal weight function. Let $(\beta_{i_0}^{ij})_i$ be the corresponding weight. In the notation of 1.52 $-(\chi_\lambda^{i_0 j}(\gamma_r^{r^{jk}}) - \chi_\lambda^{i_0 j}(\gamma_r^{r^{jk}_{coh}}))$ corresponds to $\sum_{i=1}^{s^j} \delta_{i_0}^{ij}(f^{ijk} - f^{ijk}_{coh})r$.

2

The Moduli space of Projective Parabolic Higgs Bundles

The second chapter studies projective parabolic ς-Higgs bundles and their moduli space.

2.1. Principal Bundles. An algebraic (resp. holomorphic) principal G-bundle on the Riemann surface X is a $\mathbb{C}$-scheme (resp. complex space) P with a right action $\sigma : P \times G \to P$ and a G-invariant projection $\pi : P \to X$ such that P is locally trivial in the étale topology (resp. strong topology). For algebraic G-bundles we may equivalently choose a trivialization in the fppf-topology or that locally X admits an unramified cover $V \to U \subset X$ such that the local pullback of P is trivial, i. e. $P \times_X V \simeq V \times G$ ([Mi80] 4.10, [Sch08] p. 101f). Note that the category of holomorphic G-bundles (with G-equivariant holomorphic maps) on X is equivalent to the category of algebraic G-bundles (with G-equivariant X-morphisms).[1]

We are mostly concerned with connected reductive algebraic groups, for which the trivialization may be chosen in the Zariski topology ([Sch08] 2.1.1.17) on X. More generally, for a scheme of finite type Y a principal G-bundle with connected reductive structure group on $Y \times X$ is trivial w. r. t. the product of the étale and the Zariski topology on $Y \times X$. Bundles with respect to not necessarily connected reductive algebraic groups may however occur when we consider H-bundles for $H \subset G$ a subgroup.

Given a parabolic subgroup $P^j \subset G$ for every punctures $x^j \in S$ a parabolic (principal) G-bundle is a pair $(P,\ (s^j)_{j[|S|]})$ with P a principal G-bundle and $s^j : \{x^j\} \to P \times_X \{x^j\}/P^j$ reductions.

2.2. Projective Higgs Bundles. Let W be a vector space and $\mathbb{P}(W)$ the corresponding projective space. Let P be a principal G-bundle of fixed topological type on X and $\phi \in H^0(X, \mathbb{P}(P_\varsigma))$ for a fixed homogeneous[2] representation

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[1] see for example [GAGA] and [Ser58].

[2] ς homogeneous $:\Leftrightarrow$ $\underline{\varsigma}$ homogeneous in 2.12. See as well 2.14.

$\varsigma : G \to \mathrm{Gl}(W)$ and $P_\varsigma = P \times_\varsigma W$, $\mathbb{P}(P_\varsigma) \simeq P \times_\varsigma \mathbb{P}(W)$.[3] To give ϕ is equivalent to the choice of a line bundle L and a surjection $\varphi : P_\varsigma \to L$ ([Ha77], II.7.12). In order for our construction to work, we will allow φ to be arbitrary non-trivial for now. Once a (projective) parameter scheme is constructed parametrizing non-trivial homomorphisms $\varphi : P_\varsigma \to L$, the surjective φ will form an open invariant subset thereof.
The triple $(P,\ \varphi,\ L)$ is called a projective ς-Higgs bundle. A projective parabolic ς-Higgs bundle is a quadruple $(P,\ (s^j)_{j[|S|]},\ \varphi,\ L)$ with additional reductions s^j for every puncture.

2.1. THE SEMISTABILITY CONCEPT OF PARABOLIC G-BUNDLES

In this first section we will define a semistability concept for projective parabolic ς-Higgs bundles. We will then rewrite the semistability criterion in terms of an associated parabolic Higgs tuple.

2.3. Let P be a principal G-bundle on X. Fix a faithful representation $\iota : G \to \mathrm{Gl}(U)$, U a vector space. Denote by P_ι the principal $\mathrm{Gl}(U)$-bundle induced by ι. Let P^j, $1 \leq j \leq |S|$ be a tuple of parabolic subgroups of G - one for each puncture $x^j \in S$ - and choose reductions $s^j : \{x^j\} \to P \times_X \{x^j\}/P^j$.
We follow the approach by [Bra91] (see as well [HS10].4) to define the concept of (semi)stability for tuples $(P,\ (s^j)_{j[|S|]},\ \varphi)$.
For a one-parameter subgroup $\lambda \in \mathrm{Hom}(\mathbb{C}^*, G)$ denote

$$P_G(\lambda) := \{g \in G| \lim_{z\to 0} \lambda(z) g \lambda(z)^{-1} \text{ exists in } G\}, \quad Q_G(\lambda) := P_G(-\lambda).^4$$

Let $T_\iota \subset \mathrm{Gl}(U)$ be a maximal torus corresponding to a basis $(u^i)_{i[\dim(U)]}$ and denote $(\ \cdot\ ,\ \cdot\ )$ the symmetric $\mathbb{Q}-$bilinear map

$$\hat{T}_\iota \times \hat{T}_\iota \to \mathbb{Q}, \quad \hat{T}_\iota = \mathrm{Hom}(\mathbb{C}^*, T_\iota) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

induced by

$$\mathbb{Z}^{\dim(U)} \times \mathbb{Z}^{\dim(U)} \ni ((a^i)_{i[\dim(U)]}, (b^j)_{j[\dim(U)]}) \mapsto \sum_{i=1}^{\dim(U)} a^i b^i \in \mathbb{Z}.$$

Furthermore we find the dual pairing $\langle\ \cdot\ ,\ \cdot\ \rangle : \hat{T}_\iota \times \check{T}_\iota \to \mathbb{Q}$ (cf. 1.34) for $\check{T}_\iota := \mathrm{Hom}(T_\iota, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ the rational character group. Hence for every rational one-parameter subgroup $\lambda_{\mathrm{Gl}} \in \hat{T}_\iota$ there is a rational character $\chi_{\lambda,\mathrm{Gl}} \in \check{T}_\iota$ such that

$$(\lambda', \lambda_{\mathrm{Gl}}) = \langle \lambda', \chi_{\lambda,\mathrm{Gl}} \rangle, \quad \forall \lambda' \in \hat{T}_\iota.$$

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[3] In abuse of notation we wrote ς for both the action on W and the induced action on $\mathbb{P}(W)$.

[4] $-\lambda$ denotes the inverse element of λ in the group $\hat{T}_\iota$, i. e. $z \mapsto \lambda(z)^{-1}$.

In fact λ_{Gl} defines a character in $\mathrm{Hom}(Q_{\mathrm{Gl}}(\lambda_{\mathrm{Gl}}), \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$, which we call in abuse of notation $\chi_{\lambda,\mathrm{Gl}}$, too. Further if T_ι is the extension of a maximal torus $T \subset G$ and $\lambda_{\mathrm{Gl}} = \iota \circ \lambda,\ \lambda \in \mathrm{Hom}(\mathbb{C}^*, G) \otimes_{\mathbb{Z}} \mathbb{Q}$, then the pairing

$$\langle\ \cdot\ ,\ \cdot\ \rangle : \hat{T} \times \check{T} \to \mathbb{Q},\ \hat{T} := \mathrm{Hom}(\mathbb{C}^*, T) \otimes_{\mathbb{Z}} \mathbb{Q},\ \check{T} := \mathrm{Hom}(T, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is induced by the canonical pairing for the group $\mathrm{Gl}(U)$. Observe that this map is independent of the chosen extension T_ι of T.[5] Analogously we find χ_λ and we have $Q_G(\lambda) = Q_{\mathrm{Gl}}(\lambda_{\mathrm{Gl}}) \cap G$ as well as $\chi_{\lambda,\mathrm{Gl}}|_{Q_G(\lambda)} = \chi_\lambda$.

2.4. Definition. A character $\chi : Q_{\mathrm{Ad}} \to \mathbb{C}^*,\ Q_{\mathrm{Ad}} \subset \mathrm{Ad}(G)$[6] a parabolic subgroup, is called anti-dominant if the line bundle $P_{Q_{Ad}}(\chi_{Ad})$ is ample. Here $P_{Q_{Ad}}$ denotes the Q_{Ad}-bundle $\mathrm{Ad}(G) \to \mathrm{Ad}(G)/Q_{Ad}$ and $P_{Q_{Ad}}(\chi_{Ad})$ the line bundle associated by χ_{Ad}.
If $Q \subset G$ is a parabolic subgroup and $Q_{\mathrm{Ad}} \subset \mathrm{Ad}(G)$ is the induced parabolic subgroup, then $\chi : Q \to \mathbb{C}^*$ is called anti-dominant, if $\chi = \mathrm{Ad} \circ \chi_{Ad},\ \mathrm{Ad} : Q \to Q_{Ad}$ holds for an anti-dominant character χ_{Ad} of Q_{Ad}.
If G is semisimple, χ is anti-dominant, if $P_Q(\chi)$ is ample.

2.5. Proposition. *Let G be a semi-simple linear algebraic group. The map $\hat{G} \ni \lambda \to (P_G(\lambda), \chi_{-\lambda})$ into the set of pairs of a parabolic subgroup and a dominant character $\chi_{-\lambda}$ is surjective.*[7]
$\hat{G} \ni \lambda \to (Q_G(\lambda), \chi_\lambda)$ into the set of pairs of a parabolic subgroup and an anti-dominant character χ_λ is surjective, too.
Every parabolic subgroup of a (connected) reductive group is of the form $Q_G(\lambda)$ for some one-parameter subgroup λ of G.

Proof. [GLSS08], section 3.2 or [Sch04], Example 2.1.8. The last statement is proven in Springer [Sp81], Proposition 8.4.5. □

Remark. For future reference note that if G is generally reductive a (anti-)dominant character vanishes on the radical $\mathscr{R}ad(G)$ ([Ram96i], 2.14).

2.6. For $\lambda : \mathbb{C}^* \to \mathrm{Sl}(U)$ with strictly ascending weights $\gamma^1, \ldots, \gamma^m$ we get

$$Q_{\mathrm{Gl}}(\lambda) = \{\mathrm{diag}(A^1, \ldots, A^m) + N : A^j \in \mathrm{Gl}(r^j - r^{j-1}, \mathbb{C}),$$
$$N \text{ a strictly block upper triangular matrix}\}.$$

Then $\prod_{j=1}^m \det(A^j)^{\gamma^j}$ is an anti-dominant character ([Sch08], 2.4.9).

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[5] Lemma 2.8 in Chapter II of [MFK].

[6] Recall that $\mathrm{Ad}(G)$ is semisimple for G reductive.

[7] For the definition of a dominant character see for example [Ram96i], 2.14.

2.7. Let P^j be a parabolic subgroup and $T^j \subset P^j$ a maximal torus.[8] Let $\tau^j \in \hat{T}^j_+ = \{\tau \in \hat{T}^j |\ Q_G(\tau) = P^j\}$[9] and $s^j : \{x^j\} \to P|_{x^j}/Q_G(\tau^j)$. Choose a stability parameter $(\tau^j)_{j[|S|]}$, $\tau^j \in \hat{T}^j_+$. Let $\lambda : \mathbb{C} \to G$ be a one-parameter subgroup and $(Q_G(\lambda), \chi_\lambda)$ the corresponding pair of a parabolic subgroup and a character. Let $\mathscr{R}^j = \mathscr{R}|_{x^j} : \{x^j\} \to P|_{x^j}/Q_G(\lambda)$ be a reduction of the structure group.[10] We will write $\mathscr{R}^j_{\mathrm{rep}}(x^j)$, $s^j_{\mathrm{rep}}(x^j)$ for (a choice of) representatives in $P|_{x^j}$, i. e. $[\mathscr{R}^j_{\mathrm{rep}}(x^j)] = \mathscr{R}^j(x^j) \in P|_{x^j}/Q_G(\lambda)$ and $[s^j_{\mathrm{rep}}(x^j)] = s^j(x^j) \in P|_{x^j}/Q_G(\tau^j)$. Then we find an element $g_j \in G : \mathscr{R}^j_{\mathrm{rep}}(x^j)g_j = s^j_{\mathrm{rep}}(x^j)$. Now we may shift the orbit $\mathscr{R}^j_{\mathrm{rep}}(x^j)Q_G(\lambda)$ by g_j, so that it intersects with $s^j(x^j)_{\mathrm{rep}}Q_G(\tau^j)$. The intersection of two Borel (and hence of two parabolic) subgroups always contains a maximal torus. Denote such a torus by $T^j \subset Q_G(\lambda) \cap g_j^{-1}Q_G(\tau^j)g_j$. Then we find elements $h_j \in Q_G(\tau^j)$, $h \in Q_G(\lambda)$ such that $g_j h_j \tau^j(\mathbb{C}^*)h_j^{-1}g_j^{-1}$, $h\lambda(\mathbb{C}^*)h^{-1} \subset T^j$. Let $\tau^{s^j_{\mathrm{rep}}} = g_j h_j \tau^j h_j^{-1} g_j^{-1}$ and $\lambda^{\mathscr{R}^j_{\mathrm{rep}}} = h\lambda h^{-1}$ be the corresponding one-parameter subgroups of T^j. Now we may define $\langle \tau^{s^j_{\mathrm{rep}}}, \lambda^{\mathscr{R}^j_{\mathrm{rep}}} \rangle$. Observe that $\langle \tau^{s^j_{\mathrm{rep}}}, \lambda^{\mathscr{R}^j_{\mathrm{rep}}} \rangle$ is independent of the choices made. In fact if $\mathscr{N}(T^j)$ denotes the normalizer of T^j, then h is unique up to an element of $\mathscr{N}(T^j) \cap Q_G(\lambda)$ and analogously for h^j. Using the faithful representation ι the $\mathbb{Z}^{\dim(U)}$-elements corresponding to $\tau^{s^j_{\mathrm{rep}}}$, $\lambda^{\mathscr{R}^j_{\mathrm{rep}}}$ are left invariant when conjugating with one of the available permutation matrices. Thus $\langle \tau^{s^j}, \chi_\lambda^{\mathscr{R}^j} \rangle = \langle \tau^{s^j}, \lambda^{\mathscr{R}^j} \rangle := \langle \tau^{s^j_{\mathrm{rep}}}, \lambda^{\mathscr{R}^j_{\mathrm{rep}}} \rangle$ is well-defined and depends only on the class $g_j \in Q_G(\tau^j)\backslash G/Q_G(\lambda)$.[11]

2.8. Proposition. *Fix a one-parameter subgroup τ^j as well as $\tau^j_{\mathrm{Gl}} = \iota \circ \tau^j$ for every $x^j \in S$. Let $(P,\ (s^j)_{j[|S|]})$ be a principal G-bundle and $(P_\iota,\ (s^j_{\mathrm{Gl}})_j)$ with*

$$s^j_{\mathrm{Gl}} : \{x^j\} \xrightarrow{s^j} P \times_X \{x^j\}/Q(\tau^j) \hookrightarrow P_\iota \times_X \{x^j\}/Q(\tau^j_{\mathrm{Gl}}),$$

$$\mathscr{R}^j_{\mathrm{Gl}} : \{x^j\} \xrightarrow{s^j} P \times_X \{x^j\}/Q(\lambda) \hookrightarrow P_\iota \times_X \{x^j\}/Q(\lambda_{\mathrm{Gl}})$$

for a one-parameter subgroup $\lambda : \mathbb{C}^* \to G$. Then

$$\langle \tau^{s^j}_{\mathrm{Gl}}, \chi^{\mathscr{R}^j}_{\iota\circ\lambda,\mathrm{Gl}} \rangle = \langle \tau^{s^j}, \chi^{\mathscr{R}^j}_\lambda \rangle,\ \forall 1 \le j \le |S|.$$

Proof. Obvious by definition of the inner product. See as well [HS10], 5.1.2. □

2.9. Let $\lambda : \mathbb{C}^* \to G$ be a one-parameter subgroup and χ_λ the associated anti-dominant character. Consider the principal $Q_G(\lambda)$-bundle $P \to P/Q_G(\lambda)$ and

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[8] By Borel, [Bo91] IV.11.3 Corollary, we know that maximal tori in G coincide with the maximal tori in the various Borel subgroups, and by IV.11.17 that every parabolic subgroup is conjugated to exactly one-parabolic subgroup containing a given Borel subgroup B.

[9] See [HS10], section 4.1 for an equivalent definition of $\hat{T}^j_+$.

[10] For an equivalent definition of reductions of the structure group see e. g. [KN63], I.5, fiber bundles.

[11] $\chi_\lambda^{\mathscr{R}^j}$ denotes the character to $\lambda^{\mathscr{R}^j}$.

$P_{Q_G(\lambda)}(\chi_\lambda)$ the χ_λ-associated line bundle on $P/Q_G(\lambda)$. Let $\mathscr{R} : X \to P/Q_G(\lambda)$ be a reduction and $P_{Q_G}(\chi_\lambda, \mathscr{R}) = \mathscr{R}^*(P_{Q_G(\lambda)}(\chi_\lambda))$.
Observe, that $\mathscr{R}$ extends to a reduction

$$\mathscr{R}_{\mathrm{Gl}} : X \to P/Q_G(\lambda) \hookrightarrow P_\iota/Q_{\mathrm{Gl}}(\iota \circ \lambda)$$

and that every parabolic subgroup $Q_{\mathrm{Gl}}(\iota \circ \lambda) \subset \mathrm{Gl}(U)$ by definition stabilizes a flag. Let $(F^j)_{j[r]}$ be the flag of rank $(r^j)_{j[m]}$ subbundles of $E = P_\iota$[12] induced by $\lambda_{\mathrm{Gl}} = \iota \circ \lambda$ and $(\gamma^j)_{j[r]}$ resp. $(\alpha^j)_{j[r]}$ the corresponding weights. Note that $(F^j)_{j[r]}$ depends on the reduction $\mathscr{R}_{\mathrm{Gl}}$. We get the following relation

$$\deg P_Q(\chi_\lambda, \mathscr{R}) = \deg P_{Q_{\mathrm{Gl}}}(\chi_{\lambda_{\mathrm{Gl}}}, \mathscr{R}_{\mathrm{Gl}}) = \sum_{j=1}^{m-1} \alpha^j (\deg(E)\,\mathrm{rk}(F^j) - \deg(F^j)\,\mathrm{rk}(E)).$$

Proof. Since we have a reduction of the structure group to $Q_G(\lambda)$ we find $Q_G(\lambda)$-valued transition function $(g^{ij})_{ij}$ of our principal G-bundle P ([KN63] Pro. 5.3 and Pro. 5.6.). If ι is our embedding of $G \hookrightarrow \mathrm{Sl}(U)$ we get the transition functions of $\mathscr{R}^*(P_{Q_G(\lambda)}(\chi_\lambda))$ as $(\chi_\lambda(\iota \circ g^{ij}))_{ij}$ w. r. t. the induced trivializations. $\iota \circ g^{ij} \in Q_{\mathrm{Sl}(U)}(\iota \circ \lambda)$ is a block upper triangular matrix of the form

$$\iota \circ g^{ij} = \begin{pmatrix} h_1^{ij} & & * \\ & \ddots & \\ 0 & & h_m^{ij} \end{pmatrix}.$$

Hence we have

$$\chi_\lambda(\iota \circ g^{ij}) = \prod_{k=1}^{m} \det(h_k^{ij})^{\gamma^k}.$$

On the other hand consider the vector bundle $\bigoplus_{k=1}^m (E \otimes (F^k)^\vee)^{\alpha^k r}$, where F^k is the subbundle with transition functions

$$H_k^{ij} = \begin{pmatrix} h_1^{ij} & & * \\ & \ddots & \\ 0 & & h_k^{ij} \end{pmatrix}.$$

The determinant of $\bigoplus_{k=1}^m (E \otimes (F^k)^\vee)^{\alpha^k r}$ has thus transition functions

$$\begin{aligned} \prod_{k=1}^{m} (\det(g^{ij})^{r^k} \cdot \det(((H_k^{ij})^t)^{-1})^r)^{\alpha^k r} &= \prod_{k=1}^{m} \left(\prod_{l=1}^{m} \det(h_l^{ij})^{\alpha^k r r^k} \cdot \prod_{l=1}^{k} (\det(h_l^{ij}))^{-r^2 \alpha^k} \right) \\ &= \prod_{l=1}^{m} (\det(h_l^{ij}))^{r \sum_{k=1}^m \alpha^k r^k - r^2 \sum_{k=l}^m \alpha^k}. \end{aligned}$$

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[12] More precisely: E the vector bundle corresponding to the $\mathrm{Gl}(U)-$bundle P_ι.

Using $\frac{\sum_{j=1}^{m} \gamma^j (r^j - r^{j-1})}{r} = 0$, $r^0 = 0$ from 1.3 we see that $r\sum_{k=1}^{m-1} \alpha^k r^k = \gamma^m r$. Furthermore $-r^2 \sum_{k=l}^{m-1} \alpha^k = -r(\gamma^m - \gamma^l) = -\gamma^m r + \gamma^l r$. Putting both formulas together we get $\prod_{l=1}^{m}(\det(h_l^{ij}))^{\gamma^l r}$. Therefore $\det(\bigoplus_{k=1}^{m}(E \otimes (F^k)^{\vee})^{\alpha^k r})$ and $\mathscr{R}^*(P_{Q_G(\lambda)}(\chi_\lambda))^{\otimes r}$ are isomorphic bundles and hence have the same degree, i. e.

$$\deg(\mathscr{R}^*(P_{Q_G(\lambda)}(\chi_\lambda))) = \sum_{k=1}^{m} \alpha^k \deg(E \otimes (F^k)^{\vee}) = \sum_{k=1}^{m} \alpha^k (\deg(E) r^k - r \deg(F^k)).$$

□

Remark. See as well [HS10] 5.1 or [GS05] by Tomás Gómez and Ignacio Sols, Lemma 5.6. for a proof in the case of a higher dimensional base variety.
Fritzsche, Grauert [FG02] or Kobayashi, Nomizu [KN63] give an excellent account of the connection between fiber bundles and transition functions on a Riemann surface X. An algebraic disussion of this relation is given for example in [Mi80]. As transition functions are particularly easy to work with, we will use this description again in section 3.5 as well as chapter 4. It should be mentioned however that some of our results can be proved without using cocycles.

Furthermore by the calculation in 1.38, 2.6, 2.8 and an embedding ι into $\mathrm{Sl}(U)$ we see that

$$\begin{aligned} &\sum_{k=1}^{m-1} \alpha^k \sum_{i=1}^{s^j} \beta^{ij} \left((r^{ij} - r^{i-1,j}) r^k - (r^{ijk} - r^{i-1,j,k}) r\right) \\ &= -\langle \tau^{s^j}, \chi_\lambda^{\mathscr{R}^j} \rangle, \quad \delta^{ij} = r\alpha^i(\tau_{\mathrm{Gl}}^{s^j}) = r\alpha^i(\tau^j), \quad \mathscr{R}^j = \mathscr{R}|_{\{x^j\}}{}^{13} \end{aligned}$$

is the parabolic contribution. More precisely, by 2.8 we get[14]

$$\begin{aligned} \langle \tau^{s^j}, \chi_\lambda^{\mathscr{R}^j} \rangle &= \sum_{i=1}^{s^j-1} \alpha^i(\tau^{s^j}) \cdot r \cdot \sum_{k=1}^{m} \gamma^k (r^k - r^{k-1} - (r^{ij,k} - r^{ij,k-1})) \\ &= -\sum_{i=1}^{s^j-1} \alpha^i(\tau^{s^j}) \cdot r \cdot \sum_{k=1}^{m} \alpha^k \cdot r(r^k - r^{ij,k}) \\ &= -\sum_{i=1}^{s^j-1} \alpha^i(\tau^{s^j}) \cdot r \cdot \sum_{k=1}^{m} \alpha^k \cdot (r(r^k - r^{ij,k}) - r^k(r - r^{ij})) \\ &= -\sum_{i=1}^{s^j-1} \delta^{ij} \cdot \sum_{k=1}^{m} \alpha^k \cdot (r^{ij} r^k - r^{ij,k} r). \end{aligned}$$

---

[13] $\alpha^i(\lambda^{s^j})$ is the α-weight of the one-parameter subgroup λ^{s^j}. Further note that the weights α^i are left invariant when conjugating the corresponding one-parameter subgroup, i. e. $\alpha^i(\tau_{\mathrm{Gl}}^{s^j}) = \alpha^i(\tau^j)$.

[14] Set $V_k = F^k|_{x^j}/F^{k-1}|_{x^j}$, $\chi_k = \gamma^k$ and $V^i = E^{ij}$ in 1.38.

Finally we use 1.8 for the transition to $(\beta^{ij})_{i[s^j]}$. Putting both results together we receive

$$\begin{aligned}&\deg P_Q(\chi_\lambda, \mathscr{R}) - \sum_{j:x^j \in S} \langle \tau^{s^j}, \chi_\lambda^{\mathscr{R}^j} \rangle \\ &= \sum_{k=1}^{m-1} \alpha^k (\text{par-deg}(E)\,\mathrm{rk}(F^k) - \text{par-deg}(F^k)\,\mathrm{rk}(E)),\end{aligned}$$

where λ corresponds to the filtration $(F^k)_{k[m]}$ plus the weights $(\alpha^k)_{k[m]}$ and τ^{s^j} to the filtration $(E^{ij})_{i[s^j]}$ plus the weights $(\delta^{ij}/r)_{i[s^j]}$ as above.

Remark. Note that occasionally in the literature τ is replaced by $-\tau$.

2.10. Definition. A stability parameter $\tau^j \in \hat{T}^j_+$ is called ι-admissible if the corresponding weights $r\alpha^i(\tau^j)$ are admissible, i. e. $r\sum_{i=1}^{s^j} \alpha^i(\tau^j) < 1$ holds for every $1 \leq j \leq |S|$. The definition extends to arbitrary representations $G \to \mathrm{Gl}(W)$ for some vector space W.

2.11. Definition. A parabolic principal G-bundle $(P,\ (s^j)_{j[|S|]})$ over the marked surface $(X,\ S)$ is called τ-semistable, if for every one-parameter subgroup $\lambda : \mathbb{C}^* \to G$ and every reduction $\mathscr{R} : X \to P/Q_G(\lambda)$

$$\deg P_Q(\chi_\lambda, \mathscr{R}) - \sum_{j:x^j \in S} \langle \tau^{s^j}, \chi_\lambda^{\mathscr{R}^j} \rangle \geq 0$$

holds.

Before we define a weight function for the Higgs field $\varphi : P_\varsigma \to L$ we should state a few general facts about the representations used. Consequentially we will be able to express the intrinsic definition of semistability in terms of the associated vector bundle and an associated homomorphism.

2.12. Let G be a reductive algebraic group. Then there is a representation $\iota : G \to \mathrm{Gl}(U)$ for a vector space U s. t. ι is a closed embedding (Borel, [Bo91], Corollary 1.4). Furthermore if $\varsigma : G \to \mathrm{Gl}(W)$, W vector space is another representation, then we find representations $\underline{\varsigma} : \mathrm{Gl}(U) \to \mathrm{Gl}(W)$ and $\tilde{\varsigma} : G \to \mathrm{Gl}(\tilde{U})$, $W = U \oplus \tilde{U}$ such that $\underline{\varsigma} \circ \iota = \varsigma \oplus \tilde{\varsigma}$ ([KP00], 5.4, Prop. 1).
Observe, that we can modify ι to $\iota' := \iota \oplus (\det^{-1} \circ \iota) : G \to \mathrm{Sl}(U \oplus \mathbb{C}) \subset \mathrm{Gl}(U \oplus \mathbb{C})$ which is still faithful.

2.13. Lemma. *Let $\iota : G \to \mathrm{Gl}(U)$ be a faithful representation, then there is a decomposition of U into G-modules U_a, $a \in A$ finite, s. t. $\iota(\mathscr{R}ad(G)) \subset \mathscr{Z}\left(\times_{a \in A} \mathrm{Gl}(U_a)\right)$, i. e. the radical maps to the center.*

Proof. $\mathscr{R}ad(G)$ is a torus and hence induces a decomposition $(U_a)_{a[|A|]}$ into eigenspaces to characters $\chi_{a,\mathscr{R}ad(G)} : \mathscr{R}ad(G) \to \mathbb{C}^*$ ([Bo91], Proposition before Definition 11.22). Since $\mathscr{R}ad(G) \subset \mathscr{Z}(G)$ we have for all $r \in \mathscr{R}ad(G)$, $\forall g \in G$, $\forall u_a \in U_a$

$$\begin{aligned}\iota(r, \iota(g, u_a)) &= \iota(rg, u_a) = \iota(gr, u_a) = \iota(g, \iota(r, u_a)) \\ &= \iota(g, \chi_{a,\mathscr{R}ad}(r)u_a) = \chi_{a,\mathscr{R}ad}(r)\iota(g, u_a).\end{aligned}$$

Therefore $\iota(g, u_a) \in U_a$, i. e. G preserves U_a and we have a decomposition of U into G-modules U_a. By definition $\iota(\mathscr{R}ad(G)) \subset \mathscr{Z}\left(\bigtimes_{a\in A} \mathrm{Gl}(U_a)\right)$ ([Sch08], 2.6.1). □

Notation. From now on let ι denote a faithful representation $G \hookrightarrow \mathrm{Gl}(U_a)_{a[|A|]} \cap \mathrm{Sl}(U)$, $U := \bigoplus_{a\in A} U_a$ (see 2.12 and 2.13).

2.14. Definition. A representation $\varsigma : H \to \mathrm{Gl}(W)$, $H = \mathrm{Gl}(U), \bigtimes_{a\in A} \mathrm{Gl}(U_a)$ is called polynomial, if the matrix coefficients ς^{ij} are polynomial functions. It is called rational if $\det^r \cdot \varsigma^{ij}$ is polynomial for some r. ς is called homogeneous of degree r if $\varsigma(z \cdot \mathrm{id}_U) = z^r \cdot \mathrm{id}_W$ resp. $\varsigma(z \cdot \mathrm{id}_{\bigtimes_{a\in A} U_a}) = z^r \cdot \mathrm{id}_W$. In particular homogeneous representations are rational.

Remark to 2.14. (i) When we talk about representations without further specification, we refer to rational representations.

(ii) The standard representation of $\mathrm{Gl}(U)$ on $U^{\otimes u}$ for a vector space U and an integer u is polynomial.

(iii) The definition is independent of the chosen basis of W.

(iv) The determinant representation $\det^{\otimes w} : \mathrm{Gl}(U) \to \mathbb{C}^*$ is polynomial for $w \geq 0$.

(v) The tensor product, the direct sum, exterior powers, symmetric powers, subrepresentations and quotient representations of polynomial (resp. rational) representations are polynomial (resp. rational).

(vi) The dual representation of a rational representation is rational. Every irreducible representation is homogeneous.

(vii) The representation $\underline{\varsigma}$ in 2.12 is rational by (ii)-(vi).

For more details see [KP00] sections 5.1 and 5.2.

2.15. Proposition. (i) *For every representation $\varsigma : \mathrm{Gl}(U) \to \mathrm{Gl}(W)$ there are integers u^j, v, w, $1 \leq j \leq v$ such that ς is direct summand of the standard representation*

$$\mathrm{Gl}(U) \to \mathrm{Gl}\left(\left(\bigoplus_{j=1}^{v} U^{\otimes u^j}\right) \otimes \left(\left(\bigwedge^{\dim U} U\right)^{\otimes w}\right)^{\vee}\right).$$

If ς is homogeneous, $u := u_j$, $\forall 1 \leq j \leq v$.

(ii) *Fix* $\kappa_a \in \mathbb{N}_+$, $a \in A$. *For every representation* $\varsigma : \bigtimes_{a\in A} \mathrm{Gl}(U_a) \to \mathrm{Gl}(W)$ *there are integers* u^j, v, w, $1 \leq j \leq v$ *such that* ς *is direct summand of the standard representation*

$$\bigtimes_{a\in A} \mathrm{Gl}(U_a) \to \mathrm{Gl}\left(\left(\bigoplus_{j=1}^{v}(U(\kappa_a))^{\otimes u^j}\right) \otimes \left(\left(\bigwedge^{\dim U(\kappa_a)} U(\kappa_a)\right)^{\otimes w}\right)^{\vee}\right),$$

$$U(\kappa_a) := \bigoplus_{a\in A} U_a^{\oplus \kappa_a}.$$

If ς *is homogeneous,* $u := u_j$, $\forall 1 \leq j \leq v$.

Proof. (i) is proved by the proposition in [KP00] 5.3 as well as in [CMS], Theorem 14.3. (ii) is precisely the statement of [KP00], 5.4, Proposition 1 already used in 2.12. Note that 2.12 and the remark to 2.14 provide us with a representation $\underline{\varsigma} : \mathrm{Gl}(U(\kappa_a)) \to \mathrm{Gl}(W)$ such that $\underline{\varsigma} \circ \underline{\iota} = \varsigma \oplus \tilde{\varsigma}$ for some suitable representation $\tilde{\varsigma}$ and $\underline{\iota} : \bigtimes_{a\in A} \mathrm{Gl}(U_a) \hookrightarrow \mathrm{Gl}(U(\kappa_a))$ an embedding.
The special property of homogeneous representations follows directly from the definition (of homogeneity). $\square$

2.16. Higgs Field. We still need to define a semistability condition for the Higgs field. Let ι be our faithful representation (cf. 2.12, 2.13) and $\underline{\varsigma}$ the corresponding homogeneous representation such that $\underline{\varsigma} \circ \iota = \varsigma \oplus \tilde{\varsigma}$ holds for some representation $\tilde{\varsigma}$ (cf. 2.12, remark to 2.14). Now $E_{\underline{\varsigma}} = P_{\underline{\varsigma}\circ\iota} = P_\varsigma \oplus P_{\tilde{\varsigma}}$ and $E = \bigoplus_{a\in A} E_a$ the tuple of vector bundles associated by ι. Consequentially the morphism $\varphi : P_\varsigma \to L$ induces a morphism $\varphi \circ \mathrm{pr}_1 : E_{\underline{\varsigma}} \to L$. We call $(E,\ \varphi,\ L)$: the pseudo $(\underline{\varsigma} \circ \iota)$-Higgs bundle induced from $(P,\ \varphi,\ L)$. Now we may extend φ by 2.15 to $(E^{\otimes u})^{\oplus v} \otimes (\det E^{\otimes w})^{\vee} = E_{\underline{\varsigma}} \oplus E_{\hat{\varsigma}}$ for yet another representation $\hat{\varsigma}$.

2.17. Note that given a one-parameter subgroup $\lambda : \mathbb{C}^* \to G$, ι as before, $\mathscr{R} : X \to P/Q_G(\lambda)$ a reduction and $\pi : P \to X$ the bundle projection, we can pull back the $Q_G(\lambda)$-bundle with projection $\pi_{\mathscr{R}}$ to a $Q_G(\lambda)$-bundle $Q_{\mathscr{R}}$ over X

$$\begin{tikzcd}
Q_{\mathscr{R}} \arrow[r] \arrow[d] & P \arrow[dl, "\pi"'] \arrow[d, "\pi_{\mathscr{R}}"] \\
X \arrow[r, "\mathscr{R}"'] & P/Q_G(\lambda).
\end{tikzcd}$$

Observe that $(Q_{\mathscr{R}})_{\varsigma|_{Q_G(\lambda)}} \simeq P_\varsigma$.[15] Now $Q_{\mathrm{Gl}(W)}(\varsigma \circ \lambda)$ induces a filtration $(F_\varsigma^k)_{k[m]}$ of P_ς. As $\varphi \neq 0$

$$\mu(\lambda, \mathscr{R}, \varphi) := -\min\{\gamma^j |\ \varphi|_{F_\varsigma^j} \neq 0,\ 1 \leq j \leq m\},$$

---

[15] P admits local trivializations with $Q_G(\lambda)$-valued transition functions.

is well-defined.
On the other hand λ induces a one-parameter subgroup $\iota \circ \lambda : \mathbb{C}^* \to \mathrm{Gl}(U_a)_a := \times_{a \in A} \mathrm{Gl}(U_a)$ with associated filtration $(F^j)_{j[m]}$, $F^j \subset E$. Hence using 2.15 we obtain a filtration $(\bigotimes_j^u F^{i_j})^{\oplus v}$ of $(E^{\otimes u})^{\oplus v}$. The two filtrations $((\bigotimes_j^u F^{i_j})^{\oplus v} \cap P_\varsigma)_i$ and $(F_\varsigma^k)_k$ identify under the identification $(E^{\otimes u})^{\oplus v} \otimes (\det E^{\otimes w})^\vee = P_\varsigma \oplus P_{\tilde{\varsigma}} \oplus E_{\hat{\varsigma}}$. Thus $\mu(F^j,\ \alpha^j,\ \mathrm{pr}_1 \circ \varphi) = \mu(\lambda,\ \mathscr{R},\ \varphi)$. To simplify notation in future we will usually omit the projection pr_1.

2.18. We may further include the choice of a character ξ of G into the semistability concept as follows: a character of G induces a character of the radical $\mathscr{R}ad(G)$. If ι as in 2.16 maps the radical $\mathscr{R}ad(G)$ to $\mathscr{Z}(\times_{a \in A} \mathrm{Gl}(U_a))$, then $\xi|_{\mathscr{R}ad(G)}$ comes from a character of $\mathscr{Z}(\times_{a \in A} \mathrm{Gl}(U_a)) \subset \times_{a \in A} \mathrm{Gl}(U_a)$, therefore from a choice of rational numbers ξ_a with $\sum_{a \in A} \xi_a r_a = 0$.[16] The identical calculation as in the parabolic case shows that for every one-parameter subgroup λ of G with associated weighted flag $(F^k, \alpha^k)_k$, $\mathrm{rk}(F^k) = r^k$, $\mathrm{rk}(F_a^k) = r_a^k$: $\langle \lambda, \xi \rangle = \sum_{k=1}^m \alpha^k \sum_{a \in A} \xi_a (r_a r^k - r r_a^k) = -\sum_{k=1}^m \alpha^k \sum_{a=1}^r \xi_a r r_a^k$.

2.19. Definition. Let Y be a scheme of finite type over $\mathbb{C}$, $\mathscr{P}^l \to \mathrm{Jac}^l \times X$ a Poincaré line bundle and τ^j fixed parabolic weights to given parabolic subgroups $P^j \subset G$. A Y-family of projective ς-Higgs bundles (of given topological type (ϑ, l)) is a tuple $(\mathscr{P}_Y,\ (s_Y^j)_{j[|S|]},\ \varphi_Y,\ \upsilon_Y,\ \mathscr{H}_Y)$ where

1. $\mathscr{P}_Y$ is principal G bundle (of topological type ϑ) over every point $\{y\}$.
2. $\upsilon_Y : Y \to \mathrm{Jac}^l$ is a morphism, $\mathscr{H}_Y \to Y$ a line bundle.
3. $\varphi_Y : \mathscr{P}_{Y,\varsigma} \to (\upsilon_Y \times \mathrm{id}_X)^*(\mathscr{P}^l) \otimes \pi_Y^*(\mathscr{H}_Y)$ is a homomorphism non-trivial on fibers over $y \in Y$.
4. $s_Y^j : Y \times \{x^j\} \to \mathscr{P}_Y \times_X (Y \times \{x^j\})/Q_G(\tau^j)$ for all $x^j \in S$.

An isomorphism of projective Y-families is an isomorphism of the underlying principal G-bundles that extends in the natural way to the associated objects such that it commutes with an isomorphism of the line bundles $\mathscr{H}_Y$.

2.20. Definition. A parabolic principal ς-Higgs bundle $(P,\ (s^j)_{j[|S|]},\ \varphi,\ L)$ over the marked surface $(X,\ S)$ is called $(\xi,\ \tau,\ \delta)$-(semi)stable, if for every one-parameter subgroup $\lambda : \mathbb{C}^* \to G$ and every reduction $\mathscr{R} : X \to P/Q_G(\lambda)$

$$\deg P_Q(\chi_\lambda, \mathscr{R}) - \sum_{j : x^j \in S} \langle \tau^{s^j}, \chi_\lambda^{\mathscr{R}^j} \rangle + \delta \mu(\lambda, \mathscr{R}, \varphi) + \langle \lambda, \xi \rangle \quad (\geq) \quad 0.$$

---

[16] $\iota(\mathscr{R}ad(G))$ is a torus, hence identifies with $(\mathbb{C}^*)^m$, thus ι looks component-by-component as $\prod_i z_i^{a_{ij}}$. Now finding a character that extends ξ equals solving an inhomogeneous system of linear equations with a highest rank matrix $A = (a_{ij})_{i[m]j[m]}$.

