The Asymptotic Behavior of the Term Structure of Interest Rates

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Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständing, ohne unerlaubte Beihilfe angeferteigt ist.

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Zusammenfassung

Langfristige Zinssätze werden für die Bewertung und Absicherung von festverzinslichen Finanzprodukten und Derivaten mit langer Laufzeit benötigt, sowie bei der Preisberechnung von Zahlungen, die in weiter Zukunft liegen. Solche Zahlungen kann es beispielsweise bei langfristig angelegten Infrastrukturprojekten geben oder bei Ausgleichsregelungen im Falle eines Unfalls oder einer Scheidung. Gerade im Zuge der weltweiten Finanzkrise von 2008 wuchs das Interesse von Anlegern an Investments mit langem Zeithorizont und damit auch die Notwendigkeit Zinskurven weiter in die Zukunft zu modellieren und das Verhalten am langen Ende der Kurven möglichst genau zu bestimmen. Die vorliegende Arbeit widmet sich der Untersuchung des asymptotischen Verhaltens von Zinskurven.

Zu diesem Zwecke werden drei verschiedene langfristige Zinssätze analysiert: der langfristige stetige Zinssatz, der langfristige diskrete Zinssatz und der langfristige Swapzinssatz. Diese langfristigen Zinsen werden definiert als Zinssätze deren Laufzeit gegen unendlich geht im Rahmen eines Zinsmarktes, der auf Erkenntnissen basiert, die aus der Finanzkrise gewonnen werden konnten. Alle modellunabhängigen relevanten Eigenschaften dieser Zinsen werden erläutert und die Zusammenhänge zwischen ihnen werden genauestens hinsichtlich ihrer Wechselbeziehungen untersucht. Darüber hinaus ist ein wichtiger Teil dieser Dissertation der Beschreibung des asymptotischen Verhaltens von Zinskurven in speziellen Zinsmodellen gewidmet. Diese Modelle umfassen das Zinsstrukturmodell von Heath, Jarrow und Morton, genannt HJM Framework, das Flesaker-Hughston Modell sowie das linear-rationale Modell. Das HJM Framework wird aufgrund der Möglichkeit der direkten Modellierung der gesamten Zinsstrukturkurve und aller dazugehörigen Terminkurse für die Analyse verwendet. Die stochastische Komponente wird erst mittels der Brownschen Bewegung beschrieben, dann durch einen Lévy Prozess und zuletzt mit Hilfe eines affinen Prozesses auf dem Zustandsraum von positiv semidefiniten und symmetrischen Matrizen. Der Gebrauch dieser stochastischen Prozesse kann als schrittweise Weiterentwicklung des HJM Frameworks verstanden werden, da jeweils mehr, die Zinsstruktur beeinflussende, Faktoren in die Modellierung mit einfließen können. Die anderen beiden vorgestellten Modelle, das Flesaker-Hughston Modell und das linear-rationale Modell, finden, wegen einiger attraktiver Eigenschaften, Anwendung in der Analyse des asymptotischen Zinskurvenverhaltens, wie zum Beispiel einfache Formeln für alle Zinssätze, die keine negativen Werte annehmen können.

Abstract

Long-term interest rates are essential for the valuation and hedging of various fixed income products and derivatives as well as for the pricing of payments in a distant future, such as long-term infrastructure projects or compensatory adjustments in the course of an accident or a divorce. In the aftermath of the 2008 financial crisis the modeling of interest rate curves with a long time horizon became more and more important due to increased investments in long-term products. Therefore, the study of the asymptotic behavior of the term structure of interest rates has recently achieved new relevance.

In this dissertation we investigate long-term interest rates, i.e. interest rates with maturity going to infinity, in the post-crisis interest rate market. Three different concepts of long-term interest rates are considered for this purpose: the long-term yield, the long-term simple rate, and the long-term swap rate. We analyze the properties as well as the interrelations of these long-term interest rates. In particular, we study the asymptotic behavior of the term structure of interest rates in some specific models. First, we compute the three long-term interest rates in the HJM framework with different stochastic drivers, namely Brownian motions, Lévy processes, and affine processes on the state space of positive semidefinite symmetric matrices. The HJM setting presents the advantage that the entire yield curve can be modeled directly. Furthermore, by considering increasingly more general classes of drivers, we were able to take into account the impact of different risk factors and their dependence structure on the long end of the yield curve. Finally, we study the long-term interest rates and especially the long-term swap rate in the Flesaker-Hughston model and the linear-rational methodology.

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$\langle X\rangle_t$	conditional quadratic variation of a stochastic process X at time t
$[X]_t$	quadratic variation of a stochastic process X at time t
$\alpha(t,T)$	drift for $[t,T]$
(A,ν,γ)	generating triplet of a Lévy process
$a(s,t,T)$	year fraction of $[t,T]$ divided by the year fraction of $[s,T]$ for $s\leq t\leq T$
$A(t,\mathbf{T})$	annuity of an IRS at time t for tenor structure $\mathbf{T}$
B_t	OIS bank account at time t
$B(z)$	linear drift coefficient of an affine process with $z\in S_d^+$
$\mathscr{B}(\mathbb{R})$	Borel σ-algebra on $\mathbb{R}$
$\Gamma(t,T)$	process $\Sigma(t,T)\,X_t\,\Sigma(t,T)$ for an affine process X and $t\leq T$
δ	time distance between two exchange dates of an IRS with equidistance between all exchange dates
ΔX_t	jump of a stochastic process X at time t
$f^D(t,T)=f(t,T)$	instantaneous OIS forward rate at time t with maturity T
$\mathbb{F}$	filtration satisfying the usual conditions with $\mathscr{F}_0=\{\emptyset,\Omega\}$
$F(u)$	result of the first Riccati differential equation for an affine process with $u\in S_d^+$
$\eta(t,T)$	volatility for $[t,T]$ under $\mathbb{P}$ in the classical HJM framework
$G(z)$	linear operator of an affine process with $z\in S_d^+$
$\theta(u)$	logarithm of the moment-generating function of the stochastic driver at time $t=1$ for $u\in\mathbb{R}^d$
$\mathrm{IRS}_L^{cp}(t;T_{i-1},T_i;L_x)$	coupon payoff of a floating leg of an IRS at time t for $[T_i,T_{i+1}]$, where L_x is exchanged
$\mathrm{IRS}_K^{cp}(t;T_{i-1},T_i;K)$	coupon payoff of a fixed leg of an IRS at time t for $[T_i,T_{i+1}]$, where K is exchanged
$\mathrm{IRS}_L^{leg}(t;\mathbf{T};L_x)$	value of the floating leg of an IRS at time t, where L_x is exchanged
$\mathrm{IRS}_K^{leg}(t;\mathbf{S};K)$	value of the fixed leg of an IRS at time t, where K is exchanged
$\mathrm{IRS}(t;\mathbf{T},\mathbf{S};L_x,K;v)$	value an IRS at time t, where K and L_x are exchanged
$K(t,\xi)$	compensator function of $\mu^X(ds,d\xi)$ with $t\geq 0$ and $\xi\in S_d^+\setminus\{0\}$
ℓ_t	long-term yield at time t
$L(X)$	space of integrable processes with respect to the semimartingale X
$L^0(\Omega,\mathscr{F},\mathbb{P})$	space of measurable functions on $(\Omega,\mathscr{F},\mathbb{P})$
$L^1(\Omega,\mathscr{F},\mathbb{P})$	space of integrable functions on $(\Omega,\mathscr{F},\mathbb{P})$
$L^1(\mathbb{R}_+)$	space of integrable functions on $\mathbb{R}_+$

$L^D(t;T,S) = L(t;T,S)$ ...... simply compounded OIS forward at time t rate for $[T,S]$
$L^D(t,T) = L(t,T)$ .... simple OIS spot rate for $[t,T]$
$L_x(t,T)$ .... FRA rate for $[t,T]$ indexed to an IBOR with maturity x
$L_x(t;T,S)$ .... FRA rate at time t for $[T,S]$ indexed to an IBOR with maturity x
$L_{1D}(t,T)$ .... overnight rate for $[t,T]$
$\bar{L}(t,T)$ .... compounded overnight rate for $[t,T]$
$\bar{L}(t;T,S)$ .... compounded overnight rate at time t for $[T,S]$
L_t .... long-term simple rate at time t
$\mu_\infty(t)$ .... long-term drift at time t
$\mu(u)$ .... linear jump coefficient of an affine process for $u \in S_d^+ \setminus \{0\}$
$\mu^X(ds,d\xi)$ .... random measure associated with the jumps of an affine process X
$m(u)$ .... Borel measure representing the jump term of an affine process for $u \in S_d^+ \setminus \{0\}$
$\mathscr{M}_d$ .... space of all $d \times d$ matrices with entries in $\mathbb{R}$ and $d \in \mathbb{N}$
$M_{X_t}(u)$ .... moment-generating function of the stochastic process X at time t for $u \in \mathbb{R}^d$
ν .... payer-receiver flag of a FRA
$\nu(A)$ .... Lévy measure for $A \in \mathscr{B}(\mathbb{R}^d)$
$\nu(dt,d\xi)$ .... $\mathbb{P}$-compensator of the random measure $\mu^X(ds,d\xi)$
$\nu^*(dt,d\xi)$ .... $\mathbb{Q}$-compensator of the random measure $\mu^X(ds,d\xi)$
N_t^B .... Poisson process on $B \in \mathbb{R}^d$ at time t
$(\Omega,\mathscr{F},\mathbb{P})$ .... complete probability space
$(\Omega,\mathscr{F},\mathbb{F},\mathbb{P})$ .... complete filtered probability space
$\mathrm{O}(\cdot)$ .... big O notation
$\mathrm{OIS}(t;\mathbf{T},\mathbf{S};\bar{L},K;\nu)$ .... value of an OIS at time t, where K and $\bar{L}$ are exchanged
$\Pi(t,T)$ .... price at time t of a cashflow in T of a contingent claim
$\mathscr{P}$ .... σ-algebra of pedictable processes
P .... semigroup
$\mathbb{P}$ .... real-world probability measure
$\mathbb{P}_x$ .... probability measure with $\mathbb{P}_x(X_0 = x) = 1$
$P^D(t,T) = P(t,T)$ .... price of a T-OIS bond at time t
$\mathbb{Q}$ .... risk-neutral probability measure
$\mathbb{Q}^S$ .... S-forward measure corresponding to $\mathbb{Q}$
$r_t^D = r_t$ .... OIS short rate at time t
r_t^C .... collateral rate at time t
$R(u)$ .... result of the second Riccati differential equation for an affine process with $u \in S_d^+$
$R_x(t;\mathbf{T},\mathbf{S})$ .... forward swap rate at time t of an IRS, where L_x is exchanged
$R_x(t;T)$ .... forward swap rate at time t of an IRS, where L_x is exchanged and the tenor structures are equivalent
$R_{\mathrm{OIS}}(t;t,T) = R(t,T)$ .... OIS rate for $[t,T]$
$R_{\mathrm{OIS}}(t;\mathbf{T},\mathbf{S})$ .... OIS rate at time t with $\mathbf{T}$ and $\mathbf{S}$ being the legs' tenor structures

R_t .... long-term swap rate at time t
$\mathbf{S}$ .... fixed leg tenor structure
S_d .... space of symmetric $d \times d$ matrices with entries in $\mathbb{R}$ and $d \in \mathbb{N}$
S_d^+ .... space of positive semidefinite symmetric $d \times d$ matrices with entries in $\mathbb{R}$ and $d \in \mathbb{N}$
$S_n(t)$ .... n-finite bond sum at time t
$S_\infty(t)$ .... infinite bond sum at time t
$\sigma(t,T)$ .... volatility for $[t,T]$
$\Sigma(t,T)$ .... volatility of an affine process for $[t,T]$
$\sigma_1(t,T)$ .... partial derivative of $\sigma(t,T)$ in the first component
$\sigma_2(t,T)$ .... partial derivative of $\sigma(t,T)$ in the second component
$\sigma_\infty(t)$ .... long-term volatility at time t
$\mathbf{T}$ .... floating leg tenor structure
$\mathbf{T}_\infty$ .... infinite tenor structure
$\tau(s,t)$ .... time between two dates $s \leq t$, measured in years
$\tau_L(s,t)$ .... year fraction of two dates $s \leq t$ on a floating leg of an IRS
$\tau_K(s,t)$ .... year fraction of two dates $s \leq t$ on a fixed leg of an IRS
$\mathrm{Tr}\left[A^\top B\right]$ .... trace of two matrices A and B
V_t .... present value of a collateralized contract at time t
W .... $\mathbb{P}$-Brownian motion
W^* .... $\mathbb{Q}$-Brownian motion
$Y^D(t;T,S) = Y(t;T,S)$ ...... continuously compounded OIS forward rate at time t for $[T,S]$
$Y^D(t,T) = Y(t,T)$ .... yield for $[t,T]$

List of Abbreviations

BIS	Bank for International Settlements
BNP Paribas	Banque Nationale de Paris Paribas
bp	basis point
CCP	central counterparty
CCS	cross currency swap
CDO	collateral debt obligation
cf.	confer
CME	Chicago Mercantile Exchange
CoCo	contingent convertible
CSA	credit support annex
CVA	credit value adjustment
ECB	European Central Bank
ELMM	equivalent local martingale measure
EONIA rate	Euro overnight index average rate
ETH Zürich	Eidgenössische Technische Hochschule Zürich
EU	European Union
EUR	Euro (currency)
EURIBOR	European interbank offered rate
FF rate	federal funds rate
FRA	forward rate agreement
FX	foreign exchange
GmbH	Gesellschaft mit beschränkter Haftung
HJM	Heath-Jarrow-Morton
IBOR	interbank offered rate
IDS GmbH	Investment Data Services GmbH
IMF	International Monetary Fund
IRS	plain vanilla interest rate swap
ISDA	International Swaps and Derivatives Association
LIBOR	London interbank offered rate
LMU	Ludwig-Maximilians-Universität
OECD	Organization for Economic Co-operation and Development
OIS	overnight indexed swap
OTC	over the counter
p.	page
PhD	Doctor of Philosophy
PLC	public limited company
SDE	stochastic differential equation

SE	Societas Europaea
SEK	Swedish krona (currency)
SSRN	Social Science Research Network
STIBOR	Stockholm interbank offered rate
TED spread	treasury bill and eurodollar futures spread
ucp convergence	uniform convergence on compacts in probability
US	United States
USD	United States dollar (currency)

1. Introduction

1.1. Motivation

A term structure can be defined as a function that puts a financial variable in relation to its maturity. Therefore, the term structure of interest rates relates interest rates or bond yields to different terms or maturities (cf. Chapter 1 of [83]). The term structure of interest rates is also called yield curve which is defined rigorously in our setting in Definition 2.2.3 and is assumed to be continuous. This curve is of fundamental importance in macroeconomics since it puts monetary policy in perspective to investment behavior resulting in economic growth and vice versa. It reflects the expectations of market participants about future changes in interest rates (cf. Section 1.2.3 of [58]). A practical problem is the determination of a mathematical expression for the current term structure because there is only a finite number of maturities of bonds that are traded at financial markets. This problem is solved by calibrating the term structure curve to current market data, i.e. by fitting a curve to a number of points, see, for example, Chapter 6 of [3] and Chapter 3 of [83]. To take into account the uncertainty in time evolution of the yield curve, stochastic interest rate models are needed that are coherent with market data. In the literature, there have been many different proposals for the modeling of interest rates, e.g. short rate models, interbank offered rate (IBOR) market models, swap market models, or the Heath-Jarrow-Morton (HJM) framework (cf. Chapter I and II of [33]). For the calculation of the yield at long maturities, the choice of the specific stochastic model that incorporates the expectations about the future behavior of the interest rates is crucial since there is minor market data for yields of longer maturities.[1] The asymptotic behavior of the yield curve as well as interest rates with a long term are very important topics for financial institutions that invest in products depending on a long time horizon, either via a maturity in the far-away future or due to perpetual characteristics. Therefore, the modeling of long-term interest rates is the subject of several publications in economical and mathematical research. Considering the various contributions to the topic, it has to be noted that no unique definition of long-term interest rates is provided. The denomination "long-term" can be understood in several different ways such that there exist different conventions on the concept of long-term interest rates in the literature. The *European Central Bank (ECB)* considers yields of government bonds with maturities of close to 10 years as long-term (cf. [78]), in [155] high-grade bonds with a time to maturity of more than 20 years are examined for the investigation of long-term interest rates, whereas in [161] the author considers the time span between 30 and 100 years for the analysis on the long end of yield curves. However, a natural mathematical approach to the study of long-term interest rates is to examine the different rates, meaning the continuously compounded spot rate, the simply compounded spot rate, and the swap rate, with their respective maturity going to infinity. This concept of long-term is used among others in

[1]For instance the longest term for United States (US) treasury yields provided by *Bloomberg* is 30 years (cf. [28]).

[23], [24], [25], [35], [40], [73], [160], and [161]. In this thesis we adopt this definition and present different convergence results for the long-term yield, the long-term simple rate, and the long-term swap rate dependent on the underlying interest rate model as well as results on the interdependencies between these rates that are independent of the model. All of these long-term interest rates are related to the long-term zero-coupon bond prices, which we define in our setting as the theoretical price of a perpetual zero-coupon bond, see Definition 3.1.1. The long-term yield ℓ is defined as the continuously compounded spot rate with maturity tending to infinity, see Definition 3.1.13, whereas the long-term simple rate L is the simply compounded spot rate with maturity tending to infinity, see Definition 3.1.17. This particular definition of the long-term yield corresponds to its definition in some textbooks such as in Section 2.3 of [38], in Subsection 6.3.2 of [40], and in [83]. The long-term simple rate was first defined this way in [35] to propose an alternative discounting rate for long-term financial products or projects with a very long time horizon. The long-term swap rate R, defined in equation (3.1.18), can be understood as the fixed fair rate of an overnight indexed swap (OIS) that has a payment stream with infinitely many exchanges. It should be fair in the sense that the price of the receiver and the payer of this OIS equals zero. This rate was defined for the first time in [24] and the use of an OIS as special case of an interest rate swap (IRS) stems from the fact that OIS rates are used as proxy for risk-free rates in interest rate modeling, due to the last financial crisis, as explained in Section 2.1. There is an ongoing debate about the starting point of this crisis, which is difficult to determine since the development of the crisis was a gradual process from the sub-prime crisis over to the credit crunch, then to the liquidity crisis of banks, and finally to the public debt crisis, especially in Europe (cf. [74] and [158]). Nevertheless, most of the literature considers the 9th of August 2007 as initial date, when *BNP Paribas*, one of the largest banks in the world, announced the closing of three hedge funds specialized in US mortgage debt. *BNP Paribas* was not able to value the holdings, in particular the collateralized debt obligations (CDOs). In consequence, the *ECB* allowed Euro area banks to draw as much liquidity as they needed for refinancing at the prevailing overnight rate on the same day (cf., for example, Chapter II of [13], [74], Section 3 of [75], [124], and [142]). However, there is a broad consensus about the climax of the financial crisis that is dated the 15th of September 2008, when the US investment bank *Lehman Brothers* filed for bankruptcy and a shock to the international financial market followed (cf. [45], [74], and [116]). Therefore, we call this crisis the 2008 financial crisis subsequently in this thesis.

In the course of this crisis term structure modeling in general changed significantly as explained in Section 2.1, and the evaluation of long-term financial products as well as interest rates became more important (cf. [19] and [76]). Besides the more mathematical approaches, a lot of studies were published addressing the topic of long-term interest rate modeling from a macroeconomic point of view. These approaches want to take into account the importance of monetary and fiscal policy regarding this subject, especially during the time of a financial crisis. Mankiw et al. examine in [129] the impact of monetary and fiscal policies on long-term interest rates and show that interest rates with a long time horizon do not react too sensitive to short-term rates. Several other economic factors can also be characterized as macroeconomic news, for example, data releases regarding the gross domestic product, new home sales, or initial claims as well as a substantial rise or decline in the unemployment rate or of the capacity utilization rate, see Table 1 of [102] for a comprehensive list. In general, the macroeconomic approaches to the topic of long-term interest rate modeling seek to identify precisely those factors influencing the long-term rates. Gürkaynak et al. provide in [102] evidence that most of the mentioned factors significantly affect both short-term and long-term rates, and in [107] the authors use an affine function dependent on macroeconomic variables to evaluate the continuously compounded spot rate. With the help of this model the influence of macroeconomic effects on the long-term yield can be measured. A three factor model is applied for modeling the yield curve in [123], i.e. the evolution of the interest rates is described by three latent factors that are employed in order to explain the empirical observation of falling long-term yields. The construction of a model that jointly characterizes the behavior of the yield curve and macroeconomic variables, is the subject of the publications [4] and [60]. In [4] a vector autoregression model is applied for the description of the relationship between interest rates and macroeconomics, whereas [60] uses a latent factor model with the inclusion of macroeconomic variables to model the yield curve. Instead of using other economic factors to explain the behavior of long-term interest rates, these long-term rates can also be understood as one of these factors that influence asset pricing. This approach is applied by Chen et al. in [41], where the long-term yield, in terms of long-term government bonds, is one of the several economic factors.

Regardless of considering a macroeconomic approach to the modeling of long-term interest rates or a more mathematical one, the obtained rate is essentially important for the pricing and hedging of long-term fixed income securities like perpetual bonds, life and accident insurances, pension funds, or IRSs with a long time to maturity. Besides these financial instruments there are situations in which the time horizon of cashflows extends beyond the limit of the observable term structure of interest rates: for example, the valuation of required financial resources for public and private retirement systems, the funding of long-term infrastructure projects, or the determination of compensatory adjustments in the course of an accident or a divorce. Therefore, as already mentioned, the knowledge about the asymptotic behavior of the term structure of interest rates is important from an economic as well as financial point of view. It allows to model a fair discounting rate for long-term products, but also to efficiently hedge these products. For instance, the consideration of the long-term swap rate is to some extent motivated by the observation that some financial products may involve the interchange of cashflows on a possibly unbounded time horizon since this rate could be useful in hedging the interest rate risk of these products. One of the products that are increasingly offered by banks since the start of the 2008 financial crisis is the contingent convertible (CoCo) bond. It is a debt instrument with an embedded option for the issuer, mostly banks, to convert debt into equity. This possibility is typically used by credit institutes that have to overcome a period of liquidity problems (cf. [1], [32], [64], [88], and [89]). In the course of the crisis Bernanke, the chairman of the *Federal Reserve* at that time, pointed out the importance of these products for financial institutions to maintain a certain level of capital, see [22]. Dudley, the president of the *Federal Reserve Bank of New York*, stated in [63] that, to strengthen the financial system, an increase in the use of CoCo bonds should be one of three main points realized in the aftermath of the 2008 financial crisis.[2] These products can be decomposed into a portfolio consisting of bonds and exotic options, see [32]. Furthermore, in [32] a valuation formula for the price of a CoCo bond with finite maturity

[2]The other two main points the financial industry should focus on, according to Dudley, are a more thorough and complete risk capture as well as rules that encourage institutes to save money in economic boom periods.

can be found, where in [1] the authors also consider the case of an infinite maturity. This is of practical importance since, besides CoCo bonds having a finite time to maturity, there also exist some with unlimited maturity. For example, in June 2014 *Barclays PLC* issued a perpetual CoCo bond, which pays fixed coupons of 7% with the investor having a first conversion possibility in 2019 and then every 5 years until eternity (cf. [147]). This perpetual product, where the coupons of the non-optional part are floating, could lead to investors seeking for a hedging instrument offering protection against the interest rate risk involved in the non-optional part of the contract. A fixed to floating interest rate swap with infinitely many exchanges could serve as a hedging product for the interest rate risk beared by CoCo bonds, with the fixed rate of such a swap being the long-term swap rate. Another motivation of analyzing the long-term swap rate lies in the context of multiple curve bootstrapping because, according to the post-crisis market practice, OIS contracts constitute the input quotes for bootstrapping procedures, which allow for the construction of a discounting curve, as explained in Section 2.1. In view of this, the long-term swap rate can be applied for inference of information on the long-end of the discounting curve. The main results of the investigation of the long-term swap rate R are an explicit model-free formula developed in equation (3.1.19) in the case of a convergent infinite bond sum S_∞, and Theorem 3.1.29, which tells us that this interest rate is either constant or non-monotonic over time. Further, we see in Corollary 3.1.25 that R is always finite if it exists. Hence, we propose the long-term swap rate as an alternative discounting tool for long-term investments, due to the facts that R is almost always finite, non-monotonic, can be explicitly characterized, and can be inferred by products existing on the markets.

In contrast to R, the long-term yield ℓ is monotonic in the sense that it is a non-decreasing process. This was first shown in [69] by Dybvig, Ingersoll, and Ross, and consequently is referred to as DIR theorem, see Theorem 3.1.16. It has been the topic of several publications, among others [96], [108], [121], and [132]. Besides Theorems 3.1.16 and 3.1.29 we are able to provide some more model-free results concerning the interrelations between different rates. All possible different relations between the three defined long-term interest rates are analyzed in Section 3.2. In this context we state the interesting fact that a strictly positive long-term yield entails a strictly positive long-term swap rate and an exploding long-term simple rate, see Corollary 3.2.2. Another intriguing relation is pointed out in Corollary 3.2.13 that tells us that if L is strictly positive it is not possible for ℓ and R to be strictly positive. Furthermore, we see in Corollary 3.2.19 that from a strictly positive long-term swap rate it follows that ℓ and L are non-negative processes in the rather realistic case of a finite long-term bond price. Apart from these general results on long-term interest rates, we also consider specific interest rate models to develop explicit formulas for ℓ, L, and R.

There are only a few studies analyzing long-term interest rates in predetermined term structure models and we try to provide a comprehensive overview of the different approaches in this thesis. Most of these approaches use a HJM framework like [23], [25], and [73]. In [73], El Karoui et al. examine the long-term yield in a Brownian HJM framework and conclude that in the case of a finite rate, it is independent from the underlying probability measure since the Brownian part vanishes (cf. equation (4.1.24)). This specific study of the long-term yield is presented along with results on L and R in this model in Section 4.1. In the following section, an extension of the Brownian HJM framework is considered by changing the stochastic driver from a Brownian motion to a Lévy process. This approach is based on [25] and its motivation stems from the idea to incorporate also jumps into the term structure model. Concerning the long-term interest rates in a Lévy HJM framework, the main results are findings regarding the asymptotic behavior of ℓ and Theorem 4.2.10. This theorem shows that the volatility part describing the long-term yield has to vanish, as in the Brownian HJM framework, if ℓ is not supposed to explode, except for the case of a Lévy process with only negative jumps and paths of finite variation serving as random driver. Then, in Section 4.3, another generalization of the HJM framework is used as a term structure model for the analysis of the long-term interest rates. The basis for this study is [23] and here, the stochastic drivers are affine processes on the state space of symmetric positive semidefinite matrices. This particular class of stochastic processes was chosen due to its appealing features such as the possibility to model correlated factors influencing the term structure's volatility or to describe positive spreads among different curves resulting from credit and liquidity risk. We want to take into account the increased study of this state space in financial research. A literature overview of this specific topic can be found at the beginning of Section 4.3. Again, we are able to provide an explicit formula for the long-term yield in this model, see equation (4.3.59). In this setting the long-term volatility part must also vanish if ℓ is supposed to exist finitely as proven in Theorem 4.3.16. The asymptotic behavior of the other long-term interest rates L and R is concluded from ℓ. Following this approach, we consider some interest rate models that are not assigned to the HJM framework in the next two sections. These are the Flesaker-Hughston model and the linear-rational methodology. The consideration of long-term interest rates for both of these models is part of the article [24] by Biagini et al. In Section 4.4, the Flesaker-Hughston term structure model, which is named after the authors of [91], is described and applied for the valuation of long-term interest rates. This model was developed in 1996 and has been the topic of several publications (cf. e.g. [97], [133], [139], [151] and [153]), due to some favorable characteristics such as relatively simple resulting models for bond prices, short and forward rates. Other advantages are the specification of only non-negative interest rates and the high degree of tractability. The different long-term interest rates are computed in two specific examples of this approach, where the functions specifying the form of the zero-coupon bond prices are given. Then, in Section 4.5 we present the linear-rational term structure methodology, which was recently introduced in [84] by Filipović and Trolle, as basis for the investigation of long-term interest rates. This class of term structure model was chosen for our considerations of long-term interest rates since it presents various advantages: it is highly tractable, non-negative interest rates are guaranteed, unspanned factors affecting volatility and risk premia are accommodated, analytical solutions to swaptions are admitted, and it offers a very good fit to IRSs and swaptions data. The main result here is a closed-form formula for the long term swap rate, see equation (4.5.9), and the fact that ℓ is a constant process.

Altogether, this thesis presents a complete discussion of the asymptotic behavior of the term structure of interest rates. All different long-term interest rates are defined and characterized after the introduction of the necessary interest rates and fixed income products in a modern modeling framework. Then, the interrelations of the long-term interest rates and their other model-free properties are explained to finally compute the rates explicitly and study their asymptotic behavior in specific and appropriate term structure models.

The thesis is structured as follows. Chapter 2 introduces the setting for interest rates and fixed income products necessary for all further investigations. For this matter, we first distinct in Section 2.1 the interest rate modeling in the post-crisis era from former one-curve frameworks. Then, we describe in Section 2.2 all interest rates needed in the course of the thesis by considering the discount curve of a multi-curve framework since we are mainly interested in long-term interest rates as discounting tool. Section 2.3 gives insight about the pricing of collateralized contracts such as IRSs, whose evaluation formula is derived in detail in Section 2.4. In particular, the computation of an OIS rate is shown, which is needed for the definition of R. Chapter 3 deals with long-term interest rates in the sense that they are defined and analyzed with regard to universally valid properties and their relations towards each other. In Section 3.1 the reader finds the definitions and universal characteristics of the long-term bond price P, of the infinite sum of bond prices S_∞, as well as of the long-term interest rates ℓ, L, and R, whereas the interrelations are described in Section 3.2. Finally, in Chapter 4 we analyze the long-term interest rates in some selected term structure models. The structuring of the different sections within this chapter has already been explained in detail above.

1.2. Contributing Manuscripts

This thesis is based on the following manuscripts which were developed by the thesis' author M. Härtel in cooperation with co-authors:

1. F. Biagini and M. Härtel [25]: *Behavior of Long-Term Yields in a Lévy Term Structure. International Journal of Theoretical and Applied Finance, 17(3): 1-24, 2014.*

   The results of this publication on the behavior of long-term yields in a term structure model using Lévy processes as stochastic driver were devised by M. Härtel together with Prof. F. Biagini. The work was developed at the LMU Munich. The suggestion of investigating the asymptotic behavior of the yield curve in a Lévy HJM framework was made by Prof. F. Biagini in order to to generalize the approach of N. El Karoui, A. Frachot, and H. Geman in [73] who considered a Brownian HJM framework for their analysis. A significant part of the computations contained in the proofs was taken care of by M. Härtel.

2. F. Biagini, A. Gnoatto, and M. Härtel [23]: *Affine HJM Framework on S_d^+ and Long-Term Yield. LMU Mathematics Institute, Preprint, 2013.*
   Available at: http://www.fm.mathematik.uni-muenchen.de/download/publications.

   This article is a joint work of Prof. F. Biagini, Dr. A. Gnoatto, and M. Härtel. It was developed at the LMU Munich. In joint discussions, we developed the idea of considering affine processes on the state space of symmetric positive semidefinite matrices S_d^+ for an analysis of the asymptotic behavior of long-term yields. These kind of processes have appealing features for term structure modeling and are used frequently in recent publications of financial research. Sections 2 and 3, where necessary results on affine processes on S_d^+ are gathered and the HJM framework for this kind of driving process is described, were developed by M. Härtel with support by Dr. A. Gnoatto. The investigation of the long-term yield in this particular framework which is the content of Section 4 was developed in close cooperation by Prof. F. Biagini, Dr. A. Gnoatto, and M. Härtel. The examples presented in Section 5 were chosen and computed independently by M. Härtel.

3. F. Biagini, A. Gnoatto, and M. Härtel [24]: *The Long-Term Swap Rate and a General Analysis of Long-Term Interest Rates. LMU Mathematics Institute, Preprint, 2015. Available at: http://www.fm.mathematik.uni-muenchen.de/download/publications.*

   This paper defining for the first time in literature the long-term swap rate and analyzing the interrelation of long-term yield, long-term simple rate, and long-term swap rate emerged by a collaboration of Prof. F. Biagini, Dr. A. Gnoatto, and M. Härtel. The article was developed at the LMU Munich. In joint discussions we developed the idea of introducing the long-term swap rate as a new kind of long-term interest rate. Dr. A. Gnoatto and M. Härtel have embedded this approach in the context of the post-crisis interest market by using the fact that OIS rates are mainly used as discounting rates in multi-curve frameworks. Sections 2 and 3, where prerequisites for further examinations are stated, were developed by M. Härtel. Sections 4 and 5 that contain the main results of the paper by defining and characterizing the long-term swap rate as well as investigating all relations between the long-term interest rates were developed in a joint work by Prof. F. Biagini, Dr. A. Gnoatto, and M. Härtel. The analysis of the long-term interest rates in two specific term structure models in Section 6 was performed by M. Härtel with help by Dr. A. Gnoatto.

The following list indicates in which way the three publications contribute to each part of the thesis. The formulation of the statements of the corollaries, definitions, lemmas, propositions, and theorems is similar or the same as in the three manuscripts. However, the author, who has been involved in the development of all the results contained in the three articles, provides in the present thesis a more detailed version for most of the proofs.

1. Chapter 1 was developed independently by M. Härtel.

2. Chapter 2 was developed independently by M. Härtel.

3. Chapter 3 is mainly based on F. Biagini, A. Gnoatto, and M. Härtel [24]. Section 3.1 consists of Sections 4 and 5 of F. Biagini, A. Gnoatto, and M. Härtel [24] and some work that was done independently by M. Härtel. Section 3.2 is based on Section 5 of F. Biagini, A. Gnoatto, and M. Härtel [24].

4. Chapter 4 is based on all three manuscripts [23], [24], and [25]. Section 4.1 was developed independently by M. Härtel and provides the basis for the following sections by illustrating in details some results of N. El Karoui, A. Frachot, and H. Geman [73]. Section 4.2 is based on Sections 2 and 3 of F. Biagini and M. Härtel [25]. Section 4.3 is based on Sections 2 - 4 of F. Biagini, A. Gnoatto, and M. Härtel [23]. Section 4.4 is based on Subsection 6.1 of F. Biagini, A. Gnoatto, and M. Härtel [24]. Section 4.5 is based on Subsection 6.2 of F. Biagini, A. Gnoatto, and M. Härtel [24].

2. Fixed Income Basis

In this chapter we collect some basic results and notations of instruments used on fixed income markets. The purpose is to develop a common language for the remainder of the thesis. For this, we have to introduce notations to characterize prices and yields of basic fixed income market securities as zero-coupon bonds, the money-market account and different interest rates. The interrelation between these securities is addressed as well as their respective significance in fixed income markets. When speaking about these different instruments it is important to be clear about which curve is used since the main difference between the theory of interest rate modeling before and after the 2008 financial crisis is, that in the post-crisis framework there is not only one curve used for discounting and computing forward rates, which is assumed to be risk-free, but one discounting curve representing the risk-free curve and multiple curves for modeling the forward rates dependent on the respective instrument. Therefore, we first distinguish the classical single-curve approach from the modern multi-curve framework of interest rate modeling in Section 2.1 and provide the reader insight into the effects of the financial crisis on yield curve modeling. Then, in Section 2.2 the needed interest rates as well as zero-coupon bonds are explained in a multi-curve framework. In the course of our investigations on long-term interest rates, we will consider long-term swap rates that depend upon a special class of IRSs, namely OISs. For this reason, we examine IRSs with regard to valuation of their present values and corresponding forward swap rates in Section 2.4. In the preceding section the collateral is defined and a formula for the present value of collateralized financial instruments is presented since collateralization is crucial for the valuation of derivatives, especially IRSs. Throughout the whole thesis we consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with the filtration $\mathbb{F} := (\mathscr{F}_t)_{t\geq 0}$ satisfying the usual hypothesis of right-continuity and completeness, where $\mathscr{F}_\infty \subseteq \mathscr{F}$ and $\mathscr{F}_0 = \{\emptyset, \Omega\}$. The probability measure $\mathbb{P}$ denotes the real-world measure and the equivalent probability measure $\mathbb{Q}$ is the risk-neutral one.

Note, that for the valuation of fixed income instruments and derivatives the time between the observation date and a future cashflow plays an important role. This time span is always measured in years but this year fraction depends on the use of a specific calendar, the business day convention, and the day-count convention. The choice of calendar determines the holidays in the respective payment schedule, whereas the business day convention explains how to adjust for dates in the payment schedule if they fall on a day that is not a business day. The day-count convention describes the method of calculating an accrual factor that relates to a given period. That means, when $\tau(s,t)$ measures the time between two dates $0 \leq s \leq t$ it must contain the information about these time counting conventions. A full discussion of the different business day conventions and day-count conventions can be found in Section 4.3 of [42] or in Chapter 2 of [81]. In Appendix B of [106] the market conventions for IBOR indexes, overnight indexes, and different fixed income products and derivatives are listed. Another very important factor that influences the valuation of financial instruments is the use of an exchange rate if the considered instrument is denoted in another currency than its numéraire. We will only consider a unique currency in the thesis but all following results could be converted to foreign currencies by the formulas presented in Section 2.2.1 of [92].

For the description of the single-curve framework we follow the textbooks [3], [33], and [83], whereas the main references regarding the multiple curve setup are [26], [50], [92], [106], and [135]. The information about collateralization and central clearing is taken from [6], [34], [87], [104], and [146]. We primarily used results of [2] and [87] for the discussion of IRSs.

2.1. Post-Crisis Interest Rate Market

Interbank risk can be defined as lending risk in the interbank money market according to Definition A.3. The importance of interbank risk grew in the course of the recent financial crisis and influenced the modeling of interest rates significantly. It is measured as the spread between an IBOR and the rate of a maturity-matched OIS. An IBOR is the interest rate at which banks lend to and borrow from one another in the interbank market. In the USD-denominated fixed income market the main reference rate is the USD London interbank offered rate (LIBOR), whereas in the EUR-denominated fixed income market this rate is the European interbank offered rate (EURIBOR). Both of these rates are derived as a trimmed average of specific bank panels that are periodically reviewed and revised. There are also IBORs for all kinds of local rates, like for example STIBOR for the SEK rate fixed in Stockholm (cf. Section 1.1.1 of [122]). IBORs are quoted for a range of maturities with the most important being overnight, three and six months, denoted by 1D, 3M, and 6M (cf. Section 1.1.1 of [122]).[1] There are as well fixed income products or derivatives such as OISs that are tied to overnight rates. In the USD market, the main reference rate is the effective Federal funds (FF) rate and in the EUR market the benchmark is the Euro overnight index average (EONIA) rate (cf. Section 2.1 of [87]). An OIS is an IRS where a fixed rate for a period is exchanged for the geometric average of overnight rates during this period.[2] The mentioned FF rate and EONIA rate are the overnight rates used in the OIS geometric average calculations. A party can swap its overnight borrowing or lending for borrowing or lending at a fixed rate, whereby this fixed rate is referred to as OIS rate. The calculation of this rate is explained in detail in Section 2.4. To get insight how the increased interbank risk changed the way of term structure modeling, we first consider the pre-crisis modeling approach.

Before the 2008 financial crisis the term structure of interest rates was modeled via a single-curve approach. The approach's name stems from the fact that one risk-free curve was used for modeling the discounting and forward rates. The concept of risk-free refers to the absence of elements of credit and liquidity risk, not to the absence of interest rate risk.[3] After constructing this risk-free yield curve it reflects at the present the costs of future cash flows as well as the level of the forward rates (cf. [33], [83], and [109]). This approach was justified by negligible coun-

[1] For more information on LIBOR and EURIBOR, especially on the composition of the bank panels and the different maturities, refer to [79] and [113].

[2] As noted in Section 9.2 of [110] the term "geometric average of overnight rates" must be interpreted as "geometric average of one plus the overnight rates minus one".

[3] The terms credit, interest rate, and liquidity risk are explained in Definitions A.1, A.4, and A.5.

terparty and liquidity risk (cf. Section 1 of [141]).[4] The crisis caused several inevitably consequences for all financial market participants, as among others, companies were faced with credit and liquidity problems more than ever. Hence, credit and liquidity risks had to be accounted for when pricing financial products what resulted in increased spreads between different tenors and currencies as well as in increased credit spreads. These spreads were typically smaller than the bid-ask spread and therefore negligible before the start of the crisis (cf. Section 2.3 of [26]). The definitions and corresponding interpretations of these spreads can be found in Appendix B. Figures 2.1 and 2.2 capture the growth of different credit spreads, in fact the IBOR-OIS spreads and TED spreads for the USD and EUR markets. In Figure 2.1 the LIBOR-OIS and EURIBOR-OIS spreads for 3M rates are displayed in basis points (bps) over a nine-year time period beginning in the first quarter of 2006 and ending in the first quarter of 2015.[5] It can be seen that before the start of the financial crisis the spreads were very low indicating almost no default risk in the interbank market. Between the end of the first quarter of 2006 and the beginning of August 2007 the 3M LIBOR-OIS spread was never larger than 15 bps, and the 3M EURIBOR-OIS spread never exceeded 10 bps, only to jump to 39,95 bps and 17,7 bps, respectively, on the 9th of August 2007 what is often considered as the crisis' start, as explained in Section 1.1. Then, after a continuous increase, the IBOR-OIS spreads peaked in the aftermath of *Lehman Brothers*' bankruptcy with 364,43 bps for the LIBOR-OIS spread and 195,50 bps for the EURIBOR-OIS spread. Following this peak the IBOR-OIS spreads settled at a much lower, but nevertheless non-negligible, level with an interim high in December 2011 during the climax of the European sovereign crisis (cf. Section 1 of [145]).

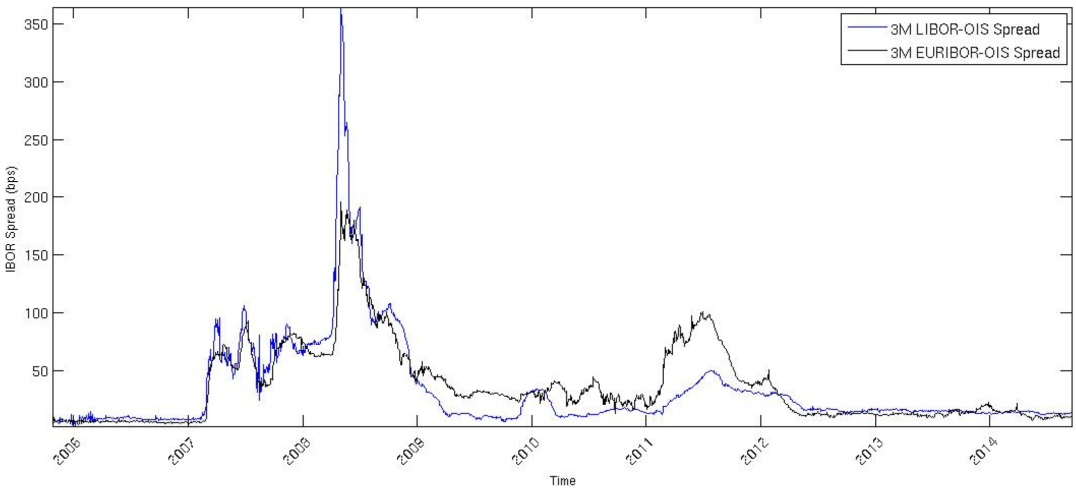

Figure 2.1.: 3M LIBOR and 3M EURIBOR spreads. Own presentation, data retrieved from *Bloomberg*.[6]

[4] A description of counterparty risk can be found in Definition A.2.

[5] One bp equates to one hundredth of a percentage point, i.e. $1\,\text{bp} = 0,01\%$ (cf. Section 13.12.1 of [33]).

[6] The author is grateful to *IDS GmbH - Analysis and Reporting Services* for providing the *Bloomberg* data.

Figure 2.2 depicts the USD and EUR TED spreads and shows the tight correlation to the respective IBOR-spread. In consequence we can make similar conclusions regarding the TED spreads as we concluded from Figure 2.1.

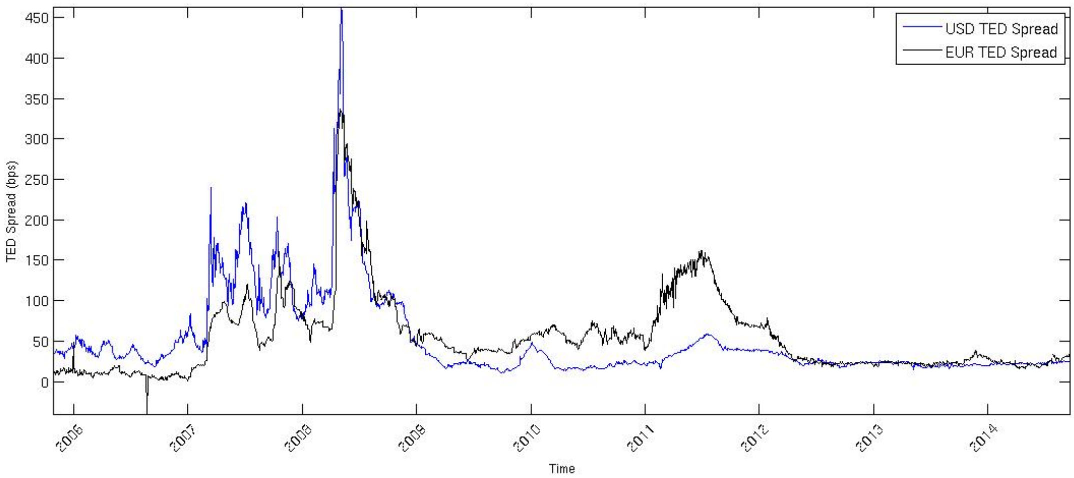

Figure 2.2.: USD and EUR TED spreads. Own presentation, data retrieved from *Bloomberg*.

In Figure 2.3 the increased counterparty risk is displayed via the curves of the tenor basis spreads of the EURIBOR and the LIBOR over the time period starting with the first quarter of 2006 till the first quarter of 2015. In both cases we consider basis swaps[7] that exchange payments based on the 6M rate semiannually and on the 3M rate quarterly with a maturity of one year. We see that before the crisis there were virtually no tenor basis spreads existent as from the first quarter of 2006 to the end of the second quarter of 2007 it was always lower than 1 bp for EURIBOR and LIBOR. It speaks for the predominant opinion on the markets during this time period that a default-free environment was present. In the course of the 2008 financial crisis tenor basis spreads raised up to 44,95 bps and 47,6 bps for EURIBOR and LIBOR, respectively, indicating the high counterparty risk. After an upward swing during the sovereign debt crisis, both tenor basis spreads stabilized on a level of around 10 bps which cannot be neglected.

In Appendix B.4 it is shown that the risky part of a cross currency swap (CCS) can be decomposed into two tenor basis swaps, hence the cross currency basis spread equals the difference of two tenor basis spreads in the respective local currencies. Consequently the cross currency basis spreads are highly correlated to tenor basis spreads. It is illustrated in Figure 2.4, which captures the spreads in CCSs between EURIBOR and LIBOR based on a 3M tenor with different maturities of one year, three years, five years, and ten years.

In the following we summarize the most important facts that are assumed in the single-curve approach, but due to the described change in the market environment could not be used any longer. Interbank credit and liquidity issues as well as collateral amounts and different funding rates do not influence the pricing of fixed income products or derivatives, IBORs serve as a

[7] See Definition B.3.1 for an explanation of basis swaps.

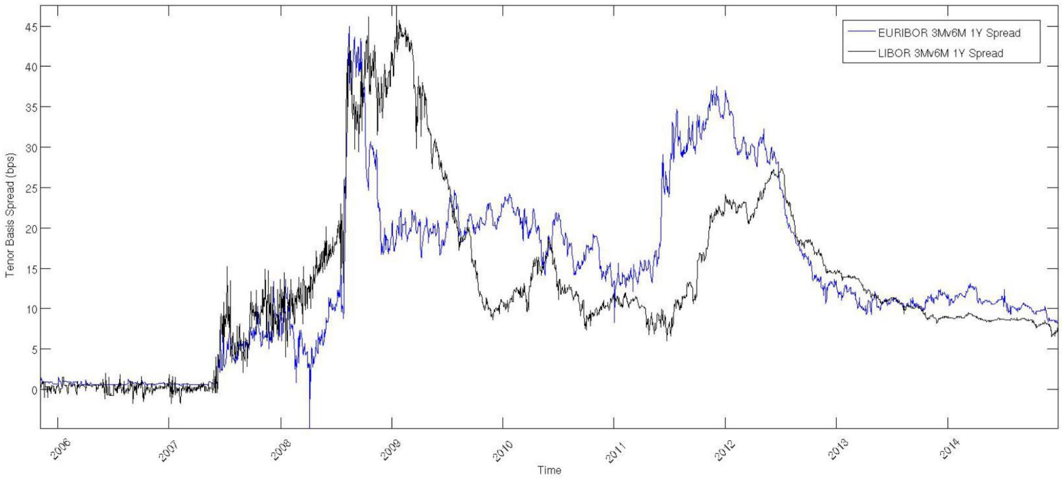

Figure 2.3.: LIBOR 3Mv6M 1Y and EURIBOR 3Mv6M 1Y spreads. Own presentation, data retrieved from *Bloomberg*.

good proxy for risk-free interest rates, and tenor basis swap spreads are insignificant. These former assumptions led to the pre-crisis standard market practice for the construction of the single yield curve which can be described as the procedure of choosing one finite set of the most liquid interest rate instruments traded in real time on the market with increasing maturity to construct one yield curve using bootstrapping techniques, and then compute on this single curve forward rate agreement (FRA) rates, discount factors, and cashflows of different kinds of financial products (cf. Section 2.1 of [2]).[8]

The new market situation, where interbank rates could not be considered risk-free any longer, entailed a new framework for the modeling of interest rates. This framework takes into account the described market developments translating into the additional requirements of homogeneity and funding. By homogeneity it is meant that interest rate derivatives have to be priced and hedged with the use of interest rate market instruments that coincide in the underlying's rate tenor. The funding requirement endures that the discount rate of any derivative's cashflow is consistent with its associated funding rate, e.g. for collateralized OTC derivatives the funding rate is the collateral rate (cf. Section 2.2 of [2]). Due to these requirements, the post-crisis market standard of interest rate modeling is the construction of multiple yield curves, hence it is called multi-curve framework or multiple curve framework (cf. e.g. [50], [106], or [135]). If an investor seeks the price of some derivatives, the construction procedure starts with choosing the appropriate funding rates for the derivatives that should be priced, then select the corresponding market instruments and build a single yield curve for discounting following the single-curve approach. Next, multiple separated FRA curves are constructed using multiple separated sets of interest rate instruments traded in real time on the market and homogeneous in the underlying tenor. Then, the relevant cashflows are computed via the FRA curves and the derivative prices

[8]FRAs are explained in Section 2.4.

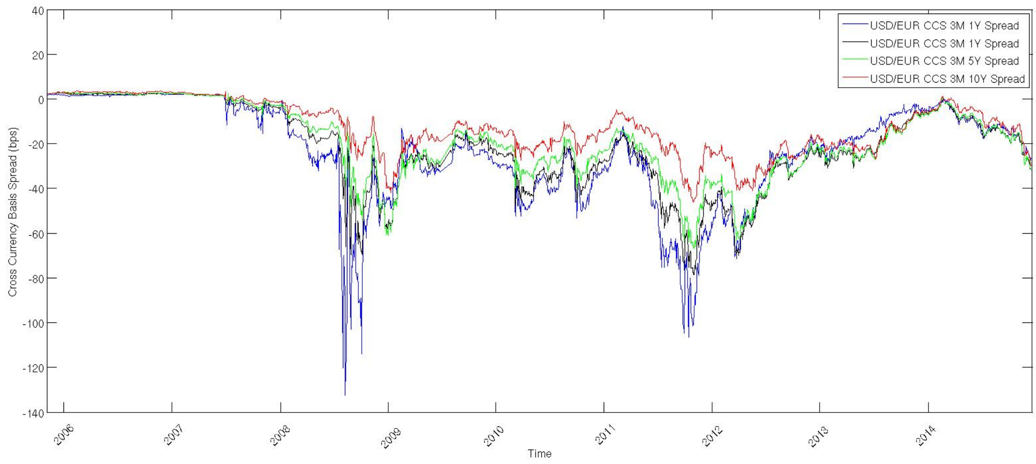

Figure 2.4.: USD/EUR cross currency 3M basis spreads. Own presentation, data retrieved from *Bloomberg*.

are derived by summing these cashflows discounted with the unique discount curve (cf. Section 2 of [106]).

2.2. Discount Curve

As we have seen in Section 2.1, different curves have to be used for the valuation of cashflows that bear different credit or liquidity risk, but we have also seen that still a discount curve dependent on the funding rates is needed. This curve is the topic of this section.

If the cashflow is valued today we use a discount curve but if the valuation time is in future, for example for the valuation of cashflows of bonds or IRSs, a suitable forward curve has to be used (this is consistent with Definitions 3 and 5 of [92]). Which curve is used depends on the respective tenor as well as on the counterparty, and its respective credit and liquidity risk, that issued the corresponding financial instrument.[9] For any counterparty the respective curves depend upon the so-called fundamental curve and the specific spread. The fundamental curve is the discounting curve that is connected to risk-free cashflows, where risk-free means that credit and liquidity risk can be neglected. There is no strict definition of this curve, the "choice of the discounting curve is by itself an open question. Different people will choose different curves" (p.9, [106]). The common proxy for these risk-free rates are OIS rates which are the market quotes for OISs, they are also often used as collateral rates in collateralized contracts (cf. Section 1.1 of [50], Section 1 of [134], and Section 2.2 of [136]). This yields OIS discounting, where the OIS zero-coupon bond curve is obtained from OIS rates by bootstrapping techniques and these bonds are considered as basic traded instruments.[10] Bonds are the primary financial instruments

[9]For discussions on credit and liquidity risk refer to Section 1 of [17] and Section 3 of [103].
[10]For explanations on bootstrapping of yield curves refer to Section 4 of [2] and Chapter 3 of [83].

that are used on the fixed income markets for trading the time value of money, which is one of the most fundamental concepts of financial mathematics and can be formulated as "a dollar today is not worth a dollar tomorrow or next year" (p.3, [62]). The basic kind of bonds do not have any periodic interest payments, the so-called zero-coupon bonds. We define an OIS zero-coupon bond the following way.

Definition 2.2.1. *A contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments and without any credit or liquidity risk, is called a T-maturity OIS zero-coupon bond. The contract value at time $t \leq T$ is denoted by $P^D(t,T)$.*

The superscript D in Definition 2.2.1 stands for *discounting* and it follows obviously that $P^D(T,T) = 1$ for all $T \geq 0$. Note that the zero-coupon bond price is a càdlàg process on $(\Omega, \mathscr{F}, \mathbb{P})$, but we write the shortened version $P^D(t,T) := P^D(\omega,t,T)$ for all $0 \leq t \leq T$. This is done for all stochastic interest rates and financial instruments throughout the thesis.

A bond's coupon payment is a periodic interest payment paid by its issuer to its holder. This interest payment can either be a fixed rate on the bond nominal or it can depend on a variable interest rate that for instance varies according to a money market index, such as LIBOR. Most bonds traded on international bond markets are coupon bearing bonds, there is only a relatively small number of zero-coupon bonds (cf. Chapter II, Section 4b of [59]). A zero-coupon bond can be understood as fixed-coupon bond with a sequence of coupons having value of zero, hence the term zero-coupon bond. In the course of this thesis the only bond type we are interested in is zero-coupon bonds, therefore when talking about bonds we always mean zero-coupon bonds, i.e. we abbreviate T-maturity OIS zero-coupon bond by T-OIS bond.

We assume that the OIS bond market fulfills the following hypothesis:

(i) There exists a frictionless market for T-OIS bonds for every maturity $T \geq 0$.

(ii) For every fixed time $t \geq 0$ the bond price $P^D(t,T)$ is differentiable in T.

(iii) The bond prices are strictly positive and finite, i.e. $0 < P^D(t,T) < \infty$ $\mathbb{P}$-a.s. for all $0 \leq t \leq T$.

(iv) There exists an OIS bank account that grows exponentially with regard to the OIS short rate.

(v) There exists an equivalent probability measure $\mathbb{Q}$ such that the OIS bonds for all maturities are $\mathbb{Q}$-martingales when denominated in units of the OIS bank account.

This framework for the bond market is more general than the ones presented in Section 6.1.1 of [3] and in Section 1.1.4 of [122] since we do not impose the bond prices to be monotonically decreasing for increasing maturity and we do not restrict the bond prices to be below 1 because we want negative rates to be possible. We only restrict the bond prices to be strictly positive and finite as we want to work in an arbitrage-free setting.

The OIS short rate process is denoted by $r^D := \left(r^D_t\right)_{t\geq 0}$ and in the framework of the thesis it is assumed that market participants can lend or borrow cash associated with the OIS bank account which is defined in the following definition. That means, we can refrain from defining a generic funding account. This is done for simplicity but without loss of generality since the funding account can be defined similar to the OIS bank account, where the bond market rate is an IBOR plus spread, cf. Section 3.2 of [2].[11] The corresponding OIS bank account is defined as follows.

Definition 2.2.2. *The OIS bank account, denoted by $(B_t)_{t\geq 0}$, is defined as the process that evolves according to the following differential equation:*

$$dB_t = r_t^D B_t dt,\ \ B_0 = 1. \tag{2.2.1}$$

As a consequence we get for all $t \geq 0$

$$B_t = \exp\left(\int_0^t r_s^D\, ds\right). \tag{2.2.2}$$

Taking the discount bond prices as basis, we can distinguish between different discounting rates with $\tau(s,t)$ measuring the time difference between two dates $0 \leq s \leq t$ in years.

The simply compounded OIS forward rate, denoted by $L^D(t;T,S)$, that is fixed at time $t \leq T$ for the time interval $[T,S]$, can be defined with the help of the following trading strategy, that can be found in Section 2.2 of [83]:

At t: Sell one T-OIS bond and buy $\frac{P^D(t,T)}{P^D(t,S)}$ S-OIS bonds.
Net investment: 0.

At T: Pay one unit of currency due to the sold T-OIS bond at time t.
Net investment: -1.

At S: Receive $\frac{P^D(t,T)}{P^D(t,S)}$ units of currency due to the bought S-OIS bonds at time t.
Net investment: $+\frac{P^D(t,T)}{P^D(t,S)}$.

This yields

$$1 + L^D(t;T,S)\,\tau(T,S) = \frac{P^D(t,T)}{P^D(t,S)} \tag{2.2.3}$$

and consequently we get for all $0 \leq t \leq T \leq S$

$$L^D(t;T,S) = \frac{1}{\tau(T,S)}\left(\frac{P^D(t,T)}{P^D(t,S)} - 1\right). \tag{2.2.4}$$

Note, that we are able to get (2.2.3) as a result of the described trading strategy only because we assume OISs as risk-free. In general, for other rates like IBORs, equation (2.2.4) does not hold due to credit and counterparty risk (cf. Section 2.2 of [136]).

Then, the simple OIS spot rate for the time interval $[t,T]$, denoted by $L^D(t,T)$, is

$$L^D(t,T) := L^D(t;t,T) = \frac{1}{\tau(t,T)}\left(\frac{1}{P^D(t,T)} - 1\right),\ t \leq T. \tag{2.2.5}$$

[11] For detailed information on funding risk and costs refer to [31], Chapter 17 of [34], and Section 4 of [140]..

Based on the same trading strategy as for the simply compounded discounting rate, we get for the continuously compounded OIS forward rate for $[T,S]$ prevailing at time $t \leq T$, denoted by $Y^D(t;T,S)$, that

$$\exp\big(Y^D(t;T,S)\,\tau(T,S)\big)\,\frac{P^D(t,S)}{P^D(t,T)} = 1\,, \tag{2.2.6}$$

what yields

$$Y^D(t;T,S) = -\frac{\log P^D(t,S) - \log P^D(t,T)}{\tau(T,S)}\,. \tag{2.2.7}$$

Accordingly, the continuously compounded OIS spot rate for $[t,T]$ is defined as

$$Y^D(t,T) := Y^D(t;t,T) = -\frac{\log P^D(t,T)}{\tau(t,T)}\,,\ t \leq T. \tag{2.2.8}$$

From now on, we will indicate the continuously compounded OIS spot rate as yield and hence define the yield curve the following way.

Definition 2.2.3. *The function $T \to Y^D(t,T)$ is referred to as the yield curve in $t \geq 0$.*

Let us recall that the term “yield curve” is used differently in the literature. As explained in the previous section, the 2008 financial crisis had consequences for interest rate modeling and in particular for yield curve constructing. Pre-crisis, it was sufficient to construct a single discounting curve, whereas nowadays a collection of interrelated curves is required. That means there are many different yield curves, and in Definition 2.2.3 we mean the OIS yield curve, or discounting yield curve respectively. For reasons of simplicity, we call this curve just yield curve. However, even pre-crisis the yield curve was not defined in a unique way. For instance, in Definition 1.3.1 of [33] it is a combination of simply compounded spot rates for maturities up to one year and annually compounded spot rates for maturities greater than one year. In this thesis, we use equation (2.2.8) as the yield in t for the time interval $[t,T]$ which is equivalent to the definitions in Section 3 of [23], in Section 2.1 of [24], in Section 2 of [25], and in Section 2.4.4 of [83].

Next, we define the instantaneous OIS forward rate which is derived from the OIS forward rates, simply and continuously compounded, when the maturity collapses towards its expiry. That means, the instantaneous OIS forward rate with maturity T prevailing at time t, denoted by $f^D(t,T)$, is

$$f^D(t,T) := \lim_{S\downarrow T} Y^D(t;T,S) = \lim_{S\downarrow T} L^D(t;T,S) = -\partial_T \log P^D(t,T)\,. \tag{2.2.9}$$

We define the forward curve as the curve of the instantaneous OIS forward rate.

Definition 2.2.4. *The function $T \to f^D(t,T)$ is referred to as the forward curve in $t \geq 0$.*

In 1992, Heath, Jarrow, and Morton proposed in [105] a framework of modeling the entire forward curve directly which we use for modeling the long-term yield as described in Sections 4.1 to 4.3.

The OIS short rate can be expressed in terms of the instantaneous OIS forward rate or in terms of the continuously compounded OIS spot rate as follows

$$r_t^D = f^D(t,t) = \lim_{T \downarrow t} Y^D(t,T)\,,\ t \geq 0. \tag{2.2.10}$$

Note, that due to equation (2.2.9) together with $P^D(T,T) = 1$, the price of a T-OIS bond at time $0 \leq t \leq T$ can be calculated as

$$P^D(t,T) = \exp\left(-\int_t^T f^D(t,u)\,du\right). \tag{2.2.11}$$

From (2.2.11) it follows with (2.2.7) that for $0 \leq t \leq T \leq S$ it holds

$$Y^D(t;T,S) = \frac{1}{\tau(T,S)} \int_T^S f(t,u)\,du. \tag{2.2.12}$$

2.3. Collateralization

Collateral can be defined in two different ways in the banking context as described in Section 2.5 of [34]. First, collateral can be understood as the traditional posting of a guarantee for a specific lending, i.e. the lender of some credit exposure receives some assets from the borrower which the lender can keep if the borrower will not be able to pay back the lended amount. This is typical for mortgages, where the collateral is the real estate property that was purchased with the help of the loan that is connected to the mortgage. The second definition of collateral, often referred to as capital market collateralization, is valid in the context of derivative securing, where the financial institutions involved in a derivative transaction, post liquid assets, in particular cash and government bonds, in a bilateral collateral account.[12] This collateral amount has to be kept in line with the mark-to-market value of all transactions between the counterparties. This form of collateralization is implemented for OTC derivatives and turned out to be more and more important in recent years. It has now become a fundamental instrument for handling counterparty risk. In the aftermath of the financial crisis of 2008 there is a special focus on counterparty risk. In the case of capital market collateralization the involved counterparties have to sign a bilateral contract that regulates the credit risk mitigation for derivative transactions, a so-called credit support annex (CSA). The CSA is a standardized agreement between counterparties governing the terms under which collateral is transferred. The following conditions are defined:[13]

- *Eligible collateral*: Assets that can be used for collateralization.
- *Haircuts*: The amount of the collateral that can applied for securitization of the derivative contract.
- *Margin period of risk:* The period from the last collateral exchange with a defaulting counterparty to the valuation time of the close-out amount.

[12] Table 3 of [115] indicates that over 90% of the exchanged collateral amounts against non-cleared OTC derivative transactions in 2013 were classified as cash or government bonds.

[13] Detailed information on CSA can be found in Section 3.2 of [7] and Section 2.5.1 of [34].

- *Minimum transfer amount*: Minimum permitted difference between collateral and mark-to-market value of the derivative.
- *Netting*: Derivative amounts that can be offset against each other.
- *Threshold*: Minimum mark-to-market amount which is allowed to hold without collateralization.

The precise terminology and further characterizations of a CSA can be found in the ISDA Standard Master Agreement (cf. [114]). The most recent ISDA market survey shows that almost all OTC derivatives transactions were subject to a CSA or another collateral agreement (cf. executive summary of [115]). For instance, this can mean that the derivative contract is cleared by a central counterparty (CCP) which is designed to reduce counterparty risk through different aspects like high collateral demands, the mutualisation of losses among collateral receivers from the CCP or multi-lateral netting agreements (cf. Section 1 of [6]). Both in the US and EU financial markets, there has been a regulatory drive towards a significantly increase of the proportion of CCP cleared derivatives (cf. Section 1 of [68] and Section 4.3 of [104]). If counterparty risk really can be mitigated in a substantial way by a CCP is still not fully answered and has been topic of several research articles, e.g. [68] and [156].

Now, we provide the generic formula for a collateralized cashflow that we will use to price IRSs. This formula has been derived among others in [87] and [146]. Let us consider a contract with nominal cashflow of X at maturity T. Its present value at time $0 \leq t \leq T$ is denoted by V_t and we assume that at any time $s \geq 0$ the posted collateral equals 100% of V_s. Under this perfect CSA the possible default of both counterparties is irrelevant, consequently there are no credit or debit value adjustments to the value of the trade. The receiver of the collateral can invest it at the discount rate r^D, but has to pay the collateral rate r^C to the poster of the collateral. Hence, under the measure $\mathbb{Q}$, the present value process is

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s^D\,ds\right)X + \int_t^T \left(r_u^D - r_u^C\right)\exp\left(-\int_t^u r_s^D\,ds\right)V_u\,du \,|\, \mathscr{F}_t\right],\ t \leq T\,. \quad (2.3.1)$$

By Proposition E.1 equation (2.3.1) can be simplified to

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s^C\,ds\right)X \,|\, \mathscr{F}_t\right],\ t \leq T\,. \quad (2.3.2)$$

Since the cashflow of a collateralized T-bond in T is 1 we get for all $0 \leq t \leq T$

$$P^c(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s^C\,ds\right) \,|\, \mathscr{F}_t\right]. \quad (2.3.3)$$

Besides the fact that OIS short rates are the best approximation for risk-free rates, they are also commonly used as collateral rates (cf. Section 1.1 of [50] and Section 1.5 of [106]). Then "the collateral rate r^C corresponds to the OIS short rate r^D, which is usually the case" (p.38, [50]). Thus, for the remainder of the thesis we define

$$r := r^D = r^C \quad (2.3.4)$$

and accordingly for all $0 \leq t \leq T$

$$P(t,T) := P^D(t,T) = P^C(t,T) \ . \tag{2.3.5}$$

From now on, we will also forego the superscript D for the yield Y, the forward rate f, and the simple rate L.

2.4. Interest Rate Swaps

The interest rate derivatives market is comprised of several different products, such as IRSs, basis swaps, caps, floors, CCSs, FRAs, swaptions, or inflation-linked swaps (cf. Section II of [90]). Regarding these instruments, we are only interested in IRSs for our further considerations concerning long-term interest rates. Therefore we describe in this section this basic kind of interest rate derivative. An IRS is a derivative that is normally traded OTC or via a CCP in which two counterparties exchange a stream of fixed-rate payments for a stream of floating-rate payments with both streams quoted in the same currency.[14] These streams are called the legs of the swap. The floating leg is typically indexed to an IBOR of a particular maturity, denoted by L_x with $x \in \{1\text{D}, 1\text{W}, 2\text{W}, 1\text{M}, 2\text{M}, 3\text{M}, 4\text{M}, 5\text{M}, 6\text{M}, 7\text{M}, 8\text{M}, 9\text{M}, 10\text{M}, 11\text{M}, 12\text{M}\}$.[15] If the floating rate is a compounded overnight rate over the payment period, then the respective swap is an OIS. This special case of IRS is addressed at the end of this section.[16] IRSs are either called payer or receiver IRS depending on the fixed rate, i.e. if the investor pays the swap's fixed coupons it is called payer IRS and vice versa.

In general it is interesting to examine IRSs since they account for the majority of the interest rate derivatives market with 78,94% of this market being attributed to IRSs. This is shown in Table 2.1, where the notional amounts of outstanding OTC interest rate derivatives at the end of December 2013 are displayed by different contract types in billion US dollars.[17]

The interest rate derivatives market is embedded in the OTC derivatives market that has experienced a tremendous growth in the last decades as a result of a higher demand for customized products that deal with financial risks. Tables 2.2 and 2.3 present the notional amounts of outstanding OTC derivatives in billion US dollars from 1998 till 2013 in total and by different asset classes. For each year the end of December data are disclosed.[18] It clearly shows the significant rise in the OTC derivatives market as a whole, as well as it indicates that interest rates are by far the most used asset class in this market.

[14] In the case of forex swaps and CCSs the currency of the two swap legs differ from each other. The main market information conveyed by these kind of swaps is the interest rate difference for a given period between two currencies, hence they are mainly interest rate products. Nevertheless they are normally not included when talking about IRSs due to market convention and form an independent class of swaps. These products are out of scope for this thesis. For more information refer to Section 2.8 of [106] and Appendix B.4.

[15] See [79] and [113] that the maturities of the EURIBOR are 1W, 2W, 1M, 2M, 3M, 6M, 9M, 12M, and of the LIBOR are 1D, 1W, 2W, 1M, 2M, 3M, 4M, 5M, 6M, 7M, 8M, 9M, 10M, 11M, 12M.

[16] For a detailed analysis on IRSs with special treatment of OISs and other specific kinds of IRSs such as LIBOR-in-arrear swaps or averaging swaps refer to Sections 5.5 - 5.7 of [3] or Section 2.4 and 2.7 of [106].

[17] Data in Table 2.1 are shown on a net basis, i.e. transactions between reporting dealers are counted only once.

[18] Data in Tables 2.2 and 2.3 are again shown on a net basis. In asset class *Other* we summed up all unallocated OTC derivatives and credit default swaps.

	Notional amounts outstanding 2013	
Contract type	Total	in %
Swaps	461.281	78,94
FRAs	73.819	12,63
Options	49.264	8,43
Total	584.364	100

Table 2.1.: Notional amounts outstanding in the global interest rate derivatives market in billions of US dollars from 2013. Own presentation, data from [16].

	Notional amounts outstanding 1998 - 2005							
	1998	1999	2000	2001	2002	2003	2004	2005
Total	80.137	88.201	95.199	111.115	141.665	197.167	257.894	297.670
Interest Rate	50.015	60.091	64.668	77.513	101.658	141.991	190.502	211.970
FX	18.001	14.344	15.666	16.748	18.448	24.475	29.289	31.364
Equity	1.488	1.809	1.891	1.881	2.309	3.787	4.385	5.793
Commodity	415	548	662	598	923	1.406	1.443	5.434
Other	10.388	11.408	12.313	14.375	18.328	25.508	32.284	43.107

Table 2.2.: Notional amounts outstanding in the global OTC derivatives market in billions of US dollars from 1998 to 2005. Own presentation, data from [8], [9], [10], and [11].

Besides the natural interest in the biggest asset class regarding derivatives, we have a special interest in IRSs and its valuation since it is current market practice to take a suitable swap curve, corresponding to the evaluated instrument, as a discount curve.[19] The swap curve is a graph of fixed coupon rates of market-quoted IRSs across different maturities in time. By capturing market perceptions of the credit quality of the banking sector, swap curves enable investors to visualize forward expectations of unsecured interbank lending rates. Swap curves are constructed and calibrated in segments to the market prices of various fixed-income instruments and it is indispensable to cover the complete term structure, i.e. short-(less than 3 months), middle-(from 3 months up to 2 years), and long-term parts (out to 10 years or more). The short end is calibrated to unsecured deposit rates, the middle area of the curve is derived from a combination of FRAs and interest rate futures, and the long end of the swap term structure is constructed from observed quotes of swap rates (cf. [152]). Then, a combination of bootstrapping and interpolation techniques is used to join these segments into a smooth and consistent swap curve.[20] Due to the

[19]For example at *IDS GmbH - Analysis and Reporting Services*, a subsidiary of *Allianz SE*, all fixed income instruments as well as all derivatives are discounted by corresponding swap curves (cf. [112]).

[20]For detailed instructions on the construction of swap curves refer to Section 3 of [152].

	Notional amounts outstanding 2006 - 2013							
	2006	2007	2008	2009	2010	2011	2012	2013
Total	414.845	595.341	598.147	603.900	601.046	647.777	632.582	710.182
Interest rate	291.582	393.138	432.657	449.875	465.260	504.117	489.706	583.364
FX	40.271	56.238	50.042	49.181	57.796	63.349	67.358	70.553
Equity	7.488	8.469	6.471	5.937	5.635	5.982	6.251	6.560
Commodity	7.115	8.455	4.427	2.944	2.922	3.091	2.587	2.206
Other	68.390	104.293	104.550	95.963	69.434	71.236	66.679	46.500

Table 2.3.: Notional amounts outstanding in the global OTC derivatives market in billions of US dollars from 2006 to 2013. Own presentation, data from [12], [14], [15], and [16].

increased usage of swap curves among market participants the definition of a long-term swap rate with examination of its properties is part of a modern research on long-term interest rates as a matter of course. Further, long-term swap rates can serve as a valuation tool for the pricing and hedging of CoCo bonds as explained in Section 1.1. For this reason, we first explain how to evaluate an IRS and its corresponding swap rate in general and then consider OISs which are important for the remainder of the thesis. Note, that most IRSs between major financial institutions are collateralized (cf. Section 3.2 of [2] and Section 2.2 of [87]).[21] This also holds for FRAs that are OTC contracts themselves (cf. Appendix A of [50]). A FRA is a forward starting deposit, i.e. a contract, where a counterparty, the lender, commits to pay a nominal amount at a future start date to another counterparty, the borrower, and then the borrower pays back at an end date that is after the start date the notional amount plus the interest accrued over the period from start date to end date.[22] In other words, in a FRA two counterparties agree to exchange two cashflows, with one being tied to a floating rate and the other to a fixed rate, both spanning the same time interval. Similar to IRSs, if the fixed rate is paid the contract is called payer FRA, if the floating rate is paid it is called a receiver FRA. Let us denote τ_L and τ_K as the year fractions with the associated floating and fixed leg calendar, business day convention, and day-count convention. In fact, the time-measurement conventions of floating and fixed leg differ normally from each other (cf. Section 1.4 of [33]). Then, the payoff at time S of a FRA, where the floating rate L_x is exchanged over the time interval $[T,S]$ is given by

$$\nu X \left(\tau_L(T,S) L_x(T,S) - \tau_K(T,S) K\right) , \tag{2.4.1}$$

with K denoting the fixed rate and X the nominal amount. In case of a payer FRA $\nu = 1$, and $\nu = -1$ for a receiver FRA. Then, the FRA rate, denoted by $L_x(t;T,S)$, is the rate K fixed at time $t \leq T$ such that the value of the FRA contract with payoff (2.4.1) at time S, has value 0. Using

[21]Duffie and Huang showed in [66] that the effect of counterparty risk on the IRS rate is extremely small and can be neglected. That means we could work with r^C for the pricing of IRSs even if we would have not presumed (2.3.4).

[22]For a detailed analysis of FRAs refer to Section 4.3.2 of [2] or Section 2.5 of [106].

equation (2.3.2) we get that for all $0 \le t \le T \le S$ it holds

$$vX\,\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^S r_u\,du\right)\left(\tau_L(T,S)\,L_x(T,S) - \tau_K(T,S)\,L_x(t;T,S)\right) \mid \mathscr{F}_t\right] = 0\,, \tag{2.4.2}$$

and it follows by (E.6) that

$$\tau_K(T,S)\,L_x(t;T,S) = \tau_L(T,S)\,\mathbb{E}^{\mathbb{Q}^S}\left[L_x(T,S) \mid \mathscr{F}_t\right]\,, \tag{2.4.3}$$

where $\mathbb{Q}^S$ denotes the S-forward measure corresponding to $\mathbb{Q}$. Now, we want to evaluate IRSs using this result since an IRS can be understood as a portfolio of FRAs.