Given a faithful representation $\iota : G \hookrightarrow \mathrm{Gl}(U_a)_a$ as before a parabolic principal ς-Higgs bundle $(P,\ (s^j)_{j[|S|]},\ \varphi,\ L)$ is $(\xi,\ \tau,\ \delta)$-semistable if and only if for every one-parameter subgroup $\lambda : \mathbb{C}^* \to G$ and every reduction $\mathscr{R} : X \to P/Q_G(\lambda)$

$$\sum_{j=1}^{m-1} \alpha^j(\text{par-deg}(E)\,\mathrm{rk}(F^j) - \text{par-deg}(F^j)\,\mathrm{rk}(E)) + \delta \cdot \mu(F^j, \alpha^j, \mathrm{pr}_1 \circ \varphi) - \\ - \sum_{k=1}^{m} \alpha^k \sum_{a=1}^{r} \xi_a\,\mathrm{rk}(E)\,\mathrm{rk}(F^k_a) \geq 0.$$

2.2. Principal Bundles as Higgs Tuples

Let P be a principal G-bundle and $\iota : G \hookrightarrow \mathrm{Gl}(U_a)_{a[|A|]} \cap \mathrm{Sl}(U),\ U := \bigoplus_{a\in A} U_a$ our faithful representation. Let $(E_a)_{a[|A|]}$ be the ι-associated Higgs tuple P_ι. The corresponding $\mathrm{Gl}(U_a)_a$-bundle is retained from $(E_a)_{a[|A|]}$ as $\mathscr{I}som(U_a, E_a)_{a[|A|]} := \mathscr{I}som(U_1 \otimes \mathcal{O}_X, E_1) \times_X \ldots \times_X \mathscr{I}som(U_{|A|} \otimes \mathcal{O}_X, E_{|A|})$. We get the following result:

2.21. Proposition. *The groupoid of principal G-bundles is isomorphic to the groupoid of pairs consisting of a tuple $(E_a)_{a[|A|]}$ and a section $s : X \to \mathscr{I}som(U_a, E_a)_{a[|A|]}/G$ where*

$$\begin{tikzcd}
P \arrow[rr, hook] \arrow[dr] & & \mathscr{I}som(U_a, E_a)_{a[|A|]} \arrow[dl] & \mapsto & P/G \arrow[rr, "s"] \arrow[dr, leftrightarrow, "\simeq"'] & & \mathscr{I}som(U_a, E_a)_{a[|A|]}/G. \arrow[dl] \\
 & X & & & & X &
\end{tikzcd}$$

P is retained as pullback of $\mathscr{I}som(U_a, E_a)_{a[|A|]} \to \mathscr{I}som(U_a, E_a)_{a[|A|]}/G$ via s.

Remark. (i) The determinant of $E = \bigtimes_{a\in A} E_a$ is trivial: If $(g^{ij})_{ij}$ are the transition functions of P, then $\det(E)$ has transition functions $(\det(\iota(g^{ij})))_{ij} = 1$ since ι maps to $\mathrm{Sl}(U)$. Note that for a semisimple group G there are no non-trivial characters, so for every faithful representation $\det(E) \simeq \mathcal{O}_X$.

(ii) Observe that $\mathscr{I}som(U_a, E_a)_{a[|A|]}/G \simeq P_\xi$ with $\xi : G \times \bigtimes_{a\in A} \mathrm{Isom}(U_a, \mathbb{C}^{r_a})/G \to \bigtimes_{a\in A} \mathrm{Isom}(U_a, \mathbb{C}^{r_a})/G,\ (g, (\bigtimes_{a\in A} B_a)G) \mapsto (\iota(g)(\bigtimes_{a\in A} B_a))G$.

2.22. Definition. Define $\mathscr{H}om(U_a, E_a)_{a[|A|]} = \mathscr{H}om(U_1 \otimes \mathcal{O}_X, E_1) \times_X \ldots \times_X \mathscr{H}om(U_{|A|} \otimes \mathcal{O}_X, E_{|A|}) = \mathrm{Spec}(\mathscr{S}ym^*(\bigoplus_{a\in A} U_a \otimes E_a^\vee))$ and a pseudo principal G-bundle as a pair of a tuple $(E_a)_{a[|A|]}$ with $\det\left(\bigoplus_{a\in A} E_a\right) = \mathcal{O}_X$ plus a section $s : X \to \mathscr{H}om(U_a, E_a)_{a[|A|]} /\!\!/ G$. Equivalently a pseudo G-bundle may be viewed as a pair of a tuple $(E_a)_{a[|A|]}$ plus a morphism $\tau : \mathscr{S}ym^*(\bigoplus_{a\in A} U_a \otimes E_a^\vee)^G \to \mathcal{O}_X$. It

is further required that s is not the trivial section resp. that τ is not the projection to the zero component of the graded sheaf $\mathscr{S}ym^*(\bigoplus_{a\in A} U_a \otimes E_a^\vee)^G$.
Let $P^j_{\mathrm{Gl}(U_a)_a}$, $1 \leq j \leq |S|$ be parabolic subgroups of $\mathrm{Gl}(U_a)_a$. We call a pair consisting of a pseudo G-bundle $((E_a)_{a[|A|]}, s)$ and a tuple of reductions $(s^j)_{j[|S|]} : \{x^j\} \to (\bigtimes_{a\in A} \mathscr{I}som(U_a \otimes \mathcal{O}_X, E_a) \times_X \{x^j\}) \,/ P^j_{\mathrm{Gl}(U_a)_a}$ a parabolic pseudo principal G-bundle. Equivalently a parabolic pseudo G-bundle $(E_a,\ (E_a^{ij})_{i[s_a^j]j[|S|]})_{a[|A|]}$ may be defined by adding parabolic filtrations $(E_a^{ij})_{i[s_a^j]}$ of fixed type $(r_a^{ij})_{i[s_a^j]}$ of $E_a|_{x^j}$ for each puncture $x^j \in S$.

We have the following result:

2.23. Lemma. ([Sch08], 2.6.3.1) *Let $((E_a)_{a[|A|]}, s)$ be a pseudo G-bundle. Then $(E_a)_{a[|A|]} = P_\iota$ for a principal G-bundle P if and only if there is a point $x \in X$ such that $s(x) \in \mathscr{I}som(U_a, E_a)_{a[|A|]}/G$.*

Proof. The local components of s satisfy $s^i\iota(G) = g^{ij}s^j\iota(G)$ for the $\mathrm{Sl}(U)$-valued transitions functions $(g^{ij})_{ij}$. Hence we get a global function $\det\circ s : X \to \mathbb{C}$ that is constant on the compact Riemann surface X ([For81], chapter I, 2.8). In particular $s(x) \in \mathscr{I}som(U_a, E_a)_{a[|A|]}/G$ for one x if and only if $s(x) \in \mathscr{I}som(U_a, E_a)_{a[|A|]}/G$ for all $x \in X$. □

2.24. A Y-family of parabolic pseudo G-bundles (for a scheme Y) is defined as the obvious extension of $((E_a)_{a[|A|]},\ \tau,\ (s^j)_{j[|S|]})$ to $Y \times X$ requiring that on $\{y\} \times X$ we retain a parabolic pseudo G-bundle as defined above. In order to use the construction of the moduli space of Higgs tuples it will be of particular importance to relate the two concepts. In a first step we are going to choose $u,\ v,\ w$ suitably.
Start by recalling [MRed], III, §8 which shows that we can choose d such that

$$\mathrm{Sym}^*(\bigoplus_{a\in A} \mathrm{Hom}(\mathbb{C}^{r_a}, U_a))^G$$

is generated by m generators of pairwise different degree $d_1, \ldots, d_m \leq d$ with $d = m \cdot \mathrm{lcm}(d_i :\ 1 \leq i \leq m)$[17] and

$$\mathrm{Sym}(d) = \bigoplus_{j\geq 0} \mathrm{Sym}^{jd}(\bigoplus_{a\in A} \mathrm{Hom}(\mathbb{C}^{r_a}, U_a))^G$$

is generated by elements of $\mathrm{Sym}(d)_1 = \mathrm{Sym}^d(\bigoplus_{a\in A} \mathrm{Hom}(\mathbb{C}^{r_a}, U_a))^G$. Therefore we find a surjective morphism $\mathbb{U}_d \to \mathrm{Sym}(d)$ with $\mathbb{U}_d =$

---

[17] If there is no chance of confusion we will sometimes write $\mathrm{lcm}(d_i)$ rather than $\mathrm{lcm}(d_i :\ 1 \leq i \leq m)$.

$\bigoplus_{\substack{(e_j)_{j[m]}\in\mathbb{Z}^m_{\geq 0}\\ \sum_{j=1}^m e_jd_j=d}} \bigotimes_{j=1}^m \mathrm{Sym}^{e_j}(\mathrm{Sym}^{d_j}(\bigoplus_{a\in A}\mathrm{Hom}(\mathbb{C}^{r_a}, U_a))^G)$. Since $\mathbb{U}_d$ is homogeneous of degree d as a $\mathrm{Gl}(U_a)_a$-module we find for every choice of $\kappa_a \in \mathbb{Z}$ integers u_{ps}, v_{ps}, w_{ps} such that $\mathbb{U}_d \subset ((\bigoplus_{a\in A}\mathbb{C}^{\kappa_a})^{\otimes u_{\mathrm{ps}}})^{\oplus v_{\mathrm{ps}}} \otimes \bigwedge^r(\bigoplus_{a\in A}\mathbb{C}^{\kappa_a})^{\vee,\otimes w_{\mathrm{ps}}}$. The induced surjective morphism of Y families is

$$\tilde{\varphi} : (\mathscr{E}_Y^{\otimes u_{\mathrm{ps}}})^{\oplus v_{\mathrm{ps}}} \otimes \det(\mathscr{E}_Y)^{\vee,\otimes w_{\mathrm{ps}}} \to \mathscr{S}ym^d(\bigoplus_{a\in A}\mathrm{Hom}(\mathscr{E}_{Y,a}, U_a \otimes \mathcal{O}_{Y\times X}))^G.$$

Therefore we may associate to a pseudo G-bundle $((\mathscr{E}_{Y,a})_{a[|A|]},\ \tau_Y)$ a tuple consisting of vector bundles $(\mathscr{E}_{Y,a})_{a[|A|]}$, a morphism $\varphi_{Y,\tau} = (\tau_Y^* \otimes \mathrm{id}_{\det(\mathscr{E}_Y)^{\otimes w}}) \circ (\tilde{\varphi} \otimes \mathrm{id}_{\det(\mathscr{E}_Y)^{\otimes w}})$, υ_Y the constant function on Y with value $[\mathcal{O}_X] \in \mathrm{Jac}^0$ and $\mathscr{H}_Y$ the invertible sheaf such that $(\upsilon_Y \times \mathrm{id}_X)^*(\mathscr{P}) = \pi_Y^*(\mathscr{H}_Y)$, $\mathscr{P}$ Poincaré line bundle on $\mathrm{Jac}^0 \times X$. ([Ha77], III. Ex. 12.4). For further use denote the representation such that $E_{\varsigma_{\mathrm{ps}}} = (E^{\otimes u_{\mathrm{ps}}})^{\oplus v_{\mathrm{ps}}} \otimes \det(E)^{\vee,\otimes w_{\mathrm{ps}}}$ as constructed above by ς_{ps}.

The astonishing feature of this construction is that it is not only an injection on isomorphism classes, but that it will allow us to relate the semistability concepts of G-bundles and Higgs tuples.

2.25. Proposition. ([Sch08], 2.6.3.2)

(i) *The map*

$$\left\{\begin{array}{c}\textit{Isomorphism classes of}\\ \textit{pseudo } G\textit{-bundles}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\textit{Isomorphism classes}\\ \textit{of Higgs tuples}\end{array}\right\}$$
$$((E_a)_{a[|A|]},\ \tau) \longmapsto ((E_a)_{a[|A|]},\ \varphi_\tau,\ \mathcal{O}_X)$$

is one-to-one (on isomorphism classes).

(ii) *A pseudo principal G-bundle $((E_a)_{a[|A|]}, \tau)$ is a principal G-bundle if and only if $\mu(F^i, \alpha^i, \varphi_\tau) \geq 0$ holds for every weighted filtration $(F^i, \alpha^i)_{i[m]}$.*

Proof. (i) Consider two pseudo G-bundles $((E_a)_{a[|A|]},\ \tau)$ and $((E_a)_{a[|A|]},\ \tilde{\tau})$ that induce the same Higgs tuple. First note, that τ is defined by the components τ_{d_j}. Then by construction we have for all $(e_j)_j \in \mathbb{Z}^m_{\geq 0}$, $\sum_j e_jd_j = d$

$$\bigotimes_{j=1}^m \mathscr{S}ym^{e_j}(\tau_{d_j}) = \bigotimes_{j=1}^m \mathscr{S}ym^{e_j}(\tilde{\tau}_{d_j}). \tag{Sym 1}$$

Restriction to the generic point implies that there is a $\left(\frac{d}{d_j}\right)$th root of unity $\zeta_{d_j} = e^{2\pi i\varphi_j}$, $\varphi_j = \frac{k_jd_j}{d}$, $k_j \in \mathbb{Z}$ such that $\tau_{d_j} = \zeta_{d_j}\tilde{\tau}_{d_j}$.

In (Sym 1) we read that $\sum_{i=1}^{m} e_i\varphi_i = b$ for some $b \in \mathbb{Z}$. Now let $(e_i)_i$ be such that b takes its maximal value B. Then

$$\sum_{i=1}^{m} e_i d_i k_i = Bd, \ \sum_{i=1}^{m} e_i d_i = d \Rightarrow \sum_{i=1}^{m} e_i d_i \left(1 - \frac{k_i}{B}\right) = 0$$

and $k_i \leq B$ for all $1 \leq i \leq m$ imply that $k_i = B$ for all $1 \leq i \leq m$.
If there is an i_0 such that $k_{i_0} > B$, then consider the case $\tilde{e}_{i_0} = \frac{d}{d_{i_0}}$ and $\tilde{e}_l = 0$ if $l \neq i_0$. For $\sum_{i=1}^{m} \tilde{e}_i\varphi_i = \tilde{b}$ we get the contradiction $1 - \frac{k_{i_0}}{\tilde{b}} = 0$ since $k_{i_0} > B \geq \tilde{b}$. Thus all k_i are the same, namely equal to B, and hence $\zeta = e^{2\pi i\varphi}$, $\varphi = \frac{B}{d}$ is a dth root of unity with $\zeta^{d_i} = \zeta_{d_i}$.
Therefore the isomorphism $\zeta \cdot \mathrm{id}_{(E_a)_{a[|A|]}}$ maps $((E_a)_{a[|A|]},\ \tau)$ to the pseudo G-bundle whose morphism coincides with $\tilde{\tau}$ over the generic point and hence everywhere.

(ii) The proof of part (ii) is described in Chapter 2.6 of [Sch08]. The weight function might be written in terms of the stalk over the generic point $\eta \in X$ and lemma 2.23[18] gives a local criterion for a pseudo G-bundle to be a principal G-bundle. [Sch08], 2.6.2.1 identifies principal G-bundles with $((\bigtimes_{a\in A} \mathrm{Gl}(\mathbb{C}^{r_a}) \cap \mathrm{Sl}(\mathbb{C}^{r}))$-semistable points[19] in $(\mathscr{I}som(U_a, E_a)_{a[|A|]}/G)|_\eta$. Finally (semi)stability is preserved by the transition s (resp. τ) to φ_τ ([Sch08], 2.6.2.3).

□

2.26. Let $E = \bigoplus_{a\in A} E_a^{\oplus\kappa_a}$ and $\varphi : E_{\underline{\varsigma}} \to L$ be a non-trivial homomorphism for a homogeneous representation $\underline{\varsigma}$; let $(E_a^{ij})_{i[s_a^j]}$ be parabolic filtrations of $E_a|_{x^j}$. Denote by ς_{ps} the representation corresponding to our pseudo G-bundle $(E,\ \tau)$. We call the tuple $((E_a,\ (E_a^{ij})_{i[s_a^j]j[|S|]})_{a[|A|]},\ \tau,\ \varphi,\ L)$ a parabolic pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundle. It is $(\xi_a,\ \delta_{\mathrm{ps}},\ \varepsilon_{\mathrm{ps}})$-(semi)stable if

$$M_{\mathrm{par}}^{\kappa,\xi}(F^k, \alpha^k) + \delta_{\mathrm{ps}} \cdot \mu(F^k, \alpha^k, \varphi) + \varepsilon_{\mathrm{ps}} \cdot \mu(F^k, \alpha^k, \varphi_\tau) \geq 0, \quad \delta_{\mathrm{ps}},\ \varepsilon_{\mathrm{ps}} \in \mathbb{Q}_+$$[20]

holds for every filtration $(F^k, \alpha^k)_{k[m]}$. Two pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundles are isomorphic if there is an isomorphism of the underlying vector bundles that extends to an isomorphism of pseudo G-bundles as well as to an isomorphism of Higgs tuples on the underlying Higgs tuples.

Remark. From now on fix $\kappa_a = 1$ for all $a \in A$. Recall that we still have the choice of a faithful representation ι. Now to fix $\kappa_a = 1$, $\forall a \in A$ and $\iota = \bigtimes_{a\in A} \iota_a^{\tilde{\kappa}_a}$ is the same as to choose $\tilde{\kappa}_a \in \mathbb{Z}$ and $\tilde{\iota} = (\bigtimes_{a\in A} \iota_a)$. For that reason we will stick with the notationally easier choice and fix $\kappa_a = 1$.

---

[18] see 2.21.(ii) as well.

[19] w. r. t. the natural action on $\left(\bigoplus_{a\in A} E_a\right)|_\eta$.

[20] Recall $\xi_a \in \mathbb{Q},\ \kappa_a \in \mathbb{Z}_+$.

In order to relate the semistability concepts for pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundles and Higgs tuples we need to combine the two μ-weight functions in the definition of semistability above in the correct way in order to obtain the semistability condition for Higgs tuples. Therefore consider the representation $\underline{\varsigma}^{\otimes b}\otimes\varsigma_{\mathrm{ps}}^{\otimes c}$ and the induced morphism

$$\varphi^{b,c}_{\mathrm{tuple}} := \varphi^{\otimes b}\otimes\varphi_\tau^{\otimes c} : E_{\underline{\varsigma}}^{\otimes b}\otimes E_{\varsigma_{\mathrm{ps}}}^{\otimes c}\to L^{\otimes b}\otimes\mathcal{O}_X^{\otimes c}\simeq L^{\otimes b}.$$

Then

2.27. Proposition. ([Sch08], Pro. 2.7.2.2, 2.7.2.3.) *Fix a parabolic pseudo* $(\underline{\varsigma}\circ\iota)$*-Higgs bundle* $\mathbf{E}=((E_a,\ (E_a^{ij})_{i[s_a^j]j[|S|]})_{a[|A|]},\ \tau,\ \varphi,\ L)$ *with associated Higgs tuple* $\mathbf{T}^{b,c}=((E_a,\ (E_a^{ij})_{i[s_a^j]j[|S|]})_{a[|A|]},\varphi^{b,c}_{\mathrm{tuple}},L)$*.*

(i) *For every weighted filtration* $(F^k,\alpha^k)_{k[m]}$ *as in 1.6*

$$\mu(F^k,\alpha^k,\varphi_{\mathrm{tuple}}) = b\cdot\mu(F^k,\alpha^k,\varphi)+c\cdot\mu(F^k,\alpha^k,\varphi_\tau).$$

(ii) *Let* $\delta,\ \delta_{\mathrm{ps}},\ \varepsilon_{\mathrm{ps}}$ *be such that* $\delta_{\mathrm{ps}}/\delta,\ \varepsilon_{\mathrm{ps}}/\delta\in\mathbb{Z}_+$*. Then* $\mathbf{E}$ *is* $(\xi_a,\ \delta_{\mathrm{ps}},\ \varepsilon_{\mathrm{ps}})$*-(semi)stable if and only if* $\mathbf{T}^{\delta_{\mathrm{ps}}/\delta,\ \varepsilon_{\mathrm{ps}}/\delta}$ *is* $(\xi_a,\ \delta)$*-(semi)stable.*

(iii) *Let* $\delta,\ \delta_{\mathrm{ps}},\ \varepsilon_{\mathrm{ps}}$ *be as in (ii). Then* $\mathbf{E}\mapsto\mathbf{T}^{\delta_{\mathrm{ps}}/\delta,\ \varepsilon_{\mathrm{ps}}/\delta}$ *is one-to-one.*

Proof. (i) This is a direct consequence of the definitions. We skip the details and refer to 3.13 for a similar argument, or to [Sch08], remark 2.7.2.1.

(ii) Obvious from part (i).

(iii) Follows from 2.25, part (i).

□

2.28. From the definition of semistability of parabolic principal $(\underline{\varsigma}\circ\iota)$-Higgs bundles we see immediately: If a pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundle is semistable, so is the underlying principal $(\underline{\varsigma}\circ\iota)$-Higgs bundle. For the reverse statement we need to find a criterion, in order to decide which weighted filtrations of the associated vector bundle E come from a reduction $\mathscr{R}:X\to P/Q_G(\lambda)$ to a one-parameter subgroup λ of G. Therefore we will need the following result by Alexander Schmitt:

2.29. Proposition. ([Sch08], 2.6.3.4.) *For a principal G-bundle with associated Higgs tuple* $(E,\ \varphi_\tau)$ *and a weighted filtration* $(F^k,\alpha^k)_{k[m]}$ *the following conditions are equivalent*

1. $\mu(F^k,\alpha^k,\varphi_\tau)=0$*;*
2. *there is a one-parameter subgroup* λ *of* G *and a corresponding reduction* $\mathscr{R}:X\to P/Q_G(\lambda)$ *such that* $(F^k,\alpha^k)_{k[m]}$ *coincides with the filtration induced by the pair* $(\lambda,\ \mathscr{R})$*.*

This criterion will be used in 3.5.

2.3. THE MODULI SPACE IN THE NON-PARABOLIC CASE

We will shortly review the construction in [Sch08] of a parameter scheme $\mathfrak{B}$ that parametrizes non-parabolic pseudo G-bundles. The idea is related to the one used to construct a parameter scheme $\mathfrak{T}_{\mathrm{par}}$ for tuples, however since we have to account for a homomorphism $\tau : \mathscr{S}ym^*(\bigoplus_{a \in A} U_a \otimes E_a^\vee) \to \mathcal{O}_X$ rather than a homomorphism $\varphi : E_\varsigma \to L$, the involved spaces differ.

2.30. Fix a $n \in \mathbb{N}$ as in the construction of the moduli space of tuples. By 2.27 we see that the pseudo G-bundles that occur, live in a bounded family. Hence we may find an open subset $\mathfrak{Q}_a$ of a Quot scheme $\overline{\mathfrak{Q}}_a$, such that $\mathfrak{Q}_a$ parametrizes quotients $q_a : V_a \otimes \mathcal{O}_X(-n) \to E_a$ with $H^0(q(n))$ an isomorphism and E_a a vector bundle of rank r_a and fiberwise degree d_a. Denote by $\mathscr{E}_{\mathfrak{Q}} = \bigoplus_{a \in A} \pi_{\mathfrak{Q}_a}^*(\mathscr{E}_{\mathfrak{Q}_a})$ the universal bundle over $\mathfrak{Q}$ where $\mathfrak{Q}$ is the fiber over $[\mathcal{O}_X]$ in $\bigtimes_{a \in A} \mathfrak{Q}_a$ of the morphism $\mathfrak{Q} \to \mathrm{Jac}^0,\ (q_a : V_a \otimes \mathcal{O}_X(-n) \to E_a)_{a \in A} \mapsto [\det(E)],\ E = \bigoplus_{a[|A|]} E_a$. Denote by $V = \bigoplus_{a \in A} V_a$ as well as $U = \bigoplus_{a \in A} U_a$.
As before we find a line bundle $\mathscr{L}$ on $\mathfrak{Q}$ such that $\det(\mathscr{E}_{\mathfrak{Q}})^\vee \simeq \pi_{\mathfrak{Q}}^*(\mathscr{L})$.[21] Now by $\mathscr{E}_{\mathfrak{Q}}^\vee \simeq \bigwedge^{r-1} \mathscr{E}_{\mathfrak{Q}} \otimes \det(\mathscr{E}_{\mathfrak{Q}})^\vee$ we find the surjective $\mathcal{O}_X$−algebra morphism

$$\begin{aligned}\psi^* : \mathscr{S}ym^*(U \otimes \bigwedge^{r-1}(V \otimes \pi_X^*(\mathcal{O}_X(-n))) \otimes \pi_{\mathfrak{Q}}^*(\mathscr{L}))^G \\ \to \mathscr{S}ym^*(\mathscr{H}om(\mathscr{E}_{\mathfrak{Q}}, U \otimes \mathcal{O}_{\mathfrak{Q} \times X}))^G \\ \to \mathscr{S}ym^*(\bigoplus_{a \in A}(\pi_{\mathfrak{Q}_a} \times \mathrm{id}_X)^* \mathscr{H}om(\mathscr{E}_{\mathfrak{Q}_a}, U_a \otimes \mathcal{O}_{\mathfrak{Q}_a \times X}))^G.\end{aligned}$$

Next using $\mathscr{S}ym^*(U \otimes \bigwedge^{r-1}(V \otimes \pi_X^*(\mathcal{O}_X(-n))))^G \simeq \mathrm{Sym}^*(U \otimes \bigwedge^{r-1} V)^G \otimes \mathcal{O}_X(* \cdot (1-r)n)$ we find a big enough number s[22] such that every $\tau : \mathscr{S}ym^*(\bigoplus_{a \in A} \mathrm{Hom}(E_a, U_a \otimes \mathcal{O}_X))^G \to \mathcal{O}_X$ comes from an element of

$$\mathfrak{Y} = \bigoplus_{k=1}^{s} \mathscr{H}om(\mathscr{S}ym^k(U \otimes \bigwedge^{r-1} V \otimes \pi_{\mathfrak{Q}}^*(\mathscr{L}))^G, H^0(\mathcal{O}_X(k(r-1)n)) \otimes \mathcal{O}_{\mathfrak{Q}}).$$

Combining the universal homomorphisms $\tilde{\tau}^k : \mathscr{S}ym^k(U \otimes \bigwedge^{r-1} V \otimes \pi_{\mathfrak{Q}}^*(\mathscr{L}))^G \to H^0(\mathcal{O}_X(k(r-1)n)) \otimes \mathcal{O}_{\mathfrak{Y} \times X}$[23] to a morphism

$$\begin{aligned}\hat{\tau} : \mathscr{U} := \bigoplus_{k=1}^{s} \mathscr{S}ym^k(U \otimes \bigwedge^{r-1}(V \otimes \pi_X^*(\mathcal{O}_X(-n))) \otimes \pi_{\mathfrak{Q}}^*(\mathscr{L}))^G \\ \to H^0(\mathcal{O}_X(k(r-1)n)) \otimes \mathcal{O}_{\mathfrak{Y} \times X} \otimes \pi_X^*(\mathcal{O}_X(-k(r-1)n)) \to \mathcal{O}_{\mathfrak{Y} \times X}\end{aligned}$$

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[21] Observe that $(\det \times \mathrm{id}_X)^*(\mathscr{P}^0) \simeq \mathcal{O}_{\mathfrak{Q} \times X}$ by the universal property of the Poincaré bundle.

[22] [MRed], §8.

[23] $\pi : \mathfrak{Y} \to \mathfrak{Q}$ the natural projection.

induces a homomorphism $\check{\tau} : \mathscr{S}ym^*(\mathscr{U}) \to \mathcal{O}_{\mathfrak{Y}\times X}$ and after combination with ψ^* a homomorphism $\check{\tau}_{\mathfrak{Y}}$ in

$$\begin{tikzcd}
\mathscr{S}ym^*(\mathscr{U}) \arrow[rr, "\check{\tau}_{\mathfrak{Y}}"] \arrow[dr, "\check{\psi}^*"'] & & \mathcal{O}_{\mathfrak{Y}\times X}. \\
& \mathscr{S}ym^*(\bigoplus_{a\in A}(\pi\circ\pi_{\mathfrak{Q}_a}\times \mathrm{id}_X)^*\mathscr{H}om(\mathscr{E}_{\mathfrak{Q}_a}, U_a\otimes\mathcal{O}_{\mathfrak{Q}_a\times X}))^G \arrow[ur, dashed, "\tau_{\mathfrak{Y}}"'] &
\end{tikzcd}$$

Thus 1.14 provides us with a closed subscheme $\mathfrak{B} \subset \mathfrak{Y}$ such that $\tau_{\mathfrak{B}} = \tau_{\mathfrak{Y}}|_{\mathfrak{B}}$ exists. By construction $\mathfrak{B}$ parametrizes all pseudo G-bundles that may occur in a $(\xi_a,\ \delta_{\mathrm{ps}},\ \varepsilon_{\mathrm{ps}})$-(semi)stable pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundle. $\mathfrak{B}$ comes with a morphism $\pi_{\mathfrak{B},\mathfrak{Q}} : \mathfrak{B} \to \mathfrak{Q}$ as well as a universal family $(\mathscr{E}_{\mathfrak{B}},\ \tau_{\mathfrak{B}})$.

We may extend the definition of a Y-family of pseudo G-bundles (cf. 2.24) to a quotient family including an additional $q_a : V_a \otimes \pi_X^*(\mathcal{O}_X(-n)) \to \mathscr{E}_{Y,a}$ that is surjective with isomorphisms $\pi_{Y,*}(q_a \otimes \mathrm{id}_{\mathcal{O}_X(n)})$ for all $a \in A$. As in the proof of 1.22 $\pi_{Y,*}(\tau_Y \circ \check{\psi}^*_Y)$ induces a lift of the unique morphism $Y \to \bigtimes_{a\in A}\mathfrak{Q}_a$ to a unique morphism $f : Y \to \mathfrak{Y}$ such that the pullback of the universal family under f is the given Y-family. Here $\check{\psi}^*_Y : \mathscr{S}ym^*(\mathscr{U}_Y) \to \mathscr{S}ym^*(\bigoplus_{a\in A}(\mathscr{H}om(\mathscr{E}_{Y,a}, U_a \otimes \mathcal{O}_{Y\times X}))^G$, $\mathscr{U}_Y = \bigoplus_{k=1}^s \mathscr{S}ym^k(U \otimes \bigwedge^{r-1}(V \otimes \pi_X^*(\mathcal{O}_X(-n))) \otimes (\det \mathscr{E}_Y)^\vee)^G$ is the surjective morphism induced by the q_a. Like in 1.22 f factorizes over $\mathfrak{B}$. Hence the universal property 1.62.(U^1) holds for $\mathfrak{B}$ too. As in the case of tuples this defines a group action of $\mathcal{G}_A$ on $\mathfrak{B}$ (resp. $\mathfrak{Y}$) and 1.62.(U^2) is readily verified.

Furthermore this $\mathcal{G}_A$-action on $\mathfrak{Y}$ leaves $\mathfrak{B}$[24] invariant and is equivariant w. r. t. the projection $\pi_{\mathfrak{B},\mathfrak{Q}}$. In particular two pseudo G-bundles parametrized by $\mathfrak{B}$ are isomorphic if and only if they lie in the same $\mathcal{G}_A$-orbit.

Even more so the local universality property is verified for $\mathfrak{B}$ (as in 1.23)[25]. The universal property of the group action holds as in 1.25.

In order to parametrize pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundles we need to account for the additional Higgs field. Therefore let $\mathfrak{T}$ be the scheme that parametrizes the non-parabolic ς-Higgs tuples $(E,\ \varphi,\ L)$ that occur in a semistable pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundle. Recall that we constructed a $\mathcal{G}_A$-action on $\mathfrak{T}$ satisfying 1.25. Combining the two parameter spaces $\mathfrak{B}$ and $\mathfrak{T}$ we find $\mathfrak{P}_{\underline{\varsigma}\circ\iota} = \mathfrak{B} \times_{\mathfrak{Q}} \mathfrak{T} \to \mathfrak{Q}$. There is a universal family $(\mathscr{E}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota}}, \tau_{\mathfrak{P}_{\underline{\varsigma}\circ\iota}}, \varphi_{\mathfrak{P}_{\underline{\varsigma}\circ\iota}}, \mathscr{H}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota}}, \upsilon_{\mathfrak{P}_{\underline{\varsigma}\circ\iota}})$[26] and a $\mathcal{G}_A$-action such that every semistable pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundle is parametrized by $\mathfrak{P}_{\underline{\varsigma}\circ\iota}$ and two of these pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundles are isomorphic if and only if they lie in the same $\mathcal{G}_A$-orbit.[27]

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[24] $gx \in g\ker(\check{\psi}^*) = \ker(g\cdot\check{\psi}^*)$ and $(g\cdot\check{\tau}_{\mathfrak{Y}})(gx) = (\check{\tau}_{\mathfrak{Y}})(g^{-1}gx) = (\check{\tau}_{\mathfrak{Y}})(x)$.

[25] Observe that we only need the local triviality of the vector bundles $\mathscr{E}_{Y,a}$.

[26] Recall that $\mathscr{E}_{\mathfrak{B}}$ and $\mathscr{E}_{\mathfrak{T}}$ are pullbacks of $\mathscr{E}_{\mathfrak{Q}}$ and hence $\mathscr{E}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota}}$ exists as pullback of $\mathscr{E}_{\mathfrak{Q}}$.

[27] Recall the definition of isomorphy of pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundles in 2.26.

2.31. Assume a group G acts on schemes X, Y, Z such that the action is equivariant to given projections $\pi^X : X \to Z$, $\pi^Y : Y \to Z$, then G acts on $X \times_Z Y$: Use the equivariance to show that all squares in the following diagram commute, then the universal property of the fiber product implies the claim.

$$\begin{tikzcd}
 & G \times (X \times_Z Y) \arrow[dl] \arrow[dr] & \\
G \times X \arrow[dr, "\mathrm{id}_G \times \pi^X"] \arrow[dd, "\mathrm{pr}_2"'] & & G \times Y \arrow[dl, "\mathrm{id}_G \times \pi^Y"'] \arrow[dd, "\mathrm{pr}_2"] \\
 & G \times Z \arrow[d, "\mathrm{pr}_2"] & \\
X \arrow[r, "\pi^X"'] & Z & Y. \arrow[l, "\pi^Y"]
\end{tikzcd}$$

2.32. We have to show that the GIT-quotient of $\mathfrak{P}_{\underline{\varsigma}\circ\iota}$ does exist. Therefore consider the surjection $\mathbb{C}^* \times \mathcal{S}_A^{\kappa_a} \to \mathcal{G}_A$, $(z,g) \mapsto zg$ and the quotient $\pi_{\mathfrak{P}_{\underline{\varsigma}\circ\iota},\overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota}}} : \mathfrak{P}_{\underline{\varsigma}\circ\iota} \to \overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota}} = \mathfrak{P}_{\underline{\varsigma}\circ\iota} /\!\!/ \mathbb{C}^*$. Since $\pi_{\mathfrak{P}_{\underline{\varsigma}\circ\iota},\mathfrak{Q}}$ is $\mathcal{G}_A$-equivariant and $\mathbb{C}^*$ acts trivially on $\mathfrak{Q}$ we get a projective morphism $\overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota}} \to \mathfrak{Q}$. Now we use 1.22 and 2.27 to construct a morphism $f : \mathfrak{P}_{\underline{\varsigma}\circ\iota} \to \overline{\mathfrak{T}}$, $\overline{\mathfrak{T}}$ parameter scheme of non-parabolic tuples corresponding to the representation $\varsigma^{\otimes(\delta_{\mathrm{ps}}/\delta)} \otimes \varsigma_{\mathrm{ps}}^{\otimes(\varepsilon_{\mathrm{ps}}/\delta)}$. The morphism descends (again by the trivial action of $\mathbb{C}^*$ on $\overline{\mathfrak{T}}$) to a morphism $\overline{f} : \overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota}} \to \overline{\mathfrak{T}}$. The projectivity of $\pi_{\mathfrak{P}_{\underline{\varsigma}\circ\iota},\mathfrak{Q}}$ implies the projectivity of $\overline{f}$ and thus 2.27 shows that $\overline{f}$ is finite. Finally we may pull back the GIT-quotient in the tuple case to $\mathfrak{P}_{\underline{\varsigma}\circ\iota}$, i. e. by 2.27 $\mathfrak{P}_{\underline{\varsigma}\circ\iota}^{(s)s} = (f \circ \pi_{\mathfrak{P}_{\underline{\varsigma}\circ\iota},\overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota}}})^{-1}(\overline{\mathfrak{T}}^{(s)s})$ to get

$$\begin{aligned}
\mathfrak{P}_{\underline{\varsigma}\circ\iota}^{(s)s} /\!\!/ \mathcal{G}_A = \mathfrak{P}_{\underline{\varsigma}\circ\iota}^{(s)s} /\!\!/ (\mathbb{C}^* \times \mathcal{S}_A^{\kappa_a}) &= (\mathfrak{P}_{\underline{\varsigma}\circ\iota}^{(s)s} /\!\!/ \mathbb{C}^*) /\!\!/ \mathcal{S}_A^{\kappa_a} \\
&= \overline{f}^{-1}(\overline{\mathfrak{T}}^{(s)s}) /\!\!/ \mathcal{S}_A^{\kappa_a\,28}
\end{aligned}$$

and the later quotient exists. Since $\pi_{\mathfrak{P}_{\underline{\varsigma}\circ\iota},\overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota}}}$ is a geometric quotient so is $\mathfrak{P}_{\underline{\varsigma}\circ\iota}^s/\mathcal{G}_A$.

Remark. By the universal property of the fiber product the universal property of 1.22 holds for $\mathfrak{P}_{\underline{\varsigma}\circ\iota}$ and the corresponding notion of quotient pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundles. Again this implies the other universal properties 1.23 and 1.25. In particular by 1.62 the moduli space of non-parabolic projective pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundles of given topological type exists as a projective scheme ([Sch08], 2.7.2.4).

2.33. Conclusion. ([Sch08], 2.7.1.4) *The moduli space $\mathfrak{P}_{\varsigma}^{(s)s} /\!\!/ \mathcal{G}_A$ of non-parabolic principal ς-Higgs bundles exists as a projective scheme.*

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[28] see [Sch08], 1.5.3.3 and 1.4.3.11 or [Bo91], 6.10 Corollary resp. [MFK].

Proof. In conclusion 3.5 we will see that semistable principal $(\underline{\varsigma} \circ \iota)$-Higgs bundles are exactly the semistable pseudo $(\underline{\varsigma} \circ \iota)$-Higgs bundles. By construction $\underline{\varsigma} \circ \iota = \varsigma \oplus \tilde{\varsigma}$. If $\varphi_{\mathfrak{P}_{\underline{\varsigma} \circ \iota}} : \mathscr{E}_{\mathfrak{P}_{\underline{\varsigma} \circ \iota}, \underline{\varsigma} \circ \iota} \to \mathscr{L}_{\mathfrak{P}_{\underline{\varsigma} \circ \iota}}$ is the universal homomorphism over $\mathfrak{P}_{\underline{\varsigma} \circ \iota} \times X$, choose in 1.14 $\psi_{\mathfrak{P}_{\underline{\varsigma} \circ \iota}} = \mathrm{pr}_2$ (the projection to the second component). Then 1.14 ensures the existence of a closed subscheme $\mathfrak{P}_{\varsigma} \subset \mathfrak{P}_{\underline{\varsigma} \circ \iota}$ and a universal homomorphism $\varphi_{\mathfrak{P}_{\varsigma}} : \mathscr{E}_{\mathfrak{P}_{\varsigma}, \varsigma} \to \mathscr{L}_{\mathfrak{P}_{\varsigma}}$. Since $\mathfrak{P}_{\varsigma}$ is $\mathcal{G}_A$-invariant, by 3.5 $\mathfrak{P}_{\varsigma}^{(s)s} /\!\!/ \mathcal{G}_A$ exists. Finally the universal properties of the parameter space $\mathfrak{P}_{\underline{\varsigma} \circ \iota}$ descend to $\mathfrak{P}_{\varsigma}$ and thus by 1.62 the moduli space exists. $\square$

2.4. CONSTRUCTION IN THE PARABOLIC CASE

Before we come to the actual definition of a parameter scheme in the parabolic setting we will state some preparatory results on proper morphisms on fiber products.

2.34. From [SGA], Expose XII, Proposition 3.2 we know that for a morphism of $\mathbb{C}$-schemes of locally finite type properness is equivalent to topological properness of the corresponding analytic map, i. e. equivalent to that map having (quasi-)compact fibers and being closed. Topological properness further implies that the preimage of every compact set is compact.[29] Let Y, E be schemes of finite type, $F \to Y$ a fiber bundle with a fiber $\boldsymbol{F}$ that is proper over $\mathbb{C}$. Furthermore let Y be proper and E be separable over $\mathbb{C}$. Then Y^{an} is a compact analytic space, E^{an} a compact Hausdorff space and we have $F^{an} \xrightarrow{\pi} Y^{an}$ the corresponding analytic fiber bundle (see [Bal10], A.10.4.1 or [GAGA]). Now if $\psi : F \to E$ is a morphism of schemes and ψ^{an} the corresponding analytic map $F^{an} \to E^{an}$ than ψ^{an} is topologically proper for example if F^{an} is compact.[30] We are going to show, that F^{an} is in fact compact. Since $F^{an} = \pi^{-1}(Y^{an})$ it will be enough to show that π is topologically proper. Since π has obviously compact fibers, we are left to show that the map is closed. Now choose a trivializing cover U_i of $F^{an} \to Y^{an}$, let $\pi|_{U_i} =: \pi_i$, $A \subset F^{an}$ be closed and $A_i = A \cap \pi^{-1}(U_i)$ the restriction (closed w. r. t. the subspace topology). If $u \notin \pi_i(A_i)$ then $\{u\} \times \boldsymbol{F}$ does not intersect A_i. Since A_i is closed we find for every pair $(u, f) \in \{u\} \times \boldsymbol{F}$ an open neighborhood $V_{u,f}$, $V_{u,f} \cap A_i = \emptyset$ and compactness of $\{u\} \times \boldsymbol{F}$ shows that $\{u\} \times \boldsymbol{F}$ is covered by finitely many open sets V_j, $V_j \cap A_i = \emptyset$. Since π is (always) open, $\bigcap_i \pi_i(V_j) \subset \pi(A)^c$ open and thus $\pi_i(A_i) = \pi(A) \cap U_i$ closed. Hence we get $\pi(A) \cap U_i$ closed in U_i for all i. Next consider the complement $V = Y^{an} \setminus \pi(A)$ which satisfies $V \cap U_i$ open in U_i. The definition of the subspace topology provides us with open sets $\tilde{U}_i$ s. t.

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[29] Note that for locally compact spaces this property is even equivalent to properness.

[30] F^{an} compact, A closed in $F^{an} \Rightarrow A$ compact $\Rightarrow \psi^{an}(A)$ compact, hence closed in a Hausdorff space. Since points in a Hausdorff space are closed, the fibers of ψ^{an} are closed and therefore are compact in the compact space F^{an}. In the algebraic category the corresponding statement results from [Ha77], II.4.8(e).

$V \cap U_i = \tilde{U}_i \cap U_i$ open in Y^{an}. Therefore $V = \bigcup_i V \cap U_i$ is open in Y^{an} and $\pi(A)$ is closed.
Collecting all results we conclude that π is closed and hence topologically proper. Thus ψ^{an} is topologically proper and the corresponding morphism of schemes ψ is proper.

Remark. To give the reader a glimpse of the actual application of these technical lemmas, consider a fiber bundle with compact fiber isomorphic to G/P for a parabolic subgroup P of G and a morphism to a compact complex Graßmannian manifold $\mathfrak{G}_a^{ij}$.

2.35. Lemma. *Given a proper morphism $f_0 : Y_0 \to T_0$ of schemes and Y_i fiber bundles over Y_0 with compact fiber F then the natural morphism $Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_n = \mathcal{Y} \to T_0$ is proper.*

Proof. By the previous paragraph, the maps $Y_i \to T_0$ are proper and therefore we find the maps $p_1, \ldots, p_n$ in the following diagram as well as p with the universal property of the fiber product.

$$\begin{tikzcd}
 & & Y_1 \arrow[rrd] \arrow[rdd, dotted, no head] & & \\
\mathrm{Spec}(K) \arrow[r, "q_K"] \arrow[dd, "\iota"'] & \mathcal{Y} \arrow[ru] \arrow[rrd] & & & Y_0 \arrow[dd, "f_0"] \\
 & & & Y_n \arrow[ru] & \\
\mathrm{Spec}(R) \arrow[rrrr, "q_R"] \arrow[ru, dashed, "\Rightarrow\exists p"] \arrow[rruuu, dashed, "p_1"'] \arrow[rrru, dashed, "p_n"'] \arrow[rrrruu, dashed, "p_0"] & & & & T_0.
\end{tikzcd}$$

We get $f_0 \circ \mathrm{pr}_0 \circ p = f_0 \circ p_0 = q_R$ and since $\mathrm{pr}_i \circ p \circ \iota = p_i \circ \iota = \mathrm{pr}_i \circ q_K$ the universal property of the fiber product shows that $p \circ \iota = q_K$. □

2.36. Lemma. *Let $Y_1, \ldots, Y_n$ be schemes over Y_0 and $T_1, \ldots, T_n$, T_0, Y_0 schemes over $\mathbb{C}$. If there are proper morphisms $f_i : Y_i \to T_i$, $i = 0, \ldots, n$ then there is a proper morphism $f : \mathcal{Y} := Y_1 \times_{Y_0} \ldots \times_{Y_0} Y_n \to T_0 \times T_1 \times \ldots \times T_n = \mathcal{T}$.*

Proof. Define $f = (f_0 \circ \mathrm{pr}_0, \ \ldots, \ f_n \circ \mathrm{pr}_n)$. We need to show, that f is proper. The existence of a lift $p : \mathrm{Spec}(R) \to \mathcal{Y}$ of $q_R : \mathrm{Spec}(R) \to \mathcal{Y}$ in the valuation criterion (below) follows from the universal property of the fiber product and the existence of the lifts $p_i : \mathrm{Spec}(R) \to Y_i,\ i = 0, \ldots, n$.

$$\begin{tikzcd}
 & & Y_1 \arrow[rr, d] \arrow[ddd, "f_1"] \arrow[ddr, dotted, no head] & & \\
\mathrm{Spec}(K) \arrow[r, "q_K"] \arrow[ddd, "\iota"'] & \mathcal{Y} \arrow[ur] \arrow[rr, d] \arrow[ddd, "f"] & & & Y_0 \arrow[ddd, "f_0"] \\
 & & & Y_n \arrow[ur] \arrow[ddd, "f_n"] & \\
 & & T_1 \arrow[ddr, dotted, no head] & & \\
\mathrm{Spec}(R) \arrow[r, "q_R"] \arrow[uuur, dashed, "\Rightarrow \exists p"] \arrow[uuuurr, dashed, "p_1"] \arrow[uuurrrr, dashed, "p_0"] \arrow[uurrr, dashed, "p_n"'] & \mathcal{T} \arrow[ur] \arrow[rrr] \arrow[rr, d] & & & T_0 \\
 & & & T_n &
\end{tikzcd}$$

The diagram commutes, since $f \circ p = (f_0 \circ \mathrm{pr}_0 \circ p, \ldots,\ f_n \circ \mathrm{pr}_n \circ p) = (f_0 \circ p_0, \ldots,\ f_n \circ p_n) = (\mathrm{pr}_0 \circ q_R, \ldots,\ \mathrm{pr}_n \circ q_R) = q_R$ by uniqueness of the lifts p_i. For the upper triangle it is enough to show, that $\mathrm{pr}_i \circ q_K = \mathrm{pr}_i \circ p \circ \iota,\ \forall\ i = 1, \ldots, n$, since then $q_K = p \circ \iota$ by the universal property of the fiber product. But $\mathrm{pr}_i \circ p = p_i$ (by construction) and $p_i \circ \iota = \mathrm{pr}_i \circ q_K$ by properness of the f_i. This proves the claim. □

2.37. If

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & X \arrow[d] & X'' \arrow[l] \arrow[d] \\
S' \arrow[r] & S & S'' \arrow[l] \\
Y' \arrow[r] \arrow[u] & Y \arrow[u] & Y'' \arrow[l] \arrow[u]
\end{tikzcd}$$

commutes, then

$$(X' \times_X X'') \times_{S' \times_S S''} (Y' \times_Y Y'') = (X' \times_{S'} Y') \times_{X \times_S Y} (X'' \times_{S''} Y'').$$

For the proof use the universal property of the fiber product.

2.38. Recall the definitions of a parabolic ς-Higgs bundle in 2.1 and of a pseudo parabolic $(\varsigma \circ \iota)$-Higgs bundle in 2.22. Below we need a slightly modified object: A ι-parabolic ς-Higgs bundle is a ς-Higgs bundle $(P,\ \varphi,\ L)$ together with reductions $s^j : \{x^j\} \to P_\iota \times_X \{x^j\}/P^j_{\mathrm{Gl}}$ for fixed parabolic subgroups $P^j_{\mathrm{Gl}} \subset \bigtimes_{a \in A} \mathrm{Gl}(U_a)$ resp. parabolic filtrations $(E^{ij}_a)_{i[s^j_a]a[|A|]}$ (of fixed type) of $P_\iota|_{x^j}$ over every puncture $x^j \in S$.

2.39. As in the non-parabolic case we are going to construct the moduli space of parabolic pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundles first. 2.27 for a parabolic pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundle implies that (semi)stable pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundles live in a bounded family. Therefore the construction above works (with a possibly different integer n). Let $\mathscr{E}_{\mathfrak{Q}_a}$, $\mathscr{E}_{\mathfrak{Q}}$ resp. $\mathscr{E}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota}}$ be the corresponding universal families over $\mathfrak{Q}_a$, $\mathfrak{Q}$ resp. $\mathfrak{P}_{\underline{\varsigma}\circ\iota}$. Fix a tuple $P^j_{\mathrm{Gl}(U_a)_a}$, $1\leq j\leq |S|$ of parabolic subgroups of $\mathrm{Gl}(U_a)_a = \bigtimes_{a\in A}\mathrm{Gl}(U_a)$. Let $\mathfrak{P}^j_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}} = \pi^*_{\mathfrak{P}_{\underline{\varsigma}\circ\iota},\mathfrak{Q}\times\{x^j\}}(\tilde{\mathfrak{P}}^j_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}})$, $\tilde{\mathfrak{P}}^j_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}} = \left(\bigtimes_{a\in A}\mathscr{I}\!som(U_a\otimes\mathcal{O}_{\mathfrak{Q}_a},\mathscr{E}_{\mathfrak{Q}_a}|_{\mathfrak{Q}_a\times\{x^j\}})\right)$, $1\leq j\leq|S|$ and

$$\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}} = \mathfrak{P}^1_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}/P^1_{\mathrm{Gl}(U_a)_a}\times_{\mathfrak{P}_{\underline{\varsigma}\circ\iota}}\cdots\times_{\mathfrak{P}_{\underline{\varsigma}\circ\iota}}\mathfrak{P}^{|S|}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}/P^{|S|}_{\mathrm{Gl}(U_a)_a}$$

where $\mathfrak{P}^j_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}/P^j_{\mathrm{Gl}(U_a)_a}$ is the bundle associated to $\mathfrak{P}^j_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}$ by the action $\mathrm{Gl}(U_a)_a\times\mathrm{Gl}(U_a)_a/P^j_{\mathrm{Gl}(U_a)_a}\to\mathrm{Gl}(U_a)_a/P^j_{\mathrm{Gl}(U_a)_a}$, $(g,[q])\mapsto[gq]$.
The scheme $\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}\to\mathfrak{P}_{\underline{\varsigma}\circ\iota}$ parametrizes parabolic principal pseudo G-bundles over the punctured Riemann surface (X,S). Analogously we define a scheme $\overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}}$ on $\overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota}} = \mathfrak{P}_{\underline{\varsigma}\circ\iota}/\!\!/\mathbb{C}^*$ as $\overline{\mathfrak{P}^j_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}} = \pi^*_{\overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota}},\mathfrak{Q}\times\{x^j\}}(\tilde{\mathfrak{P}}^j_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}})$ and

$$\overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}} = \overline{\mathfrak{P}^1_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}}/P^1_{\mathrm{Gl}(U_a)_a}\times_{\overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota}}}\cdots\times_{\overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota}}}\overline{\mathfrak{P}^{|S|}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}}/P^{|S|}_{\mathrm{Gl}(U_a)_a}.$$

Note that the fiberwise morphisms $g_{q,j}:\bigtimes_{a\in A}\mathscr{I}\!som(U_a,\mathscr{E}_{\mathfrak{Q}_a}|_{\{q\}\times\{x^j\}})/P^1_{\mathrm{Gl}(U_a)_a}\to\bigtimes_{a\in A}\bigtimes_{i=1}^{s^j_a}\mathfrak{G}^{ij}_a$ split over a flag variety of (r^{ij}_a)-dimensional filtrations of the fiber $\bigtimes_{a\in A}E_a|_{x^j}\simeq\bigtimes_{a\in A}\mathbb{C}^{r_a}$ and thus by 1.22 there are morphisms g that extend f and restrict to $g_{q,j}$ over (q,x^j). Analogously we construct a morphism $\overline{g}$ such that the following diagram over $\mathfrak{Q}$ commutes

$$\begin{tikzcd}
\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}} \arrow[rr, "g"] \arrow[dr] \arrow[dd] & & \overline{\mathfrak{T}}_{\mathrm{par}} \arrow[dd] \\
& \overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}} \arrow[ur, "\overline{g}"'] \arrow[dd] & \\
\mathfrak{P}_{\underline{\varsigma}\circ\iota} \arrow[rr, "f"] \arrow[dr] & & \overline{\mathfrak{T}} \\
& \overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota}} \arrow[ur, "\overline{f}"'] &
\end{tikzcd}$$

where $\overline{\mathfrak{T}}_{\mathrm{par}}\subset\overline{\mathfrak{T}}\times\bigtimes_{a\in A}\bigtimes_{j=1}^{|S|}\bigtimes_{i=1}^{s^j_a}\mathfrak{G}^{ij}_a$ is the parameter scheme of parabolic tuples corresponding to the representation $\varsigma^{\otimes(\delta_{\mathrm{ps}}/\delta)}\otimes\varsigma_{\mathrm{ps}}^{\otimes(\varepsilon_{\mathrm{ps}}/\delta)}$. We get

$\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}} \mathbin{/\!\!/} \mathbb{C}^* = \overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}}$: First note that by 2.37 we have $\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}} = \mathfrak{P}_{\underline{\varsigma}\circ\iota} \times_{\mathfrak{Q}} \left(\pi^*_{\mathfrak{Q}\times\{x^1\}}\tilde{\mathfrak{P}}^1_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}/P^1_{\mathrm{Gl}(U_a)_a} \times_{\mathfrak{P}_{\underline{\varsigma}\circ\iota}} \cdots\right)$ and $\overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}} = (\mathfrak{P}_{\underline{\varsigma}\circ\iota} \mathbin{/\!\!/} \mathbb{C}^*) \times_{\mathfrak{Q}} \left(\pi^*_{\mathfrak{Q}\times\{x^1\}}\tilde{\mathfrak{P}}^1_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}/P^1_{\mathrm{Gl}(U_a)_a} \times_{\mathfrak{P}_{\underline{\varsigma}\circ\iota}} \cdots\right)$. Now by the universality of the geometric quotient $\mathfrak{P}_{\underline{\varsigma}\circ\iota} \mathbin{/\!\!/} \mathbb{C}^*$[31]

$$\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}} = \mathfrak{P}_{\underline{\varsigma}\circ\iota} \times_{\mathfrak{P}_{\underline{\varsigma}\circ\iota}/\!\!/\mathbb{C}^*} \left(\mathfrak{P}_{\underline{\varsigma}\circ\iota} \mathbin{/\!\!/} \mathbb{C}^* \times_{\mathfrak{Q}} \left(\pi^*_{\mathfrak{Q}\times\{x^1\}}\tilde{\mathfrak{P}}^1_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}/P^1_{\mathrm{Gl}(U_a)_a} \times_{\mathfrak{P}_{\underline{\varsigma}\circ\iota}} \cdots\right)\right) \xrightarrow{\mathrm{pr}_2} \overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}}$$

exists, i. e. $\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}} \mathbin{/\!\!/} \mathbb{C}^* = \overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}}$.
Since the morphism $\overline{g}$ is obviously one-to-one we are left with the task to show the properness of $\overline{g}$. Fortunately 2.34 already implies that the component morphisms $\overline{g}^j : \overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}}^j/P^j_{\mathrm{Gl}(U_a)_a} \to \bigtimes_{a\in A}\bigtimes_{i=1}^{s_a^j}\mathfrak{G}_a^{ij}$ are proper. Therefore the morphism $\overline{g}$ is by 2.36 proper and thus finite. Since $\overline{g}$ is $\mathcal{S}_A^{\kappa_a}$-invariant (with respect to the natural action on $\overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}}$ (cf. 2.31) we can pull back the GIT-quotient $\overline{\mathfrak{T}}^{(s)s}_{\mathrm{par}} \mathbin{/\!\!/} \mathcal{S}_A^{\kappa_a}$ whenever it exists. We already know that the GIT-quotient $\overline{\mathfrak{T}}^{(s)s}_{\mathrm{par}} \mathbin{/\!\!/} \mathcal{S}_A^{\kappa_a}$ will exist if the stability parameters are admissible. So let us assume that this is the case. By the very definition of semistability of parabolic pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundles in 2.20 as well as 2.27 we see that g preserves semistability. Hence we may conclude as in the non-parabolic case

$$\begin{aligned}\mathfrak{P}^{(s)s}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}} \mathbin{/\!\!/} \mathcal{G}_A &= \mathfrak{P}^{(s)s}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}} \mathbin{/\!\!/} (\mathbb{C}^* \times \mathcal{S}_A^{\kappa_a})\\ &= (\mathfrak{P}^{(s)s}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}} \mathbin{/\!\!/} \mathbb{C}^*) \mathbin{/\!\!/} \mathcal{S}_A^{\kappa_a}\\ &= \overline{g}^{-1}(\overline{\mathfrak{T}}^{(s)s}_{\mathrm{par}}) \mathbin{/\!\!/} \mathcal{S}_A^{\kappa_a}.\end{aligned}$$

Again the universal geometric quotient $\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}} \to \overline{\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}}$ induces a universal geometric quotient $\mathfrak{P}^s_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}/\mathcal{G}_A$.

The transition to G-bundles in the parabolic setting is a little bit more complicated. Let us start with the extension of 2.33:

2.40. Conclusion. *The moduli space* $\mathfrak{P}^{(s)s}_{\varsigma,\iota-\mathrm{par}} \mathbin{/\!\!/} \mathcal{G}_A$ *of* τ^j*-(semi)stable pairs* $((P,\ \varphi,\ L),\ (s^j)_{j[|S|]})$ *where* $(P,\ \varphi,\ L)$ *is a principal* ς*-Higgs bundle and* $s^j : \{x^j\} \to \left(P_\iota/P^j_{\mathrm{Gl}(U_a)_a}\right)_{x^j}$, $P^j_{\mathrm{Gl}(U_a)_a} \subset \mathrm{Gl}(U_a)_a$ *parabolic subgroup, exists as a projective scheme whenever the weights* $(\beta^{ij})_{i[s^j]}$ *induced by* τ^j *are admissible (cf. 2.10).*

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[31] [MFK] 1.§4.1.10, and [Sch08] 1.4.2.13.

Proof. We may pull back the GIT quotient to $\mathfrak{P}_{\varsigma,\iota-\mathrm{par}} = \mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}|_{\mathfrak{P}_\varsigma}$. The result will then follow directly from 3.5, i. e. the fact that $\mathfrak{P}^{(s)s}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}|_{\mathfrak{P}_\varsigma}$ parametrizes already the semistable parabolic principal ς-Higgs bundles. □

Let $(P,\ (s^j)_{j[|S|]})$ be a principal G-Higgs bundle and $P^j \subset G$ fixed parabolic subgroups of G with $P^j = Q_G(\tau^j)$ for some one-parameter subgroups τ^j of P^j. First recall that $(P,(s^j)_{j[|S|]})$ is τ^j-(semi)stable if and only if $(P,(\tilde{s}^j)_{j[|S|]})$ with $\tilde{s}^j = \iota \circ s^j : \{x^j\} \to P_\iota \times_X \{x^j\}/Q_{\mathrm{Gl}(U_a)_a}(\tilde{\tau}^j)$ and $\tilde{\tau}^j = \iota \circ \tau^j$, is β^{ij}-(semi)stable (cf. 2.8). We have the following commuting diagram

$$\begin{tikzcd}
G \arrow[dr] \arrow[ddd, bend right, "\iota"'] & & \\
& G \times G/Q_G(\tau^j) \arrow[r] \arrow[ddd] & G/Q_G(\tau^j) \arrow[ddd] \\
G/Q_G(\tau^j) \arrow[ur] \arrow[ddd, bend right] & & \\
\mathrm{Gl}(U_a)_a \arrow[dr] & & \\
& \mathrm{Gl}(U_a)_a \times \mathrm{Gl}(U_a)_a/Q_{\mathrm{Gl}(U_a)_a}(\tilde{\tau}^j) \arrow[r] & \mathrm{Gl}(U_a)_a/Q_{\mathrm{Gl}(U_a)_a}(\tilde{\tau}^j) \\
\mathrm{Gl}(U_a)_a/Q_{\mathrm{Gl}(U_a)_a}(\tilde{\tau}^j) \arrow[ur] & &
\end{tikzcd}$$

(Equ 1)

where the vertical arrows are the embeddings $G/Q_G(\tau^j) \hookrightarrow \mathrm{Gl}(U_a)_a/Q_{\mathrm{Gl}(U_a)_a}(\tilde{\tau}^j)$, $gQ_G(\tau^j) \mapsto \iota(g)Q_{\mathrm{Gl}(U_a)_a}(\tilde{\tau}^j)$ using $\iota(Q_G(\tau^j)) = \iota(G) \cap Q_{\mathrm{Gl}(U_a)_a}(\tilde{\tau}^j)$. Therefore there are induced maps of fiber bundles $\mathscr{P}|_{\mathfrak{P}_\varsigma \times \{x^j\}}/Q_G(\tau^j)$ and $\mathscr{P}_\iota|_{\mathfrak{P}_\varsigma \times \{x^j\}}/Q_{\mathrm{Gl}(U_a)_a}(\tilde{\tau}^j)$ where $\mathscr{P}$ is the universal G-bundle on $\mathfrak{P}_\varsigma \times X$.[32] Thus the closed embedding $\iota : G \to \mathrm{Gl}(U_a)_a$ defines closed subschemes of the components $\mathfrak{P}^j_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}|_{\mathfrak{P}_\varsigma}$ and hence defines a closed embedding $\mathfrak{P}^{(s)s}_{\varsigma,\mathrm{par}} \hookrightarrow \mathfrak{P}^{(s)s}_{\varsigma,\iota-\mathrm{par}}$ and $\mathfrak{P}^{(s)s}_{\varsigma,\mathrm{par}}$ parametrizes (semi)stable projective parabolic ς-Higgs bundles. Since two parabolic principal ς-Higgs bundles are equivalent if and only if the associated ι-flagged ς-Higgs bundles are, we see that the embedding is equivariant and therefore $\mathfrak{P}^{ss}_{\varsigma,\mathrm{par}} /\!\!/ \mathcal{G}_A$ exists as a projective scheme as well as $\mathfrak{P}^{s}_{\varsigma,\mathrm{par}}/\mathcal{G}_A$ exists as an open subscheme.

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[32] By the previous diagram, the morphisms locally defined by ι are compatible with the transition functions induced by ι.

3

THE MODULI SPACE OF AFFINE PARABOLIC HIGGS BUNDLES

3.1. ASYMPTOTIC BEHAVIOR

We study the asymptotic behavior of the various semistability concepts. The results of this section will not only fill the gaps in the proof of 2.40, but will also enable us to treat the affine case in the next section. Let's start with another proposition:

3.1. Proposition. *Given* $r_a,\ d_a,\ l$ *as well as* $\kappa_a,\ \xi_a,\ \delta_{\mathrm{ps}}$*, there is a* $\varepsilon_{\mathrm{ps}}^{\infty} \geq 0$ *such that for all* $\varepsilon_{\mathrm{ps}} > \varepsilon_{\mathrm{ps}}^{\infty}$ *and every [parabolic] pseudo* $(\underline{\varsigma} \circ \iota)$*-Higgs bundle* $\mathbf{E} = ((E_a,\ [(E_a^{ij})_{i[s_a^j]j[|S|]}])_{a[|A|]},\ \tau,\ \varphi,\ L)$*, the following two conditions are equivalent:*

I. $\mathbf{E}$ *is* $(\kappa_a,\ \xi_a,\ \delta_{\mathrm{ps}},\ \varepsilon_{\mathrm{ps}},\ [\beta_a^{ij}])$*-(semi)stable.*

II. *For every weighted filtration* $(F^k,\ \alpha^k)_{k[r]}$ *as in 1.6:*

   A. $\mu(F^k, \alpha^k, \varphi_\tau) \geq 0$

   B. $M_{[\mathrm{par}]}^{\kappa,\xi}(F^k, \alpha^k) + \delta_{\mathrm{ps}} \cdot \mu(F^k, \alpha^k, \varphi) \ \ (\geq)\ \ 0$ *whenever* $\mu(F^k, \alpha^k, \varphi_\tau) = 0.$

Remark. A [parabolic] pseudo $(\underline{\varsigma} \circ \iota)$-Higgs bundle that satisfies II. is called asymptotically (semi)stable.

The rather evolved proof of the non-parabolic version in [Sch08], Theorem 2.7.2.5 uses instability one-parameter subgroups.[1] It works in our situation with small modifications necessary in the parabolic setting.
We will however try a different approach. Therefore we will apply an idea of Adrian Langer [GLSS08] used by him to show that for $|A| = 1$ the family of (semi)stable Higgs tuples is bounded independent of the stability parameter. The proof uses the existence of a Harder-Narasimhan filtration; hence a direct extension thereof should probably use a "Harder-Narasimhan-Filtration for Higgs tuples". Although

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[1] cf. Ramanan and Ramanathan, [RR84].

there is a version of the Harder-Narasimhan filtration for parabolic tuples (even for (κ,ξ)-semistability), to account for the additional Higgs field φ would lead us back to instability one-parameter subgroups and thus to a discussion similar to the proof given by [Sch08], Theorem 2.7.2.5.
Instead we will take a closer look at the boundedness of the weight functions for the available classical HN-filtration. Once the stage is set, the following key result can be proven rather directly.

3.2. Lemma. *Fix r_a, d_a, l, κ_a, ξ_a, δ_{ps} but allow $\varepsilon_{\mathrm{ps}} > 0$ arbitrary.*
The family of all vector bundles E occurring as $E \simeq \bigoplus_{a\in A}(E_a)^{\oplus\kappa_a}$ in a semistable [parabolic] pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundle $\mathbf{E} = ((E_a,\ [(E_a^{ij})_{i[s_a^j]j[|S|]}])_{a[|A|]},\ \tau,\ \varphi,\ L)$ is bounded (independent of $\varepsilon_{\mathrm{ps}} > 0$).

Proof. First note that the underlying vector bundle of $\mathbf{E}$ belongs to a bounded family (independent of $\varepsilon_{\mathrm{ps}}$) if it is semistable as a vector bundle. From now on assume that our [parabolic] pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundle $\mathbf{E}$ is $(\kappa_a,\ \xi_a,\ \delta_{\mathrm{ps}},\ \varepsilon_{\mathrm{ps}},\ [\beta_a^{ij}])$-semistable, but the underlying vector bundle is unstable. Consider the Harder-Narasimhan filtration

$$\{0\} = E^0 \subsetneqq E^1 \subsetneq \cdots \subsetneq E^m = E = \bigoplus_{a\in A} E_a^{\oplus\kappa_a}$$

with $r^k = \mathrm{rk}(E^k)$, $d^k = \deg(E^k)$. Denote $\mu(E) = \deg(E)/\operatorname{rk}(E)$ the classical μ-function for vector bundles and denote $\mu^i = \mu(E^i/E^{i-1})$. Recall from 1.11 that this is in fact a tuple filtration. Fix the length m of the Harder-Narasimhan filtration. Since $\mu(E^k) > \mu(E)$ we see that $\sum_{k=1}^{m-1}\alpha^k(r^k d - d^k r) < 0$ for all real non-trivial non-negative tuples $(\alpha^k)_{k[m-1]} \in \mathbb{R}_{\geq 0}^{m-1}$. Since the map $(\alpha^k)_{k[m-1]} \mapsto \sum_{j=1}^{m-1}\alpha^j = \|(\alpha^k)_k\|_1$ is continuous, the preimage B_1 of 1 in $\mathbb{R}_{\geq 0}^{m-1}$ is closed and bounded, and therefore compact. Note that in terms of the weights γ^k we have $\sum_{j=1}^{m-1}\alpha^j = \frac{\gamma^m-\gamma^1}{r} = 1$ on B_1.
Below we are going to construct a covering of B_1 by finitely many compact sets D_i, such that either $\mu_{max}(E)$ (or $|\mu_{min}(E)|$) is already smaller than a prescribed constant c_0 or $\mu(E^k,\ \cdot\ ,\ \varphi_\tau)$ is positive on one of the D_i. This on the other hand will again give us a bound for $\mu_{max}(E)$ (or $|\mu_{min}(E)|$).
Set $\tilde\alpha^j = \frac{\mu^j-\mu^{j+1}}{r}$, $\hat\alpha^j = \frac{\tilde\alpha^j}{\|(\tilde\alpha^j)_{j[m-1]}\|_1}$ for $j = 1,\ldots,m-1$ with $\|(\tilde\alpha^j)_{j[m-1]}\|_1 = \frac{\mu_{max}(E)-\mu_{min}(E)}{r}$. Then

$$\begin{aligned} M^{\kappa,0}(E^k,\tilde\alpha^k) &= \sum_{k=1}^{m-1}\frac{\mu^k-\mu^{k+1}}{r}(dr^k - rd^k) \\ &= \frac{1}{r}\left(\sum_{k=1}^{m-1}\mu^k(dr^k - rd^k) - \sum_{k=2}^{m}\mu^k(dr^{k-1} - rd^{k-1})\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{r}\sum_{k=1}^{m} \mu^k \left((dr^k - rd^k) - (dr^{k-1} - rd^{k-1})\right) \\
&\quad - \mu^m \frac{(dr^m - rd^m)}{r} + \mu^1 \frac{(dr^0 - rd^0)}{r} \\
&= \frac{1}{r}\sum_{k=1}^{m} \frac{d^k - d^{k-1}}{r^k - r^{k-1}} \left((d(r^k - r^{k-1}) - r(d^k - d^{k-1})\right) \\
&= \sum_{k=1}^{m} \frac{(d^k - d^{k-1})d}{r} - \sum_{k=1}^{m} \frac{(d^k - d^{k-1})^2}{r^k - r^{k-1}} \\
&= \frac{d^2}{r} - \sum_{k=1}^{m} \frac{(d^k - d^{k-1})^2}{r^k - r^{k-1}} \\
&= \mu(E)^2 r - \sum_{k=1}^{m} (\mu^k)^2 (r^k - r^{k-1}).
\end{aligned}
$$

First note that there are only finitely many continuous functions $M^{\kappa,\xi}_{[\mathrm{par}]}(E^k, \ \cdot\ ) + \delta_{\mathrm{ps}} \cdot \mu(E^k, \ \cdot\ , \varphi) - M^{\kappa,0}(E^k, \ \cdot\ )$. In particular we find a real number $r_{\delta,\mathrm{ps}}$ such that $M^{\kappa,\xi}_{[\mathrm{par}]}(E^k, \ \cdot\ ) + \delta_{\mathrm{ps}}\mu(E^k, \ \cdot\ , \varphi) - M^{\kappa,0}(E^k, \ \cdot\ ) < r_{\delta,\mathrm{ps}}$ on B_1.[2] Fix a positive integer n such that $|\mu(E)| \cdot \frac{1-rn^2}{2n/r} < -r_{\delta,\mathrm{ps}}$ for $\mu(E) \neq 0$.[3] Further denote by $\mu_M = \max\{\mu^1, |\mu^m|\} \neq 0$. By the same calculation as before we see that for $\mu_M > |\mu(E)| \cdot r \cdot n, \ \mu(E) \neq 0$

$$
\begin{aligned}
M^{\kappa,0}(E^k, \hat{\alpha}^k) &\leq \frac{\mu(E)^2 r - \sum_{k=1}^{m} (\mu^k)^2 (r^k - r^{k-1})}{2\mu_M / r} \\
&\leq \frac{\mu(E)^2 r - \mu_M^2}{2\mu_M / r} < \frac{\mu_M^2/(rn^2) - \mu_M^2}{2\mu_M / r} \\
&\leq \mu_M \cdot \frac{1 - rn^2}{2n^2} < |\mu(E)| \cdot \frac{1 - rn^2}{2n/r} \leq -r_{\delta,\mathrm{ps}}. \qquad (*)
\end{aligned}
$$

In the case $\mu_M > 2r_{\delta,\mathrm{ps}}/r, \ \mu(E) = 0$ we get the same result. Now either μ_M is bounded by $\max\{|\mu(E)| \cdot r \cdot n, \ 2r_{\delta,\mathrm{ps}}/r\}$ or $(*)$ holds. Assume that $(*)$ holds, then by construction[4]

$$
M^{\kappa,\xi}_{[\mathrm{par}]}(E^k, \hat{\alpha}^k + t^k) + \delta_{\mathrm{ps}}\mu(E^k, \hat{\alpha}^k + t^k, \varphi) \leq M^{\kappa,0}(E^k, \hat{\alpha}^k + t^k) + r_{\delta,\mathrm{ps}}
$$

---

[2] We may choose $r_{\delta,\mathrm{ps}}$ such that it works in the non-parabolic case as well as in the parabolic case.

[3] Note that a non-semistable bundle is not of rank 1.

[4] Assume $\alpha^k + t^k \in B_1$.

becomes negative at $t = 0$. The latter term $M^{\kappa,0}(E^k, \hat{\alpha}^k + t^k) + r_{\delta,\mathrm{ps}}$ can become zero only if

$$\begin{aligned}
&t_{min}(\mu_{max}(E) - \mu_{min}(E))m(\mu(E) - \mu(E^1))r \\
&\geq (\mu_{max}(E) - \mu_{min}(E)) \sum_{k=1}^{m-1} t^k(\mu(E) - \mu(E^k))r^k \\
&= -\frac{r_{\delta,\mathrm{ps}}(\mu_{max}(E) - \mu_{min}(E))}{r} - \mu(E)^2 r + \sum_{k=1}^{m} (\mu^k)^2 (r^k - r^{k-1}).
\end{aligned}$$

Note that the right-hand side is by assumption positive, hence there must be a negative t^k - set $t_{min} = \min\{t^k : 1 \leq k \leq m-1\} < 0$. Therefore for $\mu_M^2 > 2\mu(E)^2 r + \frac{4|r_\delta|\mu_M}{r}$

$$\begin{aligned}
|t_{min}| &\geq \frac{\sum_{k=1}^{m} (\mu^k)^2 (r^k - r^{k-1}) - \mu(E)^2 r - \frac{r_\delta(\mu_{max}(E) - \mu_{min}(E))}{r}}{(\mu_{max}(E) - \mu_{min}(E))m(\mu(E^1) - \mu(E))r} \\
&> \frac{\mu_M^2/2}{2\mu_M m \cdot 2\mu_M r} \geq \frac{1}{8r^2}.
\end{aligned}$$

This shows that for large μ_M the term $M^{\kappa,\xi}_{[\mathrm{par}]}(E^k, \,\cdot\,) + \delta_{\mathrm{ps}} \cdot \mu(E^k, \,\cdot\,, \varphi)$ is negative on a compact ball $\overline{B_R((\hat{\alpha}^k)_{k[m-1]})} \subset B_1$ around $(\hat{\alpha})^k_{k[m-1]}$ of radius at least $R = \frac{1}{8r^2}$. Putting the results together we see that there is a constant $c > 0$ such that either $\mu_M \leq c$ or $M^{\kappa,\xi}_{[\mathrm{par}]}(E^k, \,\cdot\,) + \delta_{\mathrm{ps}} \mu(E^k, \,\cdot\,, \varphi)|_{\overline{B_R((\hat{\alpha}^k)_{k[m-1]})}} < 0$.

Since B_1 is compact it is in particular totally bounded and thus we find a finite covering by compact balls $D_i = \overline{B_{R/2}(x_i)},\ x_i \in B_1,\ i = 1, \ldots, s$ of radius smaller $R/2$. Note that this covering depends only on the initial data r and on the length of the Harder-Narasimhan filtration m. Now for $\mu_M > c$ let D_i be the compact set for which $(\hat{\alpha}^k)_{k[m-1]} \in D_i$. Then by $(\kappa_a,\ \xi_a,\ \delta_{\mathrm{ps}},\ \varepsilon_{\mathrm{ps}},\ [\beta_a^{ij}])$-(semi)stability of $\mathbf{E}$, we get $\mu(E^k, \alpha^k, \varphi_\tau) > 0$ on D_i. Hence the continuous function $\mu(E^k,\ \cdot\ ,\ \varphi_\tau)$ attains its minimum on the compact set D_i. Given our intial data $\mu(E^k,\ \cdot\ ,\ \varphi_\tau)$ must be one of finitely many possible functions and thus its minimum is bounded (from below) by a constant $\mathrm{Min}_i > 0$ which depends solely on the initial data.