Each IRS is characterized by a payment schedule that consists of two discrete tenor structures

$$T_0 < T_1 < \cdots < T_N \tag{2.4.4}$$

and

$$S_0 < S_1 < \cdots < S_M \tag{2.4.5}$$

with $T_0 = S_0$ and $T_N = S_M$. Let us denote the floating leg schedule by $\mathbf{T} := \{T_0,\ldots,T_N\}$ and the fixed leg schedule by $\mathbf{S} := \{S_0,\ldots,S_M\}$. That means that at each T_i, $i = 1,\ldots,N$, a coupon payoff with floating rate $L_x(T_{i-1},T_i)$ is exchanged and at each S_j, $j = 1,\ldots,M$, a coupon payoff with fixed rate K is exchanged. Further, τ_L and τ_K are the year fractions of the floating and fixed leg, respectively. The contract's nominal is denoted by X. The coupon payoff of the floating leg at time $t \ge 0$, where the rate L_x is exchanged for the time interval $[T_{i-1},T_i]$ is $\mathrm{IRS}_L^{cp}(t;T_{i-1},T_i;L_x)$ and the fixed leg's coupon payoff at time $s \ge 0$, where the rate K is exchanged over $[S_{j-1},S_j]$ is $\mathrm{IRS}_K^{cp}(s;S_{j-1},S_j;K)$. The corresponding floating and fixed legs are denoted by $\mathrm{IRS}_L^{leg}(t;\mathbf{T};L_x)$ and $\mathrm{IRS}_K^{leg}(t;\mathbf{S};K)$, respectively. We get the coupon payoffs at each exchange date as

$$\mathrm{IRS}_L^{cp}(T_i;T_{i-1},T_i;L_x) = X L_x(T_{i-1},T_i)\,\tau_L(T_{i-1},T_i)\,,\, i = 1,\ldots,N, \tag{2.4.6}$$

$$\mathrm{IRS}_K^{cp}(S_j;S_{j-1},S_j;K) = XK\tau_K(S_{j-1},S_j)\,,\, j = 1,\ldots,M, \tag{2.4.7}$$

and this leads to the following coupon payoffs at $t \le T_i$ or $s \le S_j$, respectively

$$\begin{aligned}
\mathrm{IRS}_L^{cp}(t;T_{i-1},T_i;L_x) &\overset{(2.3.2)}{=} \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^{T_i} r_s\,ds\right)\mathrm{IRS}_L^{cp}(T_i;T_{i-1},T_i;L_x) \mid \mathscr{F}_t\right] \\
&\overset{(2.4.6)}{=} \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^{T_i} r_s\,ds\right) X L_x(T_{i-1},T_i)\,\tau_L(T_{i-1},T_i) \mid \mathscr{F}_t\right] \\
&\overset{(E.6)}{=} X\tau_L(T_{i-1},T_i)\,P(t,T_i)\,\mathbb{E}^{\mathbb{Q}^{T_i}}\left[L_x(T_{i-1},T_i) \mid \mathscr{F}_t\right] \\
&\overset{(2.4.3)}{=} X\tau_K(T_{i-1},T_i)\,P(t,T_i)\,L_x(t;T_{i-1},T_i)\,,
\end{aligned} \tag{2.4.8}$$

$$\begin{aligned}
\mathrm{IRS}_K^{cp}(s;S_{j-1},S_j;K) &\overset{(2.3.2)}{=} \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_s^{S_j} r_u\,du\right)\mathrm{IRS}_K^{cp}(S_j;S_{j-1},S_j;K) \mid \mathscr{F}_s\right] \\
&\overset{(2.4.7)}{=} XK\tau_K(S_{j-1},S_j)\,\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_s^{S_j} r_u\,du\right) \mid \mathscr{F}_s\right] \\
&\overset{(2.3.3)}{=} XK\tau_K(S_{j-1},S_j)\,P(s,S_j)\,.
\end{aligned} \tag{2.4.9}$$

Thus, we get for the floating leg in t

$$\mathrm{IRS}_L^{leg}(t;\mathbf{T};L_x) = \sum_{i=1}^{N} \mathrm{IRS}_L^{cp}(t;T_{i-1},T_i;L_x) \overset{(2.4.8)}{=} X \sum_{i=1}^{N} \tau_K(T_{i-1},T_i)\, P(t,T_i)\, L_x(t;T_{i-1},T_i)\ , \quad (2.4.10)$$

and the fixed leg in s is

$$\mathrm{IRS}_K^{leg}(s;\mathbf{S};K) = \sum_{j=1}^{M} \mathrm{IRS}_K^{cp}(s;S_{j-1},S_j;K) \overset{(2.4.9)}{=} XK \sum_{j=1}^{M} \tau_K(S_{j-1},S_j)\, P(s,S_j)\ . \qquad (2.4.11)$$

Hence, the payoff of an IRS at time t is

$$\mathrm{IRS}(t;\mathbf{T},\mathbf{S};L_x,K;v) \underset{(2.4.11)}{\overset{(2.4.10)}{=}} vX\left(\sum_{i=1}^{N} \tau_K(T_{i-1},T_i)\, P(t,T_i)\, L_x(t;T_{i-1},T_i) - K\sum_{j=1}^{M} \tau_K(S_{j-1},S_j)\, P(t,S_j)\right)\ , \qquad (2.4.12)$$

where $v = 1$ for a payer IRS and $v = -1$ in case of a receiver IRS. For the valuation of the forward swap rate, that is the fair rate of the IRS, where the swap is in equilibrium, meaning $\mathrm{IRS}(t;\mathbf{T},\mathbf{S};L_x,K;v) = 0$, we write (2.4.12) as

$$\mathrm{IRS}(t;\mathbf{T},\mathbf{S};L_x,K;v) = vX\left(\frac{1}{A(t,\mathbf{S})}\sum_{i=1}^{N} \tau_K(T_{i-1},T_i)\, P(t,T_i)\, L_x(t;T_{i-1},T_i) - K\right) A(t,\mathbf{S}) \quad (2.4.13)$$

with the annuity in t for tenor $\mathbf{S}$ being

$$A(t,\mathbf{S}) := \sum_{j=1}^{M} \tau_K(S_{j-1},S_j)\, P(t,S_j)\ . \qquad (2.4.14)$$

Then, we see that the forward swap rate $R_x(t;\mathbf{T},\mathbf{S})$ is

$$\begin{aligned} R_x(t;\mathbf{T},\mathbf{S}) &\overset{(2.4.13)}{=} \frac{1}{A(t,\mathbf{S})}\sum_{i=1}^{N} \tau_K(T_{i-1},T_i)\, P(t,T_i)\, L_x(t;T_{i-1},T_i) \\ &\overset{(2.4.3)}{=} \frac{1}{A(t,\mathbf{S})}\sum_{i=1}^{N} \tau_L(T_{i-1},T_i)\, P(t,T_i)\, \mathbb{E}^{\mathbb{Q}^{T_i}}[L_x(T_{i-1},T_i)\,|\,\mathscr{F}_t] \\ &\overset{(E.6)}{=} \frac{1}{A(t,\mathbf{S})}\sum_{i=1}^{N} \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^{T_i} r_s\,ds\right) \tau_L(T_{i-1},T_i)\, L_x(T_{i-1},T_i)\,|\,\mathscr{F}_t\right]\ . \end{aligned} \qquad (2.4.15)$$

In the special case of an IRS with equivalent floating and fixed legs' tenor structures, with same time counting conventions for both legs, i.e. $\tau := \tau_L = \tau_K$, and with equidistance between the exchange dates $\delta := \tau(T_{i-1},T_i)\,, i = 1,\dots,N$, the forward swap rate at time t is

$$\begin{aligned} R_x(t,T_N) := R_x(t;\mathbf{T},\mathbf{T}) &\underset{(2.4.15)}{\overset{(2.4.14)}{=}} \sum_{i=1}^{N} \frac{1}{\delta P(t,T_i)}\, \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^{T_i} r_s\,ds\right) \delta\, L_x(T_{i-1},T_i)\,|\,\mathscr{F}_t\right] \\ &= \frac{\delta \sum_{i=1}^{N} \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^{T_i} r_s\,ds\right) L_x(T_{i-1},T_i)\,|\,\mathscr{F}_t\right]}{S_N(t)}\ , \end{aligned} \qquad (2.4.16)$$

where S_n is defined for all $n \geq 1$ the following way.

Definition 2.4.1. *Considering $n \in \mathbb{N}$ and a finite discrete tenor structure $\mathbf{T}$, defined in* (2.4.4), *the process $S_n := (S_n(t))_{t\geq 0}$ is called n-finite bond sum, where*

$$S_n(t) := \int_{T_1}^{T_n} \exp(-\tau(t,T)\,Y(t,T))\,\xi(dT)\,,\; t \geq 0, \tag{2.4.17}$$

with ξ being a measure on $(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+))$.

We will use for all $t \geq 0$ the following characterization for ξ in the course of the thesis:

$$\xi(t) := \sum_{i=1}^{\infty} \tau(T_{i-1},T_i)\,1_{\{T_i\}}(t)\,. \tag{2.4.18}$$

Using (2.4.18), we get the n-finite bond sum as the annuity defined in (2.4.14) under the assumption of equidistance, i.e.

$$S_n(t) = \sum_{i=1}^{n} \tau(T_{i-1},T_i)\,P(t,T_i) = \delta \sum_{i=1}^{n} P(t,T_i)\,. \tag{2.4.19}$$

We now want to investigate a specific kind of IRS, the OIS. The investigation of OISs is interesting from a market participant's point of view since these instruments become more and more attractive for investors. That can be seen in an increase of about 47% in turnover comparing the second quarter of 2013 with the second quarter of 2014, as stated in the recent EURO money market survey published by the *ECB* (cf. [77]). On the other hand OISs are important for the valuation of fixed-income instruments and interest rate derivatives because, as mentioned earlier, most of the market participants use OIS rates for discounting (cf. Chart 6 and Table 14 in [115]). Especially collateralized derivatives are mainly valued using OIS rates what led us to the assumption (2.3.4) (cf. introduction of [111]).

In an OIS counterparties exchange a stream of floating-rate payments indexed to a compounded overnight rate for a stream of fixed-rate payments. We again consider tenor structures (2.4.4) and (2.4.5) with schedules $\mathbf{T}$ and $\mathbf{S}$ as well as year fractions τ_L and τ_K. The contract's nominal is also X and the fixed rate is K as in the examination of IRSs, but the floating rate is now a compounded overnight rate which is denoted by $\bar{L}$. Let us denote the value of this OIS at time t with $\mathrm{OIS}(t;\mathbf{T},\mathbf{S};\bar{L},K;\nu)$, and the corresponding OIS rate with $R_{\mathrm{OIS}}(t;\mathbf{T},\mathbf{S})$. For the valuation of the compounded overnight rate for a certain period $[T_{i-1},T_i]$ of tenor structure (2.4.4), we split up this period into a partition of K_i business days

$$T_{i-1} = t_0 < t_1 < \cdots < t_{K_i} = T_i\,. \tag{2.4.20}$$

Then the compounded overnight rate is given by

$$\bar{L}(T_{i-1},T_i) = \frac{1}{\tau_L(T_{i-1},T_i)} \left(\prod_{j=1}^{K_i} (1 + \tau_L(t_{j-1},t_j)\,L_{1\mathrm{D}}(t_{j-1},t_j)) - 1 \right), \tag{2.4.21}$$

where $L_{1\mathrm{D}}(t_{j-1},t_j)\,, j = 1,\ldots,K_i$, denotes the respective overnight rate. That means, an OIS is an IRS, where the floating rate is defined as the geometric average of an overnight index over every day of the payment period. As described in Section 5.5 of [3] and Section 2.5 of [87], we can approximate simple by continuous compounding and the overnight rate by the instantaneous rate, i.e. $L_{1D}(t_{j-1},t_j) \approx \lim_{\delta\to 0} L_{1D}(t_{j-1},t_{j-1}+\delta) = r_{t_{j-1}}$. Then equation (2.4.21) becomes

$$\bar{L}(T_{i-1},T_i) = \frac{1}{\tau_L(T_{i-1},T_i)}\left(\exp\left(\int_{T_{i-1}}^{T_i} r_s\,ds\right) - 1\right) \tag{2.4.22}$$

and consequently

$$\begin{aligned}
\bar{L}(t;T_{i-1},T_i) &\overset{(2.4.3)}{=} \frac{\tau_L(T_{i-1},T_i)}{\tau_K(T_{i-1},T_i)}\mathbb{E}^{\mathbb{Q}^{T_i}}[\bar{L}(T_{i-1},T_i)\,|\,\mathscr{F}_t] \\
&\overset{(2.4.22)}{=} \frac{1}{\tau_K(T_{i-1},T_i)}\left(\mathbb{E}^{\mathbb{Q}^{T_i}}\left[\exp\left(\int_{T_{i-1}}^{T_i} r_s\,ds\right)\,|\,\mathscr{F}_t\right] - 1\right) \\
&\overset{(E.6)}{=} \frac{1}{\tau_K(T_{i-1},T_i)}\left(\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_{T_{i-1}}^{T_i} r_s\,ds\right)\exp\left(-\int_{t}^{T_i} r_s\,ds\right)\,|\,\mathscr{F}_t\right]\frac{1}{P(t,T_i)} - 1\right) \\
&\overset{(2.3.3)}{=} \frac{1}{\tau_K(T_{i-1},T_i)}\left(\frac{P(t,T_{i-1})}{P(t,T_i)} - 1\right).
\end{aligned} \tag{2.4.23}$$

Note, that is the classical formula of the simple compounded forward rate (cf. equation (1.20) of [33]). Then, the value of an OIS at time $t \le T_0$ is

$$\begin{aligned}
\mathrm{OIS}(t;\mathbf{T},\mathbf{S};\bar{L},K;v) &\overset{(2.4.13)}{=} vX\left(\frac{1}{A(t,\mathbf{S})}\sum_{i=1}^{N}\tau_K(T_{i-1},T_i)P(t,T_i)\bar{L}(t;T_{i-1},T_i) - K\right)A(t,\mathbf{S}) \\
&\overset{(2.4.23)}{=} vX\left(\frac{1}{A(t,\mathbf{S})}\sum_{i=1}^{N}(P(t,T_{i-1}) - P(t,T_i)) - K\right)A(t,\mathbf{S}) \\
&= vX\left(\frac{P(t,T_0)-P(t,T_N)}{A(t,\mathbf{S})} - K\right)A(t,\mathbf{S})
\end{aligned} \tag{2.4.24}$$

and it follows for the OIS rate

$$R_{\mathrm{OIS}}(t;\mathbf{T},\mathbf{S}) \overset{(2.4.24)}{=} \frac{P(t,T_0)-P(t,T_N)}{A(t,\mathbf{S})},\ t \le T_0. \tag{2.4.25}$$

Let us again consider the special case of equidistance between the exchange dates, equivalent tenor structures for both payment streams, as well as consistent time counting conventions for fixed and floating leg. Then, the OIS rate is

$$R_{\mathrm{OIS}}(t;T_0,T_N) := R_{\mathrm{OIS}}(t;\mathbf{T},\mathbf{T}) \underset{(2.4.25)}{\overset{(2.4.14)}{=}} \frac{P(t,T_0)-P(t,T_N)}{S_N(t)}. \tag{2.4.26}$$

For the remainder of the thesis we consider this kind of OIS with tenor structure (2.4.4) for both legs, $\tau_L = \tau_K = \tau$, and $\delta = \tau(T_{i-1},T_i), i = 1,\dots,N$, for the sake of simplicity. Further, since we investigate long-term interest rates mostly in order to find a discount rate for long-term cashflows, it is justified to examine only OIS for the long-term swap rate due to the market practice of OIS discounting. That means, we define the swap rate at time $t \ge 0$ as

$$R(t,T) := R_{\mathrm{OIS}}(t;t,T_N), \tag{2.4.27}$$

with $T := T_N$.

Remark 2.4.2. *Note that in* (2.4.27) *we have set* $T_0 = t$. *This is equivalent to consider the interest rate* $R(t,T)$ *as associated to a rolling over strategy of OIS contracts. This is possible in our model since we admit the existence of bonds for any maturity* $T \geq 0$ *(see Assumption (i) on the OIS bond market).*

3. Long-Term Interest Rates

This chapter defines the different kinds of long-term interest rates that we want to examine in this thesis. Since our interest is especially in the long end of the discount curve, the long-term interest rates are defined as the different rates arising from the discount curve discussed in Section 2.2. Besides the definitions and corresponding interpretations of the rates, we present their main characteristics such as the DIR theorem in Subsection 3.1.2. The relations between the different rates is the topic of Section 3.2 in a model independent way such that the existence or non-existence of one rate provides insight into whether the other rates exist or do not. Remember that we still work on a filtrated probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with the filtration $\mathbb{F}$ as noted at the beginning of Chapter 2 and which is valid for the remainder of the thesis. All convergences of processes are uniform on compacts in probability (in ucp).[1] By $X = a$ for a process X which is adapted to $\mathbb{F}$ and $a \in \mathbb{R}$ we mean that $X_t = a$ $\mathbb{P}$-a.s. for all $t \geq 0$. For the sake of simplicity, we also use the improper notations $X = \infty$ and $X = -\infty$ to denote convergence in ucp to $+\infty$ and $-\infty$, see Definitions C.2.1 and C.2.2.

3.1. Definitions and Characteristics

For our further investigations on the asymptotic behavior of interest rates, we first need to define what we understand as "long-term" because there is no unique definition in the literature as explained in Section 1.1. Since we examine the long end of the interest rate curves it is natural to let the maturity go to infinity for defining the different long-term interest rates. Following this approach, three different long-term interest rates are defined and analyzed with respect to their main characteristics throughout Subsections 3.1.2 to 3.1.4, namely the long-term yield, the long-term simple rate, and the long-term swap rate. All of these rates are dependent upon the price of long-term bonds, i.e. zero-coupon bonds with maturity going to infinity, and therefore these bonds are analyzed firstly in Subsection 3.1.1. This is a theoretical construct that does not exist on the financial markets but it is related to quoted perpetual bonds, so-called consol bonds, which can be understood as long-term fixed coupon bonds that pay continuously a constant rate of money over an infinite time horizon but will never pay back the redemption value (cf. Subsection 3.1.2 of [36]). However, the long-term bonds do not pay any coupons and pay back the redemption in an infinite future.

The definitions and characteristics of the long-term rates are based on [23], [24], and [25], whereby the DIR theorem is taken from [69].

[1] Please refer to Appendix C for detailed information on ucp convergence.

3.1.1. Long-Term Bond Price

Definition 3.1.1. *We define the long-term bond* $P := (P_t)_{t\geq 0}$ *as*

$$P := \lim_{T\to\infty} P(\cdot, T) \tag{3.1.1}$$

if the limit exists in ucp.

Remark 3.1.2. *Note, that the long-term bond, introduced above, is a càdlàg process since we assumed that all bond prices are càdlàg. By Theorem 2 of Chapter I, Section 1 of [149], we know that for two right-continuous stochastic processes X and Y it holds that* $X_t = Y_t$ $\mathbb{P}$*-a.s. for all* $t \geq 0$ *is equivalent to* $\mathbb{P}$*-a.s. for all* $t \geq 0$, $X_t = Y_t$. *This fact is used in the sequel.*

Let us define a tenor structure with infinitely many dates $T_i, i \in \mathbb{N}$, as $\mathbf{T}_\infty := \{T_0, T_1, T_2, \dots\}$, where

$$T_0 < T_1 < T_2 < \cdots . \tag{3.1.2}$$

Definition 3.1.3. *Considering an infinite discrete tenor structure* $\boldsymbol{T}_\infty$, *the process* $S_\infty := (S_\infty(t))_{t\geq 0}$ *is defined as*

$$S_\infty(\cdot) := \lim_{n\to\infty} S_n(\cdot) = \delta \sum_{i=1}^{\infty} P(\cdot, T_i) \tag{3.1.3}$$

in ucp if the limit exists. We call S_∞ *the infinite bond sum.*

From Assumption (iii) on the OIS bond market it follows immediately for all $t \geq 0$ and all $n \in \mathbb{N}$ that $\mathbb{P}$-a.s. $0 < S_n(t) < \infty$ and $S_\infty(t) > 0$.

We are interested in the influence of a converging infinite bond sum on the a.s. properties of S_∞, described in the following two lemmas.

Lemma 3.1.4. *If* $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ *in ucp, it follows* $S_\infty(t) < \infty$ $\mathbb{P}$*-a.s. for all* $t \geq 0$.

Proof. This follows by $S_n(t) < \infty$ $\mathbb{P}$-a.s. for all $n \in \mathbb{N}$ and $t \geq 0$ and Proposition C.1.3 if S_∞ is considered as process on $\mathbb{R} \cup \{-\infty, +\infty\}$. In the case of S_∞ being a real-valued process, the result is obvious. □

Lemma 3.1.5. *If* $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ *in ucp, it holds for all* $t \geq 0$ $\mathbb{P}$*-a.s.* $S_n(t) \overset{n\to\infty}{\longrightarrow} \infty$.

Proof. The result is obtained by Corollary C.2.5 (ii) which can be applied due to $\mathbb{P}$-a.s. $S_n(t) \leq S_{n+1}(t)$ for all $t \geq 0$ and all $n \in \mathbb{N}$. □

Next, we present some results on the relation between the long-term bond price and the asymptotic behavior of the sum of bond prices.

Proposition 3.1.6. *If* $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ *in ucp, then* $P = 0$.

Proof. Define for all $\varepsilon > 0$, $n \in \mathbb{N}$ and $t \geq 0$ the set $A_1^{\varepsilon,t,n} := \left\{\omega \in \Omega : \sup_{0\leq s\leq t} |P(s,T_n)| > \varepsilon\right\}$. Then

$$\begin{aligned}
\mathbb{P}\big(A_1^{\varepsilon,t,n}\big) &= \mathbb{P}\left(\sup_{0\leq s\leq t} |S_n(s) - S_{n-1}(s)| > \varepsilon\right) \\
&= \mathbb{P}\left(\sup_{0\leq s\leq t} |S_n(s) - S_\infty(s) + S_\infty(s) - S_{n-1}(s)| > \varepsilon\right) \\
&\leq \mathbb{P}\left(\sup_{0\leq s\leq t} \left(|S_n(s) - S_\infty(s)| + |S_{n-1}(s) - S_\infty(s)|\right) > \varepsilon\right) \\
&\leq \mathbb{P}\left(\sup_{0\leq s\leq t} |S_n(s) - S_\infty(s)| + \sup_{0\leq s\leq t} |S_{n-1}(s) - S_\infty(s)| > \varepsilon\right) \\
&\leq \mathbb{P}\left(\left\{\sup_{0\leq s\leq t} |S_n(s) - S_\infty(s)| > \frac{\varepsilon}{2}\right\} \cup \left\{\sup_{0\leq s\leq t} |S_{n-1}(s) - S_\infty(s)| > \frac{\varepsilon}{2}\right\}\right) \\
&\overset{(*)}{\leq} \mathbb{P}\left(\sup_{0\leq s\leq t} |S_n(s) - S_\infty(s)| > \frac{\varepsilon}{2}\right) + \mathbb{P}\left(\sup_{0\leq s\leq t} |S_{n-1}(s) - S_\infty(s)| > \frac{\varepsilon}{2}\right) \\
&\overset{n\to\infty}{\longrightarrow} 0
\end{aligned} \tag{3.1.4}$$

since $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp. We used the fact that $S_\infty(t) < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ due to Lemma 3.1.4 and at $(*)$, Theorem 1.11 (d) of [93] was applied. □

Corollary 3.1.7. *If $\mathbb{P}(P_t > 0) > 0$ for some $t \geq 0$, then $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp.*

Proof. Since S_n converges in ucp for $n \to \infty$ to either S_∞ or $+\infty$, the statement follows directly by Proposition 3.1.6. □

Proposition 3.1.8. *It holds:*

(i) If there exist a process $z := (z_t)_{t\geq 0}$ with $\sup_{0\leq s\leq t} |z_s| < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $M \in \mathbb{N}$ such that for all $m \geq M$ it holds $\mathbb{P}$-a.s.

$$P(t,T_m) \leq z_t \frac{1}{\tau(t,T_m)^2} \tag{3.1.5}$$

for all $t \leq T_m$, then $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp.

(ii) If there exist a process $z := (z_t)_{t\geq 0}$ with $0 < \inf_{0\leq s\leq t} |z_s| < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $M \in \mathbb{N}$ such that for all $m \geq M$ it holds $\mathbb{P}$-a.s.

$$P(t,T_m) \geq z_t \frac{1}{\tau(t,T_m)} \tag{3.1.6}$$

for all $t \leq T_m$, then $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp.

Proof. Since for all $n \in \mathbb{N}$, $S_n = (S_n(t))_{t\geq 0}$ is a càdlàg semimartingale, it holds $\mathbb{P}$-a.s. that for all $n \in \mathbb{N}$ and all $t \geq 0$, $\sup_{0\leq s\leq t} S_n(s) < \infty$ (cf. Proposition C.1.9).
To (i): Let M be such that (3.1.5) holds for all $m \geq M$. Define for $t \geq 0$

$$B_1(t) := \left\{ \omega \in \Omega : \lim_{n\to\infty} \sup_{0\leq s\leq t} S_n(s) < \infty \right\}. \tag{3.1.7}$$

It holds for all $t \geq 0$

$$\begin{aligned}
\mathbb{P}(B_1(t)) &\geq \mathbb{P}\left(\left\{ \omega \in \Omega : \sup_{0\leq s\leq t} S_{M-1}(s) < \infty \right\} \cap \left\{ \omega \in \Omega : \lim_{n\to\infty} \sup_{0\leq s\leq t} \sum_{i=M}^{n} P(s,T_i) < \infty \right\} \right) \\
&= \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\leq s\leq t} \sum_{i=M}^{n} P(s,T_i) < \infty \right) \geq \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=M}^{n} \sup_{0\leq s\leq t} P(s,T_i) < \infty \right) \\
&\overset{(3.1.5)}{\geq} \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=M}^{n} \sup_{0\leq s\leq t} z_s \frac{1}{\tau(s,T_i)^2} < \infty \right) \geq \mathbb{P}\left(\sup_{0\leq s\leq t} z_s \sum_{i=M}^{\infty} \frac{1}{\tau(t,T_i)^2} < \infty \right) = 1\,.
\end{aligned}$$

Accordingly, we have $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp.
To (ii): Let M be such that (3.1.6) holds for all $m \geq M$. Define for $t \geq 0$

$$B_2(t) := \left\{ \omega \in \Omega : \lim_{n\to\infty} \inf_{0\leq s\leq t} S_n(s) = \infty \right\}. \tag{3.1.8}$$

It holds for all $t \geq 0$

$$\begin{aligned}
\mathbb{P}(B_2(t)) &\geq \mathbb{P}\left(\inf_{0\leq s\leq t} \sum_{i=1}^{M-1} P(s,T_i) + \lim_{n\to\infty} \inf_{0\leq s\leq t} \sum_{i=M}^{n} P(s,T_i) = \infty \right) \\
&= \mathbb{P}\left(\lim_{n\to\infty} \inf_{0\leq s\leq t} \sum_{i=M}^{n} P(s,T_i) = \infty \right) \geq \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=M}^{n} \inf_{0\leq s\leq t} P(s,T_i) = \infty \right) \\
&\overset{(3.1.6)}{\geq} \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=M}^{n} \inf_{0\leq s\leq t} z_s \frac{1}{\tau(s,T_i)} = \infty \right) \geq \mathbb{P}\left(\inf_{0\leq s\leq t} z_s \sum_{i=M}^{\infty} \frac{1}{\tau(0,T_i)} = \infty \right) = 1\,.
\end{aligned}$$

Accordingly, we have $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. □

Remark 3.1.9. *In the case of a $\mathbb{P}$-a.s. vanishing long-term bond price, where nor a process z exists as in (i) neither as in (ii) of Proposition 3.1.8, it is not possible to specify the asymptotic behavior of the sum of bond prices.*

The following three corollaries are direct consequences from the definition of S_∞ and the convergence in ucp, see (C.1.1). They are useful for the remainder of this thesis.

Corollary 3.1.10. *If $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp, then it holds in ucp that*

$$\frac{1}{S_n(\cdot)} \overset{n\to\infty}{\longrightarrow} \frac{1}{S_\infty(\cdot)}\,. \tag{3.1.9}$$

Proof. Since $S_n(\cdot) \neq 0$ $\mathbb{P}$-a.s. for all $n \in \mathbb{N}$, we can apply Corollary C.1.7 (i) and the result follows. □

Corollary 3.1.11. *If* $S_n \stackrel{n\to\infty}{\longrightarrow} S_\infty$ *in ucp, then it holds in ucp that*

$$\frac{P(\,\cdot\,,T_n)}{S_n(\cdot)} \stackrel{n\to\infty}{\longrightarrow} 0\,. \tag{3.1.10}$$

Proof. The bond prices are càdlàg processes, hence by Proposition C.1.9 we can apply Corollary C.1.7 (iii) to $P(\,\cdot\,,T_n)\frac{1}{S_n(\cdot)}$. Then, the result follows by Corollary 3.1.10 that gives us $\frac{1}{S_n(\cdot)} \stackrel{n\to\infty}{\longrightarrow} \frac{1}{S_\infty(\cdot)}$ in ucp, and by Proposition 3.1.6 that yields $P(\,\cdot\,,T_n) \stackrel{n\to\infty}{\longrightarrow} 0$ in ucp. □

Corollary 3.1.12. *If* $S_n \stackrel{n\to\infty}{\longrightarrow} +\infty$ *in ucp, then it holds in ucp that*

$$\frac{1}{S_n(\cdot)} \stackrel{n\to\infty}{\longrightarrow} 0\,. \tag{3.1.11}$$

Proof. That is a direct consequence of Corollary C.2.4 (i). □

3.1.2. Long-Term Yield

In this subsection we consider the long-term yield and recall the DIR theorem (cf. Theorem 2 of [69]).

Definition 3.1.13. *We define the long-term yield* $\ell := (\ell_t)_{t\geq 0}$ *as*

$$\ell := \lim_{T\to\infty} Y(\,\cdot\,,T) \tag{3.1.12}$$

if the limit exists in ucp.

Definition 3.1.13 means that the long-term yield is defined as the long end of the yield curve, see Definition 2.2.3, i.e. it is the continuously compounded OIS spot rate with maturity going to infinity. If we consider the actual observation date $t = 0$, the long-term yield corresponds with the continuously compounded OIS forward rate, where the maturity goes to infinity.

Proposition 3.1.14. *Suppose* $\lim_{T\to\infty}\sup_{0\leq s\leq t} Y(s,T)$ *exists* $\mathbb{P}$*-a.s. for all* $t \geq 0$*. Then, the long-term yield at* 0 *is*

$$\ell_0 = \lim_{T\to\infty} Y(0,T) = \lim_{T\to\infty} Y(0;t,T) \tag{3.1.13}$$

$\mathbb{P}$*-a.s. for all* $t \geq 0$*.*

Proof. From Lemma C.1.4 follows that $\ell_0 = \lim_{n\to\infty} Y(0,T_n)$ $\mathbb{P}$-a.s. with tenor structure $\mathbf{T}_\infty$, hence it remains to show that $\lim_{n\to\infty} Y(0,T_n) = \lim_{T\to\infty} Y(0;t,T)$ $\mathbb{P}$-a.s. for all $t \geq 0$. We get for all $t \geq 0$

$$\begin{aligned}
\lim_{T\to\infty} Y(0;t,T) &\stackrel{(2.2.7)}{=} -\lim_{T\to\infty} \frac{\log P(0,T) - \log P(0,t)}{\tau(t,T)} \stackrel{(2.2.11)}{=} \lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_t^T f(0,u)\,du \\
&\stackrel{(*)}{=} \lim_{T\to\infty} \frac{1}{\tau(0,T)} \int_0^T f(0,u)\,du \stackrel{(2.2.11)}{=} -\lim_{T\to\infty} \frac{1}{\tau(0,T)} \log P(0,T) \\
&\stackrel{(2.2.7)}{=} \lim_{T\to\infty} Y(0,T) = \lim_{n\to\infty} Y(0,T_n) \quad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{3.1.14}$$

At $(*)$ we used (i) of the assumptions on the OIS bond market, in fact that there exists for all $t \geq 0$ a bond price $P(0,t) = \exp\big(-\int_0^t f(0,u)\,du\big)$, therefore $\int_0^t f(0,u)\,du < \infty$ $\mathbb{P}$-a.s. and hence it holds $\mathbb{P}$-a.s. that $\lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_0^t f(0,u)\,du = 0$. □

If the long-term bond price vanishes, i.e. $P(\,\cdot\,,T)$ converges to 0 in ucp for T going to infinity, then the long-term yield is a non-negative processes.

Lemma 3.1.15. *If $P = 0$, then $\ell_t \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. Proposition 1 of [35] tells us that the long-term yield is non-negative under the prerequisite that the long-term bond price vanishes, i.e. it holds $\ell_t \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. □

Next, we recall the DIR theorem which tells us that the long-term yield is a non-decreasing process.

Theorem 3.1.16 (Dybvig-Ingersoll-Ross theorem). *For $0 \leq s < t$ it holds $\mathbb{P}$-a.s. that $\ell_s \leq \ell_t$ if the long-term yield exists finitely.*

Proof. We know by Assumption (iii) on the OIS bond market that $P(t,T) < \infty$ $\mathbb{P}$-a.s. for all $0 \leq t \leq T$ and therefore $Y(t,T) < \infty$ $\mathbb{P}$-a.s. because of (2.2.8). Then, it follows by Proposition C.1.3 that $\ell_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ since $Y(\,\cdot\,,T_n) \overset{n\to\infty}{\longrightarrow} \ell$ in ucp. Hence, $\ell_s \leq \ell_t$ $\mathbb{P}$-a.s. for $0 \leq s < t$ can now be shown according to the proof of Lemma 7.3 of [83]. □

As already mentioned in the introduction, this statement was first shown in 1996 by Dybvig et al. [69] and therefore is commonly referred to as DIR theorem. Following [69], several researchers took a closer look into this topic and additional results were stated. First, in [132] the original proof was clarified in some aspects by McCulloch. Next, Hubalek et al. generalized in [108] the DIR theorem's proof in an elegant mathematical way, where no additional assumptions to an arbitrage-free market have to be predetermined. Then, more generalizations on the DIR theorem have recently been provided in [96] and [121]. Furthermore, Kardaras and Platen discuss for $s < t$ the maximal discrepancy between $Y(s,T)$ and $Y(t,T)$ for a long-term, but finite, maturity T in [121].

Note, that the result of the long-term yield being a non-decreasing process is independent from the assumption of a frictionless OIS bond market, where all OIS T-bonds have final payoff $P(T,T) = 1$. These two conditions are not always satisfied in reality: OIS bonds are not traded for all maturities, and $P(T,T)$ might be less than one if the issuer of the OIS T-bond defaults. Nevertheless, an alteration of these conditions would not have any influence on the validity of the DIR theorem.

In the following we will see that a non-decreasing long-term yield does not contradict the realistic behavior of a bond price process, not necessarily an OIS bond process. As a consequence of the assumption of ℓ being a non-decreasing process it holds that for all $s < t$ we have $\mathbb{P}$-a.s. that $\ell_t \geq \ell_s$. This implies that there exists $M > 0$ such that for all $T > M$

$$Y(t,T) \geq Y(s,T) \quad \mathbb{P}\text{-a.s.},$$

i.e.

$$P(t,T) \leq P(s,T)^{\frac{\tau(t,T)}{\tau(s,T)}} \quad \mathbb{P}\text{-a.s.}$$

For $s < t$, we have that $0 \leq \frac{\tau(t,T)}{\tau(s,T)} = \frac{\tau(t,T)}{\tau(s,t)+\tau(t,T)} < 1$ and this yields the fact that there exists $M > 0$ such that for all $T > M$

$$P(t,T) \leq P(s,T)^{a(s,t,T)}$$

with $a(s,t,T) \in [0,1)$ for all $0 \leq s < t \leq T$. This is economically realistic because the fluctuations of the bond price will decrease if the time to maturity decreases. Moreover, for a maturity that is far away from the time of observation, it is comprehensible that the bond price $P(t,T)$ is always lower or equal than $P(s,T)^{a(s,t,T)}$ for $s < t$ because any incident that could occur between the times s and t only has minor effects on long-term observations and can be captured in $a(s,t,T)$.

3.1.3. Long-Term Simple Rate

Brody and Hughston detected in [35] that by using exponential discount factors for the valuation of cashflows that will occur in the distant future, the assigned present value will be very low. It could be regarded even as unfairly low since the present value will in most cases be insufficient to justify the costs for the project generating these cashflows. For example, this could be an infrastructure project upgrading and expanding transporting systems or a social project associated with sustainable energy, i.e. projects that are beneficial for future generations. In order to get a strictly positive probability for future projects realized in a remote future, the authors of [35] came up with the so-called “social discounting”, where the long-term simply compounded spot rate is applied in calculating the discounting rate for the distant future. We abbreviate it by long-term simple rate. It is called social discounting due to the interpretation that one has to think as a trustee for the future and not that the distant cashflows are delayed benefits to oneself. To integrate this interesting approach into our considerations, we now define the long-term simple rate process, denoted by L.

Definition 3.1.17. *We define the long-term simple rate $L := (L_t)_{t\geq 0}$ as*

$$L := \lim_{T\to\infty} L(\cdot,T) \tag{3.1.15}$$

if the limit exists in ucp.

Lemma 3.1.18. *It holds $\mathbb{P}$-a.s. $L_t \geq 0$ for all $t \geq 0$.*

Proof. If $L(\cdot,T)$ converges in ucp to $+\infty$, the result is clear. Therefore let us assume the OIS simple rate converges to L in ucp. Then, for all $\varepsilon > 0$ and all $t \geq 0$ it holds

$$\mathbb{P}\left(\sup_{0\leq s\leq t}\left|L(s,T_n) - \frac{1}{\tau(s,T_n)P(s,T_n)}\right| > \varepsilon\right) \overset{(2.2.5)}{=} \mathbb{P}\left(\sup_{0\leq s\leq t}\frac{1}{\tau(s,T_n)} > \varepsilon\right) \overset{n\to\infty}{\longrightarrow} 0\,.$$

That means $L(\cdot,T)$ converges in ucp to a non-negative process, hence it holds $\mathbb{P}$-a.s. $L_t \geq 0$ for all $t \geq 0$. □

The next proposition shows that if the long-term simple rate exists finitely, also the long-term yield exists finitely.

Proposition 3.1.19. *If $L(\cdot,T) \overset{T\to\infty}{\longrightarrow} L$ in ucp, then $\ell_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ with $\ell_t \leq L_t$.*

Proof. We know by Assumption (iii) on the OIS bond market that $0 < P(t,T)$ $\mathbb{P}$-a.s. for all $0 \leq t \leq T$ and therefore $L(t,T) < \infty$ $\mathbb{P}$-a.s. because of (2.2.5). Then, it follows by Proposition C.1.3 that $L_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$. Further, using that $\log(x) \leq x - 1$ for all $x \in \mathbb{R}$ as well as (2.2.5) and (2.2.8), we get that $Y(t,T) \leq L(t,T)$ $\mathbb{P}$-a.s. for all $0 \leq t \leq T$, and therefore

$$\ell_t \overset{(3.1.12)}{=} \lim_{T\to\infty} Y(t,T) \leq \lim_{T\to\infty} L(t,T) \overset{(3.1.15)}{=} L_t < \infty \ \ \mathbb{P}\text{-a.s.}$$

□

Now, let us consider the actual observation date $t = 0$. Then the long-term simple rate corresponds with the simply compounded OIS forward rate, where the maturity goes to infinity.