Next recall that $\varphi_\tau : (E^{\otimes u_{\mathrm{ps}}})^{\oplus v_{\mathrm{ps}}} \to \det(E)^{\otimes w_{\mathrm{ps}}}$ is non-trivial. Therefore for all tuple $(i_j)_{j[u_{\mathrm{ps}}]}$ such that φ_τ is non-trivial on $(\bigotimes_{j=1}^{u_{\mathrm{ps}}} E^{i_j})^{\oplus v_{\mathrm{ps}}}$ we get

$$\begin{aligned}
\sum_{j=1}^{u_{\mathrm{ps}}} \mu^{i_j} = \sum_{j=1}^{u_{\mathrm{ps}}} \underbrace{\mu_{min}(E^{i_j})}_{\mu^{i_j}} &= \mu_{min}\left(\left(\bigotimes_{j=1}^{u_{\mathrm{ps}}} E^{i_j}\right)^{\oplus v_{\mathrm{ps}}}\right) \\
&\leq \mu(\det(E)^{\otimes w_{\mathrm{ps}}}) = w_{\mathrm{ps}} \deg(E)
\end{aligned}$$

where we used that the μ-function decomposes suitably for semistable bundles (cf. 1.11, 1.12 resp. Huybrechts, Lehn [HL10], Theorem 3.1.4, [La04] Corollary 6.4.14). Recall that $\tilde{\gamma}^k = \mu(E) - \mu^k$ are the weights corresponding to $\tilde{\alpha}^k$. Moreover $\mu(E^k, \tilde{\alpha}^k, \varphi_\tau) = -\sum_{j=1}^{u_{\mathrm{ps}}} \tilde{\gamma}^{i_j}$ for a suitable tuple $(i_j)_{j[u_{\mathrm{ps}}]}$. Thus $-\sum_{j=1}^{u_{\mathrm{ps}}} \tilde{\gamma}^{i_j} \leq (w_{\mathrm{ps}} r - u_{\mathrm{ps}})\mu(E) = c'$. Then

$$0 < \|(\tilde{\alpha}^k)_{k[m-1]}\|_1 \cdot \mathrm{Min}_i \leq \|(\tilde{\alpha}^k)_{k[m-1]}\|_1 \cdot \mu\left(E^k, \frac{\tilde{\alpha}^k}{\|(\tilde{\alpha}^k)_{k[m-1]}\|_1}, \varphi_\tau\right) \leq c'$$

$$\Rightarrow \|(\tilde{\alpha}^k)_{k[m-1]}\|_1 \leq \frac{c'}{\mathrm{Min}_i} = c_i.$$

Since $\|(\tilde{\alpha}^k)_{k[m-1]}\|_1 \cdot r = \mu_{max}(E) - \mu_{min}(E)$ this implies boundedness. As there are only finitely many D_i for each of the finitely many m - m being the length of the Harder-Narasimhan filtration - we see that the family of vector bundles E of fixed data $(r,\ d)$ such that there is a [parabolic] pseudo $(\underline{\varsigma} \circ \iota)$-Higgs bundle $\mathbf{E}$ with $\mathbf{E} = ((E_a\ [(E_a^{ij})_{i[s_a^j]j[|S|]}])_{a[|A|]},\ \varphi,\ L)$ a $(\kappa_a,\ \xi_a,\ \delta_{\mathrm{ps}},\ \varepsilon_{\mathrm{ps}},\ [\beta_a^{ij}])$-(semi)stable pseudo $(\underline{\varsigma} \circ \iota)$-Higgs bundle and $E \simeq \bigoplus_{a\in A} E_a^{\oplus \kappa_a}$ is bounded independent of $\varepsilon_{\mathrm{ps}} > 0$. □

Remark. (i) The result holds for parabolic as well as non-parabolic tuples and for non-parabolic $(\underline{\varsigma} \circ \iota)$-Higgs bundles. For the proofs in the non-parabolic case set $\beta^{ij} = 0$ and apart from some constants that will be different, the proofs are just the same. For the tuple case we just remove one section. The calculations stay the same when we replace $\varepsilon_{\mathrm{ps}}$ by δ, δ_{ps} by 0, $\mu(\ \cdot\ ,\ \cdot\ ,\ \varphi_\tau)$ by $\mu(\ \cdot\ ,\ \cdot\ ,\ \varphi)$ and $u_{\mathrm{ps}},\ v_{\mathrm{ps}},\ w_{\mathrm{ps}}$ by $u,\ v,\ w$.

(ii) The proof of 3.2 implies that pseudo $(\underline{\varsigma} \circ \iota)$-Higgs bundles that satisfy 3.1.II live in a bounded family too. If a bundle that satisfies 3.1.II is neither semistable as a vector bundle nor $\mu(E^k,\ \cdot\ ,\ \varphi_\tau) > 0$ holds on the D_i (that contains $(\hat{\alpha}^k)_{k[m-1]}$ then by 3.1.II.B we get $M_{[\mathrm{par}]}^{\kappa,\xi}(E^k, \alpha^k) + \delta_{\mathrm{ps}} \cdot \mu(E^k,\ \alpha^k,\ \varphi) \geq 0$ for a $(\alpha^k)_{k[m-1]} \in D_i$ - a contradiction to the construction of the D_i.

(iii) As every subbundle $F_a \subset E_a$ amounts to a subbundle $\cdots \oplus 0 \oplus F_a^{\oplus \kappa_a} \oplus 0 \oplus \cdots$ of same slope, the family of vector bundles isomorphic to one of the E_a occurring in a parabolic pseudo $(\underline{\varsigma} \circ \iota)$-Higgs bundle is bounded as well.

Before we start with the proof of 3.1 we would like to add another lemma.

3.3. Lemma. *Fix two integers b and c and let $\delta_{\mathrm{ps}} = t \cdot b,\ \varepsilon_{\mathrm{ps}} = t \cdot c,\ t \in \mathbb{R}_+$. Furthermore fix $r_a,\ d_a,\ l,\ \kappa_a,\ \xi_a,\ \delta_{\mathrm{ps}},\ [\beta_a^{ij}]$ as before.*
The family of all vector bundles E occurring as $E \simeq \bigoplus_{a\in A}(E_a)^{\oplus \kappa_a}$ in a semistable [parabolic] pseudo $(\underline{\varsigma} \circ \iota)$-Higgs bundle $\mathbf{E} = ((E_a,\ [(E_a^{ij})_{i[s_a^j]j[|S|]}])_{a[|A|]},\ \tau,\ \varphi,\ L)$ is bounded (independent of $t > 0$).

Proof of Lemma 3.3: We will use the notation of the proof of 3.2. Then we find that either the pseudo $(\underline{\varsigma} \circ \iota)$-Higgs bundle is semistable as a vector bundle or there is a Harder-Narasimhan filtration $(E^k)_{k[m]}$ of length m. In the latter case either μ_M is bounded or we find a $R > 0$ such that $M^{\kappa,\xi}_{[\mathrm{par}]}(E^k, \,\cdot\,) < 0$ on $\overline{B_R(\hat{\alpha}^k)} \subset B_1$. If μ_M is not bounded yet, we get $\delta_{\mathrm{ps}} \cdot \mu(\,\cdot\,,\,\cdot\,,\,\varphi) + \varepsilon_{\mathrm{ps}} \cdot \mu(\,\cdot\,,\,\cdot\,,\,\varphi_\tau) > 0$ on $\overline{B_R(\hat{\alpha}^k)}$. But then $b \cdot \mu(\,\cdot\,,\,\cdot\,,\,\varphi) + c \cdot \mu(\,\cdot\,,\,\cdot\,,\,\varphi_\tau) > 0$ on some compact set D_i. Repeating the proof of of 3.2 we find tuple $i^{\varphi}_{j[u]}$, $i^{\tau}_{j[u_{\mathrm{ps}}]}$ such that

$$\begin{aligned} b \cdot \mu(E^k, \tilde{\alpha}^k, \varphi) + c \cdot \mu(E^k, \tilde{\alpha}^k, \varphi_\tau) &= -b \cdot \sum_{j=1}^{u} \tilde{\gamma}^{i^{\varphi}_j} - c \cdot \sum_{j=1}^{u_{\mathrm{ps}}} \tilde{\gamma}^{i^{\tau}_j} \\ &\leq \underbrace{b \cdot \deg(L) + (bwr + cw_{\mathrm{ps}}r - bu - cu_{\mathrm{ps}})\mu(E)}_{=:C_{b,c}} \end{aligned}$$

Now if $\mathrm{Min}^{b,c}_i$ is the minimum of $b\mu(\,\cdot\,,\,\cdot\,,\,\varphi) + c\mu(\,\cdot\,,\,\cdot\,,\,\varphi_\tau)$ on the compact set D_i we have $\|(\tilde{\alpha}^k)_{k[m-1]}\|_1 \leq \frac{C_{b,c}}{\mathrm{Min}^{b,c}_i}$ and this proves the claim. □

Proof of Proposition 3.1: (II. ⇒ I.): Recall that by the remark above, the family of bundles that satisfy II. is bounded. Assume that $\mu(F^k, \alpha^k, \varphi_\tau) \geq 0$ for every tuple filtration. Than we may proceed as in 1.49, i. e. for $d^k < -|d| - u\delta_{\mathrm{ps}}$

$$\begin{aligned} \alpha^k(dr^k - rd^k) - \delta_{\mathrm{ps}} \cdot \max\{0,\ \alpha^k(ur^k - v(k,\theta)r)\} - \varepsilon_{\mathrm{ps}} \cdot \alpha^k \underbrace{(ur^k - v(k,\tilde{\theta})r)}_{\leq 0} \\ \geq \alpha^k(dr^k - rd^k - \delta_{\mathrm{ps}}ru) \geq \alpha^k r(-|d| - \delta_{\mathrm{ps}}u - d^k) \geq 0 \end{aligned}$$

and therefore the function $S_{[\mathrm{par}]}(\alpha^k) = M^{\kappa,\xi}_{[\mathrm{par}]}(F^k, \alpha^k) + \delta_{\mathrm{ps}} \cdot \mu(F^k, \alpha^k, \varphi) + \varepsilon_{\mathrm{ps}} \cdot \mu(F^k, \alpha^k, \varphi_\tau)$ can be minimized only if $\alpha^k = 0$. Now the argument of 1.49 applies and we find a finite set $\Xi \subset \mathbb{Q}^r \cap [0,1]^r$ of tuples $(\alpha^i)_{i[r]}$ to be checked to guarantee the semistability. Further we find an integer z such that $z\Xi \subset \mathbb{Z}[1/r]^r$ and if $S_{[\mathrm{par}]}(\alpha^k)(\leq)0$ for any $(\alpha^k)_{k[r]} \in \mathbb{Z}[1/r]^r$, then $\min_{(\alpha^k)_{k[r]} \in z\Xi} S_{[\mathrm{par}]}(\alpha^k)(\leq)0$. Now denote by $-\infty < m_{[\mathrm{par}]}$ the minimum of $M^{\kappa,\xi}_{[\mathrm{par}]}(F^k, \alpha^k) + \delta_{\mathrm{ps}}\mu(F^k, \alpha^k, \varphi)$ over all $(\alpha^k)_{k[r]} \in [0,1]^r$ and all types of filtrations $(F^k)_{k[r]}$ and set

$$\varepsilon^{\infty,1}_{\mathrm{ps}} = rz|m_{[\mathrm{par}]}|.$$

Assume $\varepsilon_{\mathrm{ps}} > \varepsilon^{\infty,1}_{\mathrm{ps}}$. We have to check the semistability condition for every $(\alpha^k)_{k[r]} \in z\Xi$ and every filtration. If for such a $(\alpha^k)_{k[r]}$ we have $\mu(F^k, \alpha^k, \varphi_\tau) = 0$ for any filtration $(F^k)_{k[r]}$ then obviously II.B. implies $M^{\kappa,\xi}_{[\mathrm{par}]}(F^k, \alpha^k) + \delta_{\mathrm{ps}} \cdot \mu(F^k, \alpha^k, \varphi) + \varepsilon_{\mathrm{ps}} \cdot \mu(F^k, \alpha^k, \varphi_\tau)\ (\geq)\ 0$. If on the other hand $\mu(F^k, \alpha^k, \varphi_\tau) > 0$ then already $\mu(F^k, \alpha^k, \varphi_\tau) \geq 1/r$ and hence $\varepsilon_{\mathrm{ps}} \cdot \mu(F^k, \alpha^k, \varphi_\tau) > z \cdot |m_{[\mathrm{par}]}|$, therefore

$$\begin{aligned}M^{\kappa,\xi}_{[\mathrm{par}]}(F^k,\alpha^k)+\delta_{\mathrm{ps}}\cdot\mu(F^k,\alpha^k,\varphi)+\varepsilon_{\mathrm{ps}}\cdot\mu(F^k,\alpha^k,\varphi_\tau)&\\ > M^{\kappa,\xi}_{[\mathrm{par}]}(F^k,\alpha^k)+\delta_{\mathrm{ps}}\cdot\mu(F^k,\alpha^k,\varphi)+z\cdot|m_{[\mathrm{par}]}|\geq 0,&\end{aligned}$$

and thus II. implies I.
(I. $\Rightarrow$ II.): This second implication will be proven along the lines of a proof given by Alexander Schmitt in the special situation of a pseudo G-bundle without an additional Higgs field ([Sch08], 2.3.6.5). Since the family of semistable pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundles is $\varepsilon_{\mathrm{ps}}-$uniformly bounded by 3.2, we may construct the parameter scheme $\mathfrak{P}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$ big enough that it parametrizes all pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundles that are semistable for some $\varepsilon_{\mathrm{ps}}>0$. Let $\mathfrak{P}^{\varepsilon_{\mathrm{ps}}-ss}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$ denote the open subset of $\varepsilon_{\mathrm{ps}}$-semistable objects. If we set $\mathfrak{P}^{\varepsilon_{\mathrm{ps}}-ss}_{\underline{\varsigma}\circ\iota,\varepsilon_{\mathrm{ps}}\leq\hat{\varepsilon},[\mathrm{par}]}=\bigcup_{\varepsilon_{\mathrm{ps}}\leq\hat{\varepsilon}}\mathfrak{P}^{\varepsilon_{\mathrm{ps}}-ss}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$ and $\mathfrak{P}^{\infty-ss}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}=\bigcup_{\varepsilon_{\mathrm{ps}}>0}\mathfrak{P}^{\varepsilon_{\mathrm{ps}}-ss}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$, then $\mathfrak{P}^{\infty-ss}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$ is open and we find a $\overline{\varepsilon}$ such that $\mathfrak{P}^{\infty-ss}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}=\mathfrak{P}^{\varepsilon_{\mathrm{ps}}-ss}_{\underline{\varsigma}\circ\iota,\varepsilon_{\mathrm{ps}}\leq\overline{\varepsilon},[\mathrm{par}]}$. For the last statement recall that $M^{\kappa,\xi}_{[\mathrm{par}]}(F^k,\alpha^k)+\delta_{\mathrm{ps}}\cdot\mu(F^k,\alpha^k,\varphi)$ is bounded from below and that a non-semistable bundle can become semistable when increasing $\varepsilon_{\mathrm{ps}}$ only if $\mu(F^k,\alpha^k,\varphi_\tau)>0$ whenever $M^{\kappa,\xi}_{[\mathrm{par}]}(F^k,\alpha^k)+\delta_{\mathrm{ps}}\cdot\mu(F^k,\alpha^k,\varphi)<0$. As we have seen above for large enough $\varepsilon_{\mathrm{ps}}$ every such bundle is already semistable. Furthermore this observation directly implies that if $\overline{\varepsilon}\leq\varepsilon_1\leq\varepsilon_2$ then $\mathfrak{P}^{\varepsilon_2-ss}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}\subset\mathfrak{P}^{\varepsilon_1-ss}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$ since enlarging $\varepsilon_{\mathrm{ps}}\geq\overline{\varepsilon}$ further will only result in some of the bundles that fail to satisfy A. in 3.2.II to drop out. Alternatively an argument as in [Sch08] 2.3.6.6 will work, too.
Let $\mathfrak{V}^{ass}_{[\mathrm{par}]}$ be the set of all [parabolic] pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundles that satisfy 3.2.II. In order to complete the proof of 3.1 we need the following lemma:

3.4. Lemma. *The set* $\mathfrak{V}^{ass}_{[\mathrm{par}]}\subset\mathfrak{P}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$ *is open.*

Remark. In the case of $|A|=1$ Higgs tuples this is the statement of [Sch08] 2.3.6.8..

Proof of Lemma 3.4. By the Hilbert-Mumford criterion (cf. 1.31) condition A. in 3.2.II. is equivalent to the restriction of $\varphi_\tau|_\eta\in\mathbb{P}(E_{\underline{\varsigma}}|_\eta)$ to the generic point η being $\mathrm{Sl}(E_{\underline{\varsigma}}|_\eta)$-semistable w. r. t. the natural action on the $\mathbb{C}(X)$-vector space $E_{\underline{\varsigma}}|_\eta$. As usual the semistable points form an open subset. Let $\mathbb{P}^{ns}$ denote the $\times_{a\in A}\mathrm{Gl}(\mathbb{C}^{r_a})-$invariant closed set of non-semistable points in the projective fiber $\mathbb{P}(E_{\underline{\varsigma}}|_\eta)$. Since the universal homomorphism $\varphi_{\tau,\mathfrak{P}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}}$ on $\mathfrak{P}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}\times X$ is fiberwise non-trivial and since it maps into a line bundle it is thus fiberwise generically surjective. We may henceforth restrict it to the largest open subset $U\subset\mathfrak{P}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}\times X$ where it is surjective. By [Ha77] II.7.12 this yields a section $\phi:U\to\mathbb{P}(\mathscr{E}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]},\underline{\varsigma}})$. Now $\mathbb{P}^{ns}$ on the generic point induces a closed subscheme $C^{ns}\subset\mathbb{P}(\mathscr{E}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]},\underline{\varsigma}})$ and hence $\phi^{-1}(C^{ns})\subset U$. The closure $\overline{\phi^{-1}(C^{ns})}\subset\mathfrak{P}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}\times X\xrightarrow{\pi}\mathfrak{P}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$ maps properly to $\mathfrak{P}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$ (since X is projective) and hence the semi-continuity theorem in [EGA] IV.13.1.5 implies that the set $D^{ns} := \{b \in \mathfrak{P}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]} | \dim(\pi|^{-1}_{\phi^{-1}(C^{ns})}(b)) \geq 1\} \subset \mathfrak{P}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$ is closed. Finally by construction D^{ns} parametrizes those G-Higgs bundles that do not satisfy 3.2.II.A. Let $V^{ss} = \mathfrak{P}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]} \setminus D^{ns}$ be the open complement. Then by definition $\mathfrak{V}^{ass}_{[\mathrm{par}]} \subset V^{ss}$ and $\mathfrak{V}^{ass}_{[\mathrm{par}]} \supset V^{ss} \cap \mathfrak{P}^{\varepsilon_{\mathrm{ps}}-ss}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$ holds for every $\varepsilon_{\mathrm{ps}} > 0$. Moreover by the (II. $\Rightarrow$ I.)-direction of 3.1 $\mathfrak{V}^{ass}_{[\mathrm{par}]} \subset V^{ss} \cap \mathfrak{P}^{\varepsilon_{\mathrm{ps}}-ss}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$ holds for $\varepsilon_{\mathrm{ps}}$ big. Therefore as union of open sets $\mathfrak{V}^{ass}_{[\mathrm{par}]} = \bigcup_{\varepsilon_{\mathrm{ps}}>0} V^{ss} \cap \mathfrak{P}^{\varepsilon_{\mathrm{ps}}-ss}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$ is open. □

Completion of the proof of 3.1. Since $\mathfrak{P}^{\varepsilon_{\mathrm{ps}}-ss}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$, $\varepsilon_{\mathrm{ps}} \geq \overline{\varepsilon}$ is a decreasing series of open sets and $\mathfrak{V}^{ass}_{[\mathrm{par}]} = \bigcap_{\varepsilon_{\mathrm{ps}}\geq\overline{\varepsilon}} \mathfrak{P}^{\varepsilon_{\mathrm{ps}}-ss}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$ is open the series becomes stationary and we find $\varepsilon^{\infty,2}_{\mathrm{ps}}$: $\mathfrak{V}^{ass}_{[\mathrm{par}]} = \mathfrak{P}^{\varepsilon^{\infty,2}_{\mathrm{ps}}-ss}_{\underline{\varsigma}\circ\iota,[\mathrm{par}]}$. Now set $\varepsilon^{\infty}_{\mathrm{ps}} = \max\{\varepsilon^{\infty,1}_{\mathrm{ps}}, \varepsilon^{\infty,2}_{\mathrm{ps}}\}$.
Finally note that for the given $\varepsilon^{\infty}_{\mathrm{ps}}$ as above, if a bundle is even stable it is in particular semistable and thus satisfies II. with $\geq$ in part B. But then stability implies that even the strict inequality in B. has to hold. This completes the proof of the second direction and hence 3.1 is proved. □

3.5. Conclusion. *Given the same conditions as in 3.1 we find for every [parabolic] pseudo $(\underline{\varsigma}\circ\iota)$-Higgs bundle* $\mathbf{E} = ((E_a,\ [(E^{ij}_a)_{i[s^j_a]j[|S|]}])_{a[|A|]},\ \tau,\ \varphi,\ L)$ *that the condition 3.1.I is equivalent to* $\mathbf{E}$ *being a* $(\xi_a,\ \delta_{\mathrm{ps}},\ [\beta^{ij}_a])$*-semistable* $(\underline{\varsigma}\circ\iota)$*-Higgs bundle of suitable topological type.*

Proof. By Proposition 3.1 we may replace 3.1.I by 3.1.II. Proposition 2.25 and II.A in 3.1 show that $\mathbf{E}$ comes from a principal G-bundle. 2.29 implies the claim. □

Remark. The non-parabolic version is proved in [Sch08], Corollary 2.7.2.6.

For future use we will add two more theorems on asymptotic semistability now.

3.6. Lemma. *Fix a character ξ of G and parabolic subgroups* $Q_{\mathrm{Gl}(U_a)_a}(\iota\circ\tau^j) = P^j_{\mathrm{Gl}} \subset \mathrm{Gl}(U_a)_a$ *for some one-parameter subgroups τ^j of G and every puncture $x^j \in S$.*

(i) *The family of [ι-parabolic] principal ς-Higgs bundles* $(P,\ [(s^j)_{j[|S|]}],\ L,\ \varphi)$ *that satisfy the conditions A. and B. below is bounded.*

  A. *For every one-parameter subgroup λ of G and every $\mathscr{R}: X \to P/Q_G(\lambda)$:* $\mu(\lambda,\varphi) \geq 0$.

  B. *For every one-parameter subgroup λ of G and every $\mathscr{R}: X \to P/Q_G(\lambda)$:* $\mu(\lambda,\varphi) = 0 \Rightarrow M^{(1),\xi}_{[\mathrm{par}]}(F^k,\alpha^k)\ (\geq)\ 0$ *for a weighted filtration* $(F^k,\alpha^k)_{k[r]}$ *corresponding to λ, $\mathscr{R}$.*

(ii) *The family of [ι-parabolic] principal ς-Higgs bundles* $(P,\ [(s^j)_{j[|S|]}],\ L,\ \varphi)$ *that are* $(\delta_{\mathrm{ps}},\ [\tau^j])$*-semistable for some* $\delta_{\mathrm{ps}} \in \mathbb{Q}_+$ *is bounded.*

(iii) *There is a* $\delta_{\text{ps}}^{\infty} > 0$ *such that for some* $\delta_{\text{ps}} > \delta_{\text{ps}}^{\infty}$: *[*ι*-parabolic] principal* ς*-Higgs bundles* $(P,\ [(s^j)_{j[|S|]}],\ \varphi,\ L)$ *that satisfy the conditions A. and B. are exactly the [*ι*-parabolic]* $(\delta_{\text{ps}},\ [\tau^j])$*-(semi)stable principal* ς*-Higgs bundles.*[5]

(iv)^1 *Fix two rational numbers* $b,\ c \in \mathbb{Q}_+$. *There is a* $T > 0$ *such that a [*ι*-parabolic] principal* $(\underline{\varsigma} \circ \iota)$*-Higgs bundle satisfies A. and B. of part* (i), *if and only if the corresponding [parabolic] pseudo* $(\underline{\varsigma} \circ \iota)$*-Higgs-bundle is (semi)stable for every pair* $(\delta_{\text{ps}},\ \varepsilon_{\text{ps}})$ *with* $\delta_{\text{ps}} = b \cdot t,\ \varepsilon_{\text{ps}} = c \cdot t$ *whenever* $t > T$.

(iv)^2 *There is a* $\delta_{\text{ps}} > 0$ *and a* $\varepsilon_{\text{ps}} > 0$ *(which possibly depends on the chosen* δ_{ps}*) such that a [*ι*-parabolic]* $(\underline{\varsigma} \circ \iota)$*-Higgs bundle satisfies* (i) *if and only if the corresponding [parabolic] pseudo* $\underline{\varsigma}$*-Higgs-bundle is* $(\delta_{\text{ps}},\ \varepsilon_{\text{ps}})$*-(semi)stable.*

Proof. Recall first that our representations ς and ι give rise to a representation $\underline{\varsigma}$ such that $\varsigma \subset \underline{\varsigma} \circ \iota$. Therefore by 2.25 every principal $(\underline{\varsigma} \circ \iota)$-Higgs bundle P gives rise to a pseudo $(\underline{\varsigma} \circ \iota)$-Higgs bundle E if and only if $\mu(F^k, \alpha^k, \varphi_\tau) \geq 0$ holds for every weighted filtration $(F^k,\ \alpha^k)_{k[r]}$ of E. The ς-Higgs bundles form a subset of the $(\underline{\varsigma} \circ \iota)$-Higgs bundles.

(i) 2.29 implies that every filtration $(F^k,\ \alpha^k)_{k[r]}$ with $\mu(F^k, \alpha^k, \varphi_\tau) = 0$ comes from a one-parameter subgroup λ of G and a reduction $\mathscr{R} : X \to P/Q_G(\lambda)$. Thus $\mu(F^k, \alpha^k, \varphi_\tau) = 0 \Rightarrow \mu(F^k, \alpha^k, \varphi) \geq 0$ by part A. Furthermore if $\mu(F^k, \alpha^k, \varphi_\tau) = \mu(F^k, \alpha^k, \varphi) = 0$ then $M^{(1),\xi}_{[\text{par}]}(F^k, \alpha^k) \geq 0$ by part B. Now repeat the proof of 3.2, i. e. assume that a bundle is not semistable as a vector bundle. Let $(E^k, \hat{\alpha}^k)_{k[m]}$ be as in 3.2. We find a compact set D_i, $(\hat{\alpha}^k)_{k[m-1]} \in D_i$ where the continuous function $f(\ \cdot\ ) = \max\{\mu(E^k,\ \cdot\ , \varphi_\tau),\ \mu(E^k,\ \cdot\ , \varphi_\tau) + \mu(E^k,\ \cdot\ , \varphi)\}$ attains a positive minima (bounded from below by a positive constant). Observe that f becomes zero if and only if $\mu(E^k,\ \cdot\ , \varphi_\tau) = 0$ and hence $\mu(E^k,\ \cdot\ , \varphi) = 0$. This case however cannot occur by B. and the construction of D_i, i. e. $M^{(1),\xi}_{[\text{par}]}(E^k, \alpha^k) < 0$ on D_i.

(ii) Consider again the function $f(\ \cdot\ ) = \max\{\mu(E^k,\ \cdot\ , \varphi_\tau),\ \mu(E^k,\ \cdot\ , \varphi_\tau) + \mu(E^k,\ \cdot\ , \varphi)\}$ on a suitable set $D_i \ni (\hat{\alpha}^k)_{k[m-1]}$ whenever the underlying vector bundle is not semistable. f is non-negative and will have a zero on D_i if and only if $\mu(E^k,\ \cdot\ , \varphi_\tau) = 0 \Rightarrow \mu(E^k,\ \cdot\ , \varphi) \leq 0$. But then $M^{(1),\xi}_{[\text{par}]}(E^k, \alpha^k) + \delta_{\text{ps}} \cdot \mu(E^k, \alpha^k, \varphi) \leq M^{(1),\xi}_{[\text{par}]}(E^k, \alpha^k) < 0$ for every $\delta_{\text{ps}} > 0$ by construction of the D_i in contradiction to δ-semistability.

(iii) For (i) $\Rightarrow$ (ii) we first *claim* that only a finite set Ξ of $(\alpha^k)_{k[r]} \in (\mathbb{Q} \cap [0, 1])^r$ (resp. one-parameter subgroups) has to be checked to guarantee $(\delta_{\text{ps}},\ [\tau^j])$-(Semi)stability of principal ς-Higgs bundles.

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[5] Again stability corresponds to the strict inequality in B.

Proof of the claim. The argument is almost the same as in the proof of 3.1. Again we search for minimizers of some function $S_{[\mathrm{par}]}(\alpha^k) := M^{(1),\xi}_{[\mathrm{par}]}(F^k, \alpha^k) + \delta_{\mathrm{ps}}\mu(F^k, \alpha^k, \varphi)$ on $[0,1]^r$. However we need to be careful since we have the additional condition $\mu(F^k, \alpha^k, \varphi_\tau) = 0$, i. e. we have to minimize over a subvariety. Observe that $\mu(F^k, \alpha^k, \varphi_\tau) = 0$ implies that the $(\alpha^k)_{k[r]}$ lie in a subset of $\mathbb{R}^r_{\geq 0}$ defined by a finite number of equations of the form $\sum_{k=1}^r \alpha^k f^k_l = 0$ or $\sum_{k=1}^r \alpha^k f^k_l \geq 0$ for some $(f^k_l)_{k[r]}$ in the finite set $\{-u_{\mathrm{ps}}r, \dots, u_{\mathrm{ps}}r\}^r$. Furthermore note that $-(d^k + |d|)r \leq dr^k - d^k r \leq (|d| + c)r$. Here c is the upper bound on the degrees existing in our bounded family. As in 3.1 we find a constant c' such that for every choice of $(d^k)_{k[r]} \in \{\{x \in \mathbb{Z} : x < c'\}, c', \dots, c\}^r$ we find a minimizer of $S_{[\mathrm{par}]}(\,\cdot\,) := M^{(1),\xi}_{[\mathrm{par}]}(F^k, \,\cdot\,)$. We see that only those $(\alpha^k)_{k[r]}$ in a finite set $\Xi \subset \mathbb{Q}^r_{\geq 0}$ have to be checked to guarantee $(\delta_{\mathrm{ps}},\ [\tau^j])$-semistability (cf. 1.49). $\square$

Now we find a constant z such that $z(\alpha^k)_{k[r]} \in z\Xi \subset \mathbb{Z}[1/r]^r$ and thus $\delta^\infty_{\mathrm{ps}} = rz|m_{[\mathrm{par}]}|$, where $m_{[\mathrm{par}]}$ is the minimum of $M^{(1),\xi}_{[\mathrm{par}]}(F^k, \alpha^k)$ over all $(\alpha^k)_{k[r]} \in [0,1]^r$ and all types of filtrations $(F^k)_{k[r]}$. Hence for $\delta_{\mathrm{ps}} > \delta^\infty_{\mathrm{ps}}$ we get $M^{(1),\xi}_{[\mathrm{par}]}(F^k, \alpha^k) + \delta_{\mathrm{ps}} \cdot \mu(F^k, \alpha^k, \varphi) \geq M^{(1),\xi}_{[\mathrm{par}]}(F^k, \alpha^k) + z|m_{[\mathrm{par}]}| \geq 0$ for all $(\alpha^k)_{k[r]} \in z\Xi$ and all filtrations $(F^k)_{k[r]}$, i. e. δ_{ps}-semistability.
For the other direction (ii) ⇒ (i) observe that every δ_{ps}-semistable ς-Higgs bundle is semistable as a pseudo $(\underline{\varsigma} \circ \iota)$-Higgs bundle for some stability parameter $\varepsilon_{\mathrm{ps}} > 0$ (that does depend on δ_{ps}). Furthermore we find a $\overline{\delta}_{\mathrm{ps}}$ such that a δ_{ps}-semistable ς-Higgs bundle is already $\overline{\delta}_{\mathrm{ps}}$-semistable for all $\delta_{\mathrm{ps}} \geq \overline{\delta}_{\mathrm{ps}}$. Therefore we may construct the scheme $\mathfrak{P}_{\underline{\varsigma}\circ\iota,[\iota-\mathrm{par}]}$ large enough, such that it parametrizes all $(\delta_{\mathrm{ps}},\ \varepsilon_{\mathrm{ps}})$-semistable pseudo $(\underline{\varsigma} \circ \iota)$-Higgs bundles for $0 < \delta_{\mathrm{ps}} \leq \overline{\delta}_{\mathrm{ps}}$ and $0 < \varepsilon_{\mathrm{ps}}$ arbitrary. Then the δ_{ps}-semistable ς-Higgs bundles form an open subset $\mathfrak{P}^{\delta_{\mathrm{ps}}-ss}_{\varsigma,[\iota-\mathrm{par}]}$ of $\mathfrak{P}_{\underline{\varsigma}\circ\iota,[\iota-\mathrm{par}]}$ for all $\delta_{\mathrm{ps}} > 0$. Thus the proof of 3.1, (I) ⇒ (II) will work in this situation as well if we are able to show that the bundles that satisfy (i).A and B form open subsets $\mathfrak{U}^{a(s)s}_{[\iota-\mathrm{par}]}$ of $\mathfrak{P}_{\underline{\varsigma}\circ\iota,[\iota-\mathrm{par}]}$. We will need the following result by Alexander Schmitt:

3.7. Proposition. ([Sch05], Proposition 2.9). *Given two representations* $\varsigma_i : G \to \mathrm{Gl}(W_i)$, $i = 1, 2$, *there is a rational number* $\hat{\delta}_\infty$ *such that for every* $\hat{\delta} > \hat{\delta}_\infty$ *a point* $(x_1, x_2) \in \mathbb{P}(W_1) \times \mathbb{P}(W_2)$ *is semistable w. r. t. the linearization induced by* ς_1, ς_2 *on* $\mathcal{O}_{\mathbb{P}(W_1)\times\mathbb{P}(W_2)}(1, \hat{\delta})$, *if* x_2 *is semistable w. r. t. the* ς_2*-induced linearization on* $\mathcal{O}_{\mathbb{P}(W_2)}(1)$ *and for every one-parameter subgroup* λ *of* G *with* $\mu(x_2, \lambda) = 0 : \mu(x_1, \lambda) \geq 0$.

Proof. The proof is a direct consequence of $\mu_{\mathcal{O}_{\mathbb{P}(W_1)\times\mathbb{P}(W_2)}(1,\hat{\delta})}((x_1, x_2), \lambda) = \mu(x_1, \lambda) + \hat{\delta} \cdot \mu(x_2, \lambda)$ (cf. 1.34) and the fact that the μ-functions are discrete valued and that they have a finite number of minimizers. For details consult Alexander Schmitt [Sch05], Proposition 2.9. □

As in 3.4 we may replace the conditions $\mu(F^k, \alpha^k, \varphi_\tau) \geq 0$ and A. by the statement, that $(\varphi,\ \varphi_\tau)$ is semistable as an element of $\mathbb{P}(\mathscr{E}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,\varsigma}}|_\eta) \times \mathbb{P}(\mathscr{E}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,\varsigma_{\mathrm{ps}}}}|_\eta)$ w. r. t. a suitable linearization as given by the theorem; here η denotes as usual the generic point of X. Now we are in the situation where the proof of 3.1 works.

(iv^1) Use 3.3 and the same arguments as in (iii) resp. 3.4.

(iv^2) By (iii) we find a $\delta_{\mathrm{ps}} > 0$ such that (i) $\Leftrightarrow$ (ii). Furthermore by 3.1 we find a $\varepsilon_{\mathrm{ps}} > 0$ (that does depend on the choice of δ_{ps}) such that (ii) is equivalent to 3.1.I.
□

Remark. Alternative proofs for the non-parabolic version of the theorems may be found in [Sch08], 2.7.

3.8. Conclusion. *If the stability parameters are chosen such that 3.6 (iii) holds, then the moduli space $\mathfrak{U}^{ass}_{[\iota-\mathrm{par}]} /\!\!/ \mathcal{G}_A$ exists as a projective scheme and contains the geometric quotient $\mathfrak{U}^{as}_{[\iota-\mathrm{par}]}/\mathcal{G}_A$ as an open subscheme.*

Proof. This is a direct consequence of 3.6, since $\mathfrak{U}^{ass}_{[\iota-\mathrm{par}]}$ is the same as $\mathfrak{P}^{ss}_{\varsigma,[\iota-\mathrm{par}]}$. □

3.2. THE AFFINE CASE

3.9. Definition. A parabolic affine ϱ-Higgs bundle over $(X,\ S)$ is a pair $((P,\ (s^j)_{j[|S|]}),\ (\phi^i)_{i[m]})$ consisting of a parabolic principal G-bundle $(P,\ (s^j)_{j[|S|]})$ and sections $\phi^i : X \to P_{\tilde{\varrho}^i} \otimes L^i = (P \times_{\tilde{\varrho}^i} \tilde{W}^i) \otimes L^i$ given irreducible representations $\tilde{\varrho}^i : G \to \mathrm{Gl}(\tilde{W}^i)$ and line bundles $L^i \to X,\ 1 \leq i \leq m$. Equivalently we may replace $\phi^i : X \to P_{\tilde{\varrho}^i} \otimes L^i$ with a homomorphism $\varphi^i : P_{\varrho^i} \to L^i$ where ϱ^i is the contragredient representation to $\tilde{\varrho}^i$ on the dual space $(\tilde{W}^i)^\vee =: W^i$.

Remark. Below we will usually use the second description $\varphi^i : P_{\varrho^i} \to L^i$ where $\varrho^i : G \to \mathrm{Gl}(W^i),\ 1 \leq i \leq m$ is a representation. L^0 denotes $\mathcal{O}_X$.

The definition of semistability for affine bundles may be deduced from the weight function in the projective case 2.17. Let $\varrho = \bigoplus_{i=1}^m \varrho^i$, then we have the projection $\pi^i : P_\varrho \to P_{\varrho^i}$, under ϱ a one-parameter subgroup $\lambda : \mathbb{C}^* \to G$ and a reduction $\mathscr{R} : X \to P/Q_G(\lambda)$ is associated to a filtration F^k_ϱ of P_ϱ. Define

$$\mu(\lambda, \mathscr{R}, \varphi) = \begin{cases} 0 & \text{if } \varphi^i = 0, \forall\ 1 \leq i \leq m \\ -\min\{\gamma^k |\ \exists\ 1 \leq i \leq m : \varphi^i \circ \pi^i|_{F^k_\varrho} \not\equiv 0\} & \text{otherwise.} \end{cases}$$

3.10. Definition. A affine parabolic ϱ-Higgs bundle is called χ-(semi)stable for a rational character χ of G, if for every one-parameter subgroup λ of G and every reduction $\mathscr{R} : X \to P/Q_G(\lambda)$ for which $\mu(\lambda,\ \mathscr{R},\ \varphi) \leq 0$ already $M_{[\mathrm{par}]}^{(1),(0)}(F^k, \alpha^k) + \langle \lambda, \chi \rangle\ (\geq)\ 0$ holds. We will write $M_{[\mathrm{par}]} := M_{[\mathrm{par}]}^{(1),(0)}$.

3.11. Definition. Let Y be a scheme of finite type over $\mathbb{C}$, L^i fixed line bundles over X and τ^j fixed parabolic weights to given parabolic subgroups $P^j \subset G$. A Y-family of affine [parabolic] ϱ-Higgs bundles is a tuple $(\mathscr{P}_Y,\ [(s_Y^j)_{j[|S|]}],\ \varphi_Y)$ where

1. $\mathscr{P}_Y$ is a principal G-bundle on $Y \times X$ of the given topological type over every point $\{y\}$;
2. $\varphi_Y \in \bigoplus_{i=1}^m \mathrm{Hom}(\mathscr{P}_{\varrho^i,Y}, \pi_X^*(L^i))$;

[3.] $s_Y^j : Y \times \{x^j\} \to \mathscr{P}_Y \times_X \{x^j\}/Q_G(\tau^j)$ for all $x^j \in S$.

Two families are isomorphic if there is a G-bundle morphism $\psi_Y : \mathscr{P}_Y^1 \to \mathscr{P}_Y^2$ such that $\varphi_Y^2 \circ \psi_{Y,\varrho} = \varphi_Y^1$ for the induced isomorphism $\psi_{Y,\varrho} : \mathscr{P}_{Y,\varrho}^1 \to \mathscr{P}_{Y,\varrho}^2$ as well as $\psi_Y^j(s_Y^{j,1}) = s_Y^{j,2}$ for the induced isomorphism $\psi_Y^j : \mathscr{P}_Y^1/Q_G(\tau^j) \to \mathscr{P}_Y^2/Q_G(\tau^j)$.