Proposition 3.1.20. *Suppose* $\lim_{T\to\infty} \sup_{0\leq s\leq t} L(s,T)$ *exists* $\mathbb{P}$*-a.s. for all* $t \geq 0$*. Then, the long-term simple rate at* 0 *is*

$$L_0 = \lim_{T\to\infty} L(0,T) = \lim_{T\to\infty} L(0;t,T) \tag{3.1.16}$$

$\mathbb{P}$*-a.s. for all* $t \geq 0$*.*

Proof. From Lemma C.1.4 follows that $L_0 = \lim_{n\to\infty} L(0,T_n)$ $\mathbb{P}$-a.s. with tenor structure $\mathbf{T}_\infty$, hence it remains to show that $\lim_{n\to\infty} L(0,T_n) = \lim_{T\to\infty} L(0;t,T)$ $\mathbb{P}$-a.s. for all $t \geq 0$. We know from Proposition 3.1.19 that the long-term yield exists finitely $\mathbb{P}$-a.s. and therefore we can apply Proposition 3.1.14 and get for all $t \geq 0$

$$\begin{aligned}
\lim_{T\to\infty} L(0;t,T) &\overset{(*)}{=} \lim_{T\to\infty} \frac{1}{\tau(t,T)} \left(\exp(Y(0;t,T)\,\tau(t,T)) - 1\right) \\
&\overset{(3.1.13)}{=} \lim_{T\to\infty} \frac{1}{\tau(0,T)} \left(\exp(Y(0,T)\,\tau(0,T)) - 1\right) \\
&\overset{(**)}{=} \lim_{T\to\infty} L(0,T) = \lim_{n\to\infty} L(0,T_n) = L_0 \ \ \mathbb{P}\text{-a.s.}
\end{aligned} \tag{3.1.17}$$

At $(*)$ we used Proposition F.1 (i) and at $(**)$ Proposition F.1 (iii) was applied. □

If the long-term bond price explodes, i.e. $P(\cdot,T)$ converges to $+\infty$ in ucp for T going to infinity, then the long-term simple rate vanishes.

Lemma 3.1.21. *If* $P = \infty$*, then* $L = 0$*.*

Proof. By equation (2.2.5) and Definition C.2.1, we have that $L(\cdot,T_n) \overset{n\to\infty}{\longrightarrow} 0$ in ucp. □

3.1.4. Long-Term Swap Rate

The long-term swap rate can be understood as the fair fixed rate of an OIS that has a payment stream with infinitely many exchanges considering the tenor structure $\mathbf{T}_\infty$ for both legs, where fair is meant in the sense that the initial value of this OIS equals zero. Further, there exist $0 < c < C$ such that $c < \delta < C$. This long-term interest rate was defined for the first time by Biagini et al. in [24].

Definition 3.1.22. *We define the long-term swap rate* $R := (R_t)_{t\geq 0}$ *as*

$$R_\cdot := \lim_{T\to\infty} R(\,\cdot\,,T) \tag{3.1.18}$$

if the limit exists in ucp.

The next propositions show a direct connection between the existence, respectively the explosion, of S_∞ and the long-term swap rate. First, we are able to provide a model-free formula for R if the infinite bond sum exists finitely.

Proposition 3.1.23. *If* $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$*, then long-term swap rate is strictly positive* $\mathbb{P}$*-a.s. with*

$$R_t = \frac{1}{S_\infty(t)} > 0 \tag{3.1.19}$$

for all $t \geq 0$.

Proof. Let us consider the infinite tenor structure (3.1.2). Then, we get

$$\begin{aligned}\lim_{n\to\infty} R(\,\cdot\,,T_n) &\overset{(2.4.26)}{=} \lim_{n\to\infty} \frac{1-P(\,\cdot\,,T_n)}{S_n(\,\cdot\,)} \overset{(*)}{=} \lim_{n\to\infty} \frac{1}{S_n(\,\cdot\,)} - \lim_{n\to\infty} \frac{P(\,\cdot\,,T_n)}{S_n(\,\cdot\,)} \\ &\overset{(3.1.10)}{=} \lim_{n\to\infty} \frac{1}{S_n(\,\cdot\,)} \overset{(3.1.9)}{=} \frac{1}{S_\infty(\,\cdot\,)} \end{aligned} \tag{3.1.20}$$

in ucp. We used Lemma C.1.8 at $(*)$.

That means, $R(\,\cdot\,,T_n) \overset{n\to\infty}{\longrightarrow} R$ in ucp with $R_t = \frac{1}{S_\infty(t)}$ $\mathbb{P}$-a.s. for all $t \geq 0$. It holds $S_\infty(t) < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ due to Lemma 3.1.4. Hence, $R_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. □

In case of a finite long-term bond price and an exploding infinite bond sum, the long-term swap rate vanishes.

Proposition 3.1.24. *If* $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ *in ucp and the long-term bond P, defined in* (3.1.1)*, exists finitely, then it holds* $R = 0$.

Proof. Let us consider the infinite tenor structure (3.1.2). Then, we get

$$\lim_{n\to\infty} R(\,\cdot\,,T_n) \overset{(2.4.26)}{=} \lim_{n\to\infty} \frac{1-P(\,\cdot\,,T_n)}{S_n(\,\cdot\,)} \overset{(*)}{=} \lim_{n\to\infty} \frac{1}{S_n(\,\cdot\,)} - \lim_{n\to\infty} \frac{P(\,\cdot\,,T_n)}{S_n(\,\cdot\,)} \tag{3.1.21}$$

$$\overset{(3.1.11)}{=} -\lim_{n\to\infty} \frac{P(\,\cdot\,,T_n)}{S_n(\,\cdot\,)} \overset{(**)}{=} -P_\cdot \lim_{n\to\infty} \frac{1}{S_n(\cdot)} \overset{(3.1.11)}{=} 0 \tag{3.1.22}$$

in ucp. We used Lemma C.1.8 at $(*)$ and Corollary C.1.7 (iii) at $(**)$. □

Propositions 3.1.23 and 3.1.24 prove the existence of the long-term swap rate as a finite process if the long-term bond exists finitely. This result holds in any case when R exists, finitely or infinitely, as shown by the following corollary. Note that in case of $P = \infty$ it is unclear if R exists.

Corollary 3.1.25. *If the long-term swap rate exists, it cannot explode, i.e.* $\mathbb{P}(|R_t| < \infty) = 1$ *for all* $t \geq 0$.

Proof. This follows by Propositions 3.1.23 and 3.1.24 for the case when S_n converges in ucp to S_∞ and the case when S_n converges in ucp to $+\infty$ but the long-term bond price exists finitely. That means, we still have to consider the case when S_n and $P(\cdot, T_n)$ converge in ucp to $+\infty$. For this, simply note that $\mathbb{P}$-a.s.

$$0 \leq \sup_{0 \leq s \leq t} \frac{P(s,T_n)}{S_n(s)} = \sup_{0 \leq s \leq t} \left(1 - \frac{S_{n-1}(s)}{S_n(s)}\right) \leq 1$$

for all $t \geq 0$. Therefore

$$\mathbb{P}\left(\inf_{0 \leq s \leq t} \frac{P(s,T_n)}{S_n(s)} > M\right) \overset{n\to\infty}{\longrightarrow} 0$$

for all $M > 1$. Hence, also in this case we have $\mathbb{P}(|R_t| = \infty) = 0$. □

Corollary 3.1.26. *If* $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ *in ucp, then* $0 < R_t < \infty$ $\mathbb{P}$*-a.s. for all* $t \geq 0$.

Proof. This follows directly by Proposition 3.1.23 and Corollary 3.1.25. □

Corollary 3.1.27. *If* $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ *in ucp, then* $-\infty < R_t \leq 0$ $\mathbb{P}$*-a.s. for all* $t \geq 0$.

Proof. This is a consequence of Corollary 3.1.25 and equation (3.1.21). □

In the following proposition we see that if the infinite bond sum converges in ucp to $+\infty$, a non-negative OIS rate yields a non-positive bounded long-term swap rate. Even in case of $P = \infty$, we then know that the long-term swap rate exists.

Proposition 3.1.28. *Suppose* $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ *in ucp. If for all* $t \geq 0$ *it holds* $r_t \geq 0$ $\mathbb{P}$*-a.s., then*

$$R_t = -k_t$$

for a process $(k_t)_{t\geq 0}$ *with* $0 \leq k_t \leq 1$ $\mathbb{P}$*-a.s. for all* $t \geq 0$.

Proof. Let $r_t \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. Then, it follows for the infinite tenor structure (3.1.2) that

$$\frac{P(t,T_n)}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^{T_n} r_s\, ds\right)\middle|\mathscr{F}_t\right] \geq \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^{T_{n+1}} r_s\, ds\right)\middle|\mathscr{F}_t\right] = \frac{P(t,T_{n+1})}{B_t} \quad \mathbb{P}\text{-a.s.}$$

Due to the fact that for all $n \in \mathbb{N}$, $S_n(t) \leq S_{n+1}(t)$ $\mathbb{P}$-a.s. for all $t \geq 0$, it holds for all $n \in \mathbb{N}$

$$\frac{P(t,T_n)}{S_n(t)} = \frac{P(t,T_n)}{B_t}\frac{B_t}{S_n(t)} \geq \frac{P(t,T_{n+1})}{B_t}\frac{B_t}{S_{n+1}(t)} = \frac{P(t,T_{n+1})}{S_{n+1}(t)} \quad \mathbb{P}\text{-a.s.}$$

for all $t \geq 0$. Consequently we get that $\mathbb{P}$-a.s.

$$1 \geq \sup_{0 \leq s \leq t} \frac{P(s,T_n)}{S_n(s)} \geq \sup_{0 \leq s \leq t} \frac{P(s,T_{n+1})}{S_{n+1}(s)}$$

for all $t \geq 0$. Hence $\frac{P(\cdot,T_n)}{S_n(\cdot)} \overset{n\to\infty}{\longrightarrow} k$ in ucp, with $0 \leq k_t \leq 1$ $\mathbb{P}$-a.s. for all $t \geq 0$. In particular, we get $k = 0$ if $P_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$. □

The next theorem tells us that there cannot be made a similar statement for long-term swap rates as the DIR theorem for long-term yields because in an arbitrage-free market the long-term swap rate is not monotonic if it is not constant. Therefore we assume that there exists a liquid market for perpetual OIS, meaning OIS with infinitely many exchanges with the fixed rate corresponding to the long-term swap rate. We recall that we work under the hypothesis that the market is arbitrage-free in the sense that there exists a probability measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}$ such that discounted OIS bonds are $\mathbb{Q}$-martingales, see Assumption (v) on the OIS bond market, and that there do not exist any arbitrage portfolios as described in Section 4.3 of [83].

Theorem 3.1.29. *In the setting outlined in Chapter 2 the long-term swap rate is either constant or non-monotonic.*

Proof. We consider a perpetual OIS with infinite tenor structure $\mathbf{T}_\infty$ for both swap legs. From Proposition 3.1.24 and Corollaries 3.1.25 and 3.1.26 we know that $0 \leq R_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$.

As first case, we assume that $R_s \geq R_t$ $\mathbb{P}$-a.s. with $\mathbb{P}(R_s > R_t) > 0$ for $0 \leq t < s \leq T_1$, i.e. the long-term swap rate is a non-decreasing process. Now, the following investment strategy is applied. At time t we enter a payer OIS with perpetual annuity, nominal value N, and fixed-rate R_t. This investment has zero value in t, so there is no net investment so far. The following payoff is received in each T_i, $i \in \mathbb{N} \setminus \{0\}$ with the compounded overnight rate $\bar{L}$, defined in (2.4.22).

$$\text{Payoff at } T_i\text{:} \quad (\bar{L}(T_{i-1}, T_i) - R_t)\, \delta N\,. \tag{3.1.23}$$

In the next step, we enter at time s a receiver OIS with a perpetual annuity, nominal value N, and a fixed-rate of R_s. This OIS has zero value in s, that means there is still no net investment, and the payoff in each T_i, $i \in \mathbb{N} \setminus \{0\}$, resulting from this OIS is as follows.

$$\text{Payoff at } T_i\text{:} \quad (R_s - \bar{L}(T_{i-1}, T_i))\, \delta N\,. \tag{3.1.24}$$

With payoffs (3.1.23) and (3.1.24) this strategy yields the following payoff in T_i, $i \in \mathbb{N} \setminus \{0\}$.

$$\text{Payoff at } T_i\text{:} \quad (\bar{L}(T_{i-1}, T_i) - R_t)\, \delta N + (R_s - \bar{L}(T_{i-1}, T_i))\, \delta N = \delta N\, (R_s - R_t) \geq 0\,.$$

Obviously we get an arbitrage executing this strategy because of $\mathbb{P}(\delta N\, (R_s - R_t)) > 0$.

An analogue investment strategy can be used in case of a non-increasing long-term swap rate process, i.e. $R_s \leq R_t$ $\mathbb{P}$-a.s. with $\mathbb{P}(R_s < R_t) > 0$ for $0 \leq t < s \leq T_1$. The only difference between this strategy to the presented is that this time one invests in t in a receiver OIS and in s in a payer OIS.

Altogether it holds that in an arbitrage-free market setting the long-term swap rate cannot be non-decreasing or non-increasing, that means R can only be monotonic if it is constant. □

3.2. Interrelation Between Long-Term Rates

The introduction of the different kinds of long-term interest rates in Section 3.1 begs the question of interdependencies between these rates. In the following section we answer the questions if the vanishing, the explosion, or the finite existence of each long-term interest rate implies a certain behavior of the other long-term rates. The resulting relations among the rates are based on a model-free approach, hence they are valid for any term structure model, especially for the ones used in Chapter 4.

The results on the mutual influence of the rates on each other follow Section 5 of [24].

3.2.1. Influence of the Long-Term Yield on Long-Term Rates

In this subsection we investigate the way the long-term yield ℓ influences the characteristics of the long-term simple rate L and the long-term swap rate R. As a first result we get that the infinite bond sum exists as a strictly positive value in case of a strictly positive long-term yield.

Proposition 3.2.1. *If $0 < \ell_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$, then $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp.*

Proof. To prove that for all $t \geq 0$, $\lim_{n\to\infty} \sup_{0\leq s\leq t} S_n(s) < \infty$ $\mathbb{P}$-a.s., it is sufficient, because it implies $\lim_{n\to\infty} \sup_{0\leq s\leq t} S_n(s) < \infty$ in probability, hence $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp.

From (C.1.1) we know that for all $t \geq 0$ and all $\varepsilon > 0$ it holds

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |Y(s,T_n) - \ell_s| \leq \varepsilon\right) \overset{(2.2.8)}{=} \mathbb{P}\left(\sup_{0\leq s\leq t} \left|\frac{\log P(s,T_n)}{\tau(s,T_n)} + \ell_s\right| \leq \varepsilon\right) \overset{n\to\infty}{\longrightarrow} 1,$$

i.e. for all $t \geq 0$ and all $\varepsilon > 0$ there exists $N_\varepsilon^t \in \mathbb{N}$ such that for all $n \geq N_\varepsilon^t$

$$\mathbb{P}\left(\sup_{0\leq s\leq t} \left|\frac{\log P(s,T_n)}{\tau(s,T_n)} + \ell_s\right| \leq \varepsilon\right) > 1 - \delta(\varepsilon) \tag{3.2.1}$$

with $\delta(\varepsilon) \to 0$ for $\varepsilon \to 0$. Let us define for $\varepsilon > 0$, $u \geq 0$ and $n \in \mathbb{N}$ the set $A_2^{\varepsilon,u,n}$ the following way:

$$A_2^{\varepsilon,u,n} := \left\{\omega \in \Omega : \sup_{0\leq s\leq u} \left|\frac{\log P(s,T_n)}{\tau(s,T_n)} + \ell_s\right| \leq \varepsilon\right\}. \tag{3.2.2}$$

By (3.2.1), it holds $\mathbb{P}\big(A_2^{\varepsilon,u,n}\big) > 1 - \delta(\varepsilon)$ for $n \geq N_\varepsilon^u$ with $u > t$. Moreover

$$A_2^{\varepsilon,u,n} \subseteq \{\omega \in \Omega : |\log P(t,T_n) + \tau(t,T_n)\,\ell_t| \leq \varepsilon\,\tau(t,T_n)\}\,,$$

and for $n \geq N_\varepsilon^u$ on $A_2^{\varepsilon,u,n}$ we have

$$\exp[-(\varepsilon + \ell_t)\,\tau(t,T_n)] \leq P(t,T_n) \leq \exp[(\varepsilon - \ell_t)\,\tau(t,T_n)] \tag{3.2.3}$$

for all $t \in [0,u]$. It is $\ell_0 \leq \ell_t \leq \ell_u$ $\mathbb{P}$-a.s. for all $t \in [0,u]$ by Theorem 3.1.16, hence we have for $n \geq N_\varepsilon^u$ on $A_2^{\varepsilon,u,n}$

$$\exp[-(\varepsilon + \ell_u)\,\tau(u,T_n)] \leq \sup_{0\leq s\leq t} P(s,T_n) \leq \exp[(\varepsilon - \ell_0)\,\tau(0,T_n)]\,. \tag{3.2.4}$$

We obtain for $t < u$ and $n \geq N_\varepsilon^u$ with $B_1(t)$ defined as in (3.1.7)

$$\begin{aligned}
\mathbb{P}(B_1(t)) &= \mathbb{P}\left(\left\{\sup_{0\leq s\leq t} S_{N_\varepsilon^u-1}(s) < \infty\right\} \cap \left\{\lim_{n\to\infty}\sup_{0\leq s\leq t}\sum_{i=N_\varepsilon^u}^{n} P(s,T_i) < \infty\right\}\right) \\
&= \mathbb{P}\left(\lim_{n\to\infty}\sup_{0\leq s\leq t}\sum_{i=N_\varepsilon^u}^{n} P(s,T_i) < \infty\right) \\
&= \mathbb{P}\left(\lim_{n\to\infty}\sup_{0\leq s\leq t}\sum_{i=N_\varepsilon^u}^{n} P(s,T_i) < \infty \,\middle|\, A_2^{\varepsilon,u,n}\right)\mathbb{P}\big(A_2^{\varepsilon,u,n}\big) \\
&\quad + \mathbb{P}\left(\lim_{n\to\infty}\sup_{0\leq s\leq t}\sum_{i=N_\varepsilon^u}^{n} P(s,T_i) < \infty \,\middle|\, \Omega\backslash A_2^{\varepsilon,u,n}\right)\mathbb{P}\big(\Omega\backslash A_2^{\varepsilon,u,n}\big) \\
&\geq \mathbb{P}\left(\lim_{n\to\infty}\sup_{0\leq s\leq t}\sum_{i=N_\varepsilon^u}^{n} P(s,T_i) < \infty \,\middle|\, A_2^{\varepsilon,u,n}\right)\mathbb{P}\big(A_2^{\varepsilon,u,n}\big) \\
&\geq \mathbb{P}\left(\lim_{n\to\infty}\sum_{i=N_\varepsilon^u}^{n}\sup_{0\leq s\leq t} P(s,T_i) < \infty \,\middle|\, A_2^{\varepsilon,u,n}\right)\mathbb{P}\big(A_2^{\varepsilon,u,n}\big) \\
&\overset{(3.2.4)}{\geq} \mathbb{P}\left(\lim_{n\to\infty}\sum_{i=N_\varepsilon^u}^{n}\exp[(\varepsilon-\ell_0)\,\tau(0,T_i)] < \infty \,\middle|\, A_2^{\varepsilon,u,n}\right)\mathbb{P}\big(A_2^{\varepsilon,u,n}\big) \\
&\geq (1-\delta(\varepsilon)) \to 1
\end{aligned}$$

for $\varepsilon \to 0$. We used that $\mathbb{P}$-a.s.

$$\lim_{n\to\infty}\frac{\exp(-\ell_0\tau(0,T_{n+1}))}{\exp(-\ell_0\tau(0,T_n))} = \exp(-\ell_0\delta) \in (0,1)\,.$$

This implies by the ratio test that $\lim_{n\to\infty}\sum_{i=0}^{n}\exp[(\varepsilon-\ell_0)\,\tau(0,T_i)] < \infty$ $\mathbb{P}$-a.s. for $\varepsilon \to 0$. □

With the help of Proposition 3.2.1 we are now able to state the influence of the existence of a positive and finite long-term yield on the long-term swap rate and the long-term simple rate. We will derive the important result that a positive and finite long-term yield implies a positive and finite long-term swap rate.

Corollary 3.2.2. *If* $0 < \ell_t < \infty$ $\mathbb{P}$*-a.s. for all* $t \geq 0$*, then* $0 < R_t < \infty$ $\mathbb{P}$*-a.s. for all* $t \geq 0$ *and* $L = \infty$*.*

Proof. It follows by Proposition 3.2.1 that $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp and therefore by Corollary 3.1.26 we get that $0 < R_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$.

Remark 3 of [35] tells us that the long-term simple rate explodes, i.e. $L = \infty$. □

Typical market data indicate positive long-term interest rates on all bond and swap markets.[2] That means that the investigation of the impact of the existence of a negative long-term yield on the long-term swap rate and simple rate is a purely theoretical question. Nevertheless, it is also interesting from the perspective of a complete discussion.

[2] For long-term interest rate market data please refer to [29] for the USD market and to [78] for the EUR market.

Proposition 3.2.3. *If $-\infty < \ell_t < 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, then $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp.*

Proof. To prove that for all $t \geq 0$, $\lim_{n\to\infty} \sup_{0\leq s\leq t} S_n(s) = \infty$ $\mathbb{P}$-a.s., it is sufficient, because it implies $\lim_{n\to\infty} \sup_{0\leq s\leq t} S_n(s) = \infty$ in probability, hence $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp.

Let $B_2(t)$ defined as in (3.1.8). According to the proof of Proposition 3.2.1, we know that for $\varepsilon > 0$ and $u > 0$ there exists $N_\varepsilon^u \in \mathbb{N}$ such that for all $t < u$ and $n \geq N_\varepsilon^u$ inequality (3.2.3) holds on the set $A_2^{\varepsilon,u,n}$ defined in (3.2.2). Consequently we get that for $n \geq N_\varepsilon^u$ on $A_2^{\varepsilon,u,n}$ it holds

$$\exp[-(\varepsilon+\ell_u)\,\tau(0,T_n)] \leq \inf_{0\leq s\leq t} P(s,T_n) \leq \exp[\varepsilon\tau(0,T_n) - \ell_t\tau(t,T_n)] \tag{3.2.5}$$

for all $t \in [0,u]$. Due to ℓ is strictly negative and non-decreasing by Theorem 3.1.16 it follows that

$$\begin{aligned}
\mathbb{P}(B_2(t)) &= \mathbb{P}\left(\lim_{n\to\infty} \inf_{0\leq s\leq t} S_n(s) = \infty\right) = \mathbb{P}\left(\lim_{n\to\infty} \inf_{0\leq s\leq t} \sum_{i=N_\varepsilon^u}^{n} P(s,T_i) = \infty\right) \\
&= \mathbb{P}\left(\lim_{n\to\infty} \inf_{0\leq s\leq t} \sum_{i=N_\varepsilon^u}^{n} P(s,T_i) = \infty \,\middle|\, A_2^{\varepsilon,u,n}\right) \mathbb{P}\big(A_2^{\varepsilon,u,n}\big) \\
&\quad + \mathbb{P}\left(\lim_{n\to\infty} \inf_{0\leq s\leq t} \sum_{i=N_\varepsilon^u}^{n} P(s,T_i) = \infty \,\middle|\, \Omega\backslash A_2^{\varepsilon,u,n}\right) \mathbb{P}\big(\Omega\backslash A_2^{\varepsilon,u,n}\big) \\
&\geq \mathbb{P}\left(\lim_{n\to\infty} \inf_{0\leq s\leq t} \sum_{i=N_\varepsilon^u}^{n} P(s,T_i) = \infty \,\middle|\, A_2^{\varepsilon,u,n}\right) \mathbb{P}\big(A_2^{\varepsilon,u,n}\big) \\
&\geq \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\varepsilon^u}^{n} \inf_{0\leq s\leq t} P(s,T_i) = \infty \,\middle|\, A_2^{\varepsilon,u,n}\right) \mathbb{P}\big(A_2^{\varepsilon,u,n}\big) \\
&\overset{(3.2.5)}{\geq} \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\varepsilon^u}^{n} \exp[-(\varepsilon+\ell_u)\,\tau(0,T_n)] = \infty \,\middle|\, A_2^{\varepsilon,u,n}\right) \mathbb{P}\big(A_2^{\varepsilon,u,n}\big) \\
&\geq (1-\delta(\varepsilon)) \to 1
\end{aligned}$$

for $\varepsilon \to 0$. □

Proposition 3.2.4. *If $-\infty < \ell_t < 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ and the long-term bond price P exists finitely, then $R = 0$ and $L = 0$.*

Proof. It follows by Proposition 3.2.3 that $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. Since the long-term bond price P exists finitely, we can apply Proposition 3.1.24 and get $R = 0$.

To show that the long-term simple rate vanishes $\mathbb{P}$-a.s., we define the set $B_3(t)$ for all $t \geq 0$ the following way:

$$B_3(t) := \left\{\omega \in \Omega : \lim_{n\to\infty} \sup_{0\leq s\leq t} L(s,T_n) = 0\right\}. \tag{3.2.6}$$

Again, as in the proof of Proposition 3.2.3, we conclude that inequality (3.2.5) holds on the set $A_2^{\varepsilon,u,n}$ defined in (3.2.2). Hence, we get for all $u < t$

$$\begin{aligned}
\mathbb{P}(B_3(t)) &\overset{(2.2.5)}{=} \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\le s\le t} \frac{1}{\tau(s,T_n)\,P(s,T_n)} = 0 \right) \\
&= \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\le s\le t} \frac{1}{\tau(s,T_n)\,P(s,T_n)} = 0 \,\middle|\, A_2^{\varepsilon,u,n} \right) \mathbb{P}\big(A_2^{\varepsilon,u,n}\big) \\
&\quad + \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\le s\le t} \frac{1}{\tau(s,T_n)\,P(s,T_n)} = 0 \,\middle|\, \Omega\backslash A_2^{\varepsilon,u,n} \right) \mathbb{P}\big(\Omega\backslash A_2^{\varepsilon,u,n}\big) \\
&\ge \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\le s\le t} \frac{1}{\tau(s,T_n)\,P(s,T_n)} = 0 \,\middle|\, A_2^{\varepsilon,u,n} \right) \mathbb{P}\big(A_2^{\varepsilon,u,n}\big) \\
&\overset{(3.2.5)}{\ge} \mathbb{P}\left(\lim_{n\to\infty} \frac{1}{\tau(t,T_n)\exp[-(\varepsilon+\ell_u)\,\tau(0,T_n)]} = 0 \,\middle|\, A_2^{\varepsilon,u,n} \right) \mathbb{P}\big(A_2^{\varepsilon,u,n}\big) \\
&\ge (1-\delta(\varepsilon)) \to 1
\end{aligned}$$

for $\varepsilon \to 0$, consequently $L = 0$. Note, that this also holds in the case of a bond price process converging to $+\infty$. □

All the following propositions and corollaries of this subsection investigate the consequences of a vanishing or exploding long-term yield regarding the other long-term rates. It is shown that besides the asymptotic behavior of the yield, information about the long-term bond price is needed to state what happens to the other long-term rates. The case when the long-term yield is exploding as a positive value is an exception since no additional information on the behavior of the long-term bond price is required. First, we take a look at the case of ℓ_t taking the value zero for all $t \ge 0$.

Proposition 3.2.5. *If $\ell = 0$ and the long-term bond price P exists strictly positive and finitely with $\inf_{0\le s\le t} P_s > 0$ $\mathbb{P}$-a.s. for all $t \ge 0$, then $R = 0$ and $L = 0$.*

Proof. It follows by Corollary 3.1.7 that $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. Then, we can apply Proposition 3.1.24 since P exists finitely and get $R = 0$.

Considering the claim on L, we prove that for all $t \ge 0$ it holds that $\mathbb{P}(B_3(t)) = 1$ with $B_3(t)$ defined as in (3.2.6). Applying similar arguments as in the proof of Proposition 3.2.4, we have

$$\mathbb{P}(B_3(t)) \overset{(2.2.5)}{=} \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\le s\le t} \frac{1}{\tau(s,T_n)\,P(s,T_n)} = 0 \right) \ge \mathbb{P}\left(\lim_{n\to\infty} \frac{1}{\tau(t,T_n)\inf_{0\le s\le t} P_s} = 0 \right) = 1.$$

□

Proposition 3.2.6. *Let $\ell = 0$. Then:*

(i) If there exist a process $z := (z_t)_{t\ge 0}$ with $0 < \sup_{0\le s\le t} |z_s| < \infty$ $\mathbb{P}$-a.s. for all $t \ge 0$ and $M \in \mathbb{N}$ such that for all $m \ge M$ (3.1.5) holds $\mathbb{P}$-a.s. for all $t \le T_m$, then $0 < R_t < \infty$ $\mathbb{P}$-a.s. for all $t \ge 0$ and $L = \infty$.

(ii) *If the long-term bond price P exists finitely and there exist a process $z := (z_t)_{t\geq 0}$ with $0 < \inf_{0\leq s\leq t} |z_s| < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $M \in \mathbb{N}$ such that for all $m \geq M$ (3.1.6) holds $\mathbb{P}$-a.s. for all $t \leq T_m$, then $R = 0$ and $0 \leq L_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. **To (i):** Let M be such that (3.1.5) holds for $m \geq M$. Then, by Proposition 3.1.8 (i) it holds that $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp, hence applying Proposition 3.1.23 yields the long-term swap rate is $\mathbb{P}$-a.s. strictly positive.

To show that the long-term simple rate explodes, we define for all $t \geq 0$ the set $B_4(t)$ the following way:

$$B_4(t) := \left\{ \omega \in \Omega : \lim_{n\to\infty} \inf_{0\leq s\leq t} L(s,T_n) = \infty \right\}.$$

Using (3.1.5) we get

$$\mathbb{P}(B_4(t)) \overset{(3.1.5)}{\geq} \mathbb{P}\left(\lim_{n\to\infty} \inf_{0\leq s\leq t} \frac{\tau(s,T_n)}{z_s} = \infty \right) \geq \mathbb{P}\left(\lim_{n\to\infty} \frac{\tau(t,T_n)}{\sup_{0\leq s\leq t} z_s} = \infty \right) = 1.$$

That means $L(\,\cdot\,,T_n)$ converges in ucp to $+\infty$. Note, that in this case $P = 0$.
To (ii): Let M be such that (3.1.6) holds for $m \geq M$. Then, by Proposition 3.1.8 (ii) it holds that $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp, hence applying Proposition 3.1.24 yields the long-term swap rate is $\mathbb{P}$-a.s. vanishing.

To show that the long-term simple rate cannot explode, we define for all $t \geq 0$ the set $B_5(t)$ the following way:

$$B_5(t) := \left\{ \omega \in \Omega : \lim_{n\to\infty} \sup_{0\leq s\leq t} L(s,T_n) < \infty \right\}.$$

Using (3.1.6) we get

$$\mathbb{P}(B_5(t)) \overset{(3.1.6)}{\geq} \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\leq s\leq t} \frac{1}{\tau(s,T_n)} \frac{\tau(s,T_n)}{z_s} < \infty \right) = \mathbb{P}\left(\frac{1}{\inf_{0\leq s\leq t} z_s} < \infty \right) = 1.$$

That means $L(\,\cdot\,,T_n)$ converges in ucp to L and therefore $0 \leq L_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ due to Proposition C.1.3. □

Now, let us consider an exploding long-term yield and see how it influences the other long-term rates.

Proposition 3.2.7. *If $\ell = \infty$, then $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp.*

Proof. Using (C.2.2), it follows that for all $t \geq 0$ and all $\varepsilon > 0$ it holds

$$\mathbb{P}\left(\inf_{0\leq s\leq t} |Y(s,T_n)| > \varepsilon \right) \overset{(2.2.8)}{\geq} \mathbb{P}\left(\inf_{0\leq s\leq t} |\log P(s,T_n)| > \varepsilon\, \tau(0,T_n) \right) \overset{n\to\infty}{\longrightarrow} 1,$$

i.e. for all $t \geq 0$ and all $\varepsilon > 0$ there exists a $N_\varepsilon^t \in \mathbb{N}$ such that for all $n \geq N_\varepsilon^t$

$$\mathbb{P}\left(\inf_{0\leq s\leq t} |\log P(s,T_n)| > \varepsilon\, \tau(0,T_n) \right) > 1 - \delta(\varepsilon) \tag{3.2.7}$$

with $\delta(\varepsilon) \to 0$ for $\varepsilon \to 0$. Let us define for $\varepsilon > 0$, $u \geq 0$ and $n \in \mathbb{N}$ the set $A_3^{\varepsilon,u,n}$ the following way:

$$A_3^{\varepsilon,u,n} := \left\{ \omega \in \Omega : \inf_{0 \leq s \leq u} |\log P(s,T_n)| > \varepsilon\, \tau(0,T_n) \right\}. \tag{3.2.8}$$

It holds $\mathbb{P}\big(A_3^{\varepsilon,u,n}\big) > 1 - \delta(\varepsilon)$ by (3.2.7) for $n \geq N_\varepsilon^u$ with $u > t$. Moreover

$$A_3^{\varepsilon,u,n} \subseteq \{\omega \in \Omega : |\log P(t,T_n)| > \varepsilon\, \tau(0,T_n)\}.$$

Hence, we have for $n \geq N_\varepsilon^u$ on $A_3^{\varepsilon,u,n}$ that it holds

$$|\log P(t,T_n)| > \varepsilon\, \tau(0,T_n)$$

for all $t \in [0,u]$. Due to $\ell_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, it follows for $n \geq N_\varepsilon^u$ on $A_3^{\varepsilon,u,n}$

$$\sup_{0 \leq s \leq t} P(s,T_n) < \exp(-\varepsilon\, \tau(0,T_n)) \tag{3.2.9}$$

for all $t \in [0,u]$. We obtain for $t < u$ and $n \geq N_\varepsilon^u$ with $B_1(t)$ defined as in (3.1.7)

$$\begin{aligned}
\mathbb{P}(B_1(t)) &= \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\leq s\leq t} \sum_{i=N_\varepsilon^u}^{n} P(s,T_i) < \infty \right) \\
&\geq \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\varepsilon^u}^{n} \sup_{0\leq s\leq t} P(s,T_i) < \infty \,\middle|\, A_3^{\varepsilon,u,n} \right) \mathbb{P}\big(A_3^{\varepsilon,u,n}\big) \\
&\overset{(3.2.9)}{\geq} \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\varepsilon^u}^{n} \exp(-\varepsilon\, \tau(0,T_n)) < \infty \,\middle|\, A_3^{\varepsilon,u,n} \right) \mathbb{P}\big(A_3^{\varepsilon,u,n}\big) \\
&\geq (1-\delta(\varepsilon)) \to 1
\end{aligned}$$

for $\varepsilon \to 0$ due to the ratio test. □

Corollary 3.2.8. *If $\ell = \infty$, then $0 < R_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $L = \infty$.*

Proof. It holds $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp by Proposition 3.2.7. Therefore, $0 < R_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ due to Corollary 3.1.26.

The exploding long-term simple rate is a result of Remark 3 of [35]. □

Proposition 3.2.9. *If $\ell = -\infty$, then $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp.*

Proof. Using the the same argument as in Proposition 3.2.7, we get for all $t \in [0,u]$ and $n \geq N_\varepsilon^u$

$$\sup_{0 \leq s \leq t} P(s,T_n) > \exp(\varepsilon\, \tau(0,T_n)) \tag{3.2.10}$$

on $A_3^{\varepsilon,u,n}$, introduced in (3.2.8), because of $\ell_t < 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. We obtain for $t < u$ and $n \geq N_\varepsilon^u$ with $B_2(t)$ defined as in (3.1.8)

$$\begin{aligned}
\mathbb{P}(B_2(t)) &\geq \mathbb{P}\left(\lim_{n\to\infty}\sum_{i=N_\varepsilon^u}^{n}\inf_{0\leq s\leq t} P(s,T_i) = \infty \,\middle|\, A_3^{\varepsilon,u,n}\right)\mathbb{P}\big(A_3^{\varepsilon,u,n}\big) \\
&\overset{(3.2.10)}{\geq} \mathbb{P}\left(\lim_{n\to\infty}\sum_{i=N_\varepsilon^u}^{n}\exp(\varepsilon\tau(0,T_n)) = \infty \,\middle|\, A_3^{\varepsilon,u,n}\right)\mathbb{P}\big(A_3^{\varepsilon,u,n}\big) \\
&\geq (1-\delta(\varepsilon)) \to 1
\end{aligned}$$

for $\varepsilon \to 0$. □

Proposition 3.2.10. *If $\ell = -\infty$ and the long-term bond price P exists finitely, then $R = 0$ and $L = 0$.*

Proof. By Proposition 3.2.9 we get that $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. Then, we can apply Proposition 3.1.24 since P exists finitely and get $R = 0$.

Then, using the same notation as in the proof of Proposition 3.2.7, inequality (3.2.10) holds on $A_3^{\varepsilon,u,n}$ for all $t \in [0,u]$ and $n \geq N_\varepsilon^u$. We obtain for $t < u$ and $n \geq N_\varepsilon^u$ with $B_3(t)$ defined as in (3.2.6)

$$\begin{aligned}
\mathbb{P}(B_3(t)) &\geq \mathbb{P}\left(\lim_{n\to\infty}\sup_{0\leq s\leq t}\frac{1}{\tau(s,T_n)P(s,T_n)} = 0 \,\middle|\, A_3^{\varepsilon,u}\right)\mathbb{P}\big(A_3^{\varepsilon,u}\big) \\
&\overset{(3.2.10)}{\geq} \mathbb{P}\left(\lim_{n\to\infty}\sup_{0\leq s\leq t}\frac{1}{\tau(s,T_n)\exp(\varepsilon\tau(0,T_n))} = 0 \,\middle|\, A_3^{\varepsilon,u}\right)\mathbb{P}\big(A_3^{\varepsilon,u}\big) \\
&\geq (1-\delta(\varepsilon)) \to 1
\end{aligned}$$

for $\varepsilon \to 0$. That means $L = 0$. □

In Table 3.1 we summarize the influence of the long-term yield on the long-term swap rate and long-term simple rate.

If the long-term yield is	With the long-term bond price	Then the long-term swap rate is	Then the long-term simple rate is
$\ell < 0$	$0 \leq P < \infty$	$R = 0$	$L = 0$
$\ell < 0$	$P = \infty$	$-\infty < R < \infty$	$L = 0$
$\ell = 0$	$0 < P < \infty$	$R = 0$	$L = 0$
$\ell = 0$	$0 \leq P \leq \infty$	$0 \leq R < \infty$	$0 \leq L \leq \infty$
$\ell > 0$	$P = 0$	$0 < R < \infty$	$L = \infty$

Table 3.1.: Influence of the long-term yield on the other long-term rates. Own presentation.

3.2.2. Influence of the Long-Term Simple Rate on Long-Term Rates

Now, we study the influence of the long-term simple rate L on the other long-term interest rates. Since $L_t \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, see Lemma 3.1.18, we can distinguish between the three cases $L = 0$, $L = \infty$, and $0 < L_t < \infty$ for all $t \geq 0$, to complete the discussion. We start with the examination of a vanishing long-term simple rate and see that an infinite bond sum converging to $+\infty$ is a direct consequence.