3.12. In order to reduce the general affine case to the projective case we need to associate to every representation $\varrho : G \to \mathrm{Gl}(W)$ a homogeneous representation $\varsigma : G \to \mathrm{Gl}(\underline{W})$. We follow the approach of [Sch08], 2.8.2.
Since G is reductive, $\varrho = \bigoplus_{i=1}^m \varrho^i$ decomposes into irreducible representations ϱ^i. These are homogeneous (see the remark to 2.14). After fixing an embedding $\iota : G \hookrightarrow \mathrm{Gl}(U_a)_a$, 2.12 implies the existence of an irreducible extension $\underline{\varrho}^i : \mathrm{Gl}(U_a)_a \to \mathrm{Gl}(\underline{W}^i)$[6] such that $\varrho^i \subset \underline{\varrho}^i \circ \iota$ is a subrepresentation. Let $u^i \in \mathbb{Z}$ be such that $\underline{\varrho}^i(z \cdot \mathrm{id}_{\mathrm{Gl}(U_a)_a}) = z^{u^i} \cdot \mathrm{id}_{\mathrm{Gl}(W)}$. W. l. o. g. we may assume that $0 < u^0 < u^1 < \ldots < u^m$ for some u^0.[7] Define

$$\underline{\varsigma} := \bigoplus_{\substack{v \in \mathbb{Z}_{\geq 0}^{m+1}, \\ vu^t = \mathrm{lcm}(u^i)}} \bigotimes_{j=0}^{m} \underline{\varrho}^{i,\otimes v^j} : \mathrm{Gl}(U_a)_a \to \mathrm{Gl}(\underline{W}), \quad u = (u^0, \ldots, u^m),\ \underline{\varrho}^0 = \det,\ \varrho^0 = 1.$$

Now $\underline{\varsigma}$ is homogeneous.
Consequentially we may associate to every affine ϱ-family a corresponding projective ς-family with $\varsigma = \underline{\varsigma} \circ \iota$. Let Y be a scheme and $(\mathscr{P}_Y,\ [(s_Y^j)_{j[|S|]}],\ \varphi_Y)$ be a

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[6] $\hat{\varrho}^i \circ \iota = \varrho^i \oplus \tilde{\varrho}^i$, $\hat{\varrho}^i = \bigoplus_{j=1}^k \hat{\varrho}^{ij}$, $\hat{\varrho}^{ij}$ irreducible, i. e. $\varrho^i \subset \hat{\varrho}^{ij_0} \circ \iota$ for one $1 \leq j_0 \leq k$. Set $\underline{\varrho}^i = \hat{\varrho}^{ij_0}$.

[7] [Sch08], 2.8.2. In fact, the determinant representation (and any power thereof) lifts the trivial representation $\varrho : G \to \mathrm{Sl}(W)$. Thus we may replace $\underline{\varrho}^i$ with $\underline{\varrho}^i \otimes \det^{\underline{u}^i}$ for a suitable $\underline{u}^i \in \mathbb{Z}$.

Y-family of affine ϱ-Higgs bundles.
Given L^i there is a line bundle L on X that admits injective morphisms $\iota^j : L^j \to L^{\otimes u^j}$, $0 \leq j \leq m$.[8] Furthermore let $\pi^i : \mathscr{P}_{Y,\underline{\varrho}^i \circ \iota} \to \mathscr{P}_{Y,\varrho^i}$ and $\underline{\varphi}^i = \pi_X^*(\iota^i) \circ \varphi_Y^i \circ \pi^i$, $1 \leq i \leq m$ the resulting Higgs fields. Let $h_Y : Y \times X \to \mathbb{C}$ be a morphism non-trivial over $y \in Y$ and set also $\underline{\varphi}^0 = h_Y \cdot \pi_X^*(\iota^0) : \mathscr{P}_{Y,\underline{\varrho}^0 \circ \iota} = \mathcal{O}_{Y \times X} \to \pi_X^*(L^{\otimes u^0})$.[9] Then

$$\mathscr{P}_{Y,\underline{\varsigma} \circ \iota} = \bigoplus_{\substack{v \in \mathbb{Z}_{\geq 0}^{m+1}, \\ vu^t = \operatorname{lcm}(u^i)}} \bigotimes_{j=0}^{m} \mathscr{P}_{Y,\underline{\varrho}^j \circ \iota}^{\otimes v^j} \to \pi_X^*(L^{\otimes (v^t u)}) = \pi_X^*(L^{\otimes \operatorname{lcm}(u^i)}),$$

$$\varphi_{Y,\underline{\varsigma} \circ \iota} := \bigoplus_{\substack{v \in \mathbb{Z}_{\geq 0}^{m+1}, \\ vu^t = \operatorname{lcm}(u^i)}} \bigotimes_{j=0}^{m} \underline{\varphi}_Y^{j,\otimes v^j}.$$

Thus we constructed a family $(\mathscr{P}_Y,\ (s_Y^j)_{j[|S|]},\ \varphi_{Y,\underline{\varsigma} \circ \iota},\ \upsilon_Y,\ \mathscr{H}_Y)$ where $\mathscr{H}_Y$ is a suitable line bundle on Y such that $(\upsilon_Y \times \mathrm{id}_X)^*(\mathscr{P}^l) \otimes \mathscr{H}_Y = \pi_X^*(L^{\otimes \operatorname{lcm}(u^i)})$ holds for a suitably fixed Poincaré line bundle $\mathscr{P}^l \to \operatorname{Jac}^l \times X$, $l = \operatorname{lcm}(u^i) \cdot \deg(L)$, and $\upsilon_Y(y) := [L^{\otimes \operatorname{lcm}(u^i)}]$.[10] In particular by choosing a non-trivial function $h : X \to \mathbb{C}$ we may assign to every affine [parabolic] ϱ-Higgs bundle $(P,\ (s^j)_{j[|S|]},\ (\varphi^i)_{i[m]})$ a projective [parabolic] $(\underline{\varsigma} \circ \iota)$-Higgs bundle $(P,\ (s^j)_{j[|S|]},\ \varphi_{\underline{\varsigma} \circ \iota},\ L^{\otimes \operatorname{lcm}(u^i)})$.

Remark. The function h_Y is introduced here to serve as a technical tool later in the construction of the moduli space. We will use it again in 5.2.

3.13. Proposition. (i) *The map*

$$\left\{ \begin{array}{c} \textit{Isomorphism classes} \\ \textit{of affine [parabolic]} \\ \varrho\textit{-Higgs bundles} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{Isomorphism classes} \\ \textit{of projective [parabolic]} \\ (\underline{\varsigma} \circ \iota)\textit{-Higgs bundles} \end{array} \right\}$$
$$(P,\ [(s^j)_{j[|S|]}],\ (\varphi^i)_{i[m]}) \longmapsto (P,\ [(s^j)_{j[|S|]}],\ \varphi_{\underline{\varsigma} \circ \iota},\ L^{\otimes \operatorname{lcm}(u^i)})$$

has finite fibers for every non-trivial map h.

(ii) *An affine [parabolic] ϱ-Higgs bundle $(P,\ [(s^j)_{j[|S|]}],\ (\varphi^i)_{i[m]})$ is $(\chi,\ \tau^j)$-(semi)stable if and only if for the associated projective [parabolic] ς-Higgs bundle $(P,\ \varphi_{\underline{\varsigma} \circ \iota},\ [(s^j)_{j[|S|]}],\ L^{\otimes \operatorname{lcm}(u^i)})$ the following properties hold:*

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[8] L ample, u^0 big $\Rightarrow H^0(X, L^{\otimes u^0}) \neq 0 \Rightarrow\ \exists\, \mathcal{O}_X \to L^{\otimes u^0}$ one-to-one. Then inductively for some $u^1 \geq u^0$: $H^0(L^1, \pi^{1,*}(L^{\otimes u^1})) \neq 0\ \Rightarrow\ \exists\, L^1 \to L^1 \times_X L^{\otimes u^1} \to L^{\otimes u^1}$ one-to-one, a. s. o..

[9] Recall that $\underline{\varrho}^0 \circ \iota = \det(\iota) = 1$ and $\mathcal{O}_{Y \times X}$ is associated to the trivial representation.

[10] [Ha77], III.Ex.12.4.

A. $\mu(\lambda, \mathscr{R}, \varphi_{\underline{\varsigma}\circ\iota}) \geq 0$ *holds for an arbitrary one-parameter subgroup* λ *of* G *and every reduction* $\mathscr{R} : X \to P/Q_G(\lambda)$.

B. If $\mu(\lambda, \mathscr{R}, \varphi_{\underline{\varsigma}\circ\iota}) = 0$ *then* $M^{(1),(0)}_{[\mathrm{par}]}(F^k, \alpha^k) + \langle \lambda, \chi \rangle \ (\geq)\ 0$.

Proof. (i) If two affine ϱ-Higgs bundles are isomorphic so are the associated projective $(\underline{\varsigma} \circ \iota)$-Higgs bundles. Furthermore the parabolic filtrations stay invariant under the assignment. Hence it is enough to consider the underlying non-parabolic objects. In the non-parabolic case a proof may be found in [Sch08], 2.8.2.1. The proof is identical to that of 2.25. First note that if the classes represented by $(P,\ (\varphi^i_1)_{i[m]})$ and $(P,\ (\varphi^i_2)_{i[m]})$ have the same image, then for all $v \in \mathbb{Z}^{m+1}_{\geq 0}$ with $vu^t = \mathrm{lcm}(u^i)$: $\bigotimes^m_{j=0} \underline{\varphi}^{i,\otimes v^j}_1 = \bigotimes^m_{j=0} \underline{\varphi}^{i,\otimes v^j}_2$. As in 2.25 we restrict to the generic point and see that there is a $\mathrm{lcm}(u^i)^{\mathrm{th}}$-root of unity ζ s. t. $\underline{\varphi}^i_1 = \zeta^{u^i} \underline{\varphi}^i_2$. Since the π^i are surjective and the ι^i injective, we get $\varphi^i_1 = \zeta^{u^i} \varphi^i_2$.[11]

(ii) Consider the summand $\underline{\varphi}^{0,\otimes\,\mathrm{lcm}(u^i)/r} = h^{\mathrm{lcm}(u^i)/r} \cdot \iota^{0,\otimes\,\mathrm{lcm}(u^i)/r} \not\equiv 0$ of $\varphi_{\underline{\varsigma}\circ\iota}$. The induced filtration on $\mathcal{O}_X$ is trivial and the induced weight is therefore 0. Thus $\mu(\lambda, \mathscr{R}, \varphi_{\underline{\varsigma}\circ\iota}) \geq 0$ for an arbitrary one-parameter subgroup λ of G and every reduction $\mathscr{R} : X \to P/Q_G(\lambda)$. Hence it will be enough to show that $\mu(\lambda, \mathscr{R}, \varphi_{\underline{\varsigma}\circ\iota}) = 0$ if and only if $\mu(\lambda, \mathscr{R}, \varphi_{\underline{\varsigma}\circ\iota}) \leq 0$ if and only if $\mu(\lambda, \mathscr{R}, \varphi) \leq 0$. Assume that $\mu(\lambda, \mathscr{R}, \varphi) \leq 0$, i. e. $\varphi^j|_{F^i_j} \not\equiv 0$ implies $\gamma^i \geq 0$. Here F^i_j denotes the ϱ^j-induced filtration with weights γ^i_j, $\tilde{F}^i_j$ the $\underline{\varrho}^j \circ \iota$-induced filtration and the weights γ^i correspond to the filtration F^i w. r. t. ϱ (cf. 1.4). Observe that $\varphi_{\underline{\varsigma}\circ\iota}|_{\bigotimes^m_{j=0} \tilde{F}^{i_j,\otimes v^j}_j} \not\equiv 0 \Leftrightarrow \underline{\varphi}^j|_{\tilde{F}^{i_j}_j} \not\equiv 0,\ \forall 0 \leq j \leq m\ (v^j \neq 0)$ $\Leftrightarrow \varphi^j|_{F^{i_j}_j} \not\equiv 0,\ \forall 0 \leq j \leq m\ (v^j \neq 0)$ directly follows from the definition of $\underline{\varsigma} \circ \iota$ and $\varphi_{\underline{\varsigma}\circ\iota}$. Therefore $\mu(\lambda, \mathscr{R}, \varphi) \leq 0$ implies that $\varphi_{\underline{\varsigma}\circ\iota}|_{\bigotimes^m_{j=0} \tilde{F}^{i_j,\otimes v^j}_j} \not\equiv 0 \Rightarrow$ $\gamma^{i_j}_j \geq 0,\ \forall 0 \leq j \leq m\ (v^j \neq 0) \Rightarrow \sum^m_{j=0} v^j \gamma^{i_j}_j \geq 0$, i. e. $\mu(\lambda, \mathscr{R}, \varphi_{\underline{\varsigma}\circ\iota}) \leq 0$. On the other hand if $\mu(\lambda, \mathscr{R}, \varphi) > 0$, there is a $\gamma^i_j < 0$ with $\varphi^j|_{F^i_j} \not\equiv 0$. But then $\varphi_{\underline{\varsigma}\circ\iota}|_{(\tilde{F}^i_j)^{\otimes\,\mathrm{lcm}(u^i)/u^j}} \not\equiv 0$ and $\frac{\mathrm{lcm}(u^i)}{u^j} \cdot \gamma^i_j < 0 \Rightarrow \mu(\lambda, \mathscr{R}, \varphi_{\underline{\varsigma}\circ\iota}) > 0$.

□

3.14. 3.13 implies that the family of $(\chi, [\tau^j])$-semistable affine [parabolic] ϱ-Higgs bundles is bounded if the corresponding family of projective [parabolic] $(\underline{\varsigma}\circ\iota)$-Higgs bundles, that satisfy *A.* and *B.*, is bounded. Though these live in a bounded family by 3.6.

3.15. We already know from 3.13 that affine [parabolic] semistable ϱ-Higgs bundles have associated projective [parabolic] $(\underline{\varsigma}\circ\iota)$-Higgs bundles that satisfy *A.* and

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[11] $h \cdot \iota^0$ is generically injective.

B. in 3.13. Furthermore the GIT-Quotients for these objects do exist by 3.8. In order to pull these quotients back we need to construct a parameter scheme together with a semistability preserving equivariant affine morphism to $\mathfrak{P}_{\underline{\varsigma}\circ\iota,[\iota-\mathrm{par}]}$. We follow the approach by Alexander Schmitt in 2.8 of [Sch08].
First note again that by 3.6 the affine ϱ-Higgs bundles live in a bounded family and we can hence choose n big enough such that all constructions done previously hold. Recall that we already found a parameter scheme $\mathfrak{P}_{\underline{\varsigma}\circ\iota} \to \mathfrak{B}$ that does parametrize certain projective $(\underline{\varsigma}\circ\iota)$-Higgs bundles. Furthermore recall that on $\mathfrak{B}\times X$ we have a universal vector bundle $\mathscr{E}_{\mathfrak{B}}$. For every $m \in \mathfrak{B}$, $\mathscr{E}_{\mathfrak{B}}|_{\{m\}\times X}$ is a principal G-bundle on X (more precisely on $\{m\}\times X$). Since $\mathscr{E}_{\mathfrak{B}}$ is locally trivial over $\mathfrak{B}\times X$, we consequentially see that the reduction induced by $\tau_{\mathfrak{B}}$ fiberwise extends to a reduction of $\mathscr{E}_{\mathfrak{B}}$ to a principal G-bundle over $\mathfrak{B}\times X$. To this principal G-bundle we may associate vector bundles $\mathscr{E}_{\mathfrak{B},\varrho_i}$ on $\mathfrak{B}\times X$ and for k big enough ([Ha77], III.12.11)

$$\mathscr{F}^i_k = \mathrm{Hom}(\pi_{\mathfrak{B},*}(\mathscr{E}_{\varrho_i,\mathfrak{B}} \otimes \pi_X^*(\mathcal{O}_X(k)), \pi_{\mathfrak{B},*}(\pi_X^*(L^i(k)))), \quad 1 \le i \le m$$

$$\mathscr{F}_k := \mathscr{F}^1_k \times_{\mathfrak{B}} \cdots \times_{\mathfrak{B}} \mathscr{F}^m_k$$

is locally free over $\mathfrak{B}$. Let

$$\mathscr{F}^0_k = \mathcal{O}_{\mathfrak{B}\times X} \times_{\mathfrak{B}} \mathscr{F}_k \simeq \mathbb{C}\times\mathscr{F}_k.$$

Then using the usual $\mathcal{G}_A$-action on $\mathscr{E}_{\mathfrak{B},\varrho_i}$ induces a $\mathcal{G}_A$-action on $\mathscr{F}_k$ and thus an action on $\mathscr{F}^0_k$ as $\mathscr{E}_{\mathfrak{B},\varrho_0} \simeq \mathcal{O}_{\mathfrak{B}\times X}$ (2.31). While $\mathscr{F}^0_k$ accounts for the additional choice of $h_{\mathfrak{B}}$ used to associate affine and projective objects, $\mathscr{F}_k$ is the space that contains the closed parameter scheme $\mathfrak{A}$ over which the morphisms $\varphi^i_{\mathfrak{A}} : \mathscr{E}_{\mathfrak{A},\varrho^i} \to \pi_X^*(L^i)$ exist for all $1 \le i \le m$ (again use 1.14). The $\mathcal{G}_A$-action descends to the invariant subscheme $\mathfrak{A}$.[12] Now $\mathfrak{A}$ together with the $\mathcal{G}_A$-action fulfill the usual universal properties (1.62). Unfortunately we may not proceed as in the projective case since we are not guaranteed that $\mathfrak{A}/\!\!/\mathbb{C}^*$ does exist. Therefore we will construct a slightly bigger space (inside $\mathscr{F}^0_k$) which admits a $\mathbb{C}^*$-quotient, prove the existence of the moduli space there and subsequently realize $\mathfrak{A}$ as a subscheme thereof.
Here the morphism $h_{\mathfrak{A}}$ comes into play. We will choose it to depend only on $\mathfrak{A}$, i. e. to be constant on X. By our construction leading up to 3.13 we can now associate to our universal family of affine objects parametrized by $\mathfrak{A}$ a projective family. Hence we find the induced morphism $f : \mathfrak{A}^0 = \mathbb{C}\times\mathfrak{A} \to \mathfrak{P}_{\underline{\varsigma}\circ\iota}$ over $\mathfrak{B}$. We need to find a $\mathbb{C}^*$-action that leaves $\mathfrak{A}^0$ invariant, such that $\overline{\mathfrak{A}^0} = (\mathfrak{A}^0 \setminus 0)/\mathbb{C}^*$ is a closed subscheme of the (weighted) projective bundle $(\mathscr{F}^0 \setminus 0)/\mathbb{C}^*$ and such that f is $\mathbb{C}^*$-invariant. Therefore consider the fiberwise actions

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[12] See footnote 24 of chapter 2.

$$\begin{array}{ccc} \mathbb{C}^* \times \mathscr{O}_{\mathfrak{B}\times X} \to \mathscr{O}_{\mathfrak{B}\times X}, & \mathbb{C}^* \times \mathscr{F}^i_k \to \mathscr{F}^i_k, & 1 \le i \le m \\ (z,(m,f)) \mapsto (m, z^{-u^0} \cdot f) & (z,(m,f)) \mapsto (m, z^{-u^i} \cdot f) & \end{array}$$

which induce a $\mathbb{C}^*$-action on $\mathscr{F}^0_k$ (cf. 2.31). By construction of f it is invariant w. r. t. this action. By construction of $\mathfrak{A}$ (in particular its $\mathcal{G}_A$-invariance) it is $\mathbb{C}^*$-invariant. Observe that the $\mathbb{C}^*$-action and the $\mathcal{G}_A$ action commute (since fiberwise the $\mathbb{C}^*$-action is just the composition of the $\mathcal{G}_A$-action and $\mathbb{C}^* \to \mathcal{G}_A,\ z \mapsto z^{u^i} \cdot \mathrm{id}_{\mathcal{G}_A}$); thus the induced morphism

$$\begin{tikzcd} \overline{\mathfrak{A}^0} \arrow[rr, "\overline{f}"] \arrow[dr] & & \mathfrak{P}_{\underline{\varsigma}\circ\iota} \arrow[dl] \\ & \mathfrak{B} & \end{tikzcd}$$

is $\mathcal{G}_A$-equivariant w. r. t. to the induced $\mathcal{G}_A$-action on $\overline{\mathfrak{A}^0}$. Since $\overline{\mathfrak{A}^0} \to \mathfrak{B}$ is projective (by construction) so is $\overline{f}$ using [Ha77].II.4.8.(e). Unfortunately this morphism $\overline{f}$ does not have to be quasi-finite. The obstruction here are the points with a vanishing first component (compare to 3.13), i. e. we have to take a closer look at the first component of $\varphi_{\underline{\varsigma}\circ\iota}$. By construction f maps to the fiber $\mathfrak{P}_{\underline{\varsigma}\circ\iota,L}$ (of $\mathfrak{P}_{\underline{\varsigma}\circ\iota} \to \mathrm{Jac}^l$) over $[L^{\otimes \mathrm{lcm}(u^i)}]$, hence

$$\varphi_{\underline{\varsigma}\circ\iota,\mathfrak{P}_{\underline{\varsigma}\circ\iota}} : \underbrace{\mathscr{E}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota},\underline{\varsigma}\circ\iota}|_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,L}}}_{\mathscr{E}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,L},\underline{\varsigma}\circ\iota}} \to \pi^*_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,L}}(\underbrace{\mathscr{H}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota}}|_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,L}}}_{\mathscr{H}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,L}}}) \otimes \pi^*_X(L^{\otimes \mathrm{lcm}(u^i)}).$$

Recall that the representation $\underline{\varsigma} \circ \iota$ on $\underline{W}$ contains the trivial representation and therefore we find a G-submodule $\overline{W}$ such that $\underline{W} = \mathbb{C} \oplus \overline{W}$ and such that pr_1 is G-invariant, i. e. an element of $\mathrm{Sym}(\underline{W}^\vee)^G$. As in 2.24 we find a closed embedding $\mathrm{Proj}(\mathrm{Sym}^*(\underline{W}^\vee)^G) \hookrightarrow \mathbb{P}^s,\ [\underline{w}] \mapsto [\tau^0(\underline{w}), \ldots, \tau^s(\underline{w})]$ for some $d > 0$ and $s+1$ homogeneous degree d G-equivariant functions $\tau^j \in \mathrm{Sym}^d(\underline{W}^\vee)^G,\ 0 \le j \le s$ (see [MRed], III.§8). Choose a local trivialization U_i of $\mathfrak{P}_{\underline{\varsigma}\circ\iota,L} \times X$, then the universal homomorphism $\varphi_{\underline{\varsigma}\circ\iota,\mathfrak{P}_{\underline{\varsigma}\circ\iota}}$ induces maps $\varphi_{\underline{\varsigma}\circ\iota,i} : U_i \to \mathrm{Hom}(\underline{W},\mathbb{C}) \simeq \underline{W}^\vee$. Combining these with the τ^j leads to sections $\sigma^j \in H^0(\mathfrak{P}_{\underline{\varsigma}\circ\iota,L} \times X, (\pi^*_{\mathfrak{P}_{\underline{\varsigma}\circ\iota_L}}(\mathscr{H}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,L}}) \otimes \pi^*_X(L^{\otimes \mathrm{lcm}(u^i)}))^{\otimes d})$: Recall that the τ^j were G-invariant and of degree d, thus the $(\mathrm{Gl}(\underline{W}) \otimes \mathbb{C}^*)$-valued transition functions $c_{ik} \cdot g^{-t}_{ik}$ of $\mathscr{H}om(\mathscr{E}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,L},\underline{\varsigma}\circ\iota}, \pi^*_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,L}}(\mathscr{H}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,L}}) \otimes \pi^*_X(L^{\otimes \mathrm{lcm}(u^i)}))$ satisfy $\tau^j\varphi_{\varsigma,i} = \tau^j(c_{ik} \cdot g^{-t}_{ik}\varphi_{\underline{\varsigma}\circ\iota,k}) = c^d_{ik}\tau^j\varphi_{\underline{\varsigma}\circ\iota,k}$. Consequentially by [Ha77], III.7.12 we find a $\mathcal{G}_A$-invariant morphism $H : \mathfrak{P}_{\underline{\varsigma}\circ\iota,L} \to \mathbb{P}(H^0(X,(L^{\otimes d\,\mathrm{lcm}(u^i)})^{\oplus s+1})^\vee)$ such that

$H^*(\mathcal{O}_{\mathbb{P}(H^0(X,(L^{\otimes d\,\mathrm{lcm}(u^i)})^{\oplus s+1})^\vee)}(1)) = \mathscr{H}^{\otimes d}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,L}}$.[13] Observe that if we choose $\tau^0 = \mathrm{pr}_1^d$ then $\sigma^0 = (\varphi^0_{\underline{\varsigma}\circ\iota})^d = h_{\mathfrak{A}}^{d\cdot\mathrm{lcm}(u^i)/u^0} \cdot \iota^{0,\otimes d\cdot\mathrm{lcm}(u^i)/u^0}$. Hence the set we are interested in lies over $\mathbb{P}(\mathbb{H}^\vee)_- = \{[h^0,\dots,h^t] \in \mathbb{P}(\mathbb{H}^\vee) : h^0 \neq 0\}$ where h^0 is the coordinate to the basis element $h^0 = \iota^{0,\otimes d\cdot\mathrm{lcm}(u^i)/u^0}$ of $\mathbb{H} = H^0(X, L^{\otimes d\,\mathrm{lcm}(u^i)})^{\oplus s+1}$.

The set $H^{-1}(\mathbb{P}(\mathbb{H}^\vee)_-)$ is of course still too big to admit a GIT-quotient. Fortunately 3.13 and 3.8 already show that under f (semi)stable affine objects land inside $\mathfrak{U}^{a(s)s} \cap \mathfrak{P}_{\underline{\varsigma}\circ\iota,L}$ and the GIT-quotient thereof exists as a projective scheme. Since H is $\mathcal{G}_A$-invariant and $\mathbb{P}(\mathbb{H}^\vee)$ as well as $\mathfrak{U}^{a(s)s} \cap \mathfrak{P}_{\underline{\varsigma}\circ\iota,L} /\!\!/ \mathcal{G}_A$ are projective, so is the morphism induced by $H^{(s)s} : \mathfrak{U}^{a(s)s} \cap \mathfrak{P}_{\underline{\varsigma}\circ\iota,L} \to \mathbb{P}(\mathbb{H}^\vee)$ on $\mathfrak{U}^{a(s)s} \cap \mathfrak{P}_{\underline{\varsigma}\circ\iota,L} /\!\!/ \mathcal{G}_A$.[14] By construction we get $G^{-1}((H^{(s)s})^{-1}(\mathbb{P}(\mathbb{H}^\vee)_-)) = \mathfrak{A}^{(s)s}$ for

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "c\,\mapsto\,(1{,}c)"] \arrow[rr, bend right, "G"'] & \mathfrak{A}^0 \arrow[r, "f"] & \mathfrak{P}_{\underline{\varsigma}\circ\iota_L} \arrow[r, "H"] & (\mathbb{P}(\mathbb{H}^\vee)_-.
\end{tikzcd}$$

As the restriction of the proper map $\overline{f}$ is proper over $H^{-1}(\mathbb{P}(\mathbb{H}^\vee)_-)$, it is finite by construction. Thus by $\overline{f}^{-1}((H^{ss})^{-1}(\mathbb{P}(\mathbb{H}^\vee)_-)) = (\mathbb{C}^* \times \mathfrak{A}^{ss})/\mathbb{C}^*$

$$\begin{aligned}(\mathbb{C}^* \times \mathfrak{A}^{ss}) /\!\!/ (\mathbb{C}^* \times \mathcal{G}_A) &= ((\mathbb{C}^* \times \mathfrak{A}^{ss})/\mathbb{C}^*) /\!\!/ \mathcal{G}_A \\ &= \overline{f}^{-1}((H^{ss})^{-1}(\mathbb{P}(\mathbb{H}^\vee)_-)) /\!\!/ \mathcal{G}_A\end{aligned}$$

is a quasi-projective scheme (like $(H^{ss})^{-1}(\mathbb{P}(\mathbb{H}^\vee)_-) /\!\!/ \mathcal{G}_A$). Since $(\mathbb{C}^* \times \mathfrak{A}^{ss})$ is, as a good quotient, affine over its $(\mathbb{C}^* \times \mathfrak{A}^{ss}) /\!\!/ (\mathbb{C}^* \times \mathcal{G}_A)$-quotient and since the quotient map is trivially $\mathcal{G}_A$-equivariant, the GIT-quotient pulls back to $\mathbb{C}^* \times \mathfrak{A}^{ss}$ and therefore to its closed $\mathcal{G}_A$-invariant subscheme $\mathfrak{A}$. Finally this shows that

$$\mathfrak{A}^{ss} /\!\!/ \mathcal{G}_A$$

exists and that it is a quasi-projective scheme. Furthermore the good quotient $\mathfrak{A}^s/\mathcal{G}_A$ exists as an open subscheme since G preserves stability, is finite and $\mathfrak{U}^{as}/\mathcal{G}_A$ is a geometric quotient. It is in fact a geometric quotient if we can show that an orbit $\mathcal{G}_A \cdot c$ in $\mathfrak{A}^{ss}$ is closed, if and only if the corresponding orbit in $\mathfrak{U}^{as}$ is closed. Then as $\mathfrak{U}^{as}$ admits a geometric quotient, the orbit $\mathcal{G}_A \cdot c$ in $\mathfrak{A}^{ss}$ of a stable point c is closed as the image of c is in $\mathfrak{U}^{as}$. If there was another orbit $\overline{\mathcal{G}_A \cdot z} \cap \mathcal{G}_A \cdot c \neq \emptyset$ then z and c would map to the same point in $\mathfrak{U}^{as}/\mathcal{G}_A$, thus the corresponding

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[13] Combine the σ^i to get an element of $H^0(\mathfrak{P}_{\underline{\varsigma}\circ\iota,L} \times X, \pi^*_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,L}}(\mathscr{H}^{\otimes d}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota,L}}) \otimes (\pi_X^*(L^{\otimes d\,\mathrm{lcm}(u^i)}))^{\oplus s+1})$. Consequentially we get elements of $\mathscr{H}om((\pi_{\mathfrak{P}_{\underline{\varsigma}\circ\iota_L},*}\, \pi_X^*(L^{\otimes d\,\mathrm{lcm}(u^i)})^{\oplus s+1})^\vee, \mathscr{H}^{\otimes d}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota_L}})$ and $\mathscr{H}om(H^0((L^{\otimes d\,\mathrm{lcm}(u^i)})^{\oplus s+1})^\vee) \otimes \mathcal{O}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota_L}}, \mathscr{H}^{\otimes d}_{\mathfrak{P}_{\underline{\varsigma}\circ\iota_L}})$. Then note that [Ha77], III.7.12 may be applied by our assumption on the τ^j.

[14] This morphism will be used to construct the Hitchin map in 3.26.

orbits in $\mathfrak{U}^{as}$ equal each other; hence they are closed. So is $\mathcal{G}_A \cdot z$ and therefore $\mathcal{G}_A \cdot z = \mathcal{G}_A \cdot c$ as the unique closed orbit in our good quotient.
To prove that closed orbits are exactly the closed orbits under our assignment, recall that by definition of the $\mathbb{C}^*$-action there is an isomorphism $(\mathbb{C}^* \times \mathfrak{A})/\mathbb{C}^* \simeq (\{1\} \times \mathfrak{A})/\{\zeta_{u_0}^k : 1 \leq k \leq u_0\}$ since the group of $(u^0)^{\text{th}}$-unit roots stabilizes 1; thus we find a finite morphism

$$L : \mathfrak{A} \to \{1\} \times \mathfrak{A} \to (\mathbb{C}^* \times \mathfrak{A})/\mathbb{C}^{*}\text{[15]} \to H^{-1}(\mathbb{P}(\mathbb{H}^{\vee})_-)$$

which in particular preserves closed orbits: Finite morphisms are closed, so the image of each closed orbit is closed. On the other hand the preimage of each closed orbit is closed. Since L is finite there is a finite number of orbits in the preimage, each of which is mapped by equivariance to our closed orbit in $H^{-1}(\mathbb{P}(\mathbb{H}^{\vee})_-)$. Hence every orbit $\mathcal{G}_A \cdot c$ in the preimage must have dimension $\dim(\mathcal{G}_A)$. In particular $\mathcal{G}_A \cdot c$ is closed, since otherwise $\overline{\mathcal{G}_A \cdot c} \setminus \mathcal{G}_A \cdot c$ must contain orbits of strictly lower dimension[16] in contradiction to the previous statement about the dimension of an orbit in the preimage of a closed orbit. We particularly see, that a point in $\mathfrak{A}^{ss}/\mathcal{G}_A$ is closed if and only if its image is closed in $\mathfrak{U}^{ass}/\mathcal{G}_A$.

Remark. Observe that we could pull the GIT-quotient back directly by L only if we already knew that $\mathfrak{A}^{ss}$ resp. $\mathfrak{A}^{s}$ were $\mathbb{C}^*$-invariant subsets.

3.16. Theorem. ([Sch08], 2.8.1.2) *The moduli space of affine ϱ-Higgs bundles of given topological type exists as a quasi-projective scheme.*

3.3. Affine parabolic Higgs bundles

As in the projective case, the parabolic affine case can either be treated similarly as the non-parabolic one, given a suitable parameter scheme, or we can lift the morphism constructed in the non-parabolic case so that the GIT-Quotient $\mathfrak{U}_{\text{par}}^{a(s)s} /\!\!/ \mathcal{G}_A$ can be pulled back. To avoid repetition we will use this second approach.

3.17. Let $\mathfrak{A}$ be the parameter scheme constructed before and choose parabolic subgroups $P^1_{\mathrm{Gl}(U_a)_a}, \ldots, P^{|S|}_{\mathrm{Gl}(U_a)_a} \subset \mathrm{Gl}(U_a)_a$.
Define $\mathfrak{A}^j_{\iota-\text{par}} = \pi^*_{\mathfrak{A},\mathfrak{Q}\times\{x^j\}}(\mathfrak{P}^j_{\underline{\varsigma}\circ\iota,\iota-\text{par}})$, $\mathfrak{P}^j_{\underline{\varsigma}\circ\iota,\iota-\text{par}} = \left(\bigtimes_{a\in A} \mathscr{I}som(U_a \otimes \mathcal{O}_{\mathfrak{Q}_a}), \mathscr{E}_{\mathfrak{Q}_a}|_{\mathfrak{Q}_a\times\{x^j\}}\right)\right)$, $1 \leq j \leq |S|$ and

$$\mathfrak{A}_{\iota-\text{par}} = \mathfrak{A}^1_{\iota-\text{par}}/P^1_{\mathrm{Gl}(U_a)_a} \times_{\mathfrak{A}} \cdots \times_{\mathfrak{A}} \mathfrak{A}^{|S|}_{\iota-\text{par}}/P^{|S|}_{\mathrm{Gl}(U_a)_a}.$$

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[15] $(\mathbb{C}^* \times \mathfrak{A})/\mathbb{C}^* \subset \mathfrak{A}^0/\mathbb{C}^*$ is open as the restriction of the universal quotient to a $\mathbb{C}^*$-invariant open subset, thus the projection is proper, so is L after composition with $\overline{f}$.

[16] If $\mathcal{G}_A \cdot c$ is not closed, then it shows that the orbit is open in $\overline{\mathcal{G}_A \cdot c}$, therefore the complement is closed and of strictly smaller dimension.

We use the maps constructed in the non-parabolic case. Let $\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par},L}$ be the restriction of $\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}}$ to $\mathfrak{P}_{\underline{\varsigma}\circ\iota,L}$ and note that by 2.34, 2.35 $\pi : \mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par},L} \to \mathfrak{P}_{\underline{\varsigma}\circ\iota,L}$ is proper. Hence we get a morphism $H_{\mathrm{par}} : \mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par},L} \to \mathbb{P}(H^0(X, (L^{\otimes d\,\mathrm{lcm}(u^i)})^{\oplus s+1})^{\curlyvee})$ that descends to a proper morphism $\mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par},L} /\!\!/ \mathcal{G}_A \to \mathbb{P}(\mathbb{H}^{\curlyvee})$. Note that as in the case of projective parabolic Higgs bundles we get

$$\begin{aligned}(\mathbb{C}^* \times \mathfrak{A}_{\iota-\mathrm{par}}) /\!\!/ \mathbb{C}^* &= \pi^*_{(\mathbb{C}^*\times\mathfrak{A})/\!\!/\mathbb{C}^*,\mathfrak{Q}\times\{x^1\}}(\mathfrak{P}^1_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}})/P^{|1|}_{\mathrm{Gl}(U_a)_a} \times_{(\mathbb{C}^*\times\mathfrak{A})/\!\!/\mathbb{C}^*} \cdots \\ &\quad \cdots \times_{(\mathbb{C}^*\times\mathfrak{A})/\!\!/\mathbb{C}^*} \pi^*_{(\mathbb{C}^*\times\mathfrak{A})/\!\!/\mathbb{C}^*,\mathfrak{Q}\times\{x^{|S|}\}}(\mathfrak{P}^{|S|}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par}})/P^{|S|}_{\mathrm{Gl}(U_a)_a}.\end{aligned}$$

We get a commuting diagram

$$\begin{tikzcd}
 & \mathbb{C}^* \times \mathfrak{A}_{\iota-\mathrm{par}} \arrow[dd] \arrow[dr] & & \mathfrak{P}_{\underline{\varsigma}\circ\iota,\iota-\mathrm{par},L} \arrow[dd] \\
\mathfrak{A}_{\iota-\mathrm{par}} \arrow[ur] \arrow[dd] & & (\mathbb{C}^* \times \mathfrak{A}_{\iota-\mathrm{par}})/\mathbb{C}^* \arrow[ur] \arrow[dd] & \\
 & \mathbb{C}^* \times \mathfrak{A} \arrow[dr] \arrow[ddr] & & \mathfrak{P}_{\underline{\varsigma}\circ\iota,L} \arrow[ddl] \\
\mathfrak{A} \arrow[ur] \arrow[drr] & & (\mathbb{C}^* \times \mathfrak{A})/\mathbb{C}^* \arrow[ur] \arrow[d] & \\
 & & \mathfrak{B} \arrow[d] & \\
 & & \mathfrak{Q}. &
\end{tikzcd}$$

Furthermore the induced morphisms are just base extensions (over $\mathfrak{B}$ resp. $\mathfrak{Q}$) of the underlying morphisms constructed in the previous section. We restrict our attention as above to $H^{-1}_{\mathrm{par}}(\mathbb{P}(\mathbb{H}^{\curlyvee})_-)$. By construction this set lies over $H^{-1}(\mathbb{P}(\mathbb{H}^{\curlyvee})_-)$. Denote by H^{ss}_{par} the restriction of H_{par} to the open subset $\mathfrak{U}^{ass}_{\mathrm{par}}$ (cf. 3.8).

Since affine/proper/quasi-finite/finite morphisms are stable under base extension ([EGA] II, 1.6.2(iii) resp. [Ha77] II.4.6, [EGA] II, 5.4.2(iii) resp. [Ha77] II.4.8, [EGA] II, 6.2.4(iii) and [EGA] II, 6.1.5(iii)) the induced morphisms have the same properties and thus the GIT-quotients pull back as in the non-parabolic case. We find $\mathfrak{A}^{ss}_{\iota-\mathrm{par}} /\!\!/ \mathcal{G}_A$ as a quasi-projective scheme. The finite morphism L extends accordingly and hence we get $\mathfrak{A}^{s}_{\iota-\mathrm{par}}/\mathcal{G}_A$ as the geometric quotient. This proves

3.18. Conclusion. *Let $P^j_{\mathrm{Gl}(U_a)_a} \subset \mathrm{Gl}(U_a)_a$ be parabolic subgroups for every puncture $x^j \in S$. The moduli space $\mathfrak{A}^{(s)s}_{\iota-\mathrm{par}} /\!\!/ \mathcal{G}_A$ of (semi)stable pairs $((P,\ \varphi),\ (s^j)_{j[|S|]})$ where $(P,\ \varphi)$ is an affine principal $\underline{\varrho}$-Higgs bundle and $s^j : \{x^j\} \to P_\iota \times_X$*

$\{x^j\}/P^j_{\mathrm{Gl}(U_a)_a}$ *are reductions for every puncture* $x^j \in S$, *exists as a quasi-projective scheme whenever the* τ^j*-induced weights* $(\beta^{ij})_{i[s^j]}$ *are admissible* (*cf.* 2.10).