Proposition 3.2.11. *If $L = 0$, then $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp.*

Proof. By (C.1.1) we know that for all $t \geq 0$ and all $\varepsilon > 0$ it holds

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |L(s,T_n)| \leq \varepsilon\right) \overset{n\to\infty}{\longrightarrow} 1\,.$$

Therefore, it follows by equation (2.2.5) that for all $t \geq 0$ and all $\varepsilon > 0$ there exists $N_\varepsilon^t \in \mathbb{N}$ such that for all $n \geq N_\varepsilon^t$

$$\mathbb{P}\left(\sup_{0\leq s\leq t} \left|\frac{1}{\tau(s,T_n)}\left(\frac{1}{P(s,T_n)} - 1\right)\right| \leq \varepsilon\right) > 1 - \delta(\varepsilon) \tag{3.2.11}$$

with $\delta(\varepsilon) \to 0$ for $\varepsilon \to 0$. Then, we define the set $A_4^{\varepsilon,u,n}$ for $\varepsilon > 0$, $u \geq 0$ and $n \in \mathbb{N}$ the following way:

$$A_4^{\varepsilon,u,n} := \left\{\omega \in \Omega : \sup_{0\leq s\leq u} \left|\frac{1}{\tau(s,T_n)}\left(\frac{1}{P(s,T_n)} - 1\right)\right| \leq \varepsilon\right\}.$$

By (3.2.11), it holds $\mathbb{P}\big(A_4^{\varepsilon,u,n}\big) > 1 - \delta(\varepsilon)$ for $n \geq N_\varepsilon^u$ with $u > t$. Moreover

$$A_4^{\varepsilon,u,n} \subseteq \left\{\omega \in \Omega : \left|\frac{1}{P(t,T_n)} - 1\right| \leq \varepsilon\,\tau(t,T_n)\right\},$$

and for $n \geq N_\varepsilon^u$ on $A_4^{\varepsilon,u,n}$ we have

$$1 - \varepsilon\,\tau(t,T_n) \leq \frac{1}{P(t,T_n)} \leq 1 + \varepsilon\,\tau(t,T_n)$$

for all $t \in [0,u]$. It follows for $n \geq N_\varepsilon^u$ on $A_4^{\varepsilon,u,n}$

$$\frac{1}{1+\varepsilon\,\tau(0,T_n)} \leq \inf_{0\leq s\leq t} P(s,T_n) \leq \frac{1}{1-\varepsilon\,\tau(t,T_n)}\,. \tag{3.2.12}$$

We obtain for $t < u$ and $n \geq N_\varepsilon^u$ with $B_2(t)$ defined as in (3.1.8)

$$\begin{aligned}
\mathbb{P}(B_2(t)) &\geq \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\varepsilon^u}^{n} \inf_{0\leq s\leq t} P(s,T_i) = \infty \,\middle|\, A_4^{\varepsilon,u,n}\right) \mathbb{P}\big(A_4^{\varepsilon,u,n}\big) \\
&\overset{(3.2.12)}{\geq} \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\varepsilon^u}^{n} \frac{1}{1+\varepsilon\,\tau(0,T_i)} = \infty \,\middle|\, A_4^{\varepsilon,u,n}\right) \mathbb{P}\big(A_4^{\varepsilon,u,n}\big) \\
&\geq (1-\delta(\varepsilon)) \to 1
\end{aligned}$$

for $\varepsilon \to 0$. □

Proposition 3.2.12. *If $0 < L_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$, then $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp.*

Proof. It holds for all $t \geq 0$ and all $\varepsilon > 0$

$$\mathbb{P}\left(\sup_{0\leq s\leq t}|L(s,T_n)-L_s| \leq \varepsilon\right) \overset{n\to\infty}{\longrightarrow} 1,$$

i.e. by equation (2.2.5) for all $t \geq 0$ and all $\varepsilon > 0$ there exists $N_\varepsilon^t \in \mathbb{N}$ such that for all $n \geq N_\varepsilon^t$

$$\mathbb{P}\left(\sup_{0\leq s\leq t}\left|\frac{1}{\tau(s,T_n)}\left(\frac{1}{P(s,T_n)}-1\right)-L_s\right| \leq \varepsilon\right) > 1-\delta(\varepsilon) \tag{3.2.13}$$

with $\delta(\varepsilon) \to 0$ for $\varepsilon \to 0$. Let us define for $\varepsilon > 0$, $u \geq 0$ and $n \in \mathbb{N}$ the set $A_5^{\varepsilon,u,n}$ the following way:

$$A_5^{\varepsilon,u,n} := \left\{\omega \in \Omega : \sup_{0\leq s\leq u}\left|\frac{1}{\tau(s,T_n)}\left(\frac{1}{P(s,T_n)}-1\right)-L_s\right| \leq \varepsilon\right\}.$$

By (3.2.13), it holds $\mathbb{P}\big(A_5^{\varepsilon,u,n}\big) > 1-\delta(\varepsilon)$ for $n \geq N_\varepsilon^u$ with $u > t$. Moreover

$$A_5^{\varepsilon,u,n} \subseteq \left\{\omega \in \Omega : \left|\frac{1}{P(t,T_n)} - 1 - L_t\,\tau(t,T_n)\right| \leq \varepsilon\,\tau(t,T_n)\right\},$$

and for $n \geq N_\varepsilon^u$ on $A_5^{\varepsilon,u,n}$ we have

$$L_t\,\tau(t,T_n)+1-\varepsilon\,\tau(t,T_n) \leq \frac{1}{P(t,T_n)} \leq L_t\,\tau(t,T_n)+1+\varepsilon\,\tau(t,T_n)$$

for all $t \in [0,u]$. It follows for $n \geq N_\varepsilon^u$ on $A_5^{\varepsilon,u,n}$

$$\frac{1}{1+(L_t+\varepsilon)\,\tau(0,T_n)} \leq \inf_{0\leq s\leq t} P(s,T_n) \leq \frac{1}{1+(L_t-\varepsilon)\,\tau(0,T_n)}. \tag{3.2.14}$$

We obtain for $t < u$ and $n \geq N_\varepsilon^u$ with $B_2(t)$ defined as in (3.1.8)

$$\begin{aligned}
\mathbb{P}(B_2(t)) &\geq \mathbb{P}\left(\lim_{n\to\infty}\sum_{i=N_\varepsilon^u}^{n}\inf_{0\leq s\leq t}P(s,T_i)=\infty\,\middle|\,A_5^{\varepsilon,u,n}\right)\mathbb{P}\big(A_5^{\varepsilon,u,n}\big)\\
&\overset{(3.2.14)}{\geq} \mathbb{P}\left(\lim_{n\to\infty}\sum_{i=N_\varepsilon^u}^{n}\frac{1}{1+(L_t+\varepsilon)\,\tau(t,T_i)}=\infty\,\middle|\,A_5^{\varepsilon,u,n}\right)\mathbb{P}\big(A_5^{\varepsilon,u,n}\big)\\
&\geq (1-\delta(\varepsilon)) \to 1
\end{aligned}$$

for $\varepsilon \to 0$. □

Corollary 3.2.13. *If $0 \leq L_t < \infty$, then $\ell_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. Furthermore, $R = 0$ if P exists finitely.*

Proof. It follows by Proposition 3.2.11 that $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. Therefore, $\ell_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ due to Propositions 3.2.1 and 3.2.7.

The vanishing of the long-term swap rate is a direct consequence of Proposition 3.1.24 if the long-term bond price exists finitely. □

So far, we have seen that from a vanishing or strictly positive long-term simple rate it follows a non-positive long-term yield and a long-term swap rate that is zero in the realistic case of a finite long-term bond price. Note that if P explodes, it is not possible to specify R more accurately than in Corollary 3.1.27.

Now, let us investigate the case of an exploding long-term simple rate. We begin with the fact that if the long-term simple rate converges to $+\infty$, the infinite bond sum exists finitely.

Proposition 3.2.14. *If $L = \infty$, then $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp.*

Proof. From (C.2.2) it holds that for all $t \geq 0$ and all $\varepsilon > 0$

$$\mathbb{P}\left(\inf_{0\leq s\leq t} L(s,T_n) > \varepsilon\right) \overset{n\to\infty}{\longrightarrow} 1.$$

Consequently we get by (2.2.5) that for all $t \geq 0$ and all $\varepsilon > 0$ there exists $N_\varepsilon^t \in \mathbb{N}$ such that for all $n \geq N_\varepsilon^t$

$$\mathbb{P}\left(\tau(0,T_n) \sup_{0\leq s\leq t} P(s,T_n) \leq \varepsilon\right) > 1-\delta(\varepsilon) \tag{3.2.15}$$

with $\delta(\varepsilon) \to 0$ for $\varepsilon \to 0$. Define for $\varepsilon > 0$, $u \geq 0$ and $n \in \mathbb{N}$ the set $A_6^{\varepsilon,u,n}$ the following way:

$$A_6^{\varepsilon,u,n} := \left\{\omega \in \Omega : T_n \sup_{0\leq s\leq u} P(s,T_n) \leq \varepsilon\right\}. \tag{3.2.16}$$

For $t < u$ and $n \geq N_\varepsilon^u$ it holds with $B_1(t)$ defined as in (3.1.7)

$$\begin{aligned}
\mathbb{P}(B_1(t)) \quad &\geq \quad \mathbb{P}\left(\lim_{n\to\infty}\sum_{i=N_\varepsilon^u}^{n}\sup_{0\leq s\leq t} P(s,T_i) < \infty \,\middle|\, A_6^{\varepsilon,u,n}\right)\mathbb{P}\left(A_6^{\varepsilon,u,n}\right) \\
&\overset{(3.2.16)}{\geq} \quad \mathbb{P}\left(\lim_{n\to\infty}\sum_{i=N_\varepsilon^u}^{n}\frac{1}{\tau(0,T_i)}\varepsilon < \infty \,\middle|\, A_6^{\varepsilon,u,n}\right)\mathbb{P}\left(A_6^{\varepsilon,u,n}\right) \\
&\geq \quad (1-\delta(\varepsilon)) \to 1
\end{aligned}$$

for $\varepsilon \to 0$. □

Corollary 3.2.15. *If $L = \infty$, then $\ell_t \geq 0$ and $R_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. It follows by Proposition 3.2.14 that $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp. Then, Propositions 3.2.3 and 3.2.9 give us that $\ell_t \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. The strictly positive long-term swap rate is a consequence of Proposition 3.1.23. □

The following table summarizes the influence of the long-term simple rate on the other long-term rates.

If the long-term simple rate is	With the long-term bond price	Then the long-term yield is	Then the long-term swap rate is
$0 \leq L < \infty$	$0 \leq P < \infty$	$\ell \leq 0$	$R = 0$
$0 \leq L < \infty$	$P = \infty$	$\ell \leq 0$	$-\infty < R \leq 0$
$L = \infty$	$P = 0$	$\ell \geq 0$	$0 < R < \infty$

Table 3.2.: Influence of the long-term simple rate on the other long-term rates. Own presentation.

3.2.3. Influence of the Long-Term Swap Rate on Long-Term Rates

After explaining the consequences of the different characteristics of the long-term yield ℓ and the long-term simple rate L for the long-term swap rate R, we finally want to know about the other direction of these relations. First, we investigate what happens if R vanishes.

Proposition 3.2.16. *If $R = 0$, then $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp.*

Proof. If S_n converges in ucp to S_∞, then $0 < R_t$ $\mathbb{P}$-a.s. for all $t \geq 0$ according to Proposition 3.1.23. That is a contradiction and therefore S_n converges in ucp to $+\infty$. □

Proposition 3.2.17. *If $R = 0$, then $\ell_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. Furthermore, it holds:*

(i) If the long-term bond price P exists finitely and there exist a process $z := (z_t)_{t\geq 0}$ with $0 < \inf_{0\leq s\leq t} |z_s| < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $M \in \mathbb{N}$ such that for all $m \geq M$ (3.1.6) holds $\mathbb{P}$-a.s. for all $t \leq T_m$, then $0 \leq L_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$.

(ii) If there exist a process $z := (z_t)_{t\geq 0}$ with $0 < \inf_{0\leq s\leq t} |z_s| < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $M \in \mathbb{N}$ such that for all $m \geq M$ it holds $\mathbb{P}$-a.s.

$$P(t, T_m) \leq z_t \frac{1}{\tau(t, T_m)} \tag{3.2.17}$$

for all $t \leq T_m$, then $L_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.

Proof. It follows by Proposition 3.2.16 that $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. Propositions 3.2.1 and 3.2.7 give us that $\ell_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.
To (i): The proof of Proposition 3.2.6 (ii) yields $0 \leq L_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$.
To (ii): We get for all $t \geq 0$ with $B_3(t)$ defined as in (3.2.6) that

$$\mathbb{P}(B_3(t)) \overset{(3.2.17)}{\leq} \mathbb{P}\left(\frac{1}{\inf_{0\leq s\leq t} z_s} = 0\right) = 0.$$

Consequently the simple rate either converges in ucp to a strictly positive process or it converges to $+\infty$ in ucp. Note that in this last case $P = 0$. □

We have seen that from a vanishing long-term swap rate follows a non-positive long-term yield and that for specifying the long-term simple rate additional information on the behavior of the bond prices is needed.

Next, we examine the behavior of the long-term rates if the long-term swap rate is strictly positive.

Corollary 3.2.18. *If* $0 < R_t < \infty$ $\mathbb{P}$*-a.s. for all* $t \geq 0$*, then* $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ *in ucp.*

Proof. If S_n converges in ucp to $+\infty$, then $R \leq 0$ by Corollary 3.1.27. That means we have for $R_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, that $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp. □

Corollary 3.2.19. *If* $0 < R_t < \infty$ $\mathbb{P}$*-a.s. for all* $t \geq 0$*, then* $\ell_t \geq 0$ *and* $L_t > 0$ $\mathbb{P}$*-a.s. for all* $t \geq 0$*.*

Proof. It is $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp, due to Corollary 3.2.18. Then, according to Propositions 3.2.3 and 3.2.9, it holds $\mathbb{P}$-a.s. $\ell_t \geq 0$ for all $t \geq 0$.

Further, it holds by Proposition 3.2.11 that $L_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. □

Now, it only remains to consider a strictly negative long-term swap rate. The following two corollaries deal with this case.

Corollary 3.2.20. *If* $-\infty < R_t < 0$ $\mathbb{P}$*-a.s. for all* $t \geq 0$*, then* $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ *in ucp.*

Proof. If S_n converges in ucp to S_∞, then $R_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ by Proposition 3.1.23. That is a contradiction to $R_t < 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. Therefore, $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. □

Corollary 3.2.21. *If* $-\infty < R_t < 0$ $\mathbb{P}$*-a.s. for all* $t \geq 0$*, then* $\ell_t \leq 0$ *and* $L_t = 0$ $\mathbb{P}$*-a.s. for all* $t \geq 0$*.*

Proof. It follows by Corollary 3.2.20 that $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. Propositions 3.2.1 and 3.2.7 give us that $\ell_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.

The vanishing long-term simple rate is a consequence of Lemma 3.1.21, since we know that P does not exist finitely by Proposition 3.1.24. □

The table below summarizes the influence of the long-term swap rate on the other long-term rates. Note, that $-\infty < R_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ by Corollary 3.1.25. Hence, only three different cases have to be distinguished, $R_t = 0$, $0 < R_t < \infty$, and $-\infty < R_t < 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.

If the long-term simple rate is	With the long-term bond price	Then the long-term yield is	Then the long-term swap rate is
$R = 0$	$0 \leq P < \infty$	$\ell \leq 0$	$0 \leq L \leq \infty$
$R = 0$	$P = \infty$	$\ell \leq 0$	$L = 0$
$0 < R < \infty$	$P = 0$	$\ell \geq 0$	$0 < L \leq \infty$
$-\infty < R < 0$	$0 \leq P \leq \infty$	$\ell \leq 0$	$L = 0$

Table 3.3.: Influence of the long-term swap rate on the other long-term rates. Own presentation.

4. Asymptotic Behavior of Interest Rates

Chapter 4 presents the asymptotic behavior of the long-term rates, discussed in Chapter 3, in some popular term structure models. The term structure of interest rates describes the relationship between interest rates or bond yields and different terms or maturities, respectively. The term structure models used in this thesis for investigating the interest rates' asymptotic behavior are the HJM framework, the Flesaker-Hughston model, and the linear-rational model.

In 1992, Heath et al. proposed in [105] a new framework for modeling the entire forward curve directly, where the forward rate is described by a continuous Itô process driven by a possibly d-dimensional Brownian motion. It soon established as one of the most used frameworks for the modeling of interest rates dynamics and therefore can be found in almost every textbook on interest rate modeling, see, for example, Sections 4.4 and 4.5 of [3], Chapter 5 of [33], Chapter 6 of [83], or Section 7.6 and Appendix A of [106]. The major difference to models of arbitrage-free short rate dynamics is, that it is an arbitrage-free framework for the stochastic evolution of the entire yield curve, where the forward rates dynamics are fully specified through their instantaneous volatility structures, whereas the volatility of the short rate models is not sufficient for characterizing all interest rate dynamics. Due to its significance for interest rate modeling, various properties were studied in academic researches like which classes of interest rate models are obtainable in [43], its geometric properties in [86], a discussion of its consistency problems in [82], or different numerical solutions for its stochastic partial differential equations in [61]. The HJM framework was also applied in finding solutions to different interest related financial products such as swaptions, captions, and floortions in [30], or bonds in a jump diffusion pricing model in [44]. It was also used for modeling the long end of the yield curve, first in the Brownian setting in [73], explained in Section 4.1, then in a setting using Lévy processes as stochastic driver in [25] which is the the topic of Section 4.2, and finally in [23], where affine processes on the state space of symmetric positive semidefinite matrices serve as stochastic drivers, presented in Section 4.3.

Another popular framework for modeling the term structure of interest rates is the Flesaker-Hughston interest rate model. It was introduced in [91] and further developed in [139] and [153]. In 1996, Flesaker and Hughston were among the first to propose a new approach to interest rate modeling besides short rate modeling and term structures developed via the HJM setup. The idea is to model directly the quantity that should be a martingale, the discounted bond price process. Main advantages of this approach are that it specifies non-negative interest rates only and has a high degree of tractability. Another appealing feature is that besides relatively simple models for bond prices, short and forward rates, there are closed-form formulas for caps, floors and swaptions available. In [24] the Flesaker-Hughston model is used for the valuation of long-term interest rates, what is presented in Section 4.4.

The class of linear-rational term structure models was introduced in [84] for the first time. This approach's basic idea is to model the pricing kernel in a way such that the bond prices become linear-rational functions of the current state. It presents several advantages: it is highly tractable and offers a very good fit to interest rate swaps and swaptions data. Further, non-negative interest rates are ensured, unspanned factors affecting volatility and risk premia are accommodated, and analytical solutions to swaptions are admitted. The linear-rational interest rate methodology in the context of the asymptotic behavior of interest rates is treated in Section 4.5, based on Section 6.2 of [24].

Note that, as throughout the thesis, we still consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ in an arbitrage-free setting, where $\mathbb{P}$ denotes the real-world measure, and the space is endowed with filtration $\mathbb{F}$. The risk-neutral measure is denoted by $\mathbb{Q}$, existing due to Assumption (v) on the OIS bond market and ensuring that the considered market is arbitrage-free.

4.1. Asymptotic Behavior of Interest Rates in a Classic HJM Framework

This section discusses the asymptotic behavior of interest rates in a classic HJM framework, where the driving process is a Brownian motion. We denote it with classic since this is the framework introduced by Heath et al. in [105] and other HJM frameworks like the ones of Sections 4.2 and 4.3 are modifications of this standard framework with other driving processes. In Subsection 4.1.1 the resulting term structure model is presented, and the corresponding long-term interest rates are calculated in Subsection 4.1.2.

The literature used for this section are mainly [73] and [105].

4.1.1. Classic HJM Term Structure

Heath et al. wanted to model the forward rates and consequently deriving other interest rates and bond prices. Therefore the initial equation for the forward rate process $f(\cdot, T)$ for every $T \geq 0$ was

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\, ds + \int_0^t \eta(s,T)\, dW_s\,,\ t \leq T\,, \tag{4.1.1}$$

where α is the drift function, η the volatility, and W a $\mathbb{P}$-Brownian motion. Equation (4.1.1) holds under some measurability and integrability conditions on the drift and the volatility that can be found in Section 6.1 of [83]. Since we want to work under the risk-neutral measure $\mathbb{Q}$, we use some results from [105] to get the $\mathbb{Q}$-dynamics of the bond prices and then only need to postulate conditions on the volatility as described below. According to equation (9) of [105], the dynamics of the price of a T-OIS bond under $\mathbb{Q}$ are given by the following stochastic differential equation (SDE) for $0 \leq t \leq T$

$$dP(t,T) = P(t,T)\,(r_t\, dt + \sigma(t,T) \cdot dW_t^*)\,, \tag{4.1.2}$$

where r is the OIS rate process, σ denotes the volatility, and $W^* := (W_t^*)_{t \geq 0}$ is a d-dimensional $\mathbb{Q}$-Brownian motion. The volatility function σ is d-dimensional as well and deterministic in every dimension, i.e. for $0 \leq t \leq T$

$$\sigma(t,T) = \left(\sigma^1(t,T), \ldots, \sigma^d(t,T)\right)\,, \tag{4.1.3}$$

where

$$\sigma^i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+, \quad (s,t) \mapsto \sigma^i(s,t) \tag{4.1.4}$$

denotes the volatility for the time interval $[s,t]$ of the i-th random component of the yield. In particular $\int_0^t \sigma(s,T)\,dW_s^* = \sum_{i=1}^d \int_0^t \sigma^i(s,T)\,dW_s^{*,i}$. The partial derivatives of σ are denoted the following way:

$$\sigma_1(s,t) := \partial_s \sigma(s,t), \quad \sigma_2(s,t) := \partial_t \sigma(s,t)\,. \tag{4.1.5}$$

We denote the Euclidean scalar product on $\mathbb{R}^d$ as $x \cdot y := \sum_{i=1}^d x^i y^i$ for $x, y \in \mathbb{R}^d$, $x := \left(x^1, \ldots, x^d\right)$, $y := \left(y^1, \ldots, y^d\right)$ and the respective norm by $\|\cdot\|$.

Assumption I. *We assume that the volatility function σ satisfies the following properties:*

(A.I.1) *For all $i \in \{1, \ldots, d\}$: $\sigma^i(s,t) > 0$ for all $t \in [0,\infty]$, $s \in [0,t[$.*

(A.I.2) *For all $i \in \{1, \ldots, d\}$: $\sigma^i(s,t) = 0$ for all $t \in [0,\infty]$, $s \geq t$.*

(A.I.3) *For all $i \in \{1, \ldots, d\}$: $\sigma^i, \sigma_1^i, \sigma_2^i$ are càglàd in both components.*

(A.I.4) *There exists a càglàd function $\phi \in L^1(\mathbb{R}_+)$ such as for all $T > 0$*

$$\frac{\|\sigma(t,T)\|}{\tau(0,T)} \leq \phi(t)$$

for all $0 \leq t \leq T$.

(A.I.5) *There exists a càglàd function $\psi \in L^1(\mathbb{R}_+)$ such as for all $T > 0$*

$$\frac{\|\sigma(t,T)\|^2}{\tau(0,T)} \leq \psi(t) \tag{4.1.6}$$

for all $0 \leq t \leq T$.

Assumptions (A.I.4) and (A.I.5) are sufficient requirements such that all subsequent integrals are well-defined and that the volatility function can converge to the long-term volatility, see Proposition 4.1.8. As well, note that (A.I.5) implies that $\exp(\sigma(s,T) \cdot W_1^*) \in L^1(\Omega, \mathscr{F}, \mathbb{Q})$ for all $0 \leq s \leq T$ because of

$$\mathbb{E}^{\mathbb{Q}}[\exp(\sigma(s,T) \cdot W_1^*)] \overset{(G.2)}{=} \exp\left(\frac{1}{2}\|\sigma(s,T)\|^2\right) \leq \exp\left(\frac{\tau(0,T)}{2}\psi(s)\right) < \infty\,.$$

This yields the important fact that the discounted bond price process $\frac{P(t,T)}{B_t}, t \leq T$, is a $\mathbb{Q}$-martingale as consequence of Lemma G.1.

Corollary 4.1.1. *Under Assumption I, it holds for all $T \geq 0$ that*

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_0^t \sigma(s,T) \cdot dW_s^*\right)\right] = \exp\left(\frac{1}{2}\int_0^t \|\sigma(s,T)\|^2\,ds\right), \; t \leq T\,. \tag{4.1.7}$$

Proof. We know that the Brownian motion W^* is a Lévy process and that σ is càglàd because of (A.I.3). Hence, if inequality (G.3) holds for all $T \geq 0$ and for some $\chi \in L^1(\mathbb{R}_+)$ and $\varepsilon \in (0,1)$, we can apply Lemma G.3. This holds due to

$$\frac{1}{\tau(0,T)}\left|\log \mathbb{E}^{\mathbb{Q}}[\exp((1+\varepsilon)\,\sigma(s,T)\cdot W_1^*)]\right| \overset{(G.2)}{=} \frac{(1+\varepsilon)^2\,\|\sigma(s,T)\|^2}{2\tau(0,T)} \leq \frac{2\,\|\sigma(s,T)\|^2}{\tau(0,T)} \overset{(A.I.4)}{\leq} 2\psi(s) =: \chi(s)\,.$$

Therefore we get by Lemma G.3

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_0^t \sigma(s,T)\cdot dW_s^*\right)\right] \overset{(G.4)}{=} \exp\left(\int_0^t \log \mathbb{E}^{\mathbb{Q}}[\exp(\sigma(s,T)\cdot W_1^*)]\,ds\right) \overset{(G.2)}{=} \exp\left(\frac{1}{2}\int_0^t \|\sigma(s,T)\|^2\,ds\right).$$

□

With the help of equation (4.1.7) it is easy to find a solution to the initial SDE of the bond prices (4.1.2). This is described in the following corollary, which can be easily proved.

Corollary 4.1.2. *Under Assumption I, it holds for all $T \geq 0$ that*

$$P(t,T) = P(0,T)\,B_t\,\frac{\exp\left(\int_0^t \sigma(s,T)\cdot dW_s^*\right)}{\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_0^t \sigma(s,T)\cdot dW_s^*\right)\right]}\,,\ t \leq T\,, \tag{4.1.8}$$

is a solution of (4.1.2)*.*

Based on this solution to the SDE (4.1.2) and using the definition of the OIS bank account (2.2.2), it is possible to write the T-OIS bond price process the following way, which is concordant with equation (2) of [73]:

$$P(t,T) = P(0,T)\exp\left(\int_0^t \left(r_s - \frac{1}{2}\,\|\sigma(s,T)\|^2\right)ds + \int_0^t \sigma(s,T)\cdot dW_s^*\right),\ t \leq T\,. \tag{4.1.9}$$

It follows a characterization of the forward rate process under the risk-neutral measure that is not dependent anymore on a drift term as under $\mathbb{P}$ in (4.1.1).

Corollary 4.1.3. *Under Assumption I, it holds for all $T \geq 0$ that*

$$f(t,T) = f(0,T) + \int_0^t \sigma_2(s,T)\cdot\sigma(s,T)\,ds - \int_0^t \sigma_2(s,T)\cdot dW_s^*\,,\ t \leq T\,. \tag{4.1.10}$$

Proof. Let $0 \leq t \leq T$. Then as consequence of (4.1.9) it holds

$$\log P(t,T) = \log P(0,T) + \int_0^t \left(r_s - \frac{1}{2}\,\|\sigma(s,T)\|^2\right)ds + \int_0^t \sigma(s,T)\cdot dW_s^*\,.$$

This expression is differentiable with respect to the second variable, which is obvious for the first two summands and for the third summand it is clear considering the following representation due to Lemma G.4

$$\int_0^t \sigma(s,T)\cdot dW_s^* = \sigma(t,T)\cdot W_t^* - \int_0^t W_s^*\cdot\sigma_1(s,T)\,ds\,. \tag{4.1.11}$$

Using (2.2.9), the "Differentiationssatz" in Chapter 2.6 of [127], and (4.1.11), we derive with (4.1.10). □

Then, we get the following characterization of the simple spot rate and the yield in the classical HJM framework.

Lemma 4.1.4. *Under Assumption I, it holds for all $T \geq 0$ that*

$$L(t,T) = \frac{1}{\tau(t,T)} \left((L(0,T)\,\tau(0,T)+1) \exp\left(\int_0^t \left(\frac{1}{2} \|\sigma(s,T)\|^2 - r_s \right) ds - \int_0^t \sigma(s,T) \cdot dW_s^* \right) - 1 \right), \ t \leq T. \tag{4.1.12}$$

Proof. Let $0 \leq t \leq T$ and note that by (2.2.5) it holds for all $T \geq 0$ that $\frac{1}{P(0,T)} = L(0,T)\,\tau(0,T) + 1$. Then, equation (4.1.12) follows immediately by (4.1.9). □

Lemma 4.1.5. *Under Assumption I, it holds for all $T \geq 0$ that*

$$Y(t,T) = Y(0;t,T) + \frac{1}{2} \int_0^t \frac{\|\sigma(s,T)\|^2 - \|\sigma(s,t)\|^2}{\tau(t,T)} ds - \int_0^t \frac{\sigma(s,T) - \sigma(s,t)}{\tau(t,T)} \cdot dW_s^*, \ t \leq T, \tag{4.1.13}$$

where $Y(0;t,T)$ is defined by (2.2.12).

Proof. Let $0 \leq t \leq T$. Then

$$\begin{aligned}
Y(t,T) &\overset{(2.2.8)}{\underset{(2.2.11)}{=}} \frac{1}{\tau(t,T)} \int_t^T f(t,s)\, ds \\
&\overset{(4.1.10)}{=} \frac{1}{\tau(t,T)} \int_t^T f(0,s)\, ds + \int_t^T \int_0^t \frac{\sigma_2(u,s) \cdot \sigma(u,s)}{\tau(t,T)} du\, ds - \int_t^T \int_0^t \frac{\sigma_2(u,s)}{\tau(t,T)} \cdot dW_u^*\, ds \\
&\overset{(2.2.12)}{=} Y(0;t,T) + \frac{1}{\tau(t,T)} \left(\int_t^T \int_0^t \sigma_2(u,s) \cdot \sigma(u,s)\, du\, ds - \int_t^T \int_0^t \sigma_2(u,s) \cdot dW_u^*\, ds \right) \\
&\overset{(*)}{\underset{(G.10)}{=}} Y(0;t,T) + \frac{1}{\tau(t,T)} \left(\int_0^t \int_t^T \sigma_2(u,s) \cdot \sigma(u,s)\, ds\, du - \int_0^t \int_t^T \sigma_2(u,s)\, ds \cdot dW_u^* \right) \\
&= Y(0;t,T) + \frac{1}{2} \int_0^t \frac{\|\sigma(s,T)\|^2 - \|\sigma(s,t)\|^2}{\tau(t,T)} ds - \int_0^t \frac{\sigma(s,T) - \sigma(s,t)}{\tau(t,T)} \cdot dW_s^*.
\end{aligned}$$

At $(*)$ we used the standard Fubini Theorem for non-negative and measurable deterministic functions (cf. Theorem 14.16 of [125]). □

The OIS rate for $[t,T]$ of an OIS with both legs having tenor structure (2.4.4) is

$$R(t,T) = \frac{1 - P(t,T_N)}{S_N(t)}$$

according to (2.4.26) and (2.4.27). In this framework, the OIS rate has to be calculated using equation (4.1.9) for the valuation of all bond prices $P(t,T_1), \dots, P(t,T_N)$.

4.1.2. Long-Term Interest Rates in a Classic HJM Framework

In this section, we now use the results of the previous subsection to calculate the different long-term interest rates in the classical HJM framework. Furthermore, we investigate the asymptotic behavior of the term structure via an examination of the long-term drift and volatility, which are defined in (4.1.15) and (4.1.16), respectively. The approach is to first compute the long-term yield and then analyze the other long-term interest rates with the use of the results on the interrelations between the rates, see Subsection 3.2.1. In the following, we will drop the superscript $*$ for the $\mathbb{Q}$-Brownian motion for the sake of simplicity.

Let us define the function $\theta : \mathbb{R}^d \to \mathbb{R}$ as the logarithm of the moment-generating function of W_1, i.e. it holds for $u \in \mathbb{R}^d$

$$\theta(u) := \log \mathbb{E}^{\mathbb{Q}}[\exp(u \cdot W_1)] \overset{\text{(G.2)}}{=} \frac{1}{2}\|u\|^2. \tag{4.1.14}$$

Definition 4.1.6. *We define the long-term drift* $\mu_\infty := (\mu_\infty(t))_{t\geq 0}$ *in the classical HJM framework as*

$$\mu_\infty(t) := \lim_{T\to\infty} \frac{\theta(\sigma(t,T))}{\tau(t,T)} \overset{(4.1.14)}{=} \lim_{T\to\infty} \frac{\|\sigma(t,T)\|^2}{2\tau(t,T)},\ t \geq 0. \tag{4.1.15}$$

Definition 4.1.7. *We define the long-term volatility* $\sigma_\infty := (\sigma_\infty(t))_{t\geq 0}$ *in the classical HJM framework as*

$$\sigma_\infty(t) := \lim_{T\to\infty} \frac{\sigma(t,T)}{\tau(t,T)},\ t \geq 0. \tag{4.1.16}$$

Note that μ_∞ and σ_∞ always exist since σ is a deterministic function that is càglàd in both components (cf. (A.I.3)).

We want to characterize the long-term yield as an integral of μ_∞ and σ_∞. For this purpose, the following results are needed.

Proposition 4.1.8. *Under Assumption I, it holds*

$$\lim_{T\to\infty} \int_0^\cdot \frac{\sigma(s,T) - \sigma(s,\cdot)}{\tau(\cdot,T)} \cdot dW_s = \int_0^\cdot \sigma_\infty(s) \cdot dW_s \tag{4.1.17}$$

in ucp.

Proof. First, we note that for all $u > t$ it holds $\mathbb{Q}$-a.s.

$$\frac{1}{\tau(t,T)} \sup_{0\leq s\leq u} \int_0^s \sigma(v,s) \cdot dW_v < \infty$$

since W is a semimartingale and σ is simple predictable as deterministic process (cf. Theorem 11 of Chapter II, Section 4 of [149]). Hence, we get

$$\frac{1}{\tau(\cdot,T)} \int_0^\cdot \sigma(s,\cdot) \cdot dW_s \overset{T\to\infty}{\longrightarrow} 0 \tag{4.1.18}$$

in ucp. Next, let us define $H^T := \left(H_s^T\right)_{s\geq 0}$ with

$$H_s^T := \frac{\sigma(s,T)}{\tau(0,T)}.$$

Then for $T \to \infty$ it holds $H_t^T \to \sigma_\infty(t)$ $\mathbb{Q}$-a.s. for all $t \geq 0$. Due to (A.I.4) there exists $\phi := (\phi(s))_{s\geq 0}$ such that for all $T \geq 0$

$$\frac{\|\sigma(s,T)\|}{\tau(0,T)} \leq \phi(s)\,,\ 0 \leq s \leq T.$$

In consequence, we get for all $0 \leq s \leq T$ that $\left\|H_s^T\right\| \leq \phi(s)$, where $\phi \in L(W)$ by Theorem 15 of Chapter IV, Section 2 of [149], since ϕ is locally bounded as a càglàd deterministic function. We can then apply the dominated convergence theorem for semimartingales (cf. Chapter IV, Section 2, Theorem 32 of [149]) and get

$$\int_0^\cdot \frac{\sigma(s,T)}{\tau(0,T)} \cdot dW_s \overset{T\to\infty}{\longrightarrow} \int_0^\cdot \sigma_\infty(s) \cdot dW_s \tag{4.1.19}$$

in ucp. Then, the result follows by Lemma C.1.8 applied on (4.1.18) and (4.1.19). □

Proposition 4.1.9. *Under Assumption I, it holds for all $t \geq 0$ that*

$$\lim_{T\to\infty} \int_0^t \frac{\|\sigma(s,T)\|^2 - \|\sigma(s,t)\|^2}{2\tau(t,T)} ds = \int_0^t \mu_\infty(s)\,ds\,. \tag{4.1.20}$$

Proof. Similar as in the proof of Proposition 4.1.8, we see that

$$\frac{1}{\tau(t,T)} \int_0^t \|\sigma(s,t)\|^2\,ds \overset{T\to\infty}{\longrightarrow} 0 \tag{4.1.21}$$

for all $t \geq 0$. To show that

$$\frac{1}{2\tau(t,T)} \int_0^t \|\sigma(s,T)\|^2\,ds \overset{T\to\infty}{\longrightarrow} \int_0^t \mu_\infty(s)\,ds \tag{4.1.22}$$

holds for all $t \geq 0$, we use the dominated convergence theorem for integrable functions (cf. Corollary 6.26 of [125]), which can be employed due to (4.1.6). □

If we assume that $\lim_{T\to\infty} \sup_{0\leq s\leq t} Y(s,T)$ exists $\mathbb{Q}$-a.s. for all $t \geq 0$, then we get by Proposition 3.1.14, Lemma 4.1.5, Propositions 4.1.8 and 4.1.9 that the long-term yield in the classical HJM framework can be represented as

$$\ell_t = \ell_0 + \int_0^t \mu_\infty(s)\,ds - \int_0^t \sigma_\infty(s) \cdot dW_s,\ \ t \geq 0. \tag{4.1.23}$$

It easily follows that if $\|\sigma_\infty(t)\| > 0$ for some $t \geq 0$, then $\ell = \infty$. If ℓ exists finitely, then

$$\ell_t = \ell_0 + \int_0^t \mu_\infty(s)\,ds \tag{4.1.24}$$

for all $t \geq 0$ with $\mu_\infty(t) \geq 0$ for all $t \geq 0$, see also Lemmas 1 and 2 of [73]. Note, that the long-term yield is independent from the underlying probability measure in the case of (4.1.24).

Let us summarize the results in Table 4.1 that tells us about the different long-term interest rates dependent on the behavior of the volatility curve. We see that only non-negative long-term interest rates are possible in the classical HJM framework.

Volatility curve	Long-term volatility	Long-term drift	Long-term yield	Long-term simple rate	Long-term swap rate
$\sigma(t,T) \sim \mathrm{O}(\tau(t,T))$	$\|\sigma_\infty\| > 0$	$\mu_\infty = \infty$	$\ell = \infty$ infinite	$L = \infty$ infinite	$0 < R < \infty$ non-monotonic
$\sigma(t,T) \sim \mathrm{O}(1)$	$\|\sigma_\infty\| = 0$	$\mu_\infty = 0$	$0 \leq \ell < \infty$ constant	$0 \leq L \leq \infty$ non-negative	$0 \leq R < \infty$ non-monotonic
$\sigma(t,T) \sim \mathrm{O}\left(\sqrt{\tau(t,T)}\right)$	$\|\sigma_\infty\| = 0$	$0 < \mu_\infty < \infty$	$0 < \ell \leq \infty$ non-decreasing	$L = \infty$ infinite	$0 < R < \infty$ non-monotonic

Table 4.1.: Asymptotic behavior of the term structure of interest rates in the classical HJM framework. Own presentation.

4.2. Asymptotic Behavior of Interest Rates in a Lévy HJM Framework

In this section we extend the HJM framework from Section 4.1 by substituting the Brownian motion with a Lévy process. That means, the term structure model is widened by a more general class of driving processes which encompass also jumps in their paths in contrast to the continuous paths of a Brownian motion. First, the different interest rates are calculated in the Lévy HJM framework, and then, in Subsection 4.2.2, the associated long-term interest rates are presented.

This section is based primarily on [25] and [72].