The final missing step will now be taken as in the projective case. Let $\mathscr{P}$ denote the universal principal G-bundle on $\mathfrak{A}_{\iota-\mathrm{par}}$. The closed embedding $G \hookrightarrow \mathrm{Gl}(U_a)_a$ induces a fiberwise closed embedding of the corresponding bundles $\mathscr{P}/Q(\tau^j_G) \hookrightarrow \mathscr{P}_\iota/Q\left(\tau^j_{\mathrm{Gl}(U_a)_a}\right)$ and thus we find a closed $\mathcal{G}_A$-invariant subscheme $\mathfrak{A}^{(s)s}_{\mathrm{par}} \subset \mathfrak{A}^{(s)s}_{\iota-\mathrm{par}}$ that parametrizes semistable affine ϱ-Higgs bundles (cf. (Equ 1)). Therefore

3.19. Theorem. *Let* $P^j \subset G$ *be parabolic subgroups for every puncture* $x^j \in S$. *The moduli space* $\mathfrak{A}^{(s)s}_{\mathrm{par}} /\!\!/ \mathcal{G}_A$ *of (semi)stable pairs* $((P,\ \varphi),\ (s^j)_{j[|S|]})$ *where* $(P,\ \varphi)$ *is an affine principal* ϱ*-Higgs bundle and* $s^j : \{x^j\} \to P \times_X \{x^j\}/P^j$ *are reductions, exists as a quasi-projective scheme whenever the* τ^j*-induced weights* $(\beta^{ij})_{i[s^j]}$ *are admissible* (*cf.* 2.10).

Remark. $\mathfrak{A}^{(s)s}_{\mathrm{par}} /\!\!/ \mathcal{G}_A$ is the moduli space for the functors

$$\begin{array}{rcc} \mathbf{M}^{s(s)} : \mathbf{Sch}_{\mathbb{C}} & \to & \mathbf{Sets} \\ Y & \mapsto & \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ Y\text{-families of (semi)stable} \\ \text{affine parabolic } \varrho\text{-Higgs bundles} \end{array} \right\}. \end{array}$$

3.4. REFINING THE SEMISTABILITY CONCEPT

The following example shows that for non-semisimple reductive groups G the moduli space of stable objects (as defined before) might be empty. In order to overcome this deficit we will slightly alter the semistability concept. Using a central isogeny, the moduli space of (semi)stable objects (with respect to this new notion of (semi)stability) is constructed from the previous results.

3.20. Example. Consider the group $\mathrm{Gl}(\mathbb{C}^{r_1}) \times \mathrm{Gl}(\mathbb{C}^{r_2})$, $r_1,\ r_2 \in \mathbb{N}_+$ and the faithful representation $\iota : \mathrm{Gl}(\mathbb{C}^{r_1}) \times \mathrm{Gl}(\mathbb{C}^{r_2}) \hookrightarrow \mathrm{Sl}(\mathbb{C}^{r_1+r_2+1})$, $(g^1,\ g^2) \mapsto (g^1,\ g^2,\ \det(g^1 \oplus g^2)^{-1})$. Observe that an element $z \cdot \mathrm{id}_{\mathbb{C}^{r_1}} \oplus z' \cdot \mathrm{id}_{\mathbb{C}^{r_2}}$, $(z,\ z' \in \mathbb{C}^*)$ of the radical of $\mathrm{Gl}(\mathbb{C}^{r_1}) \times \mathrm{Gl}(\mathbb{C}^{r_2})$ is mapped to the center of $\mathrm{Gl}(\mathbb{C}^{r_1}) \times \mathrm{Gl}(\mathbb{C}^{r_2}) \times \mathbb{C}^*$. Now if E is a vector bundle associated to a principal $(\mathrm{Gl}(\mathbb{C}^{r_1}) \times \mathrm{Gl}(\mathbb{C}^{r_2}))$-bundle P by ι, then by looking at the transition functions we see that $E = E_1 \oplus E_2 \oplus \det(E_1 \oplus E_2)^\vee$ and therefore $\deg(E) = 0$.[17] Consider the two one-parameter subgroups

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[17] The transition functions of the dual line bundle are just the inverse-valued functions.

$$\begin{array}{rl} & \mathbb{C}^* \to \mathrm{Gl}(\mathbb{C}^{r_1}) \times \mathrm{Gl}(\mathbb{C}^{r_2}) \\ \lambda_1 : & z \mapsto (z^{r_2},\ z^{-r_1}) \\ \lambda_2 : & z \mapsto (z^{-r_2},\ z^{r_1}) \end{array}$$

Now

$$\iota \circ \lambda_1(z) = (z^{r_2},\ z^{-r_1},\ \underbrace{z^{-r_1 r_2 + r_2 r_1}}_{=0}), \quad \iota \circ \lambda_2(z) = (z^{-r_2},\ z^{r_1},\ 0).$$

We get induced filtrations

$$\begin{aligned} 0 \subsetneq F_1^1 = E_2 \subsetneq F_1^2 = E_2 \oplus \det(E_1 \oplus E_2)^{\vee} \subsetneq E \text{ respectively} \\ 0 \subsetneq F_2^1 = E_1 \subsetneq F_2^2 = E_1 \oplus \det(E_1 \oplus E_2)^{\vee} \subsetneq E \end{aligned}$$

to $\iota \circ \lambda_1$ respectively $\iota \circ \lambda_2$. The corresponding α-weights are $(\alpha_1^1,\ \alpha_1^2) = (r_1/3,\ r_2/3)$ respectively $(\alpha_2^1,\ \alpha_2^2) = (r_2/3,\ r_1/3)$. Choose for example $r_1 = r_2$. Now we can calculate $M^{(1),(0)}(F_t^k, \alpha_t^k)$ for $t = 1,\ 2$. If E was (semi)stable as a vector bundle we would get $M^{(1),(0)}(F_t^k, \alpha_t^k) \geq 0$ and thus

$$\begin{aligned} t = 1 : & -(\deg(E_2) + (\deg(E_2) - (\deg(E_1) + \deg(E_2)))) \ (\geq)\ 0 \\ & \Rightarrow \deg(E_1)\ (\geq)\ \deg(E_2), \\ t = 2 : & -(\deg(E_1) + (\deg(E_1) - (\deg(E_1) + \deg(E_2)))) \ (\geq)\ 0 \\ & \Rightarrow \deg(E_2)\ (\geq)\ \deg(E_1). \end{aligned}$$

While semistability can still hold if $\deg(E_1) = \deg(E_2)$, E cannot be stable.
If E occurs in a ς-Higgs bundle it might be still stable as a ς-Higgs bundle. However for a one-parameter subgroup λ of $\ker(\varsigma|_{\mathscr{R}ad(G)})$ (and hence a one-parameter subgroup of the connected component $\mathscr{R}ad(\varsigma) = \ker(\varsigma|_{\mathscr{R}ad(G)})^0$ of the identity in $\ker(\varsigma|_{\mathscr{R}ad(G)})$, the $(\varsigma \circ \lambda)$-induced filtration of P_ς will be trivial, therefore $\mu(\lambda, \mathscr{R}) = 0$.[18] Here $\mathscr{R} = \mathrm{id}_X : X \to P/Q_{\mathrm{Gl}(\mathbb{C}^{r_1}) \times \mathrm{Gl}(\mathbb{C}^{r_2})}(\lambda)$ is the only reduction that can occur. Thus the (semi)stability discussion from above is valid for the ς-Higgs bundle as well and we will not find any stable objects. As an example take the adjoint representation $\varsigma = \underline{\iota} \circ \mathrm{Ad}_{\mathrm{Gl}(\mathbb{C}^{r_1}) \times \mathrm{Gl}(\mathbb{C}^{r_2})}$:

$$\begin{aligned} (\mathrm{Gl}(\mathbb{C}^{r_1}) \times \mathrm{Gl}(\mathbb{C}^{r_2})) \times (\mathbb{C}^{r_1 \times r_1} \times \mathbb{C}^{r_2 \times r_2} \times \mathbb{C}) &\to \mathbb{C}^{r_1 \times r_1} \times \mathbb{C}^{r_2 \times r_2} \times \mathbb{C} \\ ((g_1,\ g_2), (m_1,\ m_2,\ m_3)) &\mapsto (g_1 m_1 g_1^{-1},\ g_2 m_2 g_2^{-1},\ m_3). \end{aligned}$$

We see that $\underline{\iota} \circ \mathrm{Ad}_{\mathrm{Gl}(\mathbb{C}^{r_1}) \times \mathrm{Gl}(\mathbb{C}^{r_2})}$ splits over the obvious (homogeneous) representation $\underline{\varsigma} = \mathrm{Ad}_{\mathrm{Gl}(\mathbb{C}^{r_1}) \times \mathrm{Gl}(\mathbb{C}^{r_2}) \times \mathbb{C}^*}$ and $\underline{\iota} \circ \mathrm{Ad}_{\mathrm{Gl}(\mathbb{C}^{r_1}) \times \mathrm{Gl}(\mathbb{C}^{r_2})}(\lambda_t) = \mathrm{id}$ for $t = 1,\ 2$.

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[18] Observe that by our conventions on page V of the Introduction we only exclude trivial one-parameter subgroups λ from the (semi)stability condition; $\varsigma \circ \lambda$ may be trivial.

3.21. As a consequence of the previous example we should really check semistability against one-parameter subgroups of $G/\mathscr{R}ad(\varsigma)$ rather than against one-parameter subgroups of G. This phenomena is analogous to one that occurs for principal G-bundles without a Higgs field. There the semistability condition restricts to anti-dominant characters of parabolic subgroups. But anti-dominant characters are trivial on the connected component of the center of G, hence they do not correspond to one-parameter subgroups of $\mathscr{R}ad(G)$. In the semisimple case this situation cannot occur since by definition $\mathscr{R}ad(G) = \{e\}$. In the previous example we have $\chi_{\lambda_1}(g_1,\ g_2) = \det(g_1)^{r_2} \cdot \det(g_2)^{-r_1}$ and $\chi_{\lambda_1}(2 \cdot \mathrm{id}_{\mathrm{Gl}(\mathbb{C}^{r_1})} \oplus 1 \cdot \mathrm{id}_{\mathrm{Gl}(\mathbb{C}^{r_2})}) = 2^{r_2} \neq 1$.
Consider $\pi : G \to G/\mathscr{R}ad(\varsigma)$, then a principal G-bundle P induces a principal $G/\mathscr{R}ad(\varsigma)$-bundle $P_{\mathscr{R}ad}$. Since $\mathscr{R}ad(\varsigma) \subset \mathscr{Z}(G)$, $\mathscr{R}ad(\varsigma)$ is an abelian subgroup and $G/\mathscr{R}ad(\varsigma)$ is an algebraic group. By definition of $\mathscr{R}ad(\varsigma)$, ς factors over a representation $\varsigma_{Rad} : G/\mathscr{R}ad(\varsigma) \to \mathrm{Gl}(W)$. Therefore the associated vector bundles $P_\varsigma \simeq P_{\mathscr{R}ad, \varsigma_{\mathscr{R}ad}}$ are isomorphic since $P_{\mathscr{R}ad} = P_\kappa$ and $\kappa : G \times G/\mathscr{R}ad(\varsigma) \to G/\mathscr{R}ad(\varsigma)$, $(g, \pi(r)) \mapsto \pi(gr)$ is the natural action. Furthermore $\mathscr{R}ad(\varsigma) \subset Q$ holds for every parabolic subgroup Q of G, and thus $Q/\mathscr{R}ad(\varsigma)$ identifies with a parabolic subgroup of $G/\mathscr{R}ad(\varsigma)$: If we write $Q = Q_G(\lambda)$ for some $\lambda : \mathbb{C}^* \to G$ we get by definition $Q/\mathscr{R}ad(\varsigma) = Q_{G/\mathscr{R}ad(\varsigma)}(\pi \circ \lambda)$. On the other hand a one-parameter subgroup $\lambda_{\mathscr{R}ad}$ of $G/\mathscr{R}ad(\varsigma)$ induces a one-parameter subgroup of $G/\mathscr{Z}(G)$:

1. First recall that $\pi : [G, G] \to G/\mathscr{R}ad(G)$ is surjective[19].

2. The fibers are finite, since using our embedding ι, $\mathscr{R}ad(G)$ is mapped into the center and $[G, G]$ is mapped into $[\mathrm{Gl}(U_a)_a, \mathrm{Gl}(U_a)_a] \subset \bigtimes_{a \in A} \mathrm{Sl}(U_a)$ and therefore $\iota([G, G] \cap \mathscr{R}ad(G)) \subset \bigtimes_{a \in A} \mathrm{Sl}(U_a) \cap \mathscr{Z}(\mathrm{Gl}(U_a))$ is finite. Hence a power of $\lambda_{\mathscr{R}ad}$ lifts to a one-parameter subgroup of $[G, G] \subset G$.

3. Now let T be any subtorus of $\mathscr{R}ad(G)$, then there is another subtorus $\tilde{T}$ such that the multiplication $T \times \tilde{T} \to \mathscr{R}ad(G)$ is an isomorphism. Now $[G, G] \times \tilde{T} \to G$ has a linear algebraic group R_T as its image since $\tilde{T}$ is in the center and the morphism is a morphism of algebraic groups. For $T = \mathscr{R}ad(\varsigma)$ we have $R_{\mathscr{R}ad(\varsigma)} \to G/\mathscr{R}ad(\varsigma)$ surjective: Let $g\mathscr{R}ad(\varsigma) \in G/\mathscr{R}ad(\varsigma)$, then $g = hz,\ h \in [G, G],\ z \in \mathscr{R}ad(G)$ and $z = \tilde{r}r,\ r \in \mathscr{R}ad(\varsigma),\ \tilde{r} \in \tilde{T} \Rightarrow g = h\tilde{r}r$, i. e. $h\tilde{r} \in R_{\mathscr{R}ad(\varsigma)}$ is mapped to $g\mathscr{R}ad(\varsigma)$.

4. We still need to check, that the fibers of $R_{\mathscr{R}ad(\varsigma)} \to G/\mathscr{R}ad(\varsigma)$ are finite. This however is a direct consequence of our previous considerations, i. e. $R_{\mathscr{R}ad(\varsigma)} \ni h\tilde{r} = r \in R_{\mathscr{R}ad(\varsigma)} \Rightarrow h = \tilde{r}^{-1}r \in [G, G] \cap \mathscr{R}ad(G)$ finite.

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[19] $G/\mathscr{R}ad(G)$ is semisimple, therefore $G/\mathscr{R}ad(G) = [G/\mathscr{R}ad(G), G/\mathscr{R}ad(G)]$ by [Hum75], 27.5 Theorem and $\pi([G, G]) = [G/\mathscr{R}ad(G), G/\mathscr{R}ad(G)]$.

5. Since $Q_{G/\mathscr{R}ad(\varsigma)}(\lambda_{\mathscr{R}ad}) = Q_{G/\mathscr{R}ad(\varsigma)}(\lambda^{m}_{\mathscr{R}ad})$ for any $m \in \mathbb{N}$ we see that every parabolic subgroup of $G/\mathscr{R}ad(\varsigma)$ comes from a parabolic subgroup of $R_{\mathscr{R}ad\varsigma}$.

As a result of the previous construction we see that $P_{\mathscr{R}ad}/(Q/\mathscr{R}ad(G)) \simeq P/Q$. In particular every projective parabolic ς-Higgs bundle $(P,\ (s^j)_{j[|S|]},\ \varphi,\ L)$ induces a projective parabolic ς_{Rad}-Higgs bundle $(P_{\mathscr{R}ad},\ (s^j)_{j[|S|]},\ \varphi,\ L)$; every affine parabolic ς-Higgs bundle $(P,\ (s^j)_{j[|S|]},\ \varphi)$ induces an affine parabolic ς_{Rad}-Higgs bundle $(P_{\mathscr{R}ad},\ (s^j)_{j[|S|]},\ \varphi)$.

3.22. Definition. Let χ be a character of $G/\mathscr{R}ad(\varsigma)$ (that naturally comes from a character of G trivial on $\mathscr{R}ad(\varsigma)$). A parabolic ς-Higgs bundle is called $(\chi,\ \tau^j,\ \delta)$-(semi)stable if the associated ς_{Rad}-Higgs bundle is $(\chi,\ \tau^j,\ \delta)$-(semi)stable. With the preceding considerations, this is equivalent to the statement that a projective ς-Higgs bundle is $(\chi,\ \tau^j,\ \delta)$-(semi)stable if $M^{(1),(0)}_{[\mathrm{par}]}(F^k, \alpha^k) + \langle \lambda, \chi \rangle\ (\geq)\ 0$ holds for every one-parameter subgroup λ of $R_{\mathscr{R}ad(\varsigma)}$ and every reduction $\mathscr{R} : X \to P/Q_G(\lambda)$. $(F^k, \alpha^k)_k$ denotes as usual the weighted filtration to λ and $\mathscr{R}$.
An affine ϱ-Higgs bundle is $(\chi,\ \tau^j)$-(semi)stable if for every one-parameter subgroup λ of $R_{\mathscr{R}ad(\varrho)}$ and every reduction $\mathscr{R} : X \to P/Q_G(\lambda)$ for which $\mu(\lambda, \mathscr{R}, \varphi) \leq 0$ holds, the inequality $M^{(1),(0)}_{[\mathrm{par}]}(F^k, \alpha^k) + \langle \lambda, \chi \rangle\ (\geq)\ 0$ can be verified.

3.23. In order to construct moduli spaces for our new definition of (semi)stability we will follow [Ram96ii], 4.15 (resp. [Sch08], 2.7.5). First recall that for all constructions discussed so far, the conditions (U^1) and (U^2) of the proposition 1.62 (see as well [Ram96ii], Def. 4.6) are satisfied and thus we were able to construct moduli spaces. Now however we will not have a universal family $\mathscr{T}$ as before. Coarse moduli spaces can still be constructed using the methods of [Ram96ii] which will be outlined below in the affine case.

3.24. Central Isogenies. Consider a scheme Y over $\mathbb{C}$ as a complex space. Let $\theta : G \to H$ be a surjective group homomorphism between algebraic groups, with $\ker\theta \subset \mathscr{Z}(G)$, $\ker\theta$ finite[20], then the exact sequence $0 \to K := \ker(\theta) \to G \to H \simeq G/K \to 0$ induces an exact sequence $\cdots \to H^1(Y, D) \to H^1(Y, G) \to H^1(Y, H) \to H^2(Y, D)$ of Čech Cohomology sets. In the abelian case, these are just the singular cohomology groups.[21] Given a $H-$bundle $\mathscr{H}_Y$ on $Y \times X$ define a functor as follows:

$$\begin{array}{rcc}\Gamma(\theta, \mathscr{H}_Y) : \mathbf{Sch}_Y & \to & \mathbf{Sets} \\ T \xrightarrow{f} Y & \mapsto & \left\{ \begin{array}{c} \text{Isomorphism classes of pairs} \\ (\mathscr{G}_T, \psi_T) \text{ with a } G\text{-bundle } \mathscr{G}_T \to T \times X \\ \text{and } \psi_T : \theta_*(\mathscr{G}_T) \xrightarrow{\simeq} (f \times \mathrm{id}_X)^*(\mathscr{H}_Y) \end{array} \right\}.\end{array}$$

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[20] We call θ with the given properties a central isogeny.

[21] The famous theorem of Leray states that Čech and Singular cohomology agree on a locally contractible space, i. e. in particular on every complex space (see [BV72], lemma 3.2 using [Whi65]).

On morphisms $f \in \mathrm{Mor}(Y, Z)$ we set $\Gamma(f) = f^*$. The functor $\Gamma(\theta, \mathscr{H}_Y)$ is in general not a sheaf ([Ram96ii], 4.9 and [FGA], V, §1). Its sheafification $\tilde{\Gamma}(\theta, \mathscr{H}_Y)$ (w. r. t. the faithfully flat or étale topology or strong topology) is representable by a complex space $f : Z \to Y$ such that f is a finite étale morphism resp. unramified cover over $f(Z) = \tilde{Z}$. $\tilde{Z}$ is just the preimage of 0 of the morphism $Y \to H^2(X, D)$ induced as follows: $\mathscr{H}_Y$ defines a class in $H^1(Y \times X, H)$ hence a global section of the sheaf $R^1(\pi_{Y,*})(H)$ associated to the presheaf $U \mapsto H^1(U \times X, H)$. Combining this section with the connecting homomorphism[22] gives a section of $R^2(\pi_{Y,*})(\ker(\theta))(\,\cdot\,) = H^2(\,\cdot\, \times X, \ker(\theta))$, where the latter is the constant sheaf to $H^2(X, \ker(\theta))$. Recall $\ker(\theta) \subset \mathscr{Z}(G)$. Furthermore it can be shown, that every point in $\tilde{Z}$ has an open (contractible) neighborhood U such that $\Gamma(\theta, \mathscr{H}_Y)(U) \neq \emptyset$. A proof may be found in [Sch08], 2.4.8.7 or [Ram96ii], 4.15. A purely algebraic proof is given in Proposition 5.4.1 of [GLSS06].
Further assume that we have an action A of a reductive algebraic group say $\mathcal{G}$ on Y and a linearization of this action in $\mathscr{H}_Y$, i. e. $\hat{A} : \mathcal{G} \times \mathscr{H}_Y \to \mathscr{H}_Y$ such that the bundle map is equivariant w. r. t. the two actions and the induced morphism $\mathscr{H}_Y|_y \to \mathscr{H}_Y|_{gy}$, $g \in \mathcal{G}$ is a H-bundle morphism. Then we find a G-bundle isomorphism $\Psi : \pi^*_{Y\times X}(\mathscr{H}_Y) \to (A \times \mathrm{id}_X)^*(\mathscr{H}_Y)$ over $\mathcal{G} \times Y \times X$: Observe that $\pi^*_{Y\times X}(\mathscr{H}_Y) \simeq \mathscr{H}_Y \times_{Y\times X} (\mathcal{G} \times Y \times X)$ is isomorphic to $(A \times \mathrm{id}_X)^*(\mathscr{H}_Y) \simeq \mathscr{H}_Y \times_{Y\times X} (\mathcal{G} \times Y \times X)$ as a scheme by the universal property of the fiber product. The $\mathcal{G}$-invariance guarantees, that the isomorphism is compatible with the bundle projections. Since the induced morphisms $\mathscr{H}_Y|_y \to \mathscr{H}_Y|_{gy}$, $g \in \mathcal{G}$ are H-equivariant, so is Ψ.

Assume that we are given two equivalence relations on Y and on Z such that equivalent points of Z are mapped to equivalent points of Y. Furthermore for every two equivalent points $y \sim y'$ of Y there must be an isomorphism $\Psi_y : \mathscr{H}_Y|_y \to \mathscr{H}_Y|_{y'}$ of H-bundles. We will apply the following proposition, when equivalence corresponds to isomorphy of ϱ-Higgs bundles.

3.25. Proposition. ([Sch08], 2.4.8.9) *Consider an action $A : \mathcal{G} \times Y \to Y$ with associated isomorphism $\Psi : \pi^*_{Y\times X}(\mathscr{H}_Y) \to (A \times \mathrm{id}_X)^*(\mathscr{H}_Y)$ as above, such that for any two equivalent points $y \sim y'$, $\Psi_y : \mathscr{H}_Y|_y \to \mathscr{H}_Y|_{y'}$ there is a $g \in \mathcal{G}$ such that $gy = y'$ and $\Psi_y = \Psi_{(g,y)}$ on X. Then the action A lifts to an action A_Z of Z, such that $z' \in \mathcal{G} \cdot z$ if and only if $z \sim z'$ in Z.*

Remark. The proposition is already proved in 4.10 in [Ram96ii].
More details on isogenies can be found in the original work by Claude Chevalley, [Che58]. An easy example of a central isogeny is $\mathrm{Sl}(V) \to \mathrm{PGl}(V)$ for a vector space V.

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[22] A connected homomorphism may still be constructed under the given assumptions.

We have seen above that $\theta : G \to \mathscr{R}ad(\varrho) \times G/\mathscr{R}ad(\varrho)$ is a central isogeny in the sense of 3.25. In order to account for the torus part recall that $\mathscr{R}ad(\varrho) \simeq (\mathbb{C}^*)^m$ for some $m \in \mathbb{N}$. θ associates to every G-bundle P on X a $G/\mathscr{R}ad(\varrho)$-bundle $P_{\mathscr{R}ad(\varrho)}$ on X and m line bundles L^i of degree d^i.[23] Let $\mathfrak{A}^{ss}_{[\mathrm{par}]}$ be our parameter scheme of (semi)stable affine parabolic $\varrho_{\mathscr{R}ad}$-Higgs bundles with our action A' of $\mathcal{G}_A$ and with the universal family $(\mathscr{P}_{\mathfrak{A}^{ss}_{[\mathrm{par}]}},\ (s^j_{\mathfrak{A}^{ss}_{[\mathrm{par}]}})_{j[|S|]},\ \varphi_{\mathfrak{A}^{ss}_{[\mathrm{par}]}})$ on $\mathfrak{A}^{ss}_{[\mathrm{par}]} \times X$. By definition of the $\mathcal{G}_A$-action, there is a linearization on $\mathscr{P}_{\mathfrak{A}^{ss}_{[\mathrm{par}]}}$ and thus an isomorphism $\Psi : \pi^*_{\mathfrak{A}^{ss}_{[\mathrm{par}]}\times X}(\mathscr{H}_Y) \to (A' \times \mathrm{id}_X)^*(\mathscr{H}_Y)$. Furthermore take $\mathscr{P}^i$ Poincaré line bundles on $\mathrm{Jac}^{d^i} \times X$. We may linearize the trivial $\mathscr{R}ad(\varrho)$-action on $\bigtimes_{k=1}^m \mathrm{Jac}^{d^k}$ in the $\mathscr{R}ad(\varrho)$-bundle $\mathscr{R} = (\pi_{\mathrm{Jac}^{d^1} \times X})^*(\mathscr{P}^1) \times_{X \times \bigtimes_{k=1}^m \mathrm{Jac}^{d^k}} \cdots \times_{X\times\bigtimes_{k=1}^m \mathrm{Jac}^{d^k}} (\pi_{\mathrm{Jac}^{d^m} \times X})^*(\mathscr{P}^m)$ by

$$\mathscr{R}ad(\varrho) \times \mathscr{R} \to \mathscr{R}, \quad ((z^k)_k, (r^k)_k) \mapsto (z^k r^k)_k.$$[24]

Combining the two actions and the corresponding linearizations we get a $(\mathcal{G}_A \times \mathscr{R}ad(\varrho))$-action on $Y = \mathfrak{A}^{ss}_{[\mathrm{par}]} \times \bigtimes_{k=1}^m \mathrm{Jac}^{d^k}$ with linearization in $\pi^*_{X\times\mathfrak{A}^{ss}_{[\mathrm{par}]}}(\mathscr{P}_{\mathfrak{A}^{ss}_{[\mathrm{par}]}}) \times_{X\times Y} \pi^*_{X\times\bigtimes_{k=1}^m \mathrm{Jac}^{d^k}}(\mathscr{R})$. We still need to show that the constructed action satisfies the conditions stated in 3.25. For the $\bigtimes_{k=1}^m \mathrm{Jac}^{d^k}$-part this is clear by definition of the Poincaré line bundles.[25] For $\mathfrak{A}^{ss}_{[\mathrm{par}]}$ this follows from (the proof of) the universal property 1.25. Recall the proof of Proposition 3.25 provides us with an unbranched covering $Z \to Y$ and an action A_Z on Z such that the covering map is equivariant w. r. t. the two actions. Since the quotient $Y /\!\!/ (\mathcal{G}_A \times \mathscr{R}ad(\varrho)) = (\mathfrak{A}^{ss}_{[\mathrm{par}]} /\!\!/ \mathcal{G}_A) \times \left(\bigtimes_{k=1}^m \mathrm{Jac}^{d^k} /\!\!/ \mathscr{R}ad(\varrho)\right) = (\mathfrak{A}^{ss}_{[\mathrm{par}]} /\!\!/ \mathcal{G}_A) \times \left(\bigtimes_{k=1}^m \mathrm{Jac}^{d^k}\right)$ exists and the covering is a finite map, again by 1.57 we see that $Z /\!\!/ \mathcal{G}_A$ exists as well.
The space Z represents the functor

$$\begin{array}{rcc} \Gamma(\theta, \mathscr{P}_{\mathfrak{A}^{ss}_{[\mathrm{par}]}}) : \mathbf{Sch}_{\mathfrak{A}^{ss}_{[\mathrm{par}]}} & \to & \mathbf{Sets} \\ T \xrightarrow{f} \mathfrak{A}^{ss}_{[\mathrm{par}]} & \mapsto & \left\{ \begin{array}{c} \text{Isomorphism classes of pairs} \\ (\mathscr{G}_T, \psi_T) \text{ with a } G\text{-bundle } \mathscr{G}_T \to T \times X \\ \text{and } \psi_T : \theta_*(\mathscr{G}_T) \xrightarrow{\cong} (f \times \mathrm{id}_X)^*(\mathscr{P}_{\mathfrak{A}^{ss}_{[\mathrm{par}]}}) \end{array} \right\}. \end{array}$$

If two affine ϱ-Higgs bundles are isomorphic, so are the corresponding $\varrho_{\mathscr{R}ad}$-Higgs bundles.[26] Therefore we can apply 3.25, which tells us that two affine ϱ-Higgs

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[23] The topological types of $\mathscr{P}_{\mathfrak{A}^{ss}_{[\mathrm{par}]},\mathscr{R}ad(\varrho)}$ and the L^i are uniquely defined by the topological type of $\mathscr{P}_{\mathfrak{A}^{ss}_{[\mathrm{par}]}}$ (cf. remark 5.1 in [Ram75]).

[24] cf. [Ram96ii], 4.15.

[25] Note that an automorphism of a line bundle on a compact Riemann surface is uniquely defined by an element of $\mathbb{C}^*$.

[26] See 3.21 and the definition of isomorphy of ϱ-Higgs bundles in 3.11.

bundles are in the same A_Z-orbit if and only if they are isomorphic as ϱ-Higgs bundles.
Putting all results together we find that Z is the quasi-projective coarse moduli space of semistable ϱ-Higgs bundles. By construction we recover the geometric quotient of stable ϱ-Higgs bundles as an open subscheme of Z.

3.5. HITCHIN MORPHISM

The existence of a (generalized) Hitchin morphism, i. e. a proper morphism from our moduli space of affine parabolic objects to an affine space has been shown in the non-parabolic situation by Alexander Schmitt in [Sch08], 2.8.1.4. The following result is an easy extension thereof to the parabolic setting.

Let $\varrho^j : G \to \mathrm{Gl}(W^j)$ as before and $W = \bigoplus_{j=1}^m W^j$. Then we find T G-invariant generators $\sigma^k \in \bigotimes_{j=1}^m \mathrm{Sym}^{t^{jk}}(W^{j,\vee})$ of $\mathrm{Sym}^*(W^\vee)^G$. Denote by $\mathbb{Hit}$ the affine space $\bigoplus_{k=1}^T H^0(X, \bigotimes_{j=1}^m L^{\otimes t^{jk}})$.

3.26. Lemma. *A projective Hitchin morphism* $\mathbf{Hit} : \mathfrak{A}^{ss}_{\text{par}} /\!\!/ \mathcal{G}_A \to \mathbb{Hit}$ *exists.*

Proof. First note, that it is enough to show that **Hit** (if it exists) is proper, since $\mathfrak{A}^{ss}_{\text{par}} /\!\!/ \mathcal{G}_A$ is quasi-projective and H is affine, i. e. **Hit** is quasi-projective and proper into a quasi-compact space and therefore by [EGA], II.5.5.3 projective (see as well [EGA], II.5.3.4 (v)). Furthermore, if we can construct two morphisms **Hit** and $\tilde{G}_{\text{par}}$ such that (Hit) commutes, then already **Hit** is proper, since $H^{ss}_{\text{par}} \circ G_{\text{par}}$ is proper ([Ha77], II.4.8).

$$\begin{tikzcd}
\mathfrak{A}^{ss}_{\text{par}} /\!\!/ \mathcal{G}_A \arrow[r, "G_{\text{par}}"] \arrow[d, "\mathbf{Hit}"'] & \mathfrak{U}^{ass}_{\text{par}} \cap \mathfrak{P}_{\underline{\varsigma}\circ\iota, L} /\!\!/ \mathcal{G}_A \arrow[d, "H^{ss}_{\text{par}}"] \\
\mathbb{Hit} \arrow[r, "\tilde{G}_{\text{par}}"'] & \mathbb{P}(\mathbb{H}^*)_-
\end{tikzcd} \tag{Hit}$$

Similar to the discussion in 3.14 we define **Hit** locally on a trivializing cover $(U_i)_i$ by composition of the universal morphisms $(\varphi^j_{\mathfrak{A}^{ss}_{\text{par}}})_{j[m]}|_{U_i} =: (\varphi^j_i)_{j[m]}$ and σ^k, $1 \leq k \leq T$. If $(l^j_{in})_{in}$ are transition functions of L^j and $\varrho^j(g_{in})$ the transition functions of $\mathscr{P}_{\mathfrak{A}^{ss}_{\text{par}}, \varrho^j}$, then $\sigma^k \left(\bigoplus_{j=1}^m l^j_{in} \varrho^j (g^j_{in})^{-t} \varphi^j_n \right) = \prod_{j=1}^m (l^j_{in})^{t^{jk}} \sigma^k(\varphi^j_n)$ and the corresponding map $\mathbf{Hit}^k$ is a global section of $\bigotimes_{j=1}^m L^{\otimes t^{jk}}$. By $\mathcal{G}_A$-invariance we thus find our map $\mathbf{Hit} = \bigoplus_{k=1}^T \mathbf{Hit}^k$.
We still have to construct $\tilde{G}_{\text{par}}$. The embeddings ι^j used in the construction of a projective $(\underline{\varsigma} \circ \iota)$-Higgs bundle provide maps $\underline{\iota}^k : H^0(X, \bigotimes_{j=1}^m L^{\otimes t^{jk}}) \to$

$H^0(X, \bigotimes_{j=1}^{m} L^{\otimes u^j t^{jk}})$. Moreover by homogeneity of the σ^k the global morphism defined locally by $\sigma^k((\iota^j \circ \varphi^j)_{j[m]})$ coincides with the image of $\mathbf{Hit}^k((\varphi^j)_{j[m]})$ under $\underline{\iota}^k$. We need to account for the zero-component of the projective Higgs field. Therefore denote by σ^0 a generator of $\mathrm{Sym}(\mathbb{C})^G$ (w. r. t. the trivial action) and consider the function

$$\Psi^i : (w^j)_{j[m]} \mapsto \tau^i \left(\bigoplus_{\substack{v \in \mathbb{Z}_{\geq 0}^{m+1} \\ vu^t = \mathrm{lcm}(u^i)}} \bigotimes_{j=0}^{m} w^{j, \otimes v^j} \right).$$

As a function on $\mathbb{C} \oplus W$ it may be written in terms of the generators σ^k, i. e. there is a polynomial p^i such that $\Psi^i = p^i(\sigma^0, \dots, \sigma^T)$. By the equivariance of σ^k and $\underline{\iota}^k$ we now get $p^i((\underline{\iota}^k \circ \sigma^k)_{0 \leq k \leq T}((\varphi^j)_{j[m]})) = p^i(\sigma^0, \dots, \sigma^T)((\iota^j \circ \varphi^j)_{j[m]}) = \Psi^i((\iota^j \circ \varphi^j)_{j[m]}) = \tau^i(\varphi_{\underline{\varsigma} \circ \iota})$. In particular $p^i(\underline{\iota}^k(\,\cdot\,))$ defines $\tilde{G}_{\mathrm{par}}$ suitably.[27]

$\square$

Remark. Using the finite morphism $Z \to \mathfrak{A}^{ss}_{[\mathrm{par}]} \times \bigtimes_{k=1}^{m} \mathrm{Jac}^{d^k}$ the Hitchin morphism extends as a projective morphism to Z.

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[27] Note that the first component of $\tilde{G}_{\mathrm{par}}$ is non-trivial.

4

S-Equivalence

Unfortunately, already in the case of G-fiber bundles, the functor witch associates to each scheme over $\mathbb{C}$ the set of isomorphism classes of S-families of semistable G-bundles admits no coarse moduli space (Ramanathan [Ram96i], Proposition 3.5). In the previous sections we have constructed the categorical quotients of the open subsets of semistable and stable objects. While for stable objects we were able to construct even a geometrical quotient and henceforth these quotients become coarse moduli spaces by 1.62, the semistable objects might have fibers which contain more than one orbit. Again by 1.62 this implies that they do not form a coarse moduli space. However, to overcome this deficit, we can associate non-isomorphic semistable bundles in the same fiber over the GIT-quotient. Thus two points x, y representing semistable objects in one of our parameter schemes, say $\mathfrak{T}^{ss}_{\mathrm{par}}$, should be S-equivalent if $\overline{\mathcal{G}_A \cdot x} \ \cap \ \overline{\mathcal{G}_A \cdot y} \neq \emptyset$.[1] Since we were able to construct geometric quotients on the subset of stable objects, by (Geo^1) of 1.53 $\overline{\mathcal{G}_A \cdot x} \ \cap \ \overline{\mathcal{G}_A \cdot y} \neq \emptyset$ already implies $x \in \mathcal{G}_A \cdot y$, i. e. the S-equivalence relation is the same as the isomorphy relation. On the other hand S-equivalence guarantees that in every fiber over the GIT-quotient of semistable objects there is only one S-equivalence class, i. e. our GIT-Quotient becomes a coarse moduli space for the functor of S-equivalence classes.

To make any sense of the condition imposed by S-equivalence we have to find intrinsic definitions for the specific moduli problems we faced.

4.1. S-Equivalence of Tuples

Let λ be a one-parameter subgroup of G, $Q_G(\lambda)$ the associated parabolic subgroup,

$$\mathscr{R}ad_u(\lambda) = \{g \in G : \lim_{z \to \infty} \lambda(z) g \lambda(z)^{-1} = e\}$$

its unipotent radical and

$$\mathscr{L}ev_G(\lambda) = \{g \in G : \lambda(z) g \lambda(z)^{-1} = g, \ \forall z \in \mathbb{C}^*\}$$

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[1] Recall that if $\mathcal{G}_A \cdot y$ is closed and $\overline{\mathcal{G}_A \cdot x} \ \cap \ \overline{\mathcal{G}_A \cdot y} \neq \emptyset$ there is a one-parameter subgroup λ such that $\mu(\lambda, x) = 0$ and $\lim_{z \to \infty} \lambda(z) \cdot x \simeq y$ (cf. remark to 1.53).

a Levi subgroup. We recover $Q_G(\lambda)$ as semi-direct product $\mathscr{R}ad_u(\lambda) \rtimes \mathscr{L}ev_G(\lambda)$ ([Bo91], IV.11.22).[2] For $G = \mathrm{Gl}(V)$, $q \in Q_G(\lambda)$ is a block-upper triangular matrix, $l \in \mathscr{L}ev_G(\lambda)$ the corresponding block-diagonal matrix and $r \in \mathscr{R}ad_u(\lambda)$ is $r = ql^{-1}$. A semi-direct product corresponds to a split-exact sequence $1 \to \mathscr{R}ad_u(\lambda) \to Q_G(\lambda) \underset{i}{\overset{p}{\rightleftarrows}} \mathscr{L}ev_G(\lambda) \to 1$ ([St93], Proposition 4.7.5). Recall that given a reduction $\mathscr{R} : X \to P/Q_G(\lambda)$ we get a $Q_G(\lambda)$-bundle $\mathscr{R}^*(P)$, $P \to P/Q_G(\lambda)$. Extending the structure group by $i \circ p$ we get another principal G-bundle which we call the admissible deformation $\mathrm{df}^{\lambda}_{\mathscr{R}}(P)$ of P associated to $\mathscr{R}$. If $(F^k)_{k[m]}$ is the filtration of the vector bundle $E = P_\iota$ associated to λ and $\mathscr{R}$, then $\mathrm{df}^{\lambda}_{\mathscr{R}}(P)_\iota = \bigoplus_{k=1}^m F^k/F^{k-1}$: We use cocycles as in 2.8. Let $(g^{ij})_{ij}$ be the $Q_G(\lambda)$-valued transition functions of P given by the reduction $\mathscr{R}$. Then

$$\iota \circ g^{ij} = \begin{pmatrix} h_1^{ij} & & * \\ & \ddots & \\ 0 & & h_m^{ij} \end{pmatrix}.$$

If we define F^k as the subbundle with transition functions

$$H_k^{ij} = \begin{pmatrix} h_1^{ij} & & * \\ & \ddots & \\ 0 & & h_k^{ij} \end{pmatrix},$$

then the quotient F^k/F^{k-1} has transition functions h_k^{ij}.[3] Thus $\bigoplus_{k=1}^m F^k/F^{k-1}$ has transition functions

$$\begin{pmatrix} h_1^{ij} & & 0 \\ & \ddots & \\ 0 & & h_m^{ij} \end{pmatrix} \in \mathscr{L}ev_{\mathrm{Gl}(U_a)_a}(\iota \circ \lambda).$$

Observe, that $\mathscr{L}ev_G(\lambda) = \mathscr{L}ev_{\mathrm{Gl}(U_a)_a}(\iota \circ \lambda) \cap G$. Consequentially $\mathrm{df}^{\lambda}_{\mathscr{R}}(P)_\iota = \bigoplus_{k=1}^m F^k/F^{k-1}$. For the deformation of the Higgs field we proceed analogously. First note that $\varsigma(g^{ij}) \in Q_{\mathrm{Gl}(W)}(\varsigma \circ \lambda)$ are the induced transition functions of P_ς and hence $p'(\varsigma(g^{ij})) \in \mathscr{L}ev_{\mathrm{Gl}(W)}(\varsigma \circ \lambda)$ the transition functions of $\mathrm{df}^{\varsigma\circ\lambda}_{\varsigma\circ\mathscr{R}}(P_\varsigma)$ w. r. t. the morphism $p' : Q_{\mathrm{Gl}(W)}(\varsigma \circ \lambda) \to \mathscr{L}ev_{\mathrm{Gl}(W)}(\varsigma \circ \lambda)$ induced by p. On the other hand $\mathrm{df}^{\lambda}_{\mathscr{R}}(P)_\varsigma$ has transition functions $\varsigma \circ p(g^{ij}) \in \varsigma(\mathscr{L}ev_G(\lambda))$. We claim that $p'(\varsigma(g^{ij})) = \varsigma \circ p(g^{ij})$: This follows for example from the product decompositions

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[2] The semi-direct product for the homomorphism $\mathscr{L}ev_G(\lambda) \to \mathrm{Aut}(\mathscr{R}ad_u(\lambda))$, $l \mapsto \{r \mapsto lrl^{-1}\}$.

[3] The projection $F^k \to F^k/F^{k-1}$ looks in our local coordinates as $\pi = \begin{pmatrix} 0 & E_{r^k - r^{k-1}} \end{pmatrix}$ and $\pi H_k^{ij} = h_k^{ij}\pi$.

$Q_G(\lambda) = \mathscr{R}ad_u(\lambda) \rtimes \mathscr{L}ev_G(\lambda)$ and $Q_{\mathrm{Gl}(W)}(\varsigma \circ \lambda) = \mathscr{R}ad_u(\varsigma \circ \lambda) \rtimes \mathscr{L}ev_{\mathrm{Gl}(W)}(\varsigma \circ \lambda)$. If $g^{ij} = r^{ij}l^{ij}$ then $\varsigma \circ p(g^{ij}) = \varsigma(l^{ij})$ and $\varsigma(g^{ij}) = \varsigma(r^{ij})\varsigma(l^{ij})$, $\varsigma(r^{ij}) \in \mathscr{R}ad_u(\varsigma \circ \lambda)$, $\varsigma(l^{ij}) \in \mathscr{L}ev_{\mathrm{Gl}(W)}(\varsigma \circ \lambda)$, i. e. $p'(\varsigma(g^{ij})) = \varsigma(l^{ij})$.

If $\varsigma \circ \lambda$ is the induced one-parameter subgroup with induced filtration $(F_\varsigma^k)_k$ of P_ς, $\mathrm{rk}(P_\varsigma) = \dim(W)$, then since φ is non-trivial, we find a smallest index ι_0 such that $\varphi|_{F_\varsigma^{\iota_0}} \neq 0$. Therefore the fundamental theorem on homomorphisms provides us with a non-trivial homomorphism $\varphi^{\mathrm{df}}|_{F_\varsigma^{\iota_0}/F_\varsigma^{\iota_0 - 1}} : F_\varsigma^{\iota_0}/F_\varsigma^{\iota_0 - 1} \to L$ witch extends trivially to a φ^{df} on $\mathrm{df}_{\varsigma \circ \mathscr{R}}^{\varsigma \circ \lambda}(P_\varsigma) \simeq \mathrm{df}_{\mathscr{R}}^{\lambda}(P)_\varsigma$. We define $\mathrm{df}_{\mathscr{R}}^{\lambda}(P, \varphi, L) := (\mathrm{df}_{\mathscr{R}}^{\lambda}(P), \varphi^{\mathrm{df}}, L)$.

Now let $s^j : \{x^j\} \to P \times_X \{x^j\}/P^j$ be a parabolic reduction to a parabolic subgroup $P^j \subset G$ and P our principal G-bundle. As in 2.7 we find g^j with $\mathscr{R}^j(x^j) = s^j(x^j)g_j$ for some representatives. Let $\psi_i : U_i \times Q_G(\lambda) \to P$ be local trivializations of $P \to P/Q_G(\lambda)$, $\mathscr{R}^{-1}(U_i) = V_i$ and $\overline{\psi}_i = \psi_i \circ (\mathscr{R} \times \mathrm{id}_{Q_G(\lambda)}) : V_i \times Q_G(\lambda) \to \mathscr{R}^*(P)$ the resulting trivialization of the $Q_G(\lambda)$-bundle $\mathscr{R}^*(P)$. Now $\mathrm{pr}_2 \circ (\overline{\psi}_i)^{-1}(s^j(x^j)g_j)$ defines an element q_j of $Q_G(\lambda)$ and hence under projection with p an element of $\mathscr{L}ev_G(\lambda)$. Furthermore $Q_G(\lambda)$ and $g_j^{-1}P^jg_j$ intersect in (at least) a torus T. Denote by $\mathscr{L}ev_G(T)$ the Levi subgroup of $Q_G(\lambda)$ associated to T. Then there is a unique $r_j \in \mathscr{R}ad_u(\lambda)$ such that $\mathscr{L}ev_G(T) = r_j \mathscr{L}ev_G(\lambda) r_j^{-1}$.[4] Now we find a unique decomposition $g_j^{-1}P^jg_j \cap Q_G(\lambda) = (g_j^{-1}P^jg_j \cap \mathscr{L}ev_G(T)) \cdot (g_j^{-1}P^jg_j \cap \mathscr{R}ad_u(\lambda))$ ([DM91], 2.1 Proposition) and thus $p(g_j^{-1}P^jg_j \cap Q_G(\lambda)) = r_j^{-1}g_j^{-1}P^jg_jr_j \cap \mathscr{L}ev_G(\lambda)$. The group $\mathscr{L}ev_G(T) \cap g_j^{-1}P^jg_j$ is a parabolic subgroup of $\mathscr{L}ev_G(T)$. We find a point in $\mathscr{L}ev_G(\lambda)/(r_j^{-1}g_j^{-1}P^jg_jr_j) \cap \mathscr{L}ev_G(\lambda)$ independent of the chosen representative $s^j(x^j)$. Consequentially we get a point $s^{j,\mathrm{df}}$ in $\mathrm{df}_{\mathscr{R}}^{\lambda}(P)/P^j$ using the injection $\mathscr{L}ev_G(\lambda)/\mathscr{L}ev_G(\lambda) \cap (r_j^{-1}g_j^{-1}P^jg_jr_j) \to G/P^j$, $l(r_j^{-1}g_j^{-1}P^jg_jr_j) \mapsto (lr_j^{-1}g_j^{-1})P^j$. Note that the constructed point is independent of the chosen representatives; the independence of the choice of $s^j(x^j)$ is clear. For g_j replaced by g_jq, $q = l \cdot r$ and consequentially r_j replaced by $r^{-1}r_j$ and q_j replaced by q_jq we use that

$$\begin{aligned} p(q_jq)r_j^{-1}rq^{-1}g_j^{-1}P^jg_jqr^{-1}r_jr_j^{-1}rq^{-1}g_j^{-1} &= p(q_j)r_j^{-1}lr_jr_j^{-1}l^{-1}g_j^{-1}P^j \\ &= p(q_j)r_j^{-1}g_j^{-1}P^j. \end{aligned}$$

Note in particular that the construction does not depend on the choice of the maximal torus T.

Remark. Observe that we could have chosen g_j such that $\mathscr{L}ev_G(T) = \mathscr{L}ev_G(\lambda)$.

Furthermore for vector bundle filtrations $E^{ij} \subset E|_{x^j}$ this becomes just the induced filtration $(\bigoplus_{k=1}^m F^k \cap E^{ij}/F^{k-1} \cap E^{ij})_i$ of $\bigoplus_{k=1}^m F^k/F^{k-1}\Big|_{x^j}$.

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[4] For a proof as well as the definition of $\mathscr{L}ev_G(T)$ see [DM91], 1.17 Proposition, 1.18 Corollary.

Remark. If we replace our parabolic reductions by points in $G/P^j \hookrightarrow \mathbb{P}(V_\sigma)$ for the corresponding action σ and if $p = \sum_{i=1}^{\dim(V_\sigma)} a^i v_\lambda^{i,\vee}$ for a basis of weight vectors v_λ^i of the $(\sigma \circ \lambda)$-induced $\mathbb{C}^*$-action on V_σ, then the previous construction produces a $r_j \in \mathscr{R}ad_u(\lambda)$ such that $\sigma(r_j)p = a^{i_0 j} v_\lambda^{i,\vee}$ for $a^{i_0 j}$ the highest indexed $a^{i_0 j} \neq 0$.[5]

4.1. Definition. We call a projective parabolic ς-Higgs bundle $(P,\ (s^j)_{j[|S|]},\ \varphi,\ L)$ polystable if it is semistable and $\mathrm{df}_{\mathscr{R}}^{\lambda}(P, (s^j)_{j[|S|]}, \varphi) := \mathrm{df}_{\mathscr{R}}^{\lambda}(P, (s^{j,\mathrm{df}})_{j[|S|]}, \varphi) \simeq P$ for all reductions $\mathscr{R}$ to every one-parameter subgroup λ of G such that $M_{\mathrm{par}}^{(1),\xi}(\lambda, \mathscr{R}) + \delta \cdot \mu(\lambda, \mathscr{R}, \varphi) = 0$. Since every semistable P becomes polystable after finitely many (essentially different) admissible transformations, P defines a G-bundle $\mathrm{Gr}(P)$. Two G-bundles P_1 and P_2 are called S-equivalent if $\mathrm{Gr}(P_1) \simeq \mathrm{Gr}(P_2)$ are isomorphic as G-bundles.

Remark. The concept of S-equivalence is defined for Higgs tuples analogously.

4.2. Lemma. *Let n be as in 1.52. Let $((E_a,\ (E_a^{ij})_{i[s_a^j]j[|S|]})_{a[|A|]},\ \varphi,\ L) = t$ be a $(\delta, \xi_a, \delta^{ij})$-semistable Higgs tuple for admissible weights $(\delta^{ij})_{i[s^j]j[|S|]}$. Let λ be a one-parameter subgroup of $\mathcal{S}_A^{\kappa_a}$ with associated flag $(V^k,\ \alpha^k)_{k[m]}$ and $\mu(\lambda, \mathrm{Gies}_\chi(t)) = 0$, then $M_{\mathrm{par}}^{\kappa,\xi}(F^k, \alpha^k) + \delta \cdot \mu(F^k, \alpha^k, \varphi) = 0$ holds for the induced filtration $(F^k,\ \alpha^k)_{k[r]}$ of $E = \bigoplus_{a \in A} E_a^{\oplus \kappa_a}$ and $F^{k,coh} = F^k$, $h^1(F^k(n)) = 0$. If $(F^k, \alpha^k)_{k[r]}$ is a filtration such that $M_{\mathrm{par}}^{\kappa,\xi}(F^k, \alpha^k) + \delta \cdot \mu(F^k, \alpha^k, \varphi) = 0$ then $\mu(\lambda, \mathrm{Gies}(t)) = 0$ holds for any induced one-parameter subgroup λ with associated flag $V^k = H^0(F^k(n))$.*

Proof. If $\mu_\chi(\lambda, \mathrm{Gies}(t)) = 0$, then by part (i) $\Rightarrow$ (ii) of 1.52 we see that $0 = \mu_\chi(\lambda, \mathrm{Gies}(t)) \geq M_{\mathrm{par}}^{\kappa,\xi}(F^k, \hat{\alpha}^k) + \delta \cdot \mu(F^k, \hat{\alpha}^k, \varphi) \geq 0$. In particular since $\delta^{ij} > 0$ (Tor 1) implies $V^k = H^0(F^k(n))$ and therefore $\hat{\alpha}^k = \alpha^k$. Thus we have $F^{k,coh} = F^k$ and $h^1(F^k(n)) = 0$. On the other hand if $M_{\mathrm{par}}^{\kappa,\xi}(F^k, \alpha^k) + \delta \cdot \mu(F^k, \alpha^k, \varphi) = 0$ then F^k is globally generated for big $n \geq 0$ and the main calculation (M1) shows that $\mu_\chi(\lambda, \mathrm{Gies}(t)) = 0$ holds for any induced one-parameter subgroup λ of $\mathcal{S}_A^{\kappa_a}$. $\square$

Let $q = (q_a)_{a[|A|]} \in \times_{a \in A} \mathfrak{Q}_a$ be the underlying quotient of the vector bundle $\bigoplus_{a \in A} E_a$ corresponding to a point t in $\mathfrak{T}_{\mathrm{par}}$. Let $\lambda = (\lambda_a)_{a[|A|]}$ be a one-parameter subgroup of $\mathcal{S}_A^{\kappa_a}$ with associated flags $(V_a^k)_{k[v_a]}$ and weights $(\gamma_a^k)_{k[v_a]}$ such that $\mu_\chi(\lambda, \mathrm{Gies}(t)) = 0$. Then q_a induces a filtration of E_a by (generated) subbundles E_a^k, $1 \leq k \leq m_a$[6] and therefore a one-parameter subgroup $\tilde{\lambda} = (\tilde{\lambda}_a)_{a[|A|]} : \mathbb{C}^* \to \times_{a \in A} \mathrm{Gl}(r_a, \mathbb{C})$

$$\tilde{\lambda}_a(z) = \begin{pmatrix} z^{\gamma_a^1} & & 0 \\ & \ddots & \\ 0 & & z^{\gamma_a^{m_a}} \end{pmatrix}$$

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[5] Recall that we order weights of the representation σ descending while the γ^j are ordered ascending.

[6] Note that by the previous result 4.2 $v_a = m_a$.

together with a reduction $\mathscr{R} : X \to E/Q_{\times_{a\in A}\mathrm{Gl}(r_a,\mathbb{C})}(\tilde{\lambda})$. Now let

$$g_a^{ij} = \begin{pmatrix} g_{11,a}^{ij} & & * \\ & \ddots & \\ 0 & & g_{m_am_a,a}^{ij} \end{pmatrix}, \quad (g_a^{ij})_{a[|A|]} \in Q_{\times_{a\in A}\mathrm{Gl}(r_a,\mathbb{C})}(\tilde{\lambda})$$

be the transition functions of $\bigoplus_{a\in A} E_a$. Locally q is of the form

$$q^i = (q_a^i)_a,\ q_a^i = \begin{pmatrix} q_{11,a}^i & \cdots & q_{1m_a,a}^i \\ & \ddots & \vdots \\ 0 & & q_{m_am_a,a}^i \end{pmatrix},\ q_{kl,a}^i \in \mathbb{C}^{\mathrm{rk}(E_a^k/E_a^{k-1})\times\dim(V_a^l/V_a^{l-1})}$$

w. r. t. a suitable local trivialization over some U_i. Hence

$$\tilde{\lambda}_a(z)q_a^i\lambda_a^{-1}(z) = \begin{pmatrix} q_{11,a}^i & z^{\gamma_a^1-\gamma_a^2}\cdot q_{12,a}^i & \cdots & z^{\gamma_a^1-\gamma_a^{m_a}}\cdot q_{1m_a,a}^i \\ 0 & q_{21,a} & \cdots & z^{\gamma_a^2-\gamma_a^{m_a}}\cdot q_{2m_a,a}^i \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & q_{m_am_a,a}^i \end{pmatrix}.$$

Since $q(V_a^k) = E_a^k$ we see that $(q_{lt,a}^i)_{l[m_a]t[k]}$ is of rank $\mathrm{rk}(E_a^k)$ independent of z. In particular

$$\lim_{z\to\infty}(\tilde{\lambda}_a(z)q_a^i\lambda_a^{-1}(z)) = \begin{pmatrix} q_{11,a}^i & & 0 \\ & \ddots & \\ 0 & & q_{m_am_a,a}^i \end{pmatrix}$$

has full rank. Finally note that by construction $(\tilde{\lambda}_a(z)q_a^i\lambda_a^{-1}(z))_{a[|A|]i}$ and $(q_a^i\lambda_a^{-1}(z))_{a[|A|]i}$ define the same point in $\times_{a\in A}\mathfrak{Q}_a$ for all $z\in\mathbb{C}^*$ (since they have the same kernel) and that $(\tilde{\lambda}_a(z)q_a^i\lambda_a^{-1}(z))_i$ induces transition functions $\tilde{\lambda}_a(z)g_a^{ij}\tilde{\lambda}_a^{-1}(z)$ as

$$\tilde{\lambda}_a(z)g_a^{ij}\tilde{\lambda}_a^{-1}(z)\tilde{\lambda}_a(z)q_a^j\lambda_a^{-1}(z) = \tilde{\lambda}_a(z)g_a^{ij}q_a^j\lambda_a^{-1}(z) = \tilde{\lambda}_a(z)q_a^i\lambda_a^{-1}(z).$$

The special form of the transition functions $(g_a^{ij})_{a[|A|]} \in Q_{\times_{a\in A}\mathrm{Gl}(r_a,\mathbb{C})}(\tilde{\lambda})$ implies now that

$$\begin{pmatrix} z^{\gamma_a^1} & & 0 \\ & \ddots & \\ 0 & & z^{\gamma_a^{m_a}} \end{pmatrix} g_a^{ij} \begin{pmatrix} z^{-\gamma_a^1} & & 0 \\ & \ddots & \\ 0 & & z^{-\gamma_a^{m_a}} \end{pmatrix}$$

$$= \begin{pmatrix} g_{11,a}^{ij} & z^{\gamma_a^1-\gamma_a^2}\cdot g_{12,a}^{ij} & \cdots & z^{\gamma_a^1-\gamma_a^{m_a}}\cdot g_{1m_a,a}^{ij} \\ & g_{22,a}^{ij} & \cdots & z^{\gamma_a^2-\gamma_a^{m_a}}\cdot g_{2m_a,a}^{ij} \\ & & \ddots & \vdots \\ 0 & & & g_{m_am_a,a}^{ij} \end{pmatrix} \xrightarrow{z\to\infty} \begin{pmatrix} g_{11,a}^{ij} & & 0 \\ & \ddots & \\ 0 & & g_{m_am_a,a}^{ij} \end{pmatrix}$$

as $\gamma_a^l < \gamma_a^k$ for $l < k$. Thus $\lim_{z\to\infty}(\tilde{\lambda}(z)_a g_a^{ij} \tilde{\lambda}(z)_a^{-1})_{a[|A|]} \in \mathscr{L}ev_{(\mathrm{Gl}(r_a,\mathbb{C}))_a}(\tilde{\lambda})$ and $\lim_{z\to\infty} q\lambda(z)^{-1}$ is isomorphic to $\mathrm{df}_{\mathscr{R}}^{\tilde{\lambda}}(E)$.

Remark. A different proof without using cocycles is given in [HL10], 4.4.3.

Let φ be the underlying Higgs field of our point $t \in \mathfrak{T}$ and denote by $\lambda(z)\varphi$ the Higgs field to $\lambda(z)t$. For the Higgs field we proceed as above: $(\varsigma((g_a^{ij})_a))_{ij}$ are the induced transition functions of E_ς. If $\varsigma \circ \lambda$ is the induced one-parameter subgroup with induced filtration $(F_\varsigma^k)_k$ of E_ς, $\mathrm{rk}(E_\varsigma) = \dim(W)$ and induced weights γ_ς^k, $1 \leq k \leq m_\varsigma$ the same calculation as above shows that $q_{u,v}\lambda^{-1}(z)$ converges against $\mathrm{df}_{\varsigma\circ\mathscr{R}}^{\varsigma\circ\tilde{\lambda}}(E_\varsigma) \simeq \mathrm{df}_{\mathscr{R}}^{\tilde{\lambda}}(E)_\varsigma$. We find an index ι_0 such that $\varphi_{\iota_0}^i \neq 0$ for some i and such that $(\varphi^i)_i = \begin{pmatrix} 0 & \ldots & 0 & \varphi_{\iota_0}^i & \ldots & \varphi_{\dim(W)}^i \end{pmatrix}_i$ defines the Higgs field φ. Then $\lambda(z)\cdot\varphi$ corresponds to $\varphi^i \cdot \varsigma(\tilde{\lambda}(z)^{-1}) = \begin{pmatrix} 0 & \ldots & 0 & z^{-\gamma_\varsigma^{\iota_0}}\varphi_{\iota_0}^i & \ldots & z^{-\gamma_\varsigma^{m_\varsigma}}\varphi_{\dim(W)}^i \end{pmatrix}$.[7] Further denote by $(l^{ij})_{ij}$ the transition functions of the line bundle L. Since scalar multiplication becomes the identity on $\mathfrak{T}$[8], $z^{\gamma_\varsigma^{\iota_0}}(\lambda(z)\cdot\varphi)$ corresponds to

$$\begin{pmatrix} 0 & \ldots & 0 & \varphi_{\iota_0}^i & \ldots & z^{\gamma_\varsigma^{\iota_0}-\gamma_\varsigma^{m_\varsigma}}\varphi_{\dim(W)}^i \end{pmatrix} \xrightarrow{n\to\infty} \begin{pmatrix} 0 & \ldots & \varphi_{\iota_0}^i & 0 & \cdots & 0 \end{pmatrix} =: \varphi_{\mathrm{df}}^i \neq 0.$$

Note in particular, that

$$\begin{aligned} &\varphi^i\varsigma(g^{ij}) = l^{ij}\varphi^j \\ &\Rightarrow z^{\gamma_\varsigma^{\iota_0}}\varphi^i\varsigma(\tilde{\lambda}(z)^{-1})\varsigma(\tilde{\lambda}(z)g^{ij}\tilde{\lambda}^{-1}(z)) = z^{\gamma_\varsigma^{\iota_0}} l^{ij}\varphi^j\varsigma(\tilde{\lambda}(z)^{-1}) = l^{ij}z^{\gamma_\varsigma^{\iota_0}}\varphi^j\varsigma(\tilde{\lambda}(z)^{-1}). \end{aligned}$$

Hence $\lim_{z\to\infty}\lambda(z)\cdot t \simeq \mathrm{df}_{\mathscr{R}}^{\tilde{\lambda}}(E,\varphi,L) := (\mathrm{df}_{\mathscr{R}}^{\tilde{\lambda}}(E), \varphi_{\mathrm{df}}, L)$.
We still need to check the parabolic contribution. Again denote by $\lambda(z)q^{ij}$ the parabolic quotient of $\lambda(z)t \in \mathfrak{T}_{\mathrm{par}}$. Choose a basis corresponding to our filtration $(F_a^k)_{a[|A|]k[r_a]}$ and consider $q_a^{ij} = (q_a^{lj})_{l[r_a - r_a^{ij}]}^t$ the parabolic quotients with $E_a^{ij} = \ker(q_a^{ij})$. Let $k_a^{0j}(l)$ be the smallest index k for which the l-th row $q_a^{lj} = (q_{a,t}^{lj})_{t[r_a]}$ acts non-trivial on $F_a^k|_{x^j}$; let $m_a^{0j}(l) = \min\{t : q_{a,t}^{lj} \neq 0\}$. Let $\overline{\lambda}_a^i(z) = \mathrm{diag}(z^{\gamma^{k_a^{0j}(1)}}, \ldots, z^{\gamma^{k_a^{0j}(r_a - r_a^{ij})}})$ with respect to our chosen basis. Thus again

$$\begin{aligned} &\overline{\lambda}_a^i(z)q_a^{ij}\cdot\tilde{\lambda}_a(z)^{-1} \\ &= \begin{pmatrix} 0 & \cdots & \cdots & 0 & q_{a,m_a^{0j}(1)}^{1j} & \cdots & z^{\gamma^{k_a^{0j}(1)}-\gamma^{m^a}}\cdot q_{a,r_a}^{1j} \\ \vdots & & & \vdots & & & \vdots \\ 0 & \cdots & 0 & q_{a,m_a^{0j}(r_a-r_a^{ij})}^{r_a-r_a^{ij},j} & \cdots & \cdots & z^{\gamma^{k_a^{0j}(r_a-r_a^{ij})}-\gamma^{m^a}}\cdot q_{a,r^a}^{r_a-r_a^{ij},j} \end{pmatrix} \end{aligned}$$

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[7] $\varphi^i \circ q^i = \psi^i \Rightarrow \psi^i\lambda(z) = (\varphi^i\tilde{\lambda}(z)^{-1})(\tilde{\lambda}(z)q\lambda(z)^{-1})$ and $\varphi^i g^{ij} = \varphi^j \Rightarrow \varphi^i\tilde{\lambda}(z)^{-1}(\tilde{\lambda}(z)g^{ij}\tilde{\lambda}(z)^{-1}) = \varphi^j\tilde{\lambda}(z)^{-1}$.

[8] A global multiplication of the values of φ by a scalar is compensated by an automorphism of L.

$$\xrightarrow{z\to\infty} \begin{pmatrix} 0 & \cdots & \cdots & 0 & q^{1j}_{a,m^{0j}_a(1)_a} & \cdots & 0 \\ \vdots & & & \vdots & & & \vdots \\ 0 & \cdots & 0 & q^{r_a-r^{ij}_a,j}_{a,m^{0j}_a(r_a-r^{ij}_a)} & \cdots & \cdots & 0 \end{pmatrix}.$$

Observe that there might be more than one non-zero entry in each row.
Finally note that $\ker(\overline{\lambda}^i_a(z)q^{ij}_a\cdot\tilde{\lambda}_a(z)^{-1}) = \ker(q^{ij}_a\cdot\tilde{\lambda}_a(z)^{-1})$ for all $z\in\mathbb{C}^*$ as well as $\dim\ker(\lim_{z\to\infty}\overline{\lambda}^i_a(z)q^{ij}_a\cdot\tilde{\lambda}_a(z)^{-1}) = \dim\ker(q^{ij}_a\cdot\tilde{\lambda}_a(z)^{-1}) = \dim\ker(q^{ij}_a) = r^{ij}_a$. Therefore we receive $\lim_{z\to\infty}\lambda(z)\cdot t \simeq \mathrm{df}^{\tilde{\lambda}}_{\mathscr{R}}(E,(E^{ij})_{i[s^j]j[|S|]},\varphi,L)$.

4.2. S-Equivalence of Principal Bundles

We need the following consequence of 3.1:

4.3. Lemma. *Let $(P,(s^j)_{j[|S|]},\varphi,L)$ be a semistable projective parabolic ς-Higgs bundle with associated Higgs tuple $(E,(E^{ij})_{i[s^j]j[|S|]},\varphi_{\mathrm{tuple}},L)$ (cf. 2.25). For every filtration $(F^k,\alpha^k)_{k[m]}$ such that $M^{(1),\xi}_{\mathrm{par}}(F^k,\alpha^k)+\delta\cdot\mu(F^k,\alpha^k,\varphi_{\mathrm{tuple}})=0$ there is an associated one-parameter subgroup λ of G and a reduction $\mathscr{R}: X\to P/Q_G(\lambda)$ such that $M^{(1),\xi}_{\mathrm{par}}(\lambda,\mathscr{R})+\delta_{\mathrm{ps}}\cdot\mu(\lambda,\mathscr{R},\varphi)=0$ and vice versa.*

Proof. In the proof of 3.1 we have seen that for $\varepsilon_{\mathrm{ps}}$ big enough either $\mu(F^k,\alpha^k,\varphi_\tau)=0$ or the strict inequality $M^{(1),\xi}_{\mathrm{par}}(F^k,\alpha^k)+\delta_{\mathrm{ps}}\cdot\mu(F^k,\alpha^k,\varphi)+\varepsilon_{\mathrm{ps}}\cdot\mu(F^k,\alpha^k,\varphi_\tau)>0$ holds. Hence for a suitable $\varepsilon_{\mathrm{ps}}$ the equality $M^{(1),\xi}_{\mathrm{par}}(F^k,\alpha^k)+\delta_{\mathrm{ps}}\cdot\mu(F^k,\alpha^k,\varphi)+\varepsilon_{\mathrm{ps}}\cdot\mu(F^k,\alpha^k,\varphi_\tau)=0$ implies already $\mu(F^k,\alpha^k,\varphi_\tau)=0$. By 2.29 we get a one-parameter subgroup λ of G together with a reduction $\mathscr{R}: X\to P/Q_G(\lambda)$ that correspond to $(F^k,\alpha^k)_{k[m]}$. In particular 3.1 implies that $M_{\mathrm{par}}(\lambda,\mathscr{R})+\delta\cdot\mu(\lambda,\mathscr{R},\varphi)=0$. □

The lemma directly implies that admissible deformations of a semistable projective parabolic ς-Higgs bundle P directly correspond to admissible deformations of the associated Higgs tuple.
We still need to extend the concept to the affine case. Apart from the Higgs field we define $\mathrm{df}^{\lambda}_{\mathscr{R}}(P,\ (s^j)_{j[|S|]})$ as before. Then admissible transformations commute with the transition from affine parabolic G-bundles to projective parabolic G-bundles. This property will extend to ϱ-Higgs bundles if we can define $(\varphi^{j,\mathrm{df}})_{j[m]}$ in such a way that $(\varphi^{\mathrm{df}})_{\underline{\varsigma}\circ\iota}=(\varphi_{\underline{\varsigma}\circ\iota})^{\mathrm{df}}$. First recall that as above we get $\mathrm{df}^{\lambda}_{\mathscr{R}}(P)_{\underline{\varsigma}\circ\iota}\simeq\mathrm{df}^{\underline{\varsigma}\circ\iota\circ\lambda}_{\underline{\varsigma}\circ\iota\circ\mathscr{R}}(P_{\underline{\varsigma}\circ\iota})$. Moreover we have seen in the proof of 3.13.(ii) that $\varphi_{\underline{\varsigma}\circ\iota}|_{\bigotimes_{j=0}^{m}\tilde{F}^{i_j,\otimes v^j}_j}\not\equiv 0\Leftrightarrow\varphi^j|_{F^{i_j}_j}\not\equiv 0,\forall 1\le j\le m\ (v^j\neq 0)$ where $(\tilde{F}^i_j)_i$ is the filtration associated to $\underline{\varrho}^j\circ\iota\circ\lambda$ and $(F^i_j)_i$ the filtration associated to $\varrho^j\circ\lambda$. Thus we already see that

4.4. Lemma. *Given an affine parabolic* ϱ*-Higgs bundle* $(P,\ (s^j)_{j[|S|]},\ (\varphi^i,\ L^i)_{i[m]})$ *with associated projective parabolic* $(\underline{\varsigma} \circ \iota)$*-Higgs bundle* $(P,\ (s^j)_{j[|S|]},\ \varphi_{\underline{\varsigma}\circ\iota},\ L)$*, then the following two conditions are equivalent for every reduction* $\mathscr{R}$ *to a one-parameter subgroup* λ *of* G*:*

(i) $M^{(1),\xi}_{\text{par}}(\lambda, \mathscr{R}) = 0$ *whenever* $\mu(\lambda, \mathscr{R}, \varphi) \leq 0$*,*

(ii) $M^{(1),\xi}_{\text{par}}(\lambda, \mathscr{R}) = \mu(\lambda, \mathscr{R}, \varphi_{\underline{\varsigma}\circ\iota}) = 0$.

Furthermore we see that for $\mu(\lambda, \mathscr{R}, \varphi) < 0$, i. e. $\varphi^j|_{F_j^{i_j}} \not\equiv 0 \Rightarrow \gamma_j^{i_j} > 0$, the condition $\mu(\lambda, \mathscr{R}, (\varphi_{\underline{\varsigma}\circ\iota})^{\text{df}}) = 0$ implies that $(\varphi_{\underline{\varsigma}\circ\iota})^{\text{df}} = \iota_0^{\text{lcm}(u^i)/u^0}$.[9] We conclude that $(\varphi^{\text{df}})_{\underline{\varsigma}\circ\iota} = \iota_0^{\text{lcm}(u^{\bar{i}})/u^0}$ and thus we define $\varphi^{\text{df}} \equiv 0$. As might be expected this also implies, that for $\varphi \equiv 0$ we have to define $\varphi^{\text{df}} := 0$. We are left with the more interesting case of $\mu(\lambda,\ \mathscr{R},\ \varphi) = 0$, in which there is at least one $\gamma_j^{i_j^0} = 0$ such that $\varphi^j|_{F_j^{i_j^0}} \not\equiv 0$. Using again that $\varphi^j|_{F_j^i} \not\equiv 0$ already implies $\gamma_j^i \geq 0$ we see that in terms of the zero weight subbundle $F^{i^0}_{\underline{\varsigma}\circ\iota}$ of $P_{\underline{\varsigma}\circ\iota}$, $(\varphi_{\underline{\varsigma}\circ\iota})^{\text{df}}$ is induced by $\varphi_{\underline{\varsigma}\circ\iota}|_{F^{i^0}_{\underline{\varsigma}\circ\iota}} \not\equiv 0$. Therefore $(\varphi^{j,\text{df}})_{\underline{\varsigma}\circ\iota}$ will be defined as the trivial extension of the homomorphism on $\bigoplus_{j=1}^m F_j^{i_j^0}/F_j^{i_j^0-1}$ which is induced by $\bigoplus_{j=1}^m \varphi^j|_{\bigoplus_{j=1}^m F_j^{i_j^0}}$. Here $F_j^{i_j^0}$ is the subbundle corresponding to $\gamma_j^{i_j^0} = 0$ or 0 if no such weight exists for j. Thus $\varphi^{j,\text{df}} : P_{\varrho^j} \to L^j$ is defined.

4.5. Definition. We call an affine parabolic ϱ-Higgs bundle $(P,\ (s^j)_{j[|S|]},\ \varphi)$ polystable if it is semistable and $\text{df}^{\lambda}_{\mathscr{R}}(\mathscr{R},\ P,\ (s^j)_{j[|S|]},\ \varphi) \simeq P$ for all reductions $\mathscr{R}$ to every one-parameter subgroup λ of G such that $M^{(1),\xi}_{\text{par}}(\lambda, \mathscr{R}) = 0$ and $\mu(\lambda, \mathscr{R}, \varphi) \leq 0$. Two affine parabolic ϱ-Higgs bundles are called S-equivalent if they become isomorphic as affine parabolic ϱ-Higgs bundles after a series of admissible deformations.

By construction of the admissible deformations and the results of this chapter we get (for a suitable choice of δ_{ps} and ε_{ps}) the following theorem.

4.6. Theorem. *Two semistable affine parabolic* ϱ*-Higgs bundles are* S*-equivalent if and only if the associated asymptotically semistable projective parabolic* $(\underline{\varsigma}\circ\iota)$*-Higgs bundles are* S*-equivalent, if and only if the associated Higgs tuples are* S*-equivalent, if and only if the corresponding points in the respective parameter scheme are GIT-*S*-equivalent* (*cf. 1.58*)*.*

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[9] In this case the subbundle corresponding to the zero weight is just $\mathcal{O}_X^{\text{lcm}(u^i)/u^0} \oplus 0$ since any other subbundle in the filtration corresponds to a positive, therefore bigger weight.

5

APPLICATIONS

In this final paragraph we will reformulate the (semi)stability concept of ϱ-Higgs bundles in a way more suited to the formulation of a Kobayashi-Hitchin correspondence in 5.2. Furthermore we will recover tame parabolic Higgs bundles as well as Hitchin pairs as a special case of our construction.

5.1. REFORMULATION OF THE SEMISTABILITY CONCEPT

Let G be a reductive algebraic group and $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. Recall that for every reductive group G there is a compact real Lie group K such that G is the complexification of K. In particular $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ for $\mathfrak{k} = \mathrm{Lie}(K)$. Furthermore, the Lie algebra $\mathfrak{g}$ is reductive, i. e. it decomposes as $\mathfrak{z} \oplus \mathfrak{g}_{ss}$[1] resp. $\mathfrak{rad}(\mathfrak{g}) = \mathfrak{z}$, where $\mathfrak{g}_{ss} = [\mathfrak{g},\ \mathfrak{g}]$ is the semisimple part of $\mathfrak{g}$ and $\mathfrak{z} = \mathrm{Lie}(\mathscr{Z}(G))$ for the center $\mathscr{Z}(G) \subset G$.[2] Given a maximal torus $T \subset G$ with $\mathfrak{t} = \mathrm{Lie}(T)$ we denote by $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ the complexification. Moreover there is a non-degenerated invariant symmetric bilinear form $\langle\ \cdot\ ,\ \cdot\ \rangle$ on $\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{c}$ where $\mathfrak{c}$ is the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}} \cap \mathfrak{g}_{ss}$ of $\mathfrak{g}_{ss}$.[3]

Let λ be a one-parameter subgroup of G then $X_\lambda = \left.\frac{d\lambda(1+t)}{dt}\right|_{t=0} \in i\mathfrak{k} \cap (\mathfrak{z} \oplus \mathfrak{c})$ for a maximal compact subgroup K and a maximal torus T with $\mathfrak{t} = \mathrm{Lie}(T)$. By definition of the exponential map we have $Q_G(\lambda) = P = \{g \in G : \lim_{t\to\infty} \exp(tX_\lambda)g\exp(-tX_\lambda) \text{ exists}\}$. For the dual representation $\tilde{\varrho} := \varrho^\vee : G \to \mathrm{Gl}(W^\vee)$ recall that $\mathrm{Rad}(\varrho) = \mathrm{Rad}(\tilde{\varrho}) = \ker(\tilde{\varrho}|_{\mathrm{Rad}\, G})^0 \subset \mathrm{Rad}(G) \subset \mathscr{Z}(G)$ is a compact connected normal subgroup and hence $\mathfrak{g}_{\mathrm{Rad}(\tilde{\varrho})} \hookrightarrow \mathfrak{z}$ is an ideal. Now an element of $\mathrm{Hom}(\mathbb{C}^*, \mathrm{Rad}(\tilde{\varrho}))$ corresponds to an element of $i\mathfrak{k}_{\mathrm{Rad}(\tilde{\varrho})}$, $K_{\mathrm{Rad}(\tilde{\varrho})}$ maximal

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[1] [OV90], Chapter 1, §4, theorem 6.

[2] Note that there are (non-reductive) algebraic groups (like $\mathbb{C}$) that have a reductive Lie algebra.