4.2.1. Lévy HJM Term Structure

As an extension of the classical HJM framework, Lévy processes were proposed as stochastic driving processes in several publications such as [25], [71], [72], [85], [119], or [126], mainly in order to incorporate jumps into the term structure models. The jump of a stochastic process $(Y_t)_{t\geq 0}$ at time s is denoted by $\Delta Y_s := Y_s - Y_{s-}$ whereas $Y_{s-} := \lim_{u\uparrow s} Y_u$. We follow the approach of [25] and [72], where the starting point is equation (4.1.8) for the OIS bond prices, but the Brownian motion W^* is exchanged by a Lévy process X, defined on $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{Q})$ with $\mathbb{Q}$ being the risk-neutral measure, i.e.

$$P(t,T) = P(0,T)\, B_t \frac{\exp\left(\int_0^t \sigma(s,T)\cdot dX_s\right)}{\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_0^t \sigma(s,T)\cdot dX_s\right)\right]}\,,\ t \leq T\,. \tag{4.2.1}$$

We assume that X is a d-dimensional process which is adapted to $\mathbb{F}$ and has the following decomposition with Lévy measure ν:

$$X_t = \gamma t + W_t^* + \sum_{s\in[0,t]} \Delta X_s 1_{\{\|\Delta X_s\|>1\}} + \int_{\{\|x\|\leq 1\}} x\,(N_t(\,\cdot\,,dx) - t\,\nu(dx))\,, \tag{4.2.2}$$

where $W^* := (W_t^*)_{t\geq 0}$ is a $\mathbb{Q}$-Brownian motion on $\mathbb{R}^d$ with positive definite covariance matrix $A \in \mathbb{R}^{d\times d}$, $\gamma \in \mathbb{R}^d$, and for any set $B \in \mathbb{R}^d$, $0 \notin \bar{B}$, $N_t^B = \int_B N_t(\cdot, dx)$ is a Poisson process independent of W. The volatility σ is defined as in (4.1.3), that means it is d-dimensional and deterministic in every dimension, where its components are as in (4.1.4). The volatility's partial derivatives are as well defined as in Section 4.1, see (4.1.5), and we still consider the Euclidean scalar product on $\mathbb{R}^d$ with its norm.

Assumption II. *We assume that the volatility function σ satisfies the following properties:*

(A.II.1) *For all $i \in \{1,\dots,d\} : \sigma^i(s,t) > 0$ for all $t \in [0,\infty]$, $s \in [0,t[$.*

(A.II.2) *For all $i \in \{1,\dots,d\} : \sigma^i(s,t) = 0$ for all $t \in [0,\infty]$, $s \geq t$.*

(A.II.3) *For all $i \in \{1,\dots,d\} : \sigma^i, \sigma_1^i, \sigma_2^i$ are càglàd in both components.*

(A.II.4) *There exists a càglàd function $\phi \in L^1(\mathbb{R}_+)$ such as for all $T > 0$*

$$\frac{\|\sigma(t,T)\|}{\tau(0,T)} \leq \phi(t)$$

for all $0 \leq t \leq T$.

(A.II.5) *There exists a function $\psi \in L^1(\mathbb{R}_+)$ such as for all $T > 0$ and for an $\varepsilon \in (0,1)$*

$$\frac{\left|\log \mathbb{E}^{\mathbb{Q}}[\exp((1+\varepsilon)\,\sigma(t,T)\cdot X_1)]\right|}{\tau(0,T)} \leq \psi(t) \tag{4.2.3}$$

for all $0 \leq t \leq T$.

Just as for (A.I.4) and (A.I.5) in Subsection 4.1.1, the assumptions (A.II.4) and (A.II.5) are needed to ensure that all following integrals are well-defined and that the volatility function converges to the long-term volatility in Proposition 4.2.7. Further, we see that by (A.II.5) it holds that $\exp(\sigma(t,T)\cdot X_1) \in L^1(\Omega,\mathscr{F},\mathbb{Q})$ for all $0 \leq t \leq T$ because otherwise we would be able to find a $T > 0$ and $t \leq T$ such that $\mathbb{E}^{\mathbb{Q}}[\exp(\sigma(t,T)\cdot X_1)] = \infty$ and then ψ could not dominate the left-hand side of (4.2.3) for all T. Due to this fact it easily follows by applying Lemma G.1 to the representation of the bond price in this framework (4.2.1), that the discounted bond price process $\frac{P(t,T)}{B_t}, t \leq T$, is a $\mathbb{Q}$-martingale. This justifies the choice of the bond price dynamics.

As in Subsection 4.1.1, we will use the logarithm of the moment-generating function of the driving process at time $t = 1$, defined as

$$\theta(u) := \log \mathbb{E}^{\mathbb{Q}}[\exp(u\cdot X_1)],\ u \in \mathbb{R}^d. \tag{4.2.4}$$

The moment-generating function can be represented in the following way with the use of the Lévy-Khintchine formula (cf. Theorem 43 of Chapter I, Section 4 of [149]):

$$M_{X_t}(u) := \mathbb{E}^{\mathbb{Q}}[\exp(u\cdot X_t)] = \exp(-t\xi(-iu)),\ u \in \mathbb{R}^d, \tag{4.2.5}$$

where

$$\xi(u) := \frac{1}{2}u\cdot Au - i\gamma\cdot u + \int_{\mathbb{R}^d}\left(1 - e^{iu\cdot x} + iu\cdot x1_{\{\|x\|\leq 1\}}\right)\nu(dx)\,. \tag{4.2.6}$$

In (4.2.6), A denotes the covariance matrix of the d-dimensional Brownian motion W^*, ν is the Lévy measure on $\mathbb{R}^d$, and γ is a vector on $\mathbb{R}^d$, i.e. (A,ν,γ) is the generating triplet of X (cf. Definition 8.2 of [154]). We know that the moment-generating function of X_1 with parameter σ is well-defined as a consequence of (A.II.5) and we have that for all $0 \le t \le T$

$$M_{X_1}(\sigma(t,T)) = \exp\Bigg(\gamma\cdot\sigma(t,T) + \frac{1}{2}\sigma(t,T)\cdot A\sigma(t,T) + \int_{\{\|x\|>1\}}\left(e^{\sigma(t,T)\cdot x}-1\right)\nu(dx) + \int_{\{\|x\|\le 1\}}\left(e^{\sigma(t,T)\cdot x}-1-\sigma(t,T)\cdot x\right)\nu(dx)\Bigg). \tag{4.2.7}$$

Then, it follows for all $0 \le t \le T$

$$\begin{aligned}\theta(\sigma(t,T)) &\overset{(4.2.4)}{=} \log\mathbb{E}^{\mathbb{Q}}[\exp(\sigma(t,T)\cdot X_1)] \overset{(4.2.5)}{=} \log M_{X_1}(\sigma(t,T))\\ &\overset{(4.2.7)}{=} \gamma\cdot\sigma(t,T) + \frac{1}{2}\sigma(t,T)\cdot A\sigma(t,T) + \int_{\{\|x\|>1\}}\left(e^{\sigma(t,T)\cdot x}-1\right)\nu(dx)\\ &\quad + \int_{\{\|x\|\le 1\}}\left(e^{\sigma(t,T)\cdot x}-1-\sigma(t,T)\cdot x\right)\nu(dx)\,. \end{aligned}\tag{4.2.8}$$

Now, we formulate a corollary that helps us in further calculations.

Corollary 4.2.1. *Under Assumption II, it holds for all $T \ge 0$ that*

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_0^t \sigma(s,T)\cdot dX_s\right)\right] = \exp\left(\int_0^t \theta(\sigma(s,T))\,ds\right),\ t \le T\,. \tag{4.2.9}$$

Proof. Since σ is a càglàd function in both components due to (A.II.3) and inequality (G.3) is fulfilled by (A.II.5), the result is a consequence of Lemma G.3. □

Applying equation (4.2.9) on the representation of the OIS bond price, we derive the following compact version:

$$P(t,T) = P(0,T)\exp\left(\int_0^t (r_s - \theta(\sigma(s,T)))\,ds + \int_0^t \sigma(s,T)\cdot dX_s\right). \tag{4.2.10}$$

Lemma 4.2.2. *Under Assumption II, it holds for all $T \ge 0$ that*

$$f(t,T) = f(0,T) + \int_0^t \theta'(\sigma(s,T))\cdot\sigma_2(s,T)\,ds - \int_0^t \sigma_2(s,T)\cdot dX_s\,, \tag{4.2.11}$$

where θ' denotes the first derivative of θ.

Proof. From (4.2.1) follows that

$$\log P(t,T) \overset{(4.2.9)}{=} \log P(0,T) + \log B_t - \int_0^t \theta(\sigma(s,T))\,ds + \int_0^t \sigma(s,T)\cdot dX_s\,. \tag{4.2.12}$$

The differentiability of the first summand is clear because $P(0,T)$ is a deterministic, differentiable function. The second summand is differentiable because $\log B_t$ is a constant regarding the second variable T. Next, $\int_0^t \theta(\sigma(s,T))\,ds$ is differentiable by applying the "Differentiationssatz" in Chapter 2.6 of [127], and the last summand is differentiable by taking the form

$$\int_0^t \sigma(s,T)\cdot dX_s \overset{(G.9)}{=} \sigma(t,T)\cdot X_t - \int_0^t X_s\cdot \sigma_1(s,T)\,ds\,.$$

Then, we use (2.2.9) and get (4.2.11), where the integral $\int_0^t \theta'(\sigma(s,T))\cdot\sigma_2(s,T)\,ds$ is well-defined due to (A.II.3). □

Now, we compute the simple spot rate and the yield in the Lévy HJM term structure.

Lemma 4.2.3. *Under Assumption II, it holds for all $T \geq 0$ that*

$$L(t,T) = \frac{1}{\tau(t,T)}\left((L(0,T)\,\tau(0,T)+1)\exp\left(\int_0^t (\theta(\sigma(s,T))-r_s)\,ds - \int_0^t \sigma(s,T)\cdot dX_s\right)-1\right),\ t\leq T\,.$$

Proof. We know that for all $T\geq 0$ it is $\frac{1}{P(0,T)} = L(0,T)\,\tau(0,T)+1$. Hence, the result follows by (2.2.5) and (4.2.10). □

Lemma 4.2.4. *Under Assumption II, it holds for all $T \geq 0$ that*

$$Y(t,T) = Y(0;t,T) + \int_0^t \frac{\theta(\sigma(s,T))-\theta(\sigma(s,t))}{\tau(t,T)}\,ds - \int_0^t \frac{\sigma(s,T)-\sigma(s,t)}{\tau(t,T)}\cdot dX_s\,,\ t\leq T\,, \tag{4.2.13}$$

where $Y(0;t,T)$ is defined by (2.2.12).

Proof. The proof is similar to the proof of Lemma 4.1.5 but for the reader's convenience we formulate the calculation steps in detail below for $t \leq T$:

$$\begin{aligned}
Y(t,T) &\underset{(2.2.11)}{\overset{(2.2.8)}{=}} \frac{1}{\tau(t,T)}\int_t^T f(t,s)\,ds \\
&\overset{(4.2.11)}{=} \frac{1}{\tau(t,T)}\int_t^T f(0,s)\,ds + \int_t^T\int_0^t \frac{\theta'(\sigma(u,s))\cdot\sigma_2(u,s)}{\tau(t,T)}\,du\,ds - \int_t^T\int_0^t \frac{\sigma_2(u,s)}{\tau(t,T)}\cdot dX_u\,ds \\
&\overset{(2.2.12)}{=} Y(0;t,T) + \frac{1}{\tau(t,T)}\left(\int_t^T\int_0^t \theta'(\sigma(u,s))\cdot\sigma_2(u,s)\,du\,ds - \int_t^T\int_0^t \sigma_2(u,s)\cdot dX_u\,ds\right) \\
&\underset{(G.10)}{\overset{(*)}{=}} Y(0;t,T) + \frac{1}{\tau(t,T)}\left(\int_0^t\int_t^T \theta'(\sigma(u,s))\cdot\sigma_2(u,s)\,ds\,du - \int_0^t\int_t^T \sigma_2(u,s)\,ds\cdot dX_u\right) \\
&= Y(0;t,T) + \int_0^t \frac{\theta(\sigma(s,T))-\theta(\sigma(s,t))}{\tau(t,T)}\,ds - \int_0^t \frac{\sigma(s,T)-\sigma(s,t)}{\tau(t,T)}\cdot dX_s\,.
\end{aligned}$$

The standard Fubini Theorem for non-negative and measurable deterministic functions was employed at $(*)$. □

The OIS rate for $[t,T]$ of an OIS with both legs having tenor structure (2.4.4) has to be computed by valuing all bond prices $P(t,T_1),\ldots,P(t,T_N)$ using (4.2.10), and subsequently calculating $S_N(t)$. Then, the OIS rate is valued by (2.4.26).

4.2.2. Long-Term Interest Rates in a Lévy HJM Framework

Now, we want to analyze the long-term interest rates in the presented Lévy HJM framework. Again, the long-term yield is examined first and subsequently the long-term simple rate and long-term swap rate are investigated with the help of the results on the long-term yield's influence on them, stated in Subsection 3.2.1. For this purpose, let us define the long-term drift and long-term volatility in this framework.

Definition 4.2.5. *We define the long-term drift $\mu_\infty := (\mu_\infty(t))_{t\geq 0}$ in the Lévy HJM framework as*

$$\mu_\infty(t) := \lim_{T\to\infty} \frac{\theta(\sigma(t,T))}{\tau(t,T)},\ t \geq 0. \tag{4.2.14}$$

Definition 4.2.6. *We define the long-term volatility $\sigma_\infty := (\sigma_\infty(t))_{t\geq 0}$ in the Lévy HJM framework as*

$$\sigma_\infty(t) := \lim_{T\to\infty} \frac{\sigma(t,T)}{\tau(t,T)},\ t \geq 0. \tag{4.2.15}$$

Both limits μ_∞ and σ_∞ are well-defined since σ is a deterministic function that is càglàd in both components (cf. (A.II.3)).

To compute the long-term yield ℓ as an integral of μ_∞ and σ_∞, we will use the next two propositions.

Proposition 4.2.7. *Under Assumption II, it holds*

$$\lim_{T\to\infty} \int_0^{\cdot} \left(\frac{\sigma(s,T) - \sigma(s,\cdot)}{\tau(\cdot,T)} \right) \cdot dX_s = \int_0^{\cdot} \sigma_\infty(s) \cdot dX_s \tag{4.2.16}$$

in ucp.

Proof. The proof is the same as the proof of Proposition 4.1.8, where this time we can use (A.II.4) to dominate $\frac{\|\sigma(s,T)\|}{\tau(0,T)}$ by $\phi(s)$. The change in the stochastic driver from a Brownian motion to a Lévy process does not pose any problems, since we use Theorem 11 of Chapter II, Section 4, Theorem 15 of Chapter IV, Section 2, and Theorem 32 of Chapter IV, Section 2 of [149], which are all valid for semimartingales. □

Proposition 4.2.8. *Under Assumption II, it holds for all $t \geq 0$ that*

$$\lim_{T\to\infty} \int_0^t \frac{\theta(\sigma(s,T)) - \theta(\sigma(s,t))}{\tau(t,T)}\, ds = \int_0^t \mu_\infty(s)\, ds. \tag{4.2.17}$$

Proof. We can apply the same proof as the one of Proposition 4.1.9. The dominated convergence theorem for deterministic functions holds due to (4.2.3). □

Suppose $\lim_{T\to\infty} \sup_{0\leq s\leq t} Y(s,T)$ exists $\mathbb{Q}$-a.s. for all $t \geq 0$. Hence, Proposition 3.1.14 can be applied, and with the use of Lemma 4.2.4 as well as Propositions 4.2.7 and 4.2.8 we get that the long-term yield in the Lévy HJM framework can be represented as

$$\ell_t = \ell_0 + \int_0^t \mu_\infty(s)\, ds - \int_0^t \sigma_\infty(s) \cdot dX_s,\ \ t \geq 0. \tag{4.2.18}$$

Remark 4.2.9. *Note, that we cannot immediately conclude that σ_∞ must be identically zero in case of a finite long-term yield because X may have jumps. We will investigate the general behavior of ℓ in the following. Theorem 4.2.10 will tell us that if the long-term yield is not supposed to explode in the Lévy HJM framework, the long-term volatility process always vanishes, except in the case of a finite variation Lévy process with only negative jumps.*

Next, we provide a formula for the long-term drift that we will use in subsequent computations. We have for all $t \geq 0$

$$\begin{aligned}\mu_\infty(t) &\overset{(4.2.14)}{=} \lim_{T\to\infty} \frac{\theta(\sigma(t,T))}{\tau(t,T)} \\ &\overset{(4.2.8)}{=} \lim_{T\to\infty} \frac{1}{\tau(t,T)} \left(\gamma \cdot \sigma(t,T) + \frac{1}{2}\sigma(t,T)\cdot A\sigma(t,T) + \int_{\{\|x\|>1\}} \left(e^{\sigma(t,T)\cdot x} - 1\right) \nu(dx) \right. \\ &\quad \left. + \int_{\{\|x\|\leq 1\}} \left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x\right) \nu(dx) \right). \end{aligned} \tag{4.2.19}$$

Let us analyze how a strictly finite long-term volatility affects the long-term drift in this particular term structure. This is done by distinguishing different cases of Lévy processes regarding the jump sizes and the the variation of the paths.

Theorem 4.2.10. *Let $0 < \|\sigma_\infty(t)\| < \infty$ for all $t \geq 0$. Then, it holds under Assumption II that $\mathbb{Q}$-a.s. for all $t \geq 0$:*

(i) If X has only positive jumps or both positive and negative jumps, then $\mu_\infty(t) = \infty$.

(ii) If X has only negative jumps and paths of infinite variation, then $\mu_\infty(t) = \infty$.

(iii) If X has only negative jumps and paths of finite variation, then $\mu_\infty(t) \in \mathbb{R}$.

Proof. First, we notice that there exists at least one $i \in \{1,\ldots,d\}$ such that

$$\sigma^i(t,T) \in \mathrm{O}(\tau(t,T)), \text{ i.e. } \lim_{T\to\infty} \sigma^i(t,T) = \infty. \tag{4.2.20}$$

As a direct consequence we get for all $t \geq 0$ that

$$\lim_{T\to\infty} e^{\sigma(t,T)\cdot x} = \infty \text{ if } x \in \mathbb{R}^d_+ \tag{4.2.21}$$

and

$$\lim_{T\to\infty} e^{\sigma(t,T)\cdot x} = 0 \text{ if } x \in \mathbb{R}^d_- . \tag{4.2.22}$$

For the analysis of the long-term drift we consider the different summands of equation (4.2.19). Due to (4.2.20) the first summand is

$$\lim_{T\to\infty} \frac{\gamma\cdot\sigma(t,T)}{\tau(t,T)} \overset{(4.2.15)}{=} \gamma\cdot\sigma_\infty(t) \in \mathbb{R} . \tag{4.2.23}$$

Next, we get by (4.2.20)

$$\frac{1}{2}\lim_{T\to\infty} \frac{\sigma(t,T)\cdot A\sigma(t,T)}{\tau(t,T)} = \begin{cases} \infty & \text{if } A \neq 0, \\ 0 & \text{if } A = 0, \end{cases} \tag{4.2.24}$$

since A is a positive definite matrix. Let us define the sets $A^{t,T}$ and $B^{t,T}$ for $0 \le t \le T$ as

$$A^{t,T} := \left\{ x \in \mathbb{R}^d \mid \|x\| > 1, \sigma(t,T) \cdot x < 0 \right\},$$

$$B^{t,T} := \left\{ x \in \mathbb{R}^d \mid \|x\| > 1, \sigma(t,T) \cdot x \ge 0 \right\}.$$

We write the third summand of (4.2.19) as follows

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_{\{\|x\|>1\}} \left(e^{\sigma(t,T)\cdot x} - 1 \right) \nu(dx) \\ &= \lim_{T\to\infty} \frac{1}{\tau(t,T)} \left(\int_{A^{t,T}} \left(e^{\sigma(t,T)\cdot x} - 1 \right) \nu(dx) + \int_{B^{t,T}} \left(e^{\sigma(t,T)\cdot x} - 1 \right) \nu(dx) \right). \end{aligned}$$

Then, we have that for all $x \in A^{t,T}$ it holds $-1 \le e^{\sigma(t,T)\cdot x} - 1 \le 0$ and this yields

$$-\frac{\nu(A^{t,T})}{\tau(t,T)} \le \frac{1}{\tau(t,T)} \int_{A^{t,T}} \left(e^{\sigma(t,T)\cdot x} - 1 \right) \nu(dx) \le 0$$

in the case of $x \in A^{t,T}$. Due to $A^{t,T} \subseteq \{\|x\| > 1\}$ and the fact that ν is a Lévy measure, it follows $\nu(\{\|x\| > 1\}) < \infty$. Therefore, we have that for all $t \ge 0$

$$\lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_{A^{t,T}} \left(e^{\sigma(t,T)\cdot x} - 1 \right) \nu(dx) = 0. \tag{4.2.25}$$

Since for all $t \ge 0$ it is

$$\frac{1}{\tau(t,T)} \left(e^{\sigma(t,T)\cdot x} - 1 \right) \ge \frac{1}{\tau(t,T)} \left(\sigma(t,T) \cdot x \right) \ge 0$$

if $x \in \mathbb{R}^d_+$, Fatou's lemma can be applied for $\left\{ x \in \mathbb{R}^d_+ \mid \|x\| > 1 \right\}$ (cf. Theorem 4.21 of [125]). Using $\left\{ x \in \mathbb{R}^d_+ \mid \|x\| > 1 \right\} \subseteq B^{t,T}$, it holds for all $t \ge 0$ that

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_{B^{t,T}} \left(e^{\sigma(t,T)\cdot x} - 1 \right) \nu(dx) &\ge \lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_{\left\{ x \in \mathbb{R}^d_+ \mid \|x\|>1 \right\}} \left(e^{\sigma(t,T)\cdot x} - 1 \right) \nu(dx) \\ &\overset{(*)}{\ge} \int_{\left\{ x \in \mathbb{R}^d_+ \mid \|x\|>1 \right\}} \liminf_{T\to\infty} \frac{1}{\tau(t,T)} \left(e^{\sigma(t,T)\cdot x} - 1 \right) \nu(dx) = \infty \end{aligned} \tag{4.2.26}$$

because of (4.2.21) and the fact that $\nu\left(\left\{ x \in \mathbb{R}^d_+ \mid \|x\| > 1 \right\}\right) > 0$. At $(*)$ we used Fatou's lemma. Regarding the fourth summand of (4.2.19) we only need to note that for all $t \ge 0$

$$\lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_{\{\|x\|\le 1\}} \left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T) \cdot x \right) \nu(dx) \ge 0. \tag{4.2.27}$$

To (i): If X is a Lévy process with only positive or both positive and negative jumps, then we get $\mu_\infty(t) = \infty$ for all $t \ge 0$ as a consequence of (4.2.23), (4.2.24), (4.2.25), (4.2.26), and (4.2.27).

To (ii): Let X be a Lévy process with only negative jumps and paths of infinite variation. Then it holds $A \neq 0$ or $\int_{\{\|x\|\leq 1\}} \|x\| \, \nu(dx) = \infty$ (cf. Proposition 3 of [70]). If $A \neq 0$, then $\mu_\infty(t) = \infty$ for all $t \geq 0$ because of (4.2.24) and since all other terms are non-negative. If $A = 0$, then $\int_{\{\|x\|\leq 1\}} \|x\| \, \nu(dx) = \infty$ because X has paths of infinite variation. In the latter case inequality (4.2.27) still holds if X has only negative jumps and we get:

$$\begin{aligned}
&\lim_{T\to\infty} \int_{\{x\in\mathbb{R}^d_- \,|\, \|x\|\leq 1\}} \frac{1}{\tau(t,T)} \left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x\right) \nu(dx) \\
&\overset{(*)}{\geq} \int_{\{x\in\mathbb{R}^d_- \,|\, \|x\|\leq 1\}} \liminf_{T\to\infty} \frac{1}{\tau(t,T)} \left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x\right) \nu(dx) \\
&= -\sigma_\infty(t) \cdot \int_{\{x\in\mathbb{R}^d_- \,|\, \|x\|\leq 1\}} x \, \nu(dx) = \infty ,
\end{aligned}$$

where we have used Fatou's lemma at $(*)$. This implies the exploding long-term drift since all other summands are non-negative.

To (iii): Let X be a Lévy process with only negative jumps and paths of finite variation. Then it holds $A = 0$ and $\int_{\{\|x\|\leq 1\}} \|x\| \, \nu(dx) < \infty$ (cf. Proposition 3 of [70]). We have

$$X_t = \gamma^* t + \int x N_t(\cdot, dx) ,$$

where $\gamma^* := \gamma - \int_{\{\|x\|\leq 1\}} x \, \nu(dx)$, since X is of finite variation. Then, we apply Corollary 3.1 of [46] together with the fact that we consider only negative jumps, i.e. $\nu\left(\mathbb{R}^d \backslash \mathbb{R}^d_-\right) = 0$, and get for all $t \geq 0$ that

$$\mu_\infty(t) = \gamma^* \cdot \sigma_\infty(t) + \lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_{\mathbb{R}^d_-} \left(e^{\sigma(t,T)\cdot x} - 1\right) \nu(dx) ,$$

where $\gamma^* \cdot \sigma_\infty(t) \in \mathbb{R}$ due to (4.2.20). It holds $-1 \leq e^{\sigma(t,T)\cdot x} - 1 \leq 0$ for all $x \in \mathbb{R}^d_-$ and $0 \leq t \leq T$, hence

$$-\frac{\nu\left(\left\{x\in\mathbb{R}^d_- \,|\, \|x\| > 1\right\}\right)}{\tau(t,T)} \leq \int_{\{x\in\mathbb{R}^d_- \,|\, \|x\|>1\}} \frac{e^{\sigma(t,T)\cdot x} - 1}{\tau(t,T)} \nu(dx) \leq 0 .$$

Due to $\nu\left(\left\{x\in\mathbb{R}^d_- \,|\, \|x\| > 1\right\}\right) < \infty$, we have for all $t \geq 0$ that

$$\lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_{\{x\in\mathbb{R}^d_- \,|\, \|x\|>1\}} \left(e^{\sigma(t,T)\cdot x} - 1\right) \nu(dx) = 0 \tag{4.2.28}$$

and

$$\lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_{\{x\in\mathbb{R}^d_- \,|\, \|x\|\leq 1\}} \left(e^{\sigma(t,T)\cdot x} - 1\right) \nu(dx) \leq 0 . \tag{4.2.29}$$

Since $\exp(y) \geq 1 + y$ for all $y \in \mathbb{R}^d$, it follows for all $0 \leq t \leq T$ that

$$\frac{\sigma(t,T)}{\tau(t,T)} \cdot \int_{\{x\in\mathbb{R}^d_- \,|\, \|x\|\leq 1\}} x \, \nu(dx) \leq \int_{\{x\in\mathbb{R}^d_- \,|\, \|x\|\leq 1\}} \frac{e^{\sigma(t,T)\cdot x} - 1}{\tau(t,T)} \nu(dx) . \tag{4.2.30}$$

It holds that

$$-\infty < \int_{\left\{x\in\mathbb{R}^d_- \,|\, \|x\|\leq 1\right\}} x\,\nu(dx) \leq 0$$

because X has paths of finite variation, and therefore, due to (4.2.20), we have for all $t \geq 0$ that

$$-\infty < \sigma_\infty(t) \cdot \int_{\left\{x\in\mathbb{R}^d_- \,|\, \|x\|\leq 1\right\}} x\,\nu(dx) \leq 0\,. \tag{4.2.31}$$

From (4.2.29), (4.2.30), and (4.2.31), we conclude

$$\lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_{\left\{x\in\mathbb{R}^d_- \,|\, \|x\|\leq 1\right\}} \left(e^{\sigma(t,T)\cdot x} - 1\right) \nu(dx) \in \mathbb{R}_-\,. \tag{4.2.32}$$

Putting together (4.2.28) and (4.2.32), we get

$$\lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_{\mathbb{R}^d_-} \left(e^{\sigma(t,T)\cdot x} - 1\right) \nu(dx) \in \mathbb{R}_-\,. \tag{4.2.33}$$

If X is a finite activity Lévy process, i.e. $\nu\left(\mathbb{R}^d\right) < \infty$ (cf. Proposition 2 of [70]), with finite variation, we even get

$$\lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_{\mathbb{R}^d_-} \left(e^{\sigma(t,T)\cdot x} - 1\right) \nu(dx) = 0 \tag{4.2.34}$$

because of $\nu\left(\mathbb{R}^d_-\right) < \infty$ and

$$-\frac{\nu\left(\mathbb{R}^d_-\right)}{\tau(t,T)} \leq \frac{1}{\tau(t,T)} \int_{\mathbb{R}^d_-} \left(e^{\sigma(t,T)\cdot x} - 1\right) \nu\,(dx) \leq 0$$

due to the fact that $\exp(y) \geq 1 + y$ for all $y \in \mathbb{R}^d$. Finally, it holds $\mu_\infty(t) \in \mathbb{R}$ for all $t \geq 0$ by (4.2.33), (4.2.34), and the fact that $\gamma^* \cdot \sigma_\infty(t) \in \mathbb{R}$. □

Now, we want to analyze the asymptotic behavior of the long-term drift if the long-term volatility vanishes.

Proposition 4.2.11. *Let $\sigma(t,T) \in \mathrm{O}(1)$, i.e. $\sigma^i(t,T) \in \mathrm{O}(1)$ for all $i \in \{1,\ldots,d\}$, for all $0 \leq t \leq T$. Under Assumption II it holds for all $t \geq 0$ that $\mu_\infty(t) = 0$ and therefore ℓ is constant.*

Proof. We have that $\sigma^i(t,T) \leq c$ for T big enough for all $i \in \{1,\ldots,d\}$ and consequently $\|\sigma_\infty(t)\| = 0$ for all $t \geq 0$. For the summands of (4.2.19) we obtain

$$\gamma \cdot \sigma_\infty(t) = 0 \tag{4.2.35}$$

and

$$\frac{1}{2} \lim_{T\to\infty} \frac{\sigma(t,T)\cdot A\sigma(t,T)}{\tau(t,T)} = 0\,. \tag{4.2.36}$$

We know by (A.II.5) that $M_{X_1}(\sigma(t,T))$ is well-defined for all $0 \leq t \leq T$, hence we get

$$\int_{\mathbb{R}^d} \left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x 1_{\{\|x\|\leq 1\}}\right) \nu(dx) < \infty. \tag{4.2.37}$$

Putting together the third and fourth summand of (4.2.19), this yields with (4.2.37)

$$\lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_{\mathbb{R}^d} \left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x 1_{\{\|x\|\leq 1\}} \right) \nu(dx) = 0 \,. \tag{4.2.38}$$

Then, the result follows by (4.2.35), (4.2.36), and (4.2.38). □

Proposition 4.2.12. *If $\mu_\infty(t) < \infty$ for all $t \geq 0$, then, under Assumption II, $\sigma(t,T) \in \mathrm{O}(\log(\tau(t,T)))$ for all $0 \leq t \leq T$. In this case $\mu_\infty(t) \geq 0$ for all $t \geq 0$.*

Proof. Let us check at the beginning which convergence behavior of the volatility function is necessary to guarantee finiteness of the long-term drift. Considering (4.2.19), we need to have $\gamma \cdot \sigma_\infty(t) < \infty$, i.e. $\sigma^i_\infty(t) = \lim_{T\to\infty} \frac{\sigma^i(t,T)}{\tau(t,T)} < \infty$ for all $i \in \{1,\ldots,d\}$. Therefore, we get that for all $i \in \{1,\ldots,d\}$ it holds

$$\sigma^i(t,T) \in \mathrm{O}(\tau(t,T))\,. \tag{4.2.39}$$

Then, the second summand has to fulfill

$$\frac{1}{2} \lim_{T\to\infty} \frac{\sigma(t,T)\cdot A\sigma(t,T)}{\tau(t,T)} < \infty,$$

i.e. for all $i \in \{1,\ldots,d\}$ it holds

$$\sigma^i(t,T) \in \mathrm{O}\left(\sqrt{\tau(t,T)}\right). \tag{4.2.40}$$

Moreover, the following inequality has to be satisfied:

$$\lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_{\mathbb{R}^d} \left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x 1_{\{\|x\|\leq 1\}} \right) \nu(dx) < \infty. \tag{4.2.41}$$

For inequality (4.2.41) to hold, it is sufficient that

$$\lim_{T\to\infty} \frac{1}{\tau(t,T)} \left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x 1_{\{\|x\|\leq 1\}} \right) < \infty \tag{4.2.42}$$

due to Fatou's lemma. Define for all $0 \leq t \leq T$

$$C^{t,T} := \left\{ x \in \mathbb{R}^d \,|\, \sigma(t,T)\cdot x < 0 \right\},$$

$$D^{t,T} := \left\{ x \in \mathbb{R}^d \,|\, \sigma(t,T)\cdot x \geq 0 \right\}.$$

It obviously holds $C^{t,T} \cup D^{t,T} = \mathbb{R}^d$ and for $x \in C^{t,T}$ inequality (4.2.42) is equal to

$$\lim_{T\to\infty} \frac{1}{\tau(t,T)} \left(-\sigma(t,T)\cdot x 1_{\{\|x\|\leq 1\}} \right) = -\sigma_\infty(t)\cdot x 1_{\{\|x\|\leq 1\}} < \infty\,.$$

This implies (4.2.39). On the other hand, for $x \in D^{t,T}$, inequality (4.2.42) yields

$$\lim_{T\to\infty} \frac{1}{\tau(t,T)} e^{\sigma(t,T)\cdot x} < \infty\,,$$

which means that for all $i \in \{1,\dots,d\}$, $\lim_{T\to\infty} \frac{1}{\tau(t,T)} e^{\sigma^i(t,T)x^i} < \infty$. Consequently, it holds for all $i \in \{1,\dots,d\}$ that

$$\sigma^i(t,T) \in \mathrm{O}(\log(\tau(t,T)))\,. \tag{4.2.43}$$

Putting together (4.2.39), (4.2.40), and (4.2.43), we see that the condition $\mu_\infty(t) < \infty$ implies $\sigma(t,T) \in \mathrm{O}(\log(\tau(t,T)))$.

If $\sigma(t,T) \in \mathrm{O}(\log(\tau(t,T)))$, then $\mu_\infty(t) \geq 0$ since $\|\sigma_\infty(t)\| = 0$ and

$$\lim_{T\to\infty} \frac{1}{\tau(t,T)} \int_{\mathbb{R}^d} \left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x 1_{\{\|x\|\leq 1\}} \right) \nu(dx) \geq 0\,.$$

□

Corollary 4.2.13. *If* $0 < \mu_\infty(t) < \infty$ *for all* $t \geq 0$*, then under Assumption II,* $\sigma(t,T)$ *is asymptotically lower bounded and belongs to* $\mathrm{O}(\log(\tau(t,T)))$ *for all* $0 \leq t \leq T$.

Proof. This statement follows directly by Propositions 4.2.11 and 4.2.12. □

Finally, we summarize the results of this part on the asymptotic behavior of the term structure of interest rates in a Lévy HJM framework in Table 4.2. Only non-negative values are taken by all long-term interest rates. Except for the case of a constant long-term yield, the long-term simple rate L always explodes and the long-term swap rate R is strictly positive throughout all different cases.

4.3. Asymptotic Behavior of Interest Rates in an Affine HJM Framework on S_d^+

We present in this section the derivation of an HJM framework where the market is driven by a semimartingale taking values on the cone of positive semidefinite symmetric $d \times d$ matrices with $d \in \mathbb{N}$. This particular state space is denoted by S_d^+ and we further define $\mathscr{M}_d$ as the set of all $d \times d$ matrices with entries in $\mathbb{R}$, and S_d is the space of symmetric $d \times d$ matrices with entries in $\mathbb{R}$. The stochastic driver $X := (X_t)_{t\geq 0}$ is an affine process in the sense that it is a stochastically continuous Markov processes with the property that the Laplace transform can be represented as an exponential-affine function (cf. Definition D.1 for a rigorous characterization of affine processes on S_d^+). It is defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P}_x)$ endowed with the filtration $\mathbb{F} := (\mathscr{F}_t)_{t\geq 0}$ satisfying the usual conditions of completeness and right-continuity. In contrast to the setting of Section 4.2 where the regarded probability space includes already the risk-neutral measure, we begin here with the consideration of the real-world measure $\mathbb{P}_x$ and later change to the risk-neutral measure $\mathbb{Q}$. This is done because we want to show how to derive the HJM drift condition in the affine HJM framework on S_d^+, see Theorem 4.3.6, as this analysis was done for the first time in the literature in [23], which is the basis for the investigation of this section. However, the study of the Lévy HJM framework has already been topic of several publications, as mentioned in Subsection 4.2.1, and therefore we skipped the part on the change of measure to start right away with the risk-neutral measure. The characteristics of affine processes on the state space S_d^+ are analyzed in Appendix D. There, important results are delivered which are

Volatility curve	Long-term volatility	Long-term drift	Long-term yield	Long-term simple rate	Long-term swap rate	Lévy process
$\sigma(t,T) \sim \mathrm{O}(\tau(t,T))$	$\|\sigma_\infty\| > 0$	$\mu_\infty = \infty$	$\ell = \infty$ infinite	$L = \infty$ infinite	$0 < R < \infty$ non-monotonic	Only positive jumps
$\sigma(t,T) \sim \mathrm{O}(\tau(t,T))$	$\|\sigma_\infty\| > 0$	$\mu_\infty = \infty$	$\ell = \infty$ infinite	$L = \infty$ infinite	$0 < R < \infty$ non-monotonic	Positive and negative jumps
$\sigma(t,T) \sim \mathrm{O}(\tau(t,T))$	$\|\sigma_\infty\| > 0$	$\mu_\infty \in \mathbb{R}$	$0 < \ell \leq \infty$ non-decreasing	$L = \infty$ infinite	$0 < R < \infty$ non-monotonic	Finite variation and only negative jumps
$\sigma(t,T) \sim \mathrm{O}(\tau(t,T))$	$\|\sigma_\infty\| > 0$	$\mu_\infty = \infty$	$\ell = \infty$ infinite	$L = \infty$ infinite	$0 < R < \infty$ non-monotonic	Infinite variation and only negative jumps
$\sigma(t,T) \sim \mathrm{O}\left(\sqrt{\tau(t,T)}\right)$	$\|\sigma_\infty\| = 0$	$\mu_\infty = \infty$	$\ell = \infty$ infinite	$L = \infty$ infinite	$0 < R < \infty$ non-monotonic	Positive and negative jumps
$\sigma(t,T) \sim \mathrm{O}(1)$	$\|\sigma_\infty\| = 0$	$\mu_\infty = 0$	$0 \leq \ell < \infty$ constant	$0 \leq L \leq \infty$ non-negative	$0 \leq R < \infty$ non-monotonic	Positive and negative jumps
$\sigma(t,T) \sim \mathrm{O}(\log(\tau(t,T)))$	$\|\sigma_\infty\| = 0$	$\mu_\infty \in \mathbb{R}_+$	$0 < \ell \leq \infty$ non-decreasing	$L = \infty$ infinite	$0 < R < \infty$ non-monotonic	Positive and negative jumps

Table 4.2.: Asymptotic behavior of the term structure of interest rates in the Lévy HJM framework. Own presentation.

needed in the course of this section, such as the determination of an admissible parameter set in Definition D.4 that leads ultimately to the representation (D.11) of these processes. In view of this, we consider $d \geq 2$ and X to be regular and conservative in terms of Definitions D.2 and D.3. Note, that the scalar product of elements A, B of $\mathscr{M}_d$, S_d, or S_d^+ is the trace of A and B, i.e. $A \cdot B := \mathrm{Tr}\left[A^\top B\right]$.

This class of stochastic processes has some appealing features and is increasingly studied in financial research, especially for modeling multivariate stochastic volatilities in equity and fixed income models, see e.g. [21], [53], [54], [55], [99], [137], and [150]. It allows to model a whole family of factors which share non-linear links among each other, providing a more realistic description of the market. In many situations, the presence of stochastic correlations among factors does not come at the cost of a loss of analytical tractability, as these processes are affine. The class of affine processes on S_d^+ was introduced to applications in finance by [98] and [99] in the form of Wishart processes, a particular affine process first described by Bru in [37]. Theoretical background to affine processes on S_d^+ can be found, among other publications, in [48], [49], [51], [65], [95], [100], and [131]. Concerning term structure modeling, in [94] Wishart processes are applied in modeling the short rate, whereas Fonseca et al. construct in [52] a LIBOR model with the use of affine processes. Then, Biagini et al. considered in [23] an affine HJM framework on S_d^+ to develop formulas for forward rates, short rates, and continuously compounded spot rates as well as determine the HJM condition on the drift. The results of this article are the contents of this section.

4.3.1. Affine HJM Term Structure on S_d^+

In this subsection we derive a HJM framework to model the forward rate using affine processes on S_d^+. This setting provides a flexible and concise way of taking into account the influence of a large number of factors on interest rates dynamics and represents a further contribution in capturing the dependence structure affecting the term structure evolution. An interesting aspect of our study is the use of a matrix-valued driver, whose elements are stochastically correlated among each other. Our choice for such rich multi-dimensional dynamics can be seen as beneficial in two possible contexts. Firstly, it offers an alternative way to capture the intrinsic multivariate and dynamic nature of the yield curve and secondly it can be used in the description of positive spreads among different curves to take into account the impact of credit and liquidity risk in the post-crisis interest rate market, as explained in Section 2.1. Moreover, due to the fact that the Wishart dynamics automatically guarantee that the elements of the driving process are stochastically correlated, the correlation structure can be stated in a concise way, considering the special case of a Wishart process as driving factor.

Let us assume that the forward rates evolve for every maturity $T \geq 0$ according to

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\, ds + \int_0^t \mathrm{Tr}[\sigma(s,T)\, dX_s]\,,\ 0 \leq t \leq T, \tag{4.3.1}$$

where X is an affine conservative process with representation (D.11) for a given initial value $x \in S_d^+$. For the reader's convenience we repeat this equation here. For $t \geq 0$ it holds

$$X_t = x + \int_0^t (b + B(X_s))\, ds + \int_0^t \left(\sqrt{X_s}\, dW_s\, Q + Q^\top dW_s^\top \sqrt{X_s}\right) + \int_0^t \int_{S_d^+ \setminus \{0\}} \xi\, \mu^X(ds, d\xi)\,,$$

where $\mu^X(ds,d\xi)$ is the random measure associated with the jumps of X, having the compensator as in (D.12). Since we fix the initial value $X_0 = x$, from now on we write $\mathbb{P}$ for $\mathbb{P}_x$. As mentioned above, we consider X to be a conservative, regular, affine process on the state space S_d^+ with $d \geq 2$, and therefore equation (D.11) is a valid representation of X. In addition, the linear drift coefficient B is of the form

$$B(z) = Mz + zM^\top + G(z)\,,\, z \in S_d^+, \tag{4.3.2}$$

where $M \in \mathscr{M}_d$ and $G : S_d \to S_d$ is a linear operator satisfying $G(S_d^+) \subseteq S_d^+$ (cf. (2.30) in [49]). It should be noted that all calculations of Section 4.3 could still be performed also for X not conservative and $d = 1$ by choosing another set of admissible parameters. The most general parameter set including also the aforementioned case of affine process can be found in Definition 2.3 of [49].

For our further considerations, we impose the following conditions on the drift $\alpha : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and on the volatilities $\sigma_{ij} : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $i, j \in \{1, \ldots, d\}$. Unlike the notation in Sections 4.1 and 4.2, the partial derivatives are not denominated in subscripts to avoid confusion with the above notation of the volatilities.

Assumption III.

(A.III.1) $\alpha := \alpha(\omega, s, u) : (\Omega \times \mathbb{R}_+ \times \mathbb{R}_+, \mathscr{F} \otimes \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}(\mathbb{R}_+)) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ *is jointly measurable.*

(A.III.2) *For all* $T \geq 0$*:*
$$\int_0^T \int_0^T |\alpha(s,u)| \, ds \, du < \infty \ \mathbb{P}\text{-a.s.}$$

(A.III.3) *For all* $s, u \in \mathbb{R}_+$ *and a.e.* $\omega \in \Omega$*:* $\sigma(s,u) \in S_d^+$*, i.e.* $\sigma(s,u)$ *is a symmetric positive semidefinite* $d \times d$ *matrix.*

(A.III.4) $\sigma_{ij} := \sigma_{ij}(\omega, s, u) : (\Omega \times \mathbb{R}_+ \times \mathbb{R}_+, \mathscr{F} \otimes \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}(\mathbb{R}_+)) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ *are jointly measurable for all* $i, j \in \{1, \ldots, d\}$.

(A.III.5) *For all* $T \geq 0$*:* $(\alpha(s,T))_{s \in [0,T]}$ *and* $(\sigma(s,T))_{s \in [0,T]}$ *are adapted.*

(A.III.6) *For all* $T \geq 0$*:*
$$\sup_{s,u \leq T} \|\sigma(s,u)\| < \infty \ \mathbb{P}\text{-a.s.}$$

(A.III.7) *For all* $i, j \in \{1, \ldots, d\} : \sigma_{ij} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *is càglàd in both components.*

Assumption III guarantees that the forward rate process in (4.3.1) is well-defined. We can also choose other integrability conditions such that the integrals in (4.3.1) are well-defined. In this case, the results of this section would also hold under technical modifications of the proofs.

For further computations we need the following remark. If $f : \mathbb{R} \to S_d$ for some $n, d \in \mathbb{N}$, then it is for $a, b \in \mathbb{R}$
$$\mathrm{Tr}\left[\int_a^b f(x)^\top \partial_x f(x) \, dx\right] = \frac{1}{2}\left(\|f(b)\|^2 - \|f(a)\|^2\right). \tag{4.3.3}$$
As a first result, we state a representation of the short rate in the affine HJM framework on S_d^+.

Lemma 4.3.1. *Suppose that* $f(0,T)$*,* $\alpha(t,T)$ *and* $\sigma(t,T)$ *are differentiable in* T *for all* $t \geq 0$*,* $\partial_T \alpha(t,T)$ *is jointly measurable, adapted, and càglàd in* t*, and* $\partial_T \sigma(t,T)$ *is jointly measurable, adapted, and càglàd in* t*. Further, it holds for all* $t \geq 0$ *that*
$$\int_0^t |\partial_u f(0,u)| \, du < \infty,$$
as well as
$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |\partial_T \alpha(t,T)| \, dt \, dT < \infty.$$
Then, the OIS short rate process $(r_t)_{t \geq 0}$ *is of the form*
$$r_t = r_0 + \int_0^t \phi(u) \, du + \int_0^t \mathrm{Tr}[\sigma(u,u) \, dX_u], \tag{4.3.4}$$
where
$$\phi(u) := \alpha(u,u) + \partial_u f(0,u) + \int_0^u \partial_u \alpha(s,u) \, ds + \int_0^u \mathrm{Tr}[\partial_u \sigma(s,u) \, dX_s]. \tag{4.3.5}$$

Proof. We consider representation (4.3.4) for the short-rate process and investigate the different summands. First, we use (4.3.1) and see that

$$f(0,t) = r_0 + \int_0^t \partial_u f(0,u)\, du. \tag{4.3.6}$$

In the following calculations we can use the theorem of Fubini for integrable functions (cf. Theorem 14.16 of [125]), due to the assumption (4.3.1), and have

$$\int_0^t \alpha(s,t)\, ds = \int_0^t \alpha(s,s)\, ds + \int_0^t \int_0^u \partial_u \alpha(s,u)\, ds\, du\,,\ t \geq 0\,. \tag{4.3.7}$$

Next, we use the stochastic Fubini theorem (cf. Theorem 65 of Chapter IV, Section 6 of [149]) since $\partial_T \sigma(t,T)$ is càglàd in t for all $0 \leq t \leq T$ and get

$$\begin{aligned}\int_0^t \mathrm{Tr}[\sigma(s,t)\, dX_s] &= \mathrm{Tr}\left[\int_0^t \sigma(s,t)\, dX_s\right] + \mathrm{Tr}\left[\int_0^t \int_s^t \partial_u \sigma(s,u)\, du\, dX_s\right] \\ &= \mathrm{Tr}\left[\int_0^t \sigma(s,t)\, dX_s\right] + \int_0^t \int_0^u \mathrm{Tr}[\partial_u \sigma(s,u)\, dX_s]\, du. \end{aligned} \tag{4.3.8}$$

Putting together (4.3.6), (4.3.7), and (4.3.8), we obtain that the short-rate follows a process of the form (4.3.4) with ϕ as in (4.3.5). □

Proposition 4.3.2. *Under Assumption III, the OIS bond price follows for every maturity $T \geq 0$ a process of the form*

$$\begin{aligned} P(t,T) = P(0,T) &+ \int_0^t P(s,T)\,(r_s + A(s,T))\, ds \\ &+ 2\int_0^t P(s,T)\,\mathrm{Tr}\big[\Sigma(s,T)\, \sqrt{X_s}\, dW_s\, Q\big] \\ &+ \int_0^t P(s-,T) \int_{S_d^+\setminus\{0\}} \left(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - 1\right) \left(\mu^X - \nu\right)(ds, d\xi)\,, \end{aligned} \tag{4.3.9}$$

for $t \leq T$, where

$$\Sigma(s,T) := -\int_s^T \sigma(s,u)\, du \tag{4.3.10}$$

is the T-bond volatility and

$$A(t,T) := -\int_t^T \alpha(t,u)\, du - F(-\Sigma(t,T)) - \mathrm{Tr}[R(-\Sigma(t,T))\, X_t]\,, \tag{4.3.11}$$

where F and R are given by (D.8)*,* (D.9)*, respectively.*

Proof. Define for all $T \geq 0$

$$Z(t,T) := -\int_t^T f(t,u)\, du,\ 0 \leq t \leq T. \tag{4.3.12}$$

By (4.3.1) it follows for all $T \geq 0$

$$Z(t,T) \overset{(4.3.12)}{=} -\int_t^T f(0,u)\,du - \int_t^T \int_0^t \alpha(s,u)\,ds\,du - \int_t^T \int_0^t \mathrm{Tr}[\sigma(s,u)\,dX_s]\,du,\; t \leq T\,, \tag{4.3.13}$$

and

$$Z(0,T) = -\int_0^T f(0,u)\,du = -\int_t^T f(0,u)\,du - \int_0^t f(0,u)\,du. \tag{4.3.14}$$

By Assumption III, the Fubini theorem for integrable functions, and the stochastic Fubini theorem, we have

$$-\int_t^T \int_0^t \alpha(s,u)\,ds\,du = -\int_0^t \int_s^T \alpha(s,u)\,du\,ds + \int_0^t \int_0^u \alpha(s,u)\,ds\,du, \tag{4.3.15}$$

and similarly

$$-\int_t^T \int_0^t \mathrm{Tr}[\sigma(s,u)\,dX_s]\,du = -\int_0^t \int_s^T \mathrm{Tr}[\sigma(s,u)\;dX_s] + \int_0^t \int_0^u \mathrm{Tr}[\sigma(s,u)\,dX_s]\,du\,. \tag{4.3.16}$$

Then, we use (D.11), (4.3.4), (4.3.10), (4.3.13), (4.3.14), (4.3.15), and (4.3.16), to get the following identity

$$\begin{aligned}
Z(t,T) &\underset{(4.3.14)}{\overset{(4.3.13)}{=}} Z(0,T) + \int_0^t f(0,u)\,du - \int_t^T \int_0^t \alpha(s,u)\,ds\,du - \int_t^T \int_0^t \mathrm{Tr}[\sigma(s,u)\,dX_s]\,du \\
&\overset{(4.3.4)}{=} Z(0,T) + \int_0^t r_s\,ds - \int_0^t \int_s^T \alpha(s,u)\,du\,ds - \int_0^t \int_s^T \mathrm{Tr}[\sigma(s,u)\,du\,dX_s] \\
&\underset{(4.3.10)}{\overset{(D.11)}{=}} Z(0,T) + \int_0^t r_s\,ds - \int_0^t \int_s^T \alpha(s,u)\,du\,ds \\
&\quad + \int_0^t \mathrm{Tr}\Big[\Sigma(s,T)\,\Big(\sqrt{X_s}\,dW_s\,Q + Q^\top\,dW_s^\top\,\sqrt{X_s}\Big)\Big] \\
&\quad + \int_0^t \mathrm{Tr}[\Sigma(s,T)\,(b + B(X_s))]\,ds + \int_0^t \int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\Sigma(s,T)\,\xi]\,\mu^X(ds,d\xi)\,.
\end{aligned}$$

Since $\sigma(s,t) \in S_d$ for all $s,t \geq 0$ we can apply Lemma G.6 and get

$$\begin{aligned}
Z(t,T) &\overset{(G.11)}{=} Z(0,T) + \int_0^t r_s\,ds - \int_0^t \int_s^T \alpha(s,u)\,du\,ds + 2\int_0^t \mathrm{Tr}\big[\Sigma(s,T)\,\sqrt{X_s}\,dW_s\,Q\big] \\
&\quad + \int_0^t \mathrm{Tr}[\Sigma(s,T)\,(b + B(X_s))]\,ds + \int_0^t \int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\Sigma(s,T)\,\xi]\,\mu^X(ds,d\xi)\,.
\end{aligned} \tag{4.3.17}$$

The quadratic variation of Z is for all $0 \leq t \leq T$

$$\big[Z(\cdot,T)\big]_t = \left\langle \mathrm{Tr}\left[\int_0^\cdot \Sigma(s,T)\,\sqrt{X_s}\,dW_s\,Q\right]\right\rangle_t = 4\int_0^t \mathrm{Tr}\Big[Q\Sigma(s,T)\,X_s\,\Sigma(s,T)\,Q^\top\Big]\,ds\,, \tag{4.3.18}$$

where we used the quadratic variation of the Brownian motion

$$\big[W_{lm}, W_{ru}\big]_s = \begin{cases} s & \text{if } l = r \text{ and } m = u, \\ 0 & \text{else.} \end{cases}$$

Next, let us state that by equation (2.27) of [49] it holds for B as in (4.3.2), for all $u \in S_d^+$, and a process Y on S_d^+, that

$$\operatorname{Tr}\left[B^\top(u)\,Y\right] = \operatorname{Tr}[B(Y)\,u]\,. \tag{4.3.19}$$

We have that

$$\begin{aligned}
&\int_0^t \operatorname{Tr}[\Sigma(s,T)\,(b+B(X_s))] + 2\operatorname{Tr}\left[Q\Sigma(s,T)\,X_s\,\Sigma(s,T)\,Q^\top\right] ds \\
&\qquad + \int_0^t \int_{S_d^+\setminus\{0\}} \left(e^{\operatorname{Tr}[\Sigma(s,T)\,\xi]} - 1\right) \nu(ds,d\xi) \\
&\overset{(D.12)}{\underset{(4.3.19)}{=}} -\int_0^t \operatorname{Tr}\left[-\Sigma(s,T)\,b - 2\,\Sigma(s,T)\,\alpha\Sigma(s,T)\,X_s + B^\top(-\Sigma(s,T))\,X_s\right] ds \\
&\qquad + \int_0^t \int_{S_d^+\setminus\{0\}} \left(e^{-\operatorname{Tr}[-\Sigma(s,T)\,\xi]} - 1\right) m(d\xi)\,ds \\
&\qquad + \int_0^t \operatorname{Tr}\left[X_s \int_{S_d^+\setminus\{0\}} \left(e^{-\operatorname{Tr}[-\Sigma(s,T)\,\xi]} - 1\right) \mu(d\xi)\right] ds \\
&\overset{(D.9)}{\underset{(D.8)}{=}} \int_0^t \left(-F(-\Sigma(s,T)) - \operatorname{Tr}[R(-\Sigma(s,T))\,X_s]\right) ds\,,
\end{aligned} \tag{4.3.20}$$

where $\alpha = Q^\top Q$. Besides (4.3.20), we will use in the following calculations the fact that for all $0 \le t \le T$ it holds

$$\Delta Z(t,T) = \operatorname{Tr}\left[\Sigma(t,T)\,\Delta X_t\right]\,. \tag{4.3.21}$$

We know that $P(t,T) = \exp(Z(t,T))$ by (2.2.11), therefore we use Itô's formula to get

$$\begin{aligned}
P(t,T) &= P(0,T) + \int_0^t P(s-,T)\,dZ(s,T) + \frac{1}{2}\int_0^t P(s,T)\,d\left\langle Z(\ ,T)\right\rangle_s^c \\
&\quad + \sum_{0<s\le t}^{\Delta Z(s,T)\neq 0} \left[e^{Z(s,T)} - e^{Z(s-,T)} - \Delta Z(s,T)\,e^{Z(s-,T)}\right] \\
&\overset{(4.3.21)}{\underset{(4.3.18)}{=}} P(0,T) + \int_0^t P(s-,T)\,dZ(s,T) \\
&\quad + 2\int_0^t P(s,T)\operatorname{Tr}\left[Q\Sigma(s,T)\,X_s\,\Sigma(s,T)\,Q^\top\right] ds \\
&\quad + \sum_{0\le s\le t}^{\Delta X_s\neq 0} \left[e^{Z(s,T)} - e^{Z(s-,T)} - \operatorname{Tr}[\Sigma(s,T)\,\Delta X_s]\,e^{Z(s-,T)}\right]
\end{aligned}$$

$$
\begin{aligned}
&\overset{(4.3.17)}{=} P(0,T) + 2\int_0^t P(s,T)\,\mathrm{Tr}\big[\Sigma(s,T)\sqrt{X_s}\,dW_s Q\big] \\
&\quad + \int_0^t P(s,T)\left(r_s - \int_s^T \alpha(s,u)\,du\right) ds \\
&\quad + \int_0^t P(s,T)\,\mathrm{Tr}[\Sigma(s,T)\,(b+B(X_s))]\,ds \\
&\quad + \int_0^t P(s-,T)\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\Sigma(s,T)\,\xi]\,\mu^X(ds,d\xi) \\
&\quad + 2\int_0^t P(s,T)\,\mathrm{Tr}\Big[Q\Sigma(s,T)\,X_s\,\Sigma(s,T)\,Q^\top\Big]\,ds \\
&\quad + \sum_{0\le s\le t}^{\Delta X_s\neq 0}\Big[e^{\Delta Z(s,T)}P(s-,T) - P(s-,T) - \mathrm{Tr}[\Sigma(s,T)\,\Delta X_s]\,P(s-,T)\Big] \\
&\underset{(4.3.20)}{\overset{(4.3.21)}{=}} P(0,T) + \int_0^t P(s-,T)\,(r_s + A(s,T))\,ds + 2\int_0^t P(s,T)\,\mathrm{Tr}\big[\Sigma(s,T)\sqrt{X_s}\,dW_s Q\big] \\
&\quad + \int_0^t P(s-,T)\int_{S_d^+\setminus\{0\}}\Big(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - 1\Big)\,(\mu^X - \nu)\,(ds,d\xi)\,.
\end{aligned}
$$

Due to Proposition 1.28 of Chapter II of [118], it was possible to combine the measures $\mu^X(ds,d\xi)$ and $\nu(ds,d\xi)$ because X has only jumps of finite variation (cf. (D.10)). The finiteness of all integrals above is ensured by Assumption III. □

We have that $-\Sigma(t,T) \in S_d^+$ for all $t \le 0 \le T$ as a consequence of Assumption III because of $\sigma(t,T) \in S_d^+$, and therefore it holds that $\int_t^T \sigma(t,u)\,du \in S_d^+$. Hence, all necessary integrals are finite with respect to μ^X, ν, and the compensated jump measure $\big(\mu^X - \nu\big)$, since X has jumps of finite variation and is regular due to Theorem D.5. The existence of $F(-\Sigma(t,T))$ and $R(-\Sigma(t,T))$ follows, too.

Corollary 4.3.3. *Under Assumption III, the OIS bond price process $P(t,T), 0 \le t \le T$, can be rewritten the following way*

$$
\begin{aligned}
P(t,T) = P(0,T) &+ \int_0^t P(s,T)\,(r_s + C(s,T))\,ds + \int_0^t P(s-,T)\,\mathrm{Tr}[\Sigma(s,T)\,dX_s] \\
&+ \int_0^t\int_{S_d^+\setminus\{0\}} P(s-,T)\Big(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - 1 - \mathrm{Tr}[\Sigma(s,T)\,\xi]\Big)\big(\mu^X - \nu\big)(ds,d\xi)\,, \quad (4.3.22)
\end{aligned}
$$

with for all $0 \le t \le T$

$$
C(t,T) := A(t,T) - \mathrm{Tr}[\Sigma(t,T)\,(b+B(X_t))] - \int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\Sigma(t,T)\,\xi]\,(m(d\xi) + \mathrm{Tr}[X_t\,\mu(d\xi)])\,, \tag{4.3.23}
$$

where $A(t,T)$ is defined in (4.3.11).

Proof. First, note that all integrals in (4.3.22) are finite because of Assumption III. Then, the equation results of (D.11) and (4.3.2). We again use Proposition 1.28 of Chapter II of [118] for the combination of the measures $\mu^X(ds,d\xi)$ and $\nu(ds,d\xi)$ to $\big(\mu^X - \nu\big)(ds,d\xi)$. □

The next corollary states the form of the discounted bond price process in the affine HJM framework and is obtained by representation (4.3.9) of the OIS bond.

Corollary 4.3.4. *Under Assumption III, the discounted OIS bond price follows for every maturity $T \geq 0$ a process of the form*

$$\frac{P(t,T)}{B_t} = P(0,T) + \int_0^t \frac{P(s,T)}{B_s} A(s,T)\,ds + 2\int_0^t \frac{P(s,T)}{B_s} \mathrm{Tr}\big[\Sigma(s,T)\,\sqrt{X_s}\,dW_s\,Q\big] + \int_0^t \frac{P(s-,T)}{B_s} \int_{S_d^+\setminus\{0\}} \Big(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - 1\Big)\left(\mu^X - \nu\right)(ds,d\xi)\,, \tag{4.3.24}$$

for all $t \leq T$.

Proof. This is an immediate consequence of Proposition 4.3.2 and of the definition of the OIS bank account (2.2.2). □

Now, we want to examine the restrictions on the dynamics (4.3.1) under the assumption of no arbitrage. For this purpose, let $\mathbb{Q} \sim \mathbb{P}$ be an equivalent probability measure. Then, we know due to Theorem 3.12 of [27] that there exists $\gamma \in \mathscr{M}_d$ with $\int_0^t \|\gamma_s\|^2\,ds < \infty$ for all $t \geq 0$ such that $W_t^* = W_t - \int_0^t \gamma_s\,ds$, $t \geq 0$, is a matrix variate Brownian motion under $\mathbb{Q}$ and an $\mathscr{F}_t \otimes \mathscr{B}([0,t]) \otimes \mathscr{B}(S_d^+)$ measurable function $K : \Omega \times \mathbb{R}_+ \times S_d^+ \setminus \{0\} \to \mathbb{R}_+$ with

$$\int_0^t \int_{S_d^+\setminus\{0\}} |K(s,\xi)|\,\nu(ds,d\xi) < \infty \ \mathbb{P}\text{-a.s.}$$

for all $t \geq 0$, such that μ^X has the $\mathbb{Q}$-compensator

$$\nu^*(dt,d\xi) := K(t,\xi)\,\nu(dt,d\xi)\,. \tag{4.3.25}$$

In addition, it holds for all $t \geq 0$

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\,|_{\mathscr{F}_t} =: Z_t$$

with

$$\log Z_t = \int_0^t \gamma_s\,dW_s - \int_0^t \|\gamma_s\|^2\,ds + \int_0^t \int_{S_d^+\setminus\{0\}} \log K(s,\xi)\,\mu^X(ds,d\xi) + \int_0^t \int_{S_d^+\setminus\{0\}} (1 - K(s,\xi))\,\nu(ds,d\xi)\,. \tag{4.3.26}$$

Definition 4.3.5. *The probability measure $\mathbb{Q} \sim \mathbb{P}$ is an equivalent local martingale measure (ELMM) for the bond market if for every maturity $T \geq 0$ the discounted OIS bond price process $\frac{P(t,T)}{B_t}$, $t \leq T$, is a $\mathbb{Q}$-local martingale.*

Theorem 4.3.6 (HJM drift condition on S_d^+). *A probability measure $\mathbb{Q} \sim \mathbb{P}$ with Radon-Nikodym density* (4.3.26) *is an ELMM if and only if*

$$\alpha(t,T) = -\,\mathrm{Tr}\big[\sigma(t,T)\left(b + B(X_t) + 2\sqrt{X_t}\,\gamma_t\,Q\right)\big] - 4\,\mathrm{Tr}\Big[Q\,\sigma(t,T)\,X_t\,\Sigma(t,T)\,Q^\top\Big] - \int_{S_d^+\setminus\{0\}} \mathrm{Tr}\left[\sigma(t,T)\,\xi\right] e^{\mathrm{Tr}[\Sigma(t,T)\,\xi]} K(t,\xi)\left(m(d\xi) + \mathrm{Tr}[X_s\,\mu(d\xi)]\right) \tag{4.3.27}$$

for all $T \geq 0$, $dt \otimes d\mathbb{P}$-a.s.
In this case, the $\mathbb{Q}$-dynamics of the forward rates $f(t,T)$, $0 \leq t \leq T$, are of the form

$$\begin{aligned} f(t,T) = f(0,T) + \int_0^t \Bigg\{ & 4\,\mathrm{Tr}\left[Q\,\sigma(s,T)\,X_s \int_s^T \sigma(s,u)\,du\,Q^\top \right] \\ & - \int_{S_d^+\setminus\{0\}} K(s,\xi)\,\mathrm{Tr}[\sigma(s,T)\,\xi]\left(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1\right)(m(d\xi)+\mathrm{Tr}[X_s\,\mu(d\xi)])\Bigg\}\,ds \\ & + \int_0^t \int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi]\left(\mu^X - \nu^*\right)(ds,d\xi) \\ & + 2\int_0^t \mathrm{Tr}\left[\sigma(s,T)\,\sqrt{X_s}\,dW_s^*\,Q\right]. \end{aligned} \tag{4.3.28}$$

Proof. The discounted OIS T-bond price is for all $0 \leq t \leq T$

$$\begin{aligned} \frac{P(t,T)}{B_t} \overset{(4.3.25)}{\underset{(4.3.24)}{=}} & P(0,T) + \int_0^t \frac{P(s,T)}{B_s}\left(A(s,T) + 2\,\mathrm{Tr}\big[\Sigma(s,T)\,\sqrt{X_s}\,\gamma_s\,Q\big]\right)ds \\ & + 2\int_0^t \frac{P(s,T)}{B_s}\,\mathrm{Tr}\big[\Sigma(s,T)\,\sqrt{X_s}\,dW_s^*\,Q\big] \\ & + \int_0^t \int_{S_d^+\setminus\{0\}} \frac{P(s-,T)}{B_s}\left(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1\right)\left(\mu^X - \nu^*\right)(ds,d\xi) \\ & + \int_0^t \int_{S_d^+\setminus\{0\}} \frac{P(s-,T)}{B_s}\left(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1\right)(K(s,\xi)-1)\,\nu(ds,d\xi)\,. \end{aligned} \tag{4.3.29}$$

For $\frac{P(t,T)}{B_t}$, $t \leq T$, being a local martingale under $\mathbb{Q}$, the drift in (4.3.29) must disappear, i.e. for all $0 \leq t \leq T$

$$\begin{aligned} 0 \overset{(D.12)}{=} & \int_0^t \frac{P(s,T)}{B_s} A(s,T)\,ds + 2\int_0^t \frac{P(s,T)}{B_s}\,\mathrm{Tr}\big[\Sigma(s,T)\,\sqrt{X_s}\,\gamma_s\,Q\big]\,ds \\ & + \int_0^t \int_{S_d^+\setminus\{0\}} \frac{P(s-,T)}{B_s}\left(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}-1\right)(K(s,\xi)-1)\,(m(d\xi)+\mathrm{Tr}[X_s\,\mu(d\xi)])\,ds. \end{aligned}$$

That means

$$A(t,T) = -2\,\mathrm{Tr}\big[\Sigma(t,T)\,\sqrt{X_t}\,\gamma_t\,Q\big] - \int_{S_d^+\setminus\{0\}}\left(e^{\mathrm{Tr}[\Sigma(t,T)\,\xi]}-1\right)(K(t,\xi)-1)\,(m(d\xi)+\mathrm{Tr}[X_t\,\mu(d\xi)])$$

$dt \otimes d\mathbb{P}$-a.s., hence by Theorem 6.28 of [125] it holds

$$\begin{aligned} \alpha(t,T) & \overset{(4.3.11)}{=} -\partial_T A(t,T) - \partial_T F(-\Sigma(t,T)) - \partial_T\,\mathrm{Tr}[R(-\Sigma(t,T))\,X_t] \\ & \overset{(G.11)}{=} -\mathrm{Tr}\big[\sigma(t,T)\,\big(b + B(X_t) + 2\sqrt{X_t}\,\gamma_t\,Q\big)\big] - 4\,\mathrm{Tr}\left[Q\,\sigma(t,T)\,X_t\,\Sigma(t,T)\,Q^\top\right] \\ & \quad - \int_{S_d^+\setminus\{0\}} \mathrm{Tr}\left[\sigma(t,T)\,\xi\right] e^{\mathrm{Tr}[\Sigma(t,T)\,\xi]} K(t,\xi)\,(m(d\xi)+\mathrm{Tr}[X_t\,\mu(d\xi)]) \end{aligned}$$

$dt \otimes d\mathbb{P}$-a.s. for all $0 \leq t \leq T$. Accordingly, equation (4.3.27) is a representation of the HJM drift condition in the affine HJM framework on S_d^+. Now, we can compute the forward rate under $\mathbb{Q}$

for all $0 \le t \le T$, using Proposition 1.28 of Chapter II of [118], as

$$
\begin{aligned}
f(t,T) &\overset{(4.3.1)}{\underset{(D.12)}{=}} f(0,T) + \int_0^t \alpha(s,T)\,ds + \int_0^t \mathrm{Tr}\big[\sigma(s,T)\,\big(b + B(X_s) + 2\sqrt{X_s}\,\gamma_s Q\big)\big]\,ds \\
&\quad + \int_0^t \int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi]\,\mu^X(ds,d\xi) + 2\int_0^t \mathrm{Tr}\left[\sigma(s,T)\,\sqrt{X_s}\,dW_s^* Q\right] \\
&\overset{(4.3.27)}{=} f(0,T) - 4\int_0^t \mathrm{Tr}\left[Q\,\sigma(s,T)\,X_s\,\Sigma(s,T)\,Q^\top\right] ds \\
&\quad - \int_0^t \int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi]\,e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} K(s,\xi)\,\nu(ds,d\xi) \\
&\quad + \int_0^t \int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi]\,\mu^X(ds,d\xi) + 2\int_0^t \mathrm{Tr}\left[\sigma(s,T)\,\sqrt{X_s}\,dW_s^* Q\right] \\
&\overset{(4.3.10)}{=} f(0,T) + 4\int_0^t \mathrm{Tr}\left[Q\,\sigma(s,T)\,X_s \int_s^T \sigma(s,u)\,du\,Q^\top\right] ds \\
&\quad + \int_0^t \int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi]\,\big(\mu^X - \nu^*\big)(ds,d\xi) \\
&\quad - \int_0^t \int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi]\,\Big(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - 1\Big)\,\nu^*(ds,d\xi) \\
&\quad + 2\int_0^t \mathrm{Tr}\left[\sigma(s,T)\,\sqrt{X_s}\,dW_s^* Q\right] \\
&\overset{(D.12)}{=} f(0,T) + \int_0^t \Bigg\{ 4\,\mathrm{Tr}\left[Q\,\sigma(s,T)\,X_s \int_s^T \sigma(s,u)\,du\,Q^\top\right] \\
&\quad - \int_{S_d^+\setminus\{0\}} K(s,\xi)\,\mathrm{Tr}[\sigma(s,T)\,\xi]\,\Big(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - 1\Big)\,\big(m(d\xi) + \mathrm{Tr}[X_s\,\mu(d\xi)]\big) \Bigg\}\,ds \\
&\quad + \int_0^t \int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi]\,\big(\mu^X - \nu^*\big)(ds,d\xi) + 2\int_0^t \mathrm{Tr}\left[\sigma(s,T)\,\sqrt{X_s}\,dW_s^* Q\right].
\end{aligned}
$$

□

In [105] it is shown that in the classical HJM framework the forward rates only depend on the volatility in an arbitrage-free market which is considered as one of the most important results of this particular term structure model. Theorem 4.3.6 tells us that this crucial property still holds in the framework of affine processes on S_d^+.

Next, we calculate the simple spot rate and the yield under the risk-neutral measure $\mathbb{Q}$.

Lemma 4.3.7. *Under Assumption III and the ELMM $\mathbb{Q}$, the yield for $[t,T]$ can be expressed as*

$$
\begin{aligned}
Y(t,T) &= Y(0;t,T) + 2\int_0^t \mathrm{Tr}\left[Q\,\frac{\Gamma(s,T) - \Gamma(s,t)}{\tau(t,T)}\,Q^\top\right] ds + \int_0^t \int_{S_d^+\setminus\{0\}} \frac{e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]}}{\tau(t,T)}\,\nu^*(ds,d\xi) \\
&\quad - \int_0^t \int_{S_d^+\setminus\{0\}} \frac{\mathrm{Tr}\left[(\Sigma(s,T) - \Sigma(s,t))\,\xi\right]}{\tau(t,T)}\,\mu^X(ds,d\xi) - 2\int_0^t \mathrm{Tr}\left[\frac{\Sigma(s,T) - \Sigma(s,t)}{\tau(t,T)}\,\sqrt{X_s}\,dW_s^* Q\right],
\end{aligned}
\tag{4.3.30}
$$

with

$$
\Gamma(s,t) := \Sigma(s,t)\,X_s\,\Sigma(s,t) \tag{4.3.31}
$$

for all $s,t \geq 0$.