[3] Onishchick, Vinberg, [OV90], Chapter 1, §9, theorem 9.6 and the following remark.

compact subgroup in $\mathrm{Rad}(\tilde{\varrho})$ and $\mathrm{Lie}(K_{\mathrm{Rad}(\tilde{\varrho})}) = \mathfrak{k}_{\mathrm{Rad}(\tilde{\varrho})}$.[4] $K_{\mathrm{Rad}(\tilde{\varrho})}$ is contained in a maximal compact subgroup $K_{\mathscr{Z}(G)}$ of $\mathscr{Z}(G)$, $\mathfrak{k}_{\mathrm{Rad}(G)} = \mathfrak{k}_{\mathscr{Z}(G)} := \mathrm{Lie}(K_{\mathscr{Z}(G)})$, and we have $i\mathfrak{k}_{\mathrm{Rad}(\tilde{\varrho})} = \{\xi \in i\mathfrak{k}_{\mathscr{Z}(G)} : d\tilde{\varrho}(\xi) = 0\}$.[5] Thus a one-parameter subgroup of $\mathrm{Rad}(\tilde{\varrho})$ corresponds to an element of $\ker(d\tilde{\varrho}|_{\mathfrak{z}})$. Using the non-degenerated bilinear form on $\mathfrak{k}$ we get $i\mathfrak{k} = i\mathfrak{k}_{\mathrm{Rad}(\tilde{\varrho})} \oplus i\mathfrak{l}$ for some Lie algebra $\mathfrak{l}$. In particular non-trivial one-parameter subgroups $\mathbb{C}^* \to G/\mathrm{Rad}(\varrho)$ correspond to elements of the complement $i\mathfrak{k} \setminus \ker(d\tilde{\varrho}|_{\mathfrak{z}})$.
Now denote by $W^{\vee}_{\lambda} = \{w^{\vee} \in W^{\vee} : \lim_{t\to\infty} \exp(tX_{\lambda})w^{\vee} \text{ exists}\}$. Since $W^{\vee}_{\lambda}$ is $Q_G(\lambda)$-invariant we get for every reduction $\mathscr{R} : X \to P/Q_G(\lambda)$ a subbundle $P_{\lambda,\tilde{\varrho}} = (\mathscr{R}^*(P)) \times_{\tilde{\varrho}|_{Q_G(\lambda)}} W^{\vee}_{\lambda} \subset (\mathscr{R}^*(P)) \times_{\tilde{\varrho}} W^{\vee} \simeq P_{\tilde{\varrho}}$. A section $\phi : X \to P_{\tilde{\varrho}} \otimes L$ maps to $P_{\lambda,\tilde{\varrho}} \otimes L$ if and only if $\mu(\lambda, \mathscr{R}, \varphi) \leq 0$ for the to ϕ associated homomorphism $\varphi \in \mathrm{Hom}(P_{\varrho}, L)$: λ and $\mathscr{R}$ induce a filtration $(F^k_{\varrho})_k$ of P_{ϱ}. Denote by w^i_j the eigenvectors of $\varrho(\lambda)$ to the eigenvalue z^{γ^i}. Then locally over some open set U we get $F^k_{\varsigma} = \langle w^i_j : i \leq k\rangle \otimes \mathcal{O}_U$ and $\mu(\lambda, \mathscr{R}, \varphi) \leq 0$ if and only if $\varphi(w^i_j) = 0$ for all $\gamma^i < 0$ and every U. But then $\lim_{z\to\infty} \tilde{\varrho}(\lambda(z))\phi_x(\sum_{i,j} \alpha^i_j w^i_j) = \lim_{z\to\infty} \phi_x(\sum_{i,j} \alpha^i_j z^{-\gamma^i} w^i_j) = \sum_{i,j:\gamma^i\geq 0}(\lim_{z\to\infty} z^{-\gamma^i}\alpha^i_j \varphi_x(w^i_j))$ exists for all $\alpha^i_j \in \mathbb{C}$, $\forall x \in U$ and every U. On the other hand if $\varphi_x(w^{i_0}_{j_0}) \neq 0$ for at least one $\gamma^{i_0} < 0$ and one j_0, then $\lim_{z\to\infty} \tilde{\varrho}(\lambda(z))\phi_x(\alpha_{i_0 j_0} w^{i_0}_{j_0}) = \lim_{z\to\infty} z^{-\gamma^{i_0}} \varphi_x(w^{i_0}_{j_0})$ does not exist.

5.1. Definition. Let $\alpha \in i\mathfrak{k}_{\mathscr{Z}(G)}$. An affine parabolic ϱ-Higgs bundle $(P,\ (s^j)_{j[|S|]},\ \varphi)$ is (α, τ^j)-semistable if for every one-parameter subgroup λ of G and every reduction $\mathscr{R} : X \to P/Q_G(\lambda)$ such that $\varphi \in H^0(X,\ P_{\lambda,\tilde{\varrho}} \otimes L)$:

$$\deg P_Q(\chi_{\lambda}, \mathscr{R}) - \sum_{j:x^j\in S} \langle \tau^{s^j}, \chi^{\mathscr{R}j}_{\lambda}\rangle + \langle \alpha, X_{\lambda}\rangle \geq 0.$$

$(P,\ (s^j)_{j[|S|]},\ \varphi)$ is (α, τ^j)-stable if it is (α, τ^j)-semistable and if $\deg P_Q(\chi_{\lambda}, \mathscr{R}) - \sum_{j:x^j\in S}\langle \tau^{s^j}, \chi^{\mathscr{R}j}_{\lambda}\rangle + \langle \alpha, X_{\lambda}\rangle > 0$ holds for every one-parameter subgroup λ of G such that $X_{\lambda} \notin \ker(d\tilde{\varrho}|_{\mathfrak{z}})$.

Remark. For some applications like the Kobayashi-Hitchin correspondence it is advantageous to check (semi)stability against (strictly) anti-dominant Lie algebra characters of $\mathfrak{p}$. Compare with [GGM12] or [LT06] for a detailed account of the semistability concept in this context. Note that not every anti-dominant character χ of $\mathfrak{p}$ comes from a character of P. However a (positive) integer multiple of χ does ([GGM12], section 2.6) and the semistability criterion stays the same.

Remark. The (semi)stability concept above directly generalizes to Γ-Higgs bundles. More precisely, let Γ be a connected real reductive Lie group with compact subgroup K and Cartan decomposition $\mathrm{Lie}(\Gamma) = \mathfrak{k} \oplus \mathfrak{p}$. Then $G = K_{\mathbb{C}}$ is

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[4] $\mathrm{Rad}(G) = \mathscr{Z}(G)^0 \subset \mathscr{Z}(G)$.

[5] For a matrix group $\xi \in \mathfrak{k}_{\mathrm{Rad}(\tilde{\varrho})}$ if and only if $\exp(t\xi) \in \ker \tilde{\varrho}|_{\mathrm{Rad}\,G}$, $\forall t \in \mathbb{R}$ if and only if $\mathrm{id} = \tilde{\varrho}|_{\mathrm{Rad}\,G} \exp(t\xi) = \exp(t \cdot d\tilde{\varrho}|_{\mathfrak{z}}(\xi))$, $\forall t \in \mathbb{R}$.

reductive. Let ϱ be the complexified isotropy representation (to the Cartan decomposition) $G \to \mathrm{Gl}(\mathfrak{p}_{\mathbb{C}})$. Then a (semi)stable affine parabolic $\varrho^{\vee}$-Higgs bundle $(P, \varphi : P_{\varrho^{\vee}} \to \omega_X, (s^j)_{j[|S|]})$ is called a (semi)stable Γ-Higgs bundle. A detailed discussion of (semi)stability in the case $\Gamma = \mathrm{Sp}(2r, \mathbb{R})$ can be found in [Sch08], 2.8.4 or [GGM13].

Example. For a specific choice of G we are sometimes able to simplify the semistability concept. For example for the groups $G = \mathrm{SO}(r, \mathbb{C})$ and $G = \mathrm{Sp}(2r, \mathbb{C})$ we consider an Ad-Higgs vector bundle (E, φ) with an additional non-degenerated bilinear form B (symmetric or alternating). We call a subbundle $F \subset E$ isotropic if $F \subset F^{\perp} := \{e \in E : B(e, f) = 0, \forall f \in F\}$ and parabolicly isotropic if additionally $F^{ij} \subset F^{ij,\perp} := \{e \in E^{ij} : B(e, f) = 0, \forall f \in F^{ij}\}$. A G-Higgs bundle is now (semi)stable if for every non-trivial parabolicly isotropic φ-preserved subbundle $F \subset E$ we have $\text{par-deg}(F) \leq \text{par-deg}(E)$.[6]

5.2. TAME PARABOLIC HIGGS BUNDLES

The attentive reader might have noticed that in all previous constructions we did not require any interaction between the Higgs field and the parabolic filtration. Parabolic Higgs vector bundles were introduced by Carlos Simpson in [Sim90] as follows: A tame parabolic Higgs vector bundle $(E,\ (E^{ij})_{i[s^j]j[|S|]},\ \phi)$ on a punctured Riemann surface $(X,\ S)$ is an algebraic/holomorphic vector bundle E on X, parabolic filtrations $(E^{ij})_{i[s^j]}$ of $E|_{x^j}$, $x^j \in S$ and a Higgs field $\phi :\ E \to E \otimes \omega_X(D)$ such that $\phi(E^{ij}) \subset E^{ij} \otimes \omega_X(D)$ for all $1 \leq i \leq s^j$, $x^j \in S$ and $D = \sum_{x^j \in S} x^j$ the divisor on X associated to S. $(E,\ (E^{ij})_{i[s^j]j[|S|]},\ \phi)$ is semistable if $\frac{\text{par-deg}\, F}{\mathrm{rk}(F)} \leq \frac{\text{par-deg}\, E}{\mathrm{rk}(E)}$ holds for every ϕ-invariant subbundle $F \subset E$, i. e. $\phi(F) \subset F \otimes \omega_X(D)$.
We would like to recover the tame parabolic Higgs vector bundles as a special case of affine parabolic Ad-Higgs bundles (resp. a slight modification thereof). First note that ϕ gives rise to a section $H^0(X,\ \mathrm{End}(E) \otimes \omega_X(D))$ and that $\mathrm{Ad}(E) \simeq \mathrm{End}(E)$.[7] Using the contragredient representation $\tilde{\mathrm{Ad}}$ we get a morphism $\varphi : P_{\tilde{\mathrm{Ad}}} \to \omega_X(D)$.[8] By 5.1, the corresponding affine parabolic $\tilde{\mathrm{Ad}}$-Higgs bundle $(E,\ (E^{ij})_{i[s^j]j[|S|]},\ \varphi)$ is (semi)stable for trivial χ if[9]

$$\sum_{k=1}^{r} \alpha^k (\text{par-deg}(E)\, \mathrm{rk}(F^k) - \text{par-deg}(F^k)\, \mathrm{rk}(E))\ (\geq)\ 0$$

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[6] see e. g. [GGM13] or Arroyo [Arr09] for the non-parabolic version. The limited choice of reducing parabolic subgroups of $\mathrm{Gl}(r, \mathbb{C})$ coming from G also restricts the choice of (pointwise) parabolic filtrations and weights.

[7] If $(g^{ij})_{ij}$ are the transition functions of E, than the $\mathrm{Ad}(g^{ij})$-action on $\mathbb{C}^{n \times n}$ is identified with the $((g^{ij})^{-1})^t \otimes (g^{ij})$ action under $\mathbb{C}^{n \times n} \simeq \mathbb{C}^{n^2}$.

[8] Note that for a semisimple Lie group Ad is a self-dual representation.

[9] see 2.20.

holds for every weighted filtration $(F^k, \alpha^k)_{k[r]}$ (as in 1.6) of E such that $\phi \in H^0(X,\ P_{\lambda,\mathrm{Ad}} \otimes \omega_X(D))$. But $\phi \in H^0(X,\ P_{\lambda,\mathrm{Ad}} \otimes \omega_X(D))$ if and only if the endomorphism associated to ϕ maps $\phi(F^k) \subset F^k \otimes \omega_X(D)$. To see the last equivalence, note that $\mathfrak{g}_{\lambda,\mathrm{Ad}} = \{Y \in \mathfrak{g} = \mathbb{C}^{n\times n} : \lim_{z\to\infty} \lambda(z) Y \lambda(z)^{-1} \text{ exists}\}$ for the one-parameter subgroup λ associated to $(F^k,\ \alpha^k)_{k[r]}$. Locally at $x \in X$ we get in a basis of eigenvectors of λ and for not necessarily different weights $(\gamma^j)_{j[r]}$

$$\lim_{z\to\infty} \lambda(z)(\phi^{kl}|_x)_{kl}\lambda(z)^{-1} = \lim_{z\to\infty}(z^{\gamma^k-\gamma^l}\phi^{kl}|_x)_{kl} \text{ exists} \Leftrightarrow \phi^{kl}|_x = 0,\ \forall\gamma^k > \gamma^l.$$

Thus ϕ maps to the $Q_{\mathrm{Gl}(\mathbb{C}^n)}(\lambda)$-subbundle given by the reduction $\mathscr{R}$ corresponding to $(F^k,\ \alpha^k)_{k[r]}$ and therefore $\phi(F^k) \subset F^k \otimes \omega_X(D)$ for all k. Since $\sum_{k=1}^r \alpha^k(\text{par-deg}(E)\operatorname{rk}(F^k) - \text{par-deg}(F^k)\operatorname{rk}(E))\ (\leq)\ 0$ if and only if at least one $\text{par-deg}(E)\operatorname{rk}(F^{k_0}) - \text{par-deg}(F^{k_0})\operatorname{rk}(E)\ (\leq)\ 0$ we see that $(E,\ (E^{ij})_{i[s^j]j[|S|]},\ \phi)$ is (semi)stable if and only if it is (semi)stable as a tame parabolic Higgs vector bundle.

To get a moduli space for tame parabolic Higgs vector bundles we need to account for the condition $\phi(E^{ij}) \subset E^{ij} \otimes \omega_X(D)$.

For $\varrho = \tilde{\mathrm{Ad}} :\ \mathrm{Gl}(\mathbb{C}^r) \to \mathrm{Gl}\left((\mathfrak{gl}(\mathbb{C}^r))^\vee\right)$ [10] there are universal morphisms $\phi_{\mathfrak{A}_{\mathrm{par}}} : \mathscr{E}_{\mathfrak{A}_{\mathrm{par}}} \to \mathscr{E}_{\mathfrak{A}_{\mathrm{par}}} \otimes \pi_X^*(\omega_X(D))$ and universal quotients $q^{ij}_{\mathfrak{A}_{\mathrm{par}}} : \mathscr{E}_{\mathfrak{A}_{\mathrm{par}}}|_{x^j} \to \mathscr{H}^{ij}_{\mathfrak{A}_{\mathrm{par}}}$. Thus there is a closed subscheme $\hat{\mathfrak{A}}_{\mathrm{par}} \subset \mathfrak{A}_{\mathrm{par}}$ where the restriction of $(q^{ij}_{\mathfrak{A}_{\mathrm{par}}} \otimes \mathrm{id}_{\pi_X^*(\omega_X(D))}) \circ \phi_{\mathfrak{A}_{\mathrm{par}}}$ to $\ker q^{ij}_{\mathfrak{A}_{\mathrm{par}}}$ vanishes. The universal properties 1.22 and 1.23 still hold for the natural extension of the concept of a Y-family to our new objects. Since the $\mathcal{G}_A$-action leaves $\hat{\mathfrak{A}}_{\mathrm{par}}$ invariant, the result of 1.25 holds for $\hat{\mathfrak{A}}_{\mathrm{par}}$, too.[11] Since $\mathfrak{A}^{(s)s}_{\mathrm{par}} /\!\!/ \mathcal{G}_A$ exists as a quasi-projective scheme, so does $\hat{\mathfrak{A}}^{(s)s}_{\mathrm{par}} /\!\!/ \mathcal{G}_A$.

The concept of a tame parabolic Higgs vector bundle extends to principal bundles as well. Given a line bundle L a tame parabolic twisted G-Higgs bundle $(P,\ (s^j)_{j[|S|]},\ \phi)$ is a parabolic principal bundle $(P, (s^j)_{j[|S|]})$ on X and a section $\phi \in H^0(X, P_{\mathrm{Ad}} \otimes L(D))$ such that $\phi|_{x^j}$ maps to $(P_{\tau^j,\mathrm{Ad}} \otimes L(D))|_{x^j}$.[12] Observe that $\mathscr{Z}(G) \subset Q_G(\tau^j)$ and therefore $s^j_{\mathfrak{A}_{\mathrm{par}}}$ defines a universal morphism $\tilde{s}^j_{\mathfrak{A}} : (P_{\mathfrak{A}_{\mathrm{par}},\mathrm{Ad}}/\operatorname{Ad}(Q(\tau^j)) \otimes \pi_X^*(L(D)))|_{x^j} \to \pi_X^*(L(D))|_{x^j}$ on $\mathfrak{A} \times \{x^j\}$. Again we find a closed subscheme where the composition with $\phi_{\mathfrak{A}_{\mathrm{par}}}$ vanishes.

Remark. Observe that like in the general construction we may associate a projective object to our affine object: a tame parabolic G-Higgs bundle $(P,\ (s^j)_{j[|S|]},\ \phi)$

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[10] Note that the adjoint representation is homogeneous.

[11] Observe that since $q_{\mathfrak{A}_{\mathrm{par}}} : V \otimes \mathcal{O}_X(-n) \to \mathscr{E}_{\mathfrak{A}_{\mathrm{par}}}$ is an epimorphism, $(q^{ij}_{\mathfrak{A}_{\mathrm{par}}} \otimes \mathrm{id}_{\pi_X^*(\omega_X(D))}) \circ \phi_{\mathscr{A}_{\mathrm{par}}}|_{\ker q^{ij}_{\mathfrak{A}_{\mathrm{par}}}} = 0 \Leftrightarrow (q^{ij}_{\mathfrak{A}_{\mathrm{par}}} \otimes \mathrm{id}_{\pi_X^*(\omega_X(D))}) \circ \phi_{\mathfrak{A}_{\mathrm{par}}} \circ q_{\mathfrak{A}_{\mathrm{par}}}|_{\ker q^{ij}_{\mathfrak{A}_{\mathrm{par}}} \circ q_{\mathfrak{A}_{\mathrm{par}}}} = 0$. Now $\mathcal{G}_A$-invariance is obvious.

[12] Sometimes these objects are called $\omega_X(D)$-pairs to distinguish them from those $(P,\ (s^j)_{j[|S|]},\ \phi)$ that satisfy the stronger (nilpotency) condition $\phi|_{x^j}$ in $((P \times_{\mathrm{Ad}} \{X \in \mathfrak{g} : \operatorname{Ad}(e^{tX_{\tau^j}})X \to 0\}) \otimes L(D))|_{x^j}$.

has an associated Hitchin pair $(P,\ (s^j)_{j[|S|]},\ \phi,\ 1)$. A Hitchin pair is a tame parabolic G-Higgs vector bundle $(P,\ (s^j)_{j[|S|]},\ \phi)$ plus a complex number h. Two Hitchin pairs $(P_1,\ (s_1^j)_j,\ \phi_1,\ h_1)$ and $(P_2,\ (s_2^j)_j,\ \phi_2,\ h_2)$ are equivalent if there is a bundle isomorphism $\psi : P_1 \to P_2$ that respects the parabolic reductions and a complex number $z \in \mathbb{C}^*$ such that $h_1 = zh_2$ as well as $z \cdot (\mathrm{Ad}(\psi) \otimes \mathrm{id}_L) \circ \phi_1 = \phi_2$. A Hitchin pair $(P,\ (s^j)_j,\ \phi,\ h)$ is (semi)stable if the associated tame parabolic Higgs vector bundle $(P,\ (s^j)_{j[|S|]},\ \phi)$ is (semi)stable and if in the case $h = 0$, there is no reduction $\mathscr{R}$ to a one-parameter subgroup λ such that $\mu(\phi, \lambda, \mathscr{R}) < 0$. If G is semisimple $\phi \in H^0(X, P_{\mathrm{Ad}} \otimes L(D))$ induces a $\varphi \in \mathrm{Hom}(P_{\mathrm{Ad}},\ L(D))$ under $P_{\mathrm{Ad}} \simeq P_{\mathrm{Ad}}^\vee$. In particular, since the adjoint representation is homogeneous, we get an associated projective $(\varrho^0 \oplus \mathrm{Ad})$-Higgs bundle with Higgs field $(h\iota^0) \oplus (\iota^1\varphi)$, which obviously satisfies asymptotic (semi)stability, i. e. 3.13.(ii). Thus the projective moduli space of Hitchin pairs exists.

Remark. Yet another application of our general construction are ν^{ij}-parabolic Higgs bundles ([IIS06i], [IIS06ii]): Let $(\nu^{ij})_{i[r]j[|S|]}$ be a tuple of complex numbers such that $\deg(E) + \sum_{j=1}^{|S|}\sum_{i=1}^{r} \nu^{ij} = 0$. A ν^{ij}-parabolic Higgs bundle is an affine parabolic Ad-Higgs bundle $(E, (E^{ij})_{i[r]j[|S|]}, \phi)$ of parabolic type $(r^k)_{k[r]} = (k)_{k[r]}$ with structure group $\mathrm{Gl}(\mathbb{C}^r)$ such that $(\mathrm{res}_{x^j}(\phi) - \nu^{ij})(E^{ij}) \subset E^{i+1,j}$ for all i, j. We consider ϕ as a homomorphism $E \to E \otimes \omega_X(D)$ and use the classical stability condition. Again we have universal morphisms $\phi_{\mathfrak{A}_{\mathrm{par}}} : \mathscr{E}_{\mathfrak{A}_{\mathrm{par}}} \to \mathscr{E}_{\mathfrak{A}_{\mathrm{par}}} \otimes \pi_X^*(\omega_X(D))$, universal quotients $q_{\mathfrak{A}_{\mathrm{par}}}^{ij} : \mathscr{E}_{\mathfrak{A}_{\mathrm{par}}}|_{x^j} \to \mathscr{H}_{\mathfrak{A}_{\mathrm{par}}}^{ij}$ and a closed subscheme $\hat{\mathfrak{A}}_{\mathrm{par}} \subset \mathfrak{A}_{\mathrm{par}}$ where the restriction of $(q_{\mathfrak{A}_{\mathrm{par}}}^{ij} \otimes \mathrm{id}_{\pi_X^*(\omega_X(D))}) \circ (\mathrm{res}_{x^j}(\phi_{\mathfrak{A}_{\mathrm{par}}}) - \nu^{ij})$ to $\ker q_{\mathfrak{A}_{\mathrm{par}}}^{ij}$ vanishes. As a result the moduli space of (semi)stable ν^{ij}-parabolic Higgs bundles $\mathfrak{A}_{\mathrm{par}}^{(s)s} /\!\!/ \mathcal{G}_A$ exists as a quasi-projective scheme.

5.3. THE KOBAYASHI-HITCHIN CORRESPONDENCE

Let $\varrho_K : K \to \mathrm{U}(W)$ denote the unitary representation to ϱ. Consider the open Riemann surface $\mathring{X} = X \backslash S$. Given a reduction $\mathscr{R} : \mathring{X} \to P/K$ we have $(\mathscr{R}^*P) \times_{\varrho_K} W \simeq P \times_\varrho W$ and hence a chosen hermitian structure on W induces a hermitian structure h on $P \times_\varrho W$. Let h_L be a hermitian metric on L, then $h \otimes h_L$ is a metric on $P_\varrho \otimes L$. Let $\phi \in H^0(\mathring{X}, P_\varrho \otimes L)$ and $\phi^{\vee, h\otimes h_l} \in H^0(\mathring{X}, (P_\varrho \otimes L)^\vee)$ the dual w. r. t. $h \otimes h_L$, then $i\phi \otimes \phi^{\vee, h\otimes h_l}$ is skew-hermitian as an element of $H^0(\mathring{X}, \mathrm{End}(P_\varrho \otimes L)^\vee) \simeq H^0(\mathring{X}, \mathrm{End}(P_\varrho)^\vee)$, i. e. defines an element of $H^0(\mathring{X}, (P_\varrho \times_{Ad} \mathfrak{u})^\vee)$. The dual homomorphism $d\varrho_K^\vee : \mathfrak{u}^\vee \to \mathfrak{k}^\vee$ defines the moment map $d\varrho_K^\vee \left(-\frac{i}{2}\phi \otimes \phi^{\vee, h\otimes h_l}\right) \in H^0(\mathring{X}, ((\mathscr{R}^*P) \times_{Ad} \mathfrak{k})^\vee)$[13] and the non-degenerated bilinear form on $\mathfrak{k}$ identifies $d\varrho_K^\vee \left(-\frac{i}{2}\phi \otimes \phi^{\vee, h\otimes h_l}\right)$ with a section $\mu_{\mathscr{R}}(\phi) \in H^0(\mathring{X}, (\mathscr{R}^*P) \times_{Ad} \mathfrak{k})$.

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[13] cf. [Kir84], Lemma 2.5. for the case of a projective action.

We are now able to state the Kobayashi-Hitchin correspondence for stable affine parabolic ϱ-Higgs bundles.

5.2. Theorem. (Kobayashi-Hitchin correspondence) *Let* $\alpha \in i\mathfrak{k}_{\mathscr{Z}(G)}$ *and* $(P,\ (s^j)_{j[|S|]},\ \varphi,\ L)$ *be a* $(\alpha,\ \tau^j)$*-stable affine parabolic* ϱ*-Higgs bundle. Then there is a unique reduction* $\mathscr{R} : \mathring{X} \to P/K$ *such that*

$$\Lambda(F_{\mathscr{R}}) + \mu_{\mathscr{R}}(\phi) = -i\alpha$$

as an equality in $(\mathscr{R}^* P) \times_{Ad} \mathfrak{k}$ *, where* $F_{\mathscr{R}}$ *denotes the curvature of the (unique) Chern connection on* P *w. r. t.* $\mathscr{R}$*.*[14]

Remark. The Kobayashi-Hitchin correspondence extends to polystable pairs. More details can be found in [GGM12]. In particular [GGM12] provides a proof of the Kobayashi-Hitchin correspondence in the non-parabolic case.

The Kobayashi-Hitchin correspondence originates in the 1960's when M. S. Narasimhan and C. S. Seshadri proved a first correspondence between irreducible flat unitary bundles and stable vector bundles of degree 0 on a compact Riemann surfaces ([NS65]). At the beginning of the 1980's Kobayashi [Kob80] (and independently Lübke [Lüb82]) proved, that a holomorphic bundle on a Kähler manifold that admits a Hermitian-Einstein metric, is already stable. The reverse statement conjectured by Kobayashi and independently by Hitchin was consequentially proved by Donaldson in the case of compact Riemann surfaces and algebraic surfaces ([Don85], [Don87]). A famous result by Uhlenbeck and Yau established the correspondence on every Kähler manifold ([UY86]). Higgs bundles where first defined in 1987 by Nigel Hitchin, who extended the until then known correspondence to relate Hermitian-Einstein metrics to stable Higgs bundles ([Hit87]). Parabolic Higgs bundles finally where introduced by Carlos Simpson in [Sim90]. There have been various further extensions and modifications of the original correspondence, e. g. [Cor88], [Bra91], [Ban96], [Biq97], [Mun00]. The interested reader may find a much more complete account of the available literature in most books on the topic. Recent results include [GGM12], [LT06], [BS11] or [Moc07]. References for geometric properties of the moduli space of Higgs bundles may be found in the introduction.

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[14] Λ is the dual Lefschetz operator.

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REFERENCES

[ADW91] S. Axelrod, Steve Della Pietra, and Edward Witten, *Geometric Quantization of Chern-Simons Gauge Theory*, J. Differential Geometry **33** (1991), 787-902.

[Arr09] M.A. Arroyo, *The Geometry of* $\mathrm{SO}(p,q)$*-Higgs Bundles*, Dissertation, Universidad de Salamanca, 2009.

[Bal10] V. Balaji, *An Introduction to Families, Deformations and Moduli*, Universitätsverlag Göttingen, Göttingen, 2010.

[Ban96] D. Banfield, *The geometry of coupled equations in gauge theory*, Dissertation, University of Oxford, 1996.

[BD12] I. Biswas and A. Day, *SYZ Duality for Parabolic Higgs Moduli Spaces*, Nuclear Physics B **862** (2012), no. 1, 327–340.

[Be14] N. Beck, *Modulräume dekorierter Prinzipalbündel auf einer projektiven Kurve*, Doktorarbeit, Freie Universität Berlin, 2014.

[Bir71] David Birkes, *Orbits of linear algebraic groups*, Annals of Mathematics **93** (1971), no. 3, 459-475.

[BH04] Indranil Biswas and Yogish I. Holla, *Harder-Narasimhan reduction of a principal bundle*, Nagoya Math. J. **174** (2004), 201-223.

[Bho89] Usha Bhosle, *Moduli of Parabolic G-Bundles* **20** (1987), no. 1.

[Biq97] Olivier Biquard, *Fibrés de Higgs et connexions intégrables : le cas logarithmique (diviseur lisse)*, Annales scientifiques de l'École Normale Supérieure **30** (1) (1997), 41-96.

[Bo91] Armand Borel, *Linear Algebraic groups*, 2nd ed., Graduate Texts in Mathematics **126**, Springer-Verlag, Berlin, 1991.

[Bra91] S. B. Bradlow, *Special metrics and stability for holomorphic bundles with global sections*, J. Diff. Geometry **33** (1991), 169-213.

[BS11] I. Biswas and M. Stemmler, *Hermitian–Einstein connections on polystable parabolic principal Higgs bundles*, Adv. Theor. Math. Phys. **15** (2011), no. 2.

[BV72] D. Burghelea and A. Verona, *Local homological properties of analytic sets*, Manuscripta mathematica **7** (1972), 55-66.

[CMS] Roger Carter, Ian Macdonald, and Graeme Segal, *Lectures on Lie Groups and Lie Algebras*, London Mathematical Society Student Texts **32**, Cambridge University Press, 1995.

[Ch46] C. Chevalley, *Theory of Lie groups*, Princeton University Press, 1946.

[Che58] C. Chevalley, *Les isogénies*, Paris: Secrétariat mathématique, 1958.

[Cor88] K. Corlette, *Flat G-bundles with canonical metrics*, J. Differential Geom. **28** (1988), no. 3, 361-382.

[DM91] Francois Digne and Jean Michel, *Representations of Finite Groups of Lie Type*, London Mathematical Society Student Texts **21**, Cambridge University Press, 1991.

[DS95] V. G. Drinfeld and Carlos Simpson, *B-structures on G-bundles and local triviality*, Mathematical Research Letters **2** (1995), 823-829.

[Don85] S. K. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. **50** (1985), no. 3, 1-26.

[Don87] S. K. Donaldson, *Infinite determinants, stable bundles, and curvature*, Duke Math. J. **54** (1987), 231-247.

[DP09] Ron Donagi and Tony Pantev, *Geometric Langlands and non-Abelian Hodge theory*, Surveys in Differential Geometry **13** (2009), 85-116.

[EGA] Alexander Grothendieck and Jean Dieudonné, *Éléments de géométrie algébrique*, Publications Mathématiques de l'IHÉS **4, 8, 11, 17, 20, 24, 28, 32**, 1960-1967.

[Fal93] Gerd Faltings, *Stable $G-$Bundles and Projektive Connections*, J. Algebraic Geometry **2** (1993), 507-568.

[FG02] Klaus Fritzsche and Hans Grauert, *From Holomorphic Functions to Complex Manifolds*, Graduate Texts in Mathematics **213**, Springer, Berlin, 2002.

[FGA] Alexander Grothendieck, *Technique de descente et théorèmes d'existence en géometrie algébrique*, Fondements de la géométrie algébrique, Extraits du Séminaire Bourbaki, 1957—1962, Secrétariat Mathématique, Paris.

[For81] Otto Forster, *Lectures on Riemann Surfaces*, Graduate Texts in Mathematics **81**, Springer, Berlin, 1981.

[GAGA] J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Annales de l'institut Fourier (1956), 1-42.

[GGM12] O. Garcia-Prada, P. B. Gothen, and I. Mundet i Riera, *The Hitchin-Kobayashi correspondence, Higgs pairs and surface group representations*, arXiv:0909.4487 [math.AG] (v3: 15 Aug 2012).

[GGM13] O. Garcia-Prada, P. B. Gothen, and I. Mundet i Riera, *Higgs bundles and surface group representations in the real symplectic group*, Journal of Topology **6** (2013).

[Gi77] D. Gieseker, *On the moduli space of vector bundles on an algebraic surface*, Ann. of Math. **106** (1977), no. 2, 45-60.

[GLSS06] T. L. Gómez, A. Langer, A. Schmitt, and I. Sols, *Moduli spaces for principal bundles in large characteristic*, Proceedings of the International Workshop on Teichmüller Theory and Moduli Problems (2006), 281–371.

[GLSS08] T. L. Gómez, A. Langer, A. Schmitt, and I. Sols, *Moduli spaces for principal bundles in arbitrary characteristic*, Adv. Math. **219** (2008), 1177-1245.

[GS00] T. L. Gómez and I. Sols, *Stable tensors and moduli spaces of orthogonal sheaves*, math.AG/0103150 (2003).

[GS05] T. L. Gómez and I. Sols, *Moduli space of principal sheaves over projective varieties*, Ann. of Math. **161** (2005), no. 2, 1037-1092.

[Gr66] John W. Gray, *Fibred and Cofibred Categories*, 1966.

[GW08] Sergei Gukov and Edward Witten, *Gauge Theory, Ramification, And The Geometric Langlands Program*, Current Developments in Mathematics **2006** (2008), 35-180.

[Ha77] Robin Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer-Verlag, New York, 1977.

[Ha80] Robin Hartshorne, *Stable reflexive sheaves*, Vol. 254, 1980.

[Hit87] N.J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), 59-126.

[Hit90] Nigel J. Hitchin, *Flat Connections and Geometric Quantization*, Commun. Math. Phys. **131** (1990), 347-380.

[HL10] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, 2nd ed., Cambridge University Press, 2010.

[HS10] A. Schmitt and J. Heinloth, *The cohomology rings of moduli stacks of principal bundles over curves*, Doc. Math. **15** (2010), 423-488.

[Hum75] James E. Humphreys, *Linear Algebraic Groups*, Graduate texts in mathmeatics **21**, Springer, New York, 1975.

[IIS06i] M. Inaba, K. Iwasaki, and M.-H. Saito, *Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painleé equation of type VI. I*, Publ. Res. Inst. Math. Sci. **42** (2006), no. 4, 987–1089.

[IIS06ii] M. Inaba, K. Iwasaki, and M.-H. Saito, *Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painleé equation of type VI. II*, Moduli spaces and arithmetic geometry (Tokyo), Adv. Stud. Pure Math. Math. Soc. Japan **45** (2006), 387–432.

[Kap06] Anton Kapustin, *Holomorphic reduction of $\mathcal{N} = 2$ gauge theories, Wilson-'t Hooft operators, and S-duality*, arXiv:hep-th/0612119v2 (2006).

[Kir84] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry* (1984).

[KN63] Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of Differential Geometry, Vol. I*, Interscience Tracts in pure and applied mathematics **15 (1)**, John Wiley & Sons, New York, 1963.

[Kob80] S. Kobayashi, *First Chern class and holomorphic tensor fields*, Nagoya Math. J. **77** (1980).

[KP00] H. Kraft and C. Procesi, *Classical Invariant Theory*, http://jones.math.unibas.ch/~kraft/Papers/KP-Primer.pdf (2000).

[La04] Robert Lazarsfeld, *Positivity in algebraic geometry II*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 49, Springer-Verlag, Berlin, 2004.

[LP97] J. Le Potier, *Lectures on Vector bundles*, Cambridge Studies in Advance Mathematics, **54**, Cambridge University Press, 1997.

[LT06] M. Lübke and A. Teleman, *The Universal Kobayashi-hitchin Correspondence on Hermitian Manifolds*, Memoirs of the American Mathematical Society, American Mathematical Society, 2006.

[Lüb82] M. Lübke, *Chernklassen von Hermitian-Einstein-Vektorbündeln*, Mathematische Annalen **260**, 133-142.

[Mar10] K. Marathe, *Topics in Physical Mathematics*, Springer-Verlag, London, 2010.

[MFK] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric Invariant Theory*, 3rd ed., Springer-Verlag, Berlin, 1994.

[Mi80] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series, **33**, Princeton University Press, 1968.

[MRed] D. Mumford, *The Red Book of Varieties and Schemes*, Lecture Notes in Mathematics **1358**, Springer-Verlag, Berlin, 2004.

[Moc07] T. Mochizuki, *Asymptotic Behaviour of Tame Harmonic Bundles and an Application to Pure Twistor D-modules, Vol. 1/2*, American Mathematical Society (2007).

[MS80] V. B. Mehta and C. S. Seshadri, *Moduli of vector bundles on curves with parabolic structures*, Math. Ann. **248** (1980), 205-39.

[Mun00] I. Mundet i Riera, *A Hitchin-Kobayashi correspondence for Kähler fibrations*, J. reine angew. Math. **528** (2000), 41-80.

[New78] P. E. Newstead, *Lecture on Introduction to Moduli problems and orbit spaces*, Tata Institute of Fundamental Research, Bombay, Lectures on Mathematics and Physics, **51**, by Springer-Verlag, New York - Berlin, 1978.

[NS65] M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. (2) **82** (1965), 540-67.

[OV90] Arkadij L. Onishchik and Ernest B. Vinberg, *Lie Groups and Algebraic Groups*, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1990.

[Ra73] S. Ramanan, *The moduli space of vector bundles over an algebraic curve*, Ann. of Math. **200** (1973), 69-84.

[Ram75] A. Ramanathan, *Stable principal bundles on a compact Riemann surface*, Ann. of Math. **213** (1975), 129-152.

[Ram96i] A. Ramanathan, *Moduli spaces of principal bundles over algebraic curves I*, Proc. Indian Acad. Sci. Math. Sci. **106** (1996), 301-328.

[Ram96ii] A. Ramanathan, *Moduli spaces of principal bundles over algebraic curves II*, Proc. Indian Acad. Sci. Math. Sci. **106** (1996), 421-449.

[Rou78] Guy Rousseau, *Immeubles sphériques et théorie des invariants*, C.R. Acad. Sc. Paris **286** (178), 247-250.

[RR84] S. Ramanan and A. Ramanathan, *Some remarks on the instability flag*, Tohoku Math. J. **36** (1984), no. 2, 269-291.

[Sch04] Alexander Schmitt, *A closer look at semistability for singular principal bundles*, Int. Math. Res. Not. **2004:62** (2004), 3327-66.

[Sch05] Alexander Schmitt, *Global boundedness for decorated sheaves*, Int. Math. Res. Not. **2004:68** (2004), 3637-3671.

[Sch08] Alexander Schmitt, *Geometric Invariant Theory and Decorated Principal Bundles*, Zurich Lectures in Advanced Mathematics, European Mathematical Society, Zürich, 2008.

[ScSc95] Peter Scheinost and Martin Schottenloher, *Metaplectic Quanitization of the Moduli of Flat and Parabolic Bundles*, J. für Reine und Angewandte Mathematik **466** (1995), 145-219.

[Ser58] J.-P. Serre, *Espaces fibrés algébriques*, Séminaire Claude Chevalley **3** (1958), 1-37.

[Ses67] C. S. Seshadri, *Space of unitary vector bundles on a compact Riemann surface*, Ann. of Math. **85** (1967), no. 2, 303-336.

[Ses77] C. S. Seshadri, *Geometric reductivity over arbitrary base*, Advances in Mathematics **26** (1977), no. 3, 225-274.

[Ses82] C. S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Société Mathématique de France, Paris, 1982.

[SGA] Alexander Grothendieck and Michel Raynaud, *Revêtements étales et groupe fondamental* (*SGA 1*), Documents mathématiques **3** (xviii + 327 pages).

[Sim90] C. T. Simpson, *Harmonic Bundles on Noncompact Curves*, Journal of the American Mathematical Society **3** (1990), no. 3.

[Sim94] C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective group variety I/II*, Publ. Math. l'I.H.É.S. **79/80** (1994), 47-129/5-79.

[Sp81] T. A. Springer, *Linear algebraic groups*, 2nd ed., Progress in Mathematics **9**, Birkhäuser, Boston, 1981.

[St51] Norman Steenrod, *Topology of fibre bundles*, Princeton University Press, 1951.

[St93] Mark Steinberger, *Algebra*, Prindle, Weber, and Schmidt Series in Advanced Mathematics, 1993.

[UY86] K. K. Uhlenbeck and S.-T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Comm. Pure and Appl. Math 39-S (1986), 257-293.

[Whi65] H. Whitney, *Tangents to an analytic variety*, Ann. of Math. **83** (1965), no. 3, 496-549.

EIDESSTATTLICHE VERSICHERUNG

(Siehe Promotionsordnung vom 12.07.11, §8, Abs. 2 Pkt. .5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbständig, ohne unerlaubte Beihilfe angefertigt ist.

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