Proof. It holds for $s \geq 0$ and $0 \leq t_1 \leq t_2$ that

$$
\begin{aligned}
\int_{t_1}^{t_2} \mathrm{Tr}\Big[Q\,\sigma(s,u)\,X_s\,\Sigma(s,u)\,Q^\top\Big]\,du &\overset{(4.3.10)}{=} -\int_{t_1}^{t_2} \mathrm{Tr}\Big[Q\,\partial_u\Sigma(s,u)\,X_s\,\Sigma(s,u)\,Q^\top\Big]\,du \\
&= -\int_{t_1}^{t_2} \mathrm{Tr}\Big[\big(Q\Sigma(s,u)\,\sqrt{X_s}\big)^\top \partial_u\big(Q\Sigma(s,u)\,\sqrt{X_s}\big)\Big]\,du \\
&\overset{(4.3.3)}{=} -\frac{1}{2}\Big(\big\|Q\Sigma(s,t_2)\,\sqrt{X_s}\big\|^2 - \big\|Q\Sigma(s,t_1)\,\sqrt{X_s}\big\|^2\Big) \\
&= -\frac{1}{2}\,\mathrm{Tr}\Big[Q\,\big(\Sigma(s,t_2)\,X_s\,\Sigma(s,t_2) - \Sigma(s,t_1)\,X_s\,\Sigma(s,t_1)\big)\,Q^\top\Big] \\
&\overset{(4.3.31)}{=} -\frac{1}{2}\,\mathrm{Tr}\Big[Q\,\big(\Gamma(s,t_2) - \Gamma(s,t_1)\big)\,Q^\top\Big]
\end{aligned}
\tag{4.3.32}
$$

and

$$
\int_{t_1}^{t_2} \sigma(s,u)\,du \overset{(4.3.10)}{=} -\left(\Sigma(s,t_2) - \Sigma(s,t_1)\right). \tag{4.3.33}
$$

Now, we get by the stochastic Fubini theorem and the one for integrable functions that the yield on $[t,T]$ is

$$
\begin{aligned}
Y(t,T) &= \frac{1}{\tau(t,T)}\left(\int_t^T f(t,u)\,du\right) \\
&\overset{(4.3.28)}{=} \int_t^T \frac{f(0,u)}{\tau(t,T)}\,du - \frac{4}{\tau(t,T)}\int_t^T\int_0^t \mathrm{Tr}\Big[Q\,\sigma(s,u)\,X_s\,\Sigma(s,u)\,Q^\top\Big]\,ds\,du \\
&\quad + \frac{1}{\tau(t,T)}\int_t^T\int_0^t\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,u)\,\xi]\,\big(\mu^X - \nu^*\big)(ds,d\xi)\,du \\
&\quad - \frac{1}{\tau(t,T)}\int_t^T\int_0^t\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma(s,T)\,\xi]\,\Big(e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - 1\Big)\,\nu^*(ds,d\xi)\,du \\
&\quad + \frac{2}{\tau(t,T)}\int_t^T\int_0^t \mathrm{Tr}\big[\sigma(s,u)\,\sqrt{X_s}\,dW_s^*\,Q\big]\,du \\
&\overset{(2.2.12)}{=} Y(0;t,T) - \frac{4}{\tau(t,T)}\int_0^t\int_t^T \mathrm{Tr}\Big[Q\,\sigma(s,u)\,X_s\,\Sigma(s,u)\,Q^\top\Big]\,du\,ds \\
&\quad - \frac{1}{\tau(t,T)}\int_0^t\int_t^T\int_{S_d^+\setminus\{0\}} \partial_u\,\mathrm{Tr}[\Sigma(s,u)\,\xi]\,\big(\mu^X - \nu^*\big)(du,d\xi)\,ds \\
&\quad + \frac{1}{\tau(t,T)}\int_0^t\int_t^T\int_{S_d^+\setminus\{0\}} \partial_u e^{\mathrm{Tr}[\Sigma(s,u)\,\xi]}\,\nu^*(du,d\xi)\,ds \\
&\quad - \frac{1}{\tau(t,T)}\int_0^t\int_t^T\int_{S_d^+\setminus\{0\}} \partial_u\,\mathrm{Tr}[\Sigma(s,u)\,\xi]\,\nu^*(du,d\xi)\,ds \\
&\quad + \frac{2}{\tau(t,T)}\int_0^t \mathrm{Tr}\Big[\int_t^T \sigma(s,u)\,du\,\sqrt{X_s}\,dW_s^*\,Q\Big] \\
&\underset{(4.3.33)}{\overset{(4.3.32)}{=}} Y(0;t,T) + 2\int_0^t \mathrm{Tr}\Big[Q\frac{\Gamma(s,T) - \Gamma(s,t)}{\tau(t,T)}Q^\top\Big]\,ds
\end{aligned}
$$

$$+\int_0^t \int_{S_d^+\setminus\{0\}} \frac{e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]}}{\tau(t,T)}\, \nu^*(ds,d\xi)$$
$$-\int_0^t \int_{S_d^+\setminus\{0\}} \frac{\mathrm{Tr}\left[(\Sigma(s,T)-\Sigma(s,t))\,\xi\right]}{\tau(t,T)}\, \mu^X(ds,d\xi)$$
$$-2\int_0^t \mathrm{Tr}\left[\frac{\Sigma(s,T)-\Sigma(s,t)}{\tau(t,T)}\sqrt{X_s}\,dW_s^*\,Q\right].$$

Note, that it was possible to apply the theorems of Fubini because of Assumption III. □

Lemma 4.3.8. *Under Assumption III and the ELMM $\mathbb{Q}$, the simple spot rate for $[t,T]$ can be expressed as*

$$L(t,T) = \frac{1}{\tau(t,T)}\Bigg(\left(\tau(t,T)\,Y(0;t,T)+1\right)\exp\Bigg(2\int_0^t \mathrm{Tr}\left[Q\left(\Gamma(s,T)-\Gamma(s,t)\right)Q^\top\right]ds$$
$$+\int_0^t \int_{S_d^+\setminus\{0\}} e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]}\, \nu^*(ds,d\xi)$$
$$-\int_0^t \int_{S_d^+\setminus\{0\}} \mathrm{Tr}\left[(\Sigma(s,T)-\Sigma(s,t))\,\xi\right]\mu^X(ds,d\xi)$$
$$-2\int_0^t \mathrm{Tr}\left[\frac{\Sigma(s,T)-\Sigma(s,t)}{\tau(t,T)}\sqrt{X_s}\,dW_s^*\,Q\right]\Bigg)\Bigg).$$

Proof. By Proposition F.1 (ii) we have that $\exp(Y(0;t,T)\,\tau(t,T)) = \tau(t,T)\,L(0;t,T)+1$. Then, the result follows by putting (4.3.30) into the equation stated in Proposition F.1 (iii). □

Corollary 4.3.9. *By* (D.12)*,* (4.3.25)*, and* (4.3.30)*, we obtain that*

$$Y(t,T) = Y(0;t,T) + \int_0^t \Bigg\{2\,\mathrm{Tr}\left[Q\,\frac{\Gamma(s,T)-\Gamma(s,t)}{\tau(t,T)}\,Q^\top\right]$$
$$+\int_{S_d^+\setminus\{0\}} \frac{M(s,t,T,\xi)\,K(s,\xi)}{\tau(t,T)}\left(m(d\xi)+\mathrm{Tr}[X_s\,\mu(d\xi)]\right)\Bigg\}ds$$
$$-\int_0^t \int_{S_d^+\setminus\{0\}} \frac{\mathrm{Tr}\left[(\Sigma(s,T)-\Sigma(s,t))\,\xi\right]}{\tau(t,T)}\left(\mu^X-\nu^*\right)(ds,d\xi)$$
$$-2\int_0^t \mathrm{Tr}\left[\frac{\Sigma(s,T)-\Sigma(s,t)}{\tau(t,T)}\sqrt{X_s}\,dW_s^*\,Q\right] \tag{4.3.34}$$

with

$$M(s,t,T,\xi) := e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]} - \mathrm{Tr}\left[(\Sigma(s,T)-\Sigma(s,t))\,\xi\right]. \tag{4.3.35}$$

Proof. For all $0 \le t \le T$ it holds

$$
\begin{aligned}
Y(t,T) &\overset{(4.3.30)}{=} Y(0;t,T) + 2\int_0^t \mathrm{Tr}\left[Q\frac{\Gamma(s,T)-\Gamma(s,t)}{\tau(t,T)}Q^\top\right] ds \\
&\quad + \int_0^t \int_{S_d^+\setminus\{0\}} \frac{e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]}}{\tau(t,T)} \nu^*(ds,d\xi) \\
&\quad - \int_0^t \int_{S_d^+\setminus\{0\}} \frac{\mathrm{Tr}\left[(\Sigma(s,T)-\Sigma(s,t))\,\xi\right]}{\tau(t,T)} \mu^X(ds,d\xi) \\
&\quad - 2\int_0^t \mathrm{Tr}\left[\frac{\Sigma(s,T)-\Sigma(s,t)}{\tau(t,T)}\sqrt{X_s}dW_s^* Q\right] \\
&= Y(0;t,T) + 2\int_0^t \mathrm{Tr}\left[Q\frac{\Gamma(s,T)-\Gamma(s,t)}{\tau(t,T)}Q^\top\right] ds \\
&\quad + \int_0^t \int_{S_d^+\setminus\{0\}} \frac{e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]} - \mathrm{Tr}\left[(\Sigma(s,T)-\Sigma(s,t))\,\xi\right]}{\tau(t,T)} \nu^*(ds,d\xi) \\
&\quad - \int_0^t \int_{S_d^+\setminus\{0\}} \frac{\mathrm{Tr}\left[(\Sigma(s,T)-\Sigma(s,t))\,\xi\right]}{\tau(t,T)} \left(\mu^X - \nu^*\right)(ds,d\xi) \\
&\quad - 2\int_0^t \mathrm{Tr}\left[\frac{\Sigma(s,T)-\Sigma(s,t)}{\tau(t,T)}\sqrt{X_s}dW_s^* Q\right] \\
&\underset{(4.3.35)}{\overset{(D.12)}{=}} Y(0;t,T) + \int_0^t \left\{ 2\,\mathrm{Tr}\left[Q\frac{\Gamma(s,T)-\Gamma(s,t)}{\tau(t,T)}Q^\top\right] \right. \\
&\quad \left. + \int_{S_d^+\setminus\{0\}} \frac{M(s,t,T,\xi)\,K(s,\xi)}{\tau(t,T)} \left(m(d\xi) + \mathrm{Tr}[X_s\mu(d\xi)]\right) \right\} ds \\
&\quad - \int_0^t \int_{S_d^+\setminus\{0\}} \frac{\mathrm{Tr}\left[(\Sigma(s,T)-\Sigma(s,t))\,\xi\right]}{\tau(t,T)} \left(\mu^X - \nu^*\right)(ds,d\xi) \\
&\quad - 2\int_0^t \mathrm{Tr}\left[\frac{\Sigma(s,T)-\Sigma(s,t)}{\tau(t,T)}\sqrt{X_s}dW_s^* Q\right].
\end{aligned}
$$

□

As in both HJM frameworks presented before, for computing the OIS rate in t with maturity $T \ge t$ of an OIS with both legs having tenor structure (2.4.4), all corresponding bond prices $P(t,T_1), \ldots, P(t,T_N)$ have to be evaluated with the help of (4.3.22). Afterwards, $S_N(t)$ can be calculated and finally the OIS rate is valued by (2.4.26).

4.3.2. Long-Term Interest Rates in an Affine HJM Framework on S_d^+

This subsection investigates the asymptotic behavior of the term structure of interest rates in the affine HJM framework on S_d^+ outlined above. First, we want to find a representation of the long-term yield and then we analyze under which conditions this rate can exist finitely. The other long-term interest rates are afterwards examined by the respective interrelations between them and ℓ, just as we did it in the previous two sections. For the sake of simplicity, we drop the superscript $*$ for ν and W but keep in mind that nevertheless we work under the risk-neutral measure $\mathbb{Q}$.

Let us define the long-term drift and long-term volatility in this framework.

Definition 4.3.10. *We define the long-term drift $\mu_\infty := (\mu_\infty(t))_{t\geq 0}$ in the affine HJM framework as*

$$\mu_\infty(\cdot) := \lim_{T\to\infty} \frac{\Gamma(\cdot,T)}{\tau(\cdot,T)} \tag{4.3.36}$$

if the limit exists in ucp with values on $\mathscr{M}_d$.

Definition 4.3.11. *We define the long-term volatility $\sigma_\infty := (\sigma_\infty(t))_{t\geq 0}$ in the affine HJM framework as*

$$\sigma_\infty(\cdot) := \lim_{T\to\infty} \frac{\Sigma(\cdot,T)}{\tau(\cdot,T)} \tag{4.3.37}$$

if the limit exists in ucp with values on $\mathscr{M}_d$.

For the characterization of the long-term yield as an integral of the long-term drift and the long-term volatility we need the following assumption.

Assumption IV. *Let $\Sigma(s,t)$ be defined as in* (4.3.10) *for all $0 \leq s \leq t$ and W a matrix variate Brownian motion. There exists a progressively measurable process $w \in L(W)$ with values in S_d^+ such that for every $i, j \in \{1,\dots,d\}$, w_{ij} is a càdlàg process with*

$$\frac{1}{\sqrt{\tau(0,t)}} \left|\Sigma(s,t)_{ij}\right| \leq w_{ij}(s) \quad \mathbb{Q}\text{-a.s.} \tag{4.3.38}$$

for all $0 \leq s \leq t$ and $t \neq 0$.

Proposition 4.3.12. *Under Assumptions III and IV, it holds*

$$\lim_{T\to\infty} 2\int_0^{\cdot} \mathrm{Tr}\left[\frac{\Sigma(s,T)-\Sigma(s,\cdot)}{\tau(\cdot,T)}\sqrt{X_s}\, dW_s\, Q\right] = 2\int_0^{\cdot} \mathrm{Tr}\left[\sigma_\infty(s)\sqrt{X_s}\, dW_s\, Q\right] \tag{4.3.39}$$

in ucp, where $\Sigma(t,T)$ is defined for all $0 \leq t \leq T$ as in (4.3.10).

Proof. Due to Assumption III it is for all $t \geq 0$ $\mathbb{Q}$-a.s.

$$\sup_{0\leq u\leq t}\left|\int_0^u \mathrm{Tr}\left[2\, Q\,\Sigma(s,u)\sqrt{X_s}\, dW_s\right]\right| < \infty,$$

hence

$$\frac{1}{\tau(t,T)} \sup_{0\leq u\leq t} \int_0^u \mathrm{Tr}\left[2\, Q\,\Sigma(s,u)\sqrt{X_s}\, dW_s\right] \overset{T\to\infty}{\longrightarrow} 0.$$

This yields

$$\frac{1}{\tau(\cdot,T)} \int_0^{\cdot} \mathrm{Tr}\left[2\, Q\,\Sigma(s,\cdot)\sqrt{X_s}\, dW_s\right] \overset{T\to\infty}{\longrightarrow} 0 \tag{4.3.40}$$

in ucp. Let us define $F^T := \left(F_t^T\right)_{t\geq 0}$ for $T \geq 0$, where

$$F_t^T := 2\,\frac{Q\,\Sigma(t,T)\sqrt{X_t}}{\tau(0,T)}. \tag{4.3.41}$$

Then, for $T \to \infty$ it holds $F_t^T \to 2\,Q\,\sigma_\infty(t)\,\sqrt{X_t}$ $\mathbb{Q}$-a.s. for all $t \geq 0$. We get by Assumption IV for all $0 \leq t \leq T$ with $\tau(0,T) \geq 1$

$$
\begin{aligned}
\left\| H_t^T \right\| &\overset{(4.3.41)}{=} \frac{2}{\tau(0,T)} \left\| Q \Sigma(t,T) \sqrt{X_t} \right\| \\
&= \frac{2}{\tau(0,T)} \left(\mathrm{Tr}\left[\sqrt{X_t}\, \Sigma(t,T)\, Q^\top Q \Sigma(t,T) \sqrt{X_t} \right] \right)^{1/2} \\
&= \frac{2}{\tau(0,T)} \left(\sum_{i,j,k,l,m,n} \sqrt{X_{ij,t}}\, \Sigma(t,T)_{jk}\, Q_{kl}^\top Q_{lm}\, \Sigma(t,T)_{mn} \sqrt{X_{ni,t}} \right)^{1/2} \\
&\overset{(4.3.38)}{\leq} \frac{2}{\tau(0,T)} \left(\sum_{i,j,k,l,m,n} \sqrt{\tau(0,T)}\, \sqrt{X_{ij,t}}\, w_{jk}(t)\, Q_{kl}^\top Q_{lm} \sqrt{\tau(0,T)}\, w_{mn}(t)\, \sqrt{X_{ni,t}} \right)^{1/2} \\
&= \frac{2}{\sqrt{\tau(0,T)}} \left(\sum_{i,j,k,l,m,n} \sqrt{X_{ij,t}}\, w_{jk}(t)\, Q_{kl}^\top Q_{lm}\, w_{mn}(t)\, \sqrt{X_{ni,t}} \right)^{1/2} \\
&\leq 2 \left(\sum_{i,j,k,l,m,n} \sqrt{X_{ij,t}}\, w_{jk}(t)\, Q_{kl}^\top Q_{lm}\, w_{mn}(t)\, \sqrt{X_{ni,t}} \right)^{1/2} \\
&= 2 \left(\mathrm{Tr}\left[\sqrt{X_t}\, w(t)\, Q^\top Q\, w(t) \sqrt{X_t} \right] \right)^{1/2} \\
&= 2 \left\| Q\, w(t) \sqrt{X_t} \right\| =: h(t)\,.
\end{aligned}
$$

We can assume $\tau(0,T) \geq 1$ since we investigate long-term interest rates. Due to Theorem 16 in Chapter IV, Section 2 of [149], it is $h \in L(W)$, which holds since $\sqrt{X} \in L(W)$ by Theorem D.7 and $w \in L(W)$ by Assumption IV. Therefore, we can use the dominated convergence theorem for semimartingales, which gives us

$$
\int_0^{\cdot} \mathrm{Tr}\left[2\, \frac{Q \Sigma(s,T) \sqrt{X_s}}{\tau(0,T)}\, dW_s \right] \overset{T \to \infty}{\longrightarrow} 2 \int_0^{\cdot} \mathrm{Tr}\left[\sigma_\infty(s) \sqrt{X_s}\, dW_s\, Q \right] \tag{4.3.42}
$$

in ucp.

Then, the result follows by Lemma C.1.8 applied for (4.3.40) and (4.3.42). □

Proposition 4.3.13. *Under Assumptions III and IV, it holds $\mathbb{Q}$-a.s. for all $t \geq 0$*

$$
\lim_{T \to \infty} 2 \int_0^t \mathrm{Tr}\left[Q\, \frac{\Gamma(s,T) - \Gamma(s,t)}{\tau(t,T)}\, Q^\top \right] ds = 2 \int_0^t \mathrm{Tr}\left[Q\, \mu_\infty(s)\, Q^\top \right] ds\,, \tag{4.3.43}
$$

where $\Gamma(t,T)$ is defined for all $0 \leq t \leq T$ as in (4.3.31).

Proof. First, notice that it holds $\mathbb{Q}$-a.s. for all $t \geq 0$

$$
\sup_{0 \leq u \leq t} \left| \int_0^u \mathrm{Tr}\left[Q\, \Gamma(s,u)\, Q^\top \right] ds \right| < \infty
$$

since Γ is càdlàg in both components, which yields

$$\frac{1}{\tau(t,T)} \sup_{0\le u\le t} \int_0^u \mathrm{Tr}\Big[Q\Gamma(s,u)\,Q^\top\Big]\,ds \overset{T\to\infty}{\longrightarrow} 0\,.$$

Accordingly, we have

$$\frac{1}{\tau(\cdot,T)} \int_0^{\cdot} \mathrm{Tr}\Big[Q\Gamma(s,\cdot)\,Q^\top\Big]\,ds \overset{T\to\infty}{\longrightarrow} 0 \tag{4.3.44}$$

in ucp. Next, define $G^T := \left(G_t^T\right)_{t\ge 0}$, where

$$G_t^T := 2\frac{Q\Gamma(t,T)\,Q^\top}{\tau(0,T)}\,. \tag{4.3.45}$$

For $T\to\infty$ it follows $G_t^T \to 2\,Q\,\mu_\infty(t)\,Q^\top$ $\mathbb{Q}$-a.s. for all $t\ge 0$. We have for all $i,j\in\{1,\dots,d\}$ and $0\le t\le T$ $\mathbb{Q}$-a.s.

$$\begin{aligned}
\Gamma(t,T)_{ij} &\overset{(4.3.31)}{=} (\Sigma(t,T)X_t\Sigma(t,T))_{ij} = \sum_{k,l}\Sigma(t,T)_{ik}X_{kl,t}\Sigma(t,T)_{lj}\\
&\overset{(4.3.38)}{\le} \sum_{k,l}\sqrt{\tau(0,T)}\,w_{ik}(t)X_{kl,t}\sqrt{\tau(0,T)}\,w_{lj}(t) = \tau(0,T)\,(w(t)X_tw(t))_{ij}\,,
\end{aligned} \tag{4.3.46}$$

hence

$$\begin{aligned}
\left\|G_t^T\right\| &\overset{(4.3.45)}{=} \frac{2}{\tau(0,T)}\left\|Q\Gamma(t,T)\,Q^\top\right\|\\
&= \frac{2}{\tau(0,T)}\left(\mathrm{Tr}\Big[Q\Gamma(t,T)\,Q^\top Q\Gamma(t,T)\,Q^\top\Big]\right)^{1/2}\\
&= \frac{2}{\tau(0,T)}\left(\sum_{i,j,k,l,m,n} Q_{ij}\,\Gamma(t,T)_{jk}\,Q_{kl}^\top\,Q_{lm}\,\Gamma(t,T)_{mn}\,Q_{ni}^\top\right)^{1/2}\\
&\overset{(4.3.46)}{\le} \frac{2}{\tau(0,T)}\left(\sum_{i,j,k,l,m,n} Q_{ij}\,\tau(0,T)\,(w(t)X_tw(t))_{jk}\,Q_{kl}^\top\,Q_{lm}\,\tau(0,T)\,(w(t)X_tw(t))_{mn}\,Q_{ni}^\top\right)^{1/2}\\
&= 2\left(\mathrm{Tr}\Big[Q\,w(t)\,X_tw(t)\,Q^\top Q\,w(t)\,X_tw(t)\,Q^\top\Big]\right)^{1/2}\\
&= 2\left\|Q\,w(t)\,X_tw(t)\,Q^\top\right\| =: g(t)\,.
\end{aligned}$$

We know by Proposition C.1.9 that $\int_0^t g(s)\,ds<\infty$ $\mathbb{Q}$-a.s. for all $t\ge 0$ since g is a càdlàg process. That means, we can apply the dominated convergence theorem for semimartingales and get

$$2\int_0^{\cdot}\mathrm{Tr}\left[Q\frac{\Gamma(s,T)-\Gamma(s,\cdot)}{\tau(\cdot,T)}\,Q^\top\right]ds \overset{T\to\infty}{\longrightarrow} 2\int_0^{\cdot}\mathrm{Tr}\Big[Q\,\mu_\infty(s)\,Q^\top\Big]\,ds \tag{4.3.47}$$

in ucp.

Using Lemma C.1.8 for (4.3.44) and (4.3.47) gives us (4.3.43). □

Proposition 4.3.14. *Under Assumptions III and IV, it holds*

$$\int_0^{\cdot}\int_{S_d^+\setminus\{0\}} \frac{\mathrm{Tr}[(\Sigma(s,T)-\Sigma(s,\cdot))\,\xi]}{\tau(\cdot,T)}\,\mu^X(ds,d\xi) \overset{T\to\infty}{\longrightarrow} \int_0^{\cdot}\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi) \quad (4.3.48)$$

in ucp, where $\Sigma(t,T)$ *is defined for all* $0\le t\le T$ *as in* (4.3.10).

Proof. We have $\mathbb{Q}$-a.s. for all $t\ge 0$

$$\begin{aligned}
\left|\int_0^t\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\Sigma(s,t)\,\xi]\,\mu^X(ds,d\xi)\right| &\le \int_0^t\int_{S_d^+} |\mathrm{Tr}[\Sigma(s,t)\,\xi]|\,\mu^X(ds,d\xi)\\
&\le \sqrt{\tau(0,t)}\int_0^t\int_{S_d^+\setminus\{0\}} \frac{1}{\sqrt{\tau(0,t)}}\,\|\Sigma(s,t)\|\,\|\xi\|\,\mu^X(ds,d\xi)\\
&\overset{(4.3.38)}{\le} \sqrt{\tau(0,t)}\int_0^t\int_{S_d^+\setminus\{0\}} \|w(s)\|\,\|\xi\|\,\mu^X(ds,d\xi)\\
&\le \sqrt{\tau(0,t)}\sup_{0\le u\le t}\|w(u)\|\int_0^t\int_{S_d^+\setminus\{0\}} \|\xi\|\,\mu^X(ds,d\xi)\,.
\end{aligned}$$

Next, let us define $q(t) := \sqrt{\tau(0,t)}\,\sup_{0\le u\le t}\|w(u)\|\,Y_t$, where

$$Y_t := \int_0^t\int_{S_d^+\setminus\{0\}} \|\xi\|\,\mu^X(ds,d\xi)\,,\ t\ge 0\,.$$

This leads to the following $\mathbb{Q}$-a.s. for all $t\ge 0$:

$$\begin{aligned}
\sup_{0\le u\le t}|q(u)| = \sup_{0\le u\le t}\left|\sqrt{\tau(0,u)}\sup_{0\le s\le u}\|w(s)\|\,Y_u\right| &\le \sqrt{\tau(0,t)}\sup_{0\le u\le t}\sup_{0\le s\le u}\|w(s)\|\sup_{0\le u\le t}Y_u\\
= \sqrt{\tau(0,t)}\sup_{0\le s\le t}\|w(s)\|\sup_{0\le u\le t}Y_u < \infty\,,
\end{aligned}$$

due to Proposition C.1.9, which can be applied since $\|w(t)\|, t\ge 0$, is a càdlàg process as well as $(Y_t)_{t\ge 0}$ by (D.10). Then, it holds $\mathbb{Q}$-a.s. for all $t\ge 0$

$$\frac{1}{\tau(0,T)}\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\Sigma(s,u)\,\xi]\,\mu^X(ds,d\xi) \overset{T\to\infty}{\longrightarrow} 0\,,$$

which yields

$$\frac{1}{\tau(0,T)}\int_0^{\cdot}\int_{S_d^+\setminus\{0\}} \mathrm{Tr}[\Sigma(s,\cdot)\,\xi]\,\mu^X(ds,d\xi) \overset{T\to\infty}{\longrightarrow} 0 \quad (4.3.49)$$

in ucp. Next, it follows for all for all $0\le t\le T$ with $\tau(0,T)\ge 1$

$$\frac{|\mathrm{Tr}\,[\Sigma(t,T)\,\xi]|}{\tau(0,T)} \le \frac{1}{\tau(0,T)}\,\|\Sigma(t,T)\|\,\|\xi\| \overset{(4.3.38)}{\le} \frac{1}{\sqrt{\tau(0,T)}}\,\|w(t)\|\,\|\xi\| \le \|w(t)\|\,\|\xi\| =: j(t,\xi)\,.$$

The assumption $\tau(0,T)\ge 1$ is justified due to the fact that we analyze long-term interest rates. By Proposition C.1.9, we have $\mathbb{Q}$-a.s. for all $t\ge 0$

$$\int_0^t\int_{S_d^+\setminus\{0\}} j(s,\xi)\,\mu^X(ds,d\xi) = \int_0^t\int_{S_d^+\setminus\{0\}} \|w(s)\|\,\|\xi\|\,\mu^X(ds,d\xi) \le \sup_{0\le u\le t}\|w(u)\|\,Z_t < \infty\,,$$

i.e. j is $\mathbb{Q}$-a.s. integrable for all $t \geq 0$ with respect to the random measure μ^X on $[0,t] \times S_d^+ \setminus \{0\}$. Consequently, the dominated convergence theorem for integrable functions is applicable and we have $\mathbb{Q}$-a.s. for all $t \geq 0$

$$\int_0^t \int_{S_d^+ \setminus \{0\}} \frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)} \mu^X(ds,d\xi) \overset{T\to\infty}{\longrightarrow} \int_0^t \int_{S_d^+ \setminus \{0\}} \mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\,.$$

With Lemma G.7 it follows $\mathbb{Q}$-a.s. for all $t \geq 0$

$$\sup_{0\leq u\leq t} \int_0^u \int_{S_d^+ \setminus \{0\}} \frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)} \mu^X(ds,d\xi) \overset{T\to\infty}{\longrightarrow} \sup_{0\leq u\leq t} \int_0^u \int_{S_d^+ \setminus \{0\}} \mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)$$

and accordingly the following convergence holds in ucp:

$$\lim_{T\to\infty} \int_0^\cdot \int_{S_d^+ \setminus \{0\}} \frac{\mathrm{Tr}\,[\Sigma(s,T)\,\xi]}{\tau(0,T)}\,\mu^X(ds,d\xi) = \int_0^\cdot \int_{S_d^+ \setminus \{0\}} \mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\,. \tag{4.3.50}$$

Now, the result is a consequence of Lemma C.1.8 used with (4.3.49) and (4.3.50). □

Proposition 4.3.15. *Under Assumptions III and IV, it holds*

$$\lim_{T\to\infty} \int_0^\cdot \int_{S_d^+ \setminus \{0\}} \frac{e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]} - e^{\mathrm{Tr}[\Sigma(s,\,\cdot\,)\,\xi]}}{\tau(\,\cdot\,,T)}\,\nu(ds,d\xi) = 0 \tag{4.3.51}$$

in ucp, where $\Sigma(t,T)$ *is defined for all* $0 \leq t \leq T$ *as in* (4.3.10).

Proof. First, we reformulate the left-hand side of (4.3.51) for all $t \geq 0$ to

$$\lim_{T\to\infty} \frac{1}{\tau(0,T)} \left(\int_0^t \int_{S_d^+ \setminus \{0\}} \left(1 - e^{\mathrm{Tr}[\Sigma(s,t)\,\xi]}\right) \nu(ds,d\xi) - \int_0^t \int_{S_d^+ \setminus \{0\}} \left(1 - e^{\mathrm{Tr}[\Sigma(s,T)\,\xi]}\right) \nu(ds,d\xi) \right). \tag{4.3.52}$$

Define for all $u \in S_d^+$

$$\tilde{F}(u) := \int_{S_d^+ \setminus \{0\}} \left(e^{-\mathrm{Tr}[u\xi]} - 1\right) m(d\xi)\,, \tag{4.3.53}$$

$$\tilde{R}(u) := \int_{S_d^+ \setminus \{0\}} \left(e^{-\mathrm{Tr}[u\xi]} - 1\right) \mu(d\xi)\,. \tag{4.3.54}$$

By (D.12), (4.3.53), and (4.3.54), it follows for all $t \geq 0$

$$\int_0^t \int_{S_d^+ \setminus \{0\}} \left(1 - e^{\mathrm{Tr}[\Sigma(s,t)\xi]}\right) \nu(ds,d\xi) = -\int_0^t \left(\tilde{F}(-\Sigma(s,t)) + \mathrm{Tr}\left[\tilde{R}(-\Sigma(s,t))\,X_s\right]\right) ds.$$

Then it holds $\mathbb{Q}$-a.s. for all $t \geq 0$

$$\sup_{0\leq u\leq t} \int_0^u \int_{S_d^+ \setminus \{0\}} \left(1 - e^{\mathrm{Tr}[\Sigma(s,u)\,\xi]}\right) \nu(ds,d\xi) < \infty$$

due to Proposition C.1.9, which holds because the process $\tilde{F}(-\Sigma(s,t))+\mathrm{Tr}\big[\tilde{R}(-\Sigma(s,t))X_s\big]$, $s\in[0,t]$, is càdlàg for all $t\geq 0$. Accordingly, we get $\mathbb{Q}$-a.s. for all $t\geq 0$

$$\frac{1}{\tau(0,T)}\sup_{0\leq u\leq t}\int_0^u\int_{S_d^+\setminus\{0\}}\Big(1-e^{\mathrm{Tr}[\Sigma(s,u)\,\xi]}\Big)\,\nu(ds,d\xi)\overset{T\to\infty}{\longrightarrow}0\,, \tag{4.3.55}$$

hence the following convergence holds in ucp:

$$\frac{1}{\tau(0,T)}\int_0^\cdot\int_{S_d^+\setminus\{0\}}\Big(1-e^{\mathrm{Tr}[\Sigma(s,\,\cdot\,)\,\xi]}\Big)\,\nu(ds,d\xi)\overset{T\to\infty}{\longrightarrow}0\,. \tag{4.3.56}$$

The next inequality holds for all $0\leq t\leq T$ and for all $\xi\in S_d^+$ $\mathbb{Q}$-a.s.

$$1-e^{\mathrm{Tr}[\Sigma(t,T)\,\xi]}=1-e^{-\mathrm{Tr}[-\Sigma(t,T)\,\xi]}\leq 1\wedge\mathrm{Tr}[-\Sigma(t,T)\,\xi]\overset{(4.3.38)}{\leq}1\wedge\sqrt{\tau(0,T)}\,\mathrm{Tr}[w(t)\,\xi] \tag{4.3.57}$$

and consequently we get for all $0\leq t\leq T$ with $\tau(0,T)\geq 1$

$$\frac{1-e^{\mathrm{Tr}[\Sigma(t,T)\,\xi]}}{\tau(0,T)}\overset{(4.3.57)}{\leq}\frac{1}{\tau(0,T)}\wedge\frac{1}{\sqrt{\tau(0,T)}}\mathrm{Tr}[w(t)\,\xi]\leq 1\wedge\mathrm{Tr}[w(t)\,\xi]\leq 1\wedge\|w(t)\|\,\|\xi\|=:i(t,\xi)\,.$$

As in the proof of Proposition 4.3.14, we choose $\tau(0,T)\geq 1$ with the same justification. It holds $\mathbb{Q}$-a.s. for all $t\geq 0$

$$\begin{aligned}
\int_0^t\int_{S_d^+\setminus\{0\}}i(s,\xi)\,\nu(ds,d\xi) &= \int_0^t\int_{S_d^+\setminus\{0\}}i(s,\xi)\left(\mathbb{1}_{\{\|w(s)\|\leq 1\}}+\mathbb{1}_{\{\|w(s)\|>1\}}\right)\nu(ds,d\xi)\\
&= \int_0^t\int_{S_d^+\setminus\{0\}}i(s,\xi)\,\mathbb{1}_{\{\|w(s)\|\leq 1\}}\,\nu(ds,d\xi)\\
&\quad+\int_0^t\int_{S_d^+\setminus\{0\}}i(s,\xi)\,\mathbb{1}_{\{\|w(s)\|>1\}}\,\nu(ds,d\xi)\\
&\leq \int_0^t\int_{S_d^+\setminus\{0\}}\mathbb{1}_{\{\|w(s)\|\leq 1\}}\,(1\wedge\|\xi\|)\,\nu(ds,d\xi)\\
&\quad+\int_0^t\int_{S_d^+\setminus\{0\}}\mathbb{1}_{\{\|w(s)\|>1\}}\,\|w(s)\|\,(1\wedge\|\xi\|)\,\nu(ds,d\xi)\\
&\overset{(D.12)}{\leq} \int_0^t\int_{S_d^+\setminus\{0\}}(1\wedge\|\xi\|)\,\nu(ds,d\xi)\\
&\quad+\int_0^t\|w(s)\|\,\mathbb{1}_{\{\|w(s)\|>1\}}\,ds\int_{S_d^+\setminus\{0\}}(1\wedge\|\xi\|)\,m(d\xi)\\
&\quad+\mathrm{Tr}\left[\int_0^t\|w(s)\|\,\mathbb{1}_{\{\|w(s)\|>1\}}X_s\,ds\int_{S_d^+\setminus\{0\}}(1\wedge\|\xi\|)\,\mu(d\xi)\right]
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \int_0^t \int_{S_d^+\setminus\{0\}} (1 \wedge \|\xi\|)\, \nu(ds, d\xi) \\
& + \int_0^t \|w(s)\|\, ds \int_{S_d^+\setminus\{0\}} (1 \wedge \|\xi\|)\, m(d\xi) \\
& + \operatorname{Tr}\left[\int_0^t \|w(s)\| X_s\, ds \int_{S_d^+\setminus\{0\}} (1 \wedge \|\xi\|)\, \mu(d\xi) \right].
\end{aligned}
$$

Then, i is $\mathbb{Q}$-a.s. integrable with respect to the random measure ν on $[0,t] \times S_d^+$ using (D.4), (D.5), and Proposition C.1.9, since the processes X and $\|w(t)\| X_t, t \geq 0$, are càdlàg. That means, we can apply the dominated convergence theorem for integrable functions and get $\mathbb{Q}$-a.s. for all $t \geq 0$

$$
\int_0^t \int_{S_d^+\setminus\{0\}} \frac{e^{\operatorname{Tr}[\Sigma(s,T)\,\xi]} - 1}{\tau(0,T)}\, \nu(ds, d\xi) \overset{T\to\infty}{\longrightarrow} 0
$$

and consequently it holds $\mathbb{Q}$-a.s. for all $t \geq 0$

$$
\sup_{0\leq u\leq t} \int_0^u \int_{S_d^+\setminus\{0\}} \frac{e^{\operatorname{Tr}[\Sigma(s,T)\,\xi]} - 1}{\tau(0,T)}\, \nu(ds, d\xi) = \int_0^t \int_{S_d^+\setminus\{0\}} \frac{e^{\operatorname{Tr}[\Sigma(s,T)\,\xi]} - 1}{\tau(0,T)}\, \nu(ds, d\xi) \overset{T\to\infty}{\longrightarrow} 0.
$$

This yields

$$
\int_0^{\cdot} \int_{S_d^+\setminus\{0\}} \frac{e^{\operatorname{Tr}[\Sigma(s,T)\,\xi]} - 1}{\tau(0,T)}\, \nu(ds, d\xi) \overset{T\to\infty}{\longrightarrow} 0 \tag{4.3.58}
$$

in ucp. Finally, (4.3.51) results from Lemma C.1.8 applied for (4.3.56) and (4.3.58). □

The representation of the long-term yield with the help of the long-term drift and long-term volatility in the affine HJM setting can now be stated. Putting together Proposition 3.1.14, Lemma 4.3.7, and Propositions 4.3.12 to 4.3.15, leads to

$$
\ell_t = \ell_0 + 2\int_0^t \operatorname{Tr}\left[Q\, \mu_\infty(s)\, Q^\top \right] ds - 2\int_0^t \operatorname{Tr}\left[\sigma_\infty(s) \sqrt{X_s}\, dW_s\, Q \right] - \int_0^t \int_{S_d^+\setminus\{0\}} \operatorname{Tr}[\sigma_\infty(s)\, \xi]\, \mu^X(ds, d\xi) \tag{4.3.59}
$$

for all $t \geq 0$, where we assumed that $\lim_{T\to\infty} \sup_{0\leq s\leq t} Y(s,T)$ exists $\mathbb{Q}$-a.s. for all $t \geq 0$.

Theorem 4.3.16. *Under Assumptions III and IV, the long-term yield is given by*

$$
\ell_t = \ell_0 + 2\int_0^t \operatorname{Tr}\left[Q\, \mu_\infty(s)\, Q^\top \right] ds,\ t \geq 0, \tag{4.3.60}
$$

with $\operatorname{Tr}\left[Q\, \mu_\infty(s)\, Q^\top \right] \geq 0$ *for all* $0 \leq s \leq t$ *if* ℓ *exists in a finite form.*

Proof. Assume $0 < \|\sigma_\infty(t)\| < \infty$ for $t \leq T$. Then $\Sigma(t,T)_{ij} \in \mathrm{O}(\tau(t,T))$ for all $i, j \in \{1, \ldots, d\}$, due to (4.3.37), and therefore $\mathbb{Q}$-a.s. for all $t \geq 0$

$$
\begin{aligned}
\operatorname{Tr}\left[Q\, \mu_\infty(t)\, Q^\top \right] &\overset{(4.3.36)}{=} \sum_{i,j,k} Q_{ij} \lim_{T\to\infty} \frac{\Gamma(t,T)_{jk}}{\tau(t,T)} Q^\top_{ki} \\
&\overset{(4.3.31)}{=} \lim_{T\to\infty} \frac{1}{\tau(t,T)} \sum_{i,j,k,l,m} Q_{ij}\, \Sigma(t,T)_{jl}\, X_{lm,t}\, \Sigma(t,T)_{mk}\, Q_{ik} = \infty.
\end{aligned}
$$

It follows that the long-term volatility must vanish if the long-term yield is supposed to exist finitely. We get in this case (4.3.60). Additionally, it holds $\mathbb{Q}$-a.s. for all $t \geq 0$

$$\begin{aligned}\operatorname{Tr}\Big[Q\,\mu_\infty(t)\,Q^\top\Big] &\overset{(4.3.36)}{=} \lim_{T\to\infty}\frac{1}{\tau(t,T)}\operatorname{Tr}\Big[Q\Gamma(t,T)\,Q^\top\Big]\\ &\overset{(4.3.31)}{=} \lim_{T\to\infty}\frac{1}{\tau(t,T)}\operatorname{Tr}\Big[Q\Sigma(t,T)\,X_t\,\Sigma(t,T)\,Q^\top\Big]\\ &= \lim_{T\to\infty}\frac{1}{\tau(t,T)}\Big\|\sqrt{X_t}\,\Sigma(t,T)\,Q^\top\Big\|^2 \geq 0\,. \end{aligned} \tag{4.3.61}$$

□

Theorem 4.3.16 shows that the long-term yield is a non-decreasing process in the affine HJM framework on S_d^+, which is clear since this holds universally by the DIR theorem, stated in Subsection 3.1.2. The more interesting conclusion is that, if ℓ exists finitely, it remains the same under a change of equivalent probability measures. This follows from the representation (4.3.60) and the fact that the long-term drift depends only on the limit of the volatility and on the affine process X. In this way, we were able to generalize this result from Subsections 4.1.2 and 4.2.2 to a multifactor setting. For comparison, the reader has to refer to equation (4.1.24) in the classical HJM framework, and to Theorem 4.2.10 (i) and (ii) in the Lévy HJM setting because we do not consider negative jumps for affine processes on S_d^+.

Proposition 4.3.17. *Let $\sigma(t,T) \in \mathrm{O}\Big(\frac{1}{\sqrt{\tau(t,T)}}\Big)$ for all $t \geq 0$, i.e. $\sigma(t,T)_{ij} \in \mathrm{O}\Big(\frac{1}{\sqrt{\tau(t,T)}}\Big)$ $\mathbb{Q}$-a.s. for all $i,j \in \{1,\dots,d\}$ and all $t \geq 0$. Then, we get under Assumptions III and IV $\mathbb{Q}$-a.s. for all $t \geq 0$*

$$\operatorname{Tr}\Big[Q\,\mu_\infty(t)\,Q^\top\Big] < \infty\,.$$

Proof. If $\sigma(t,T) \in \mathrm{O}\Big(\frac{1}{\sqrt{\tau(t,T)}}\Big)$ for all $t \geq 0$, it follows $\mathbb{Q}$-a.s.

$$\begin{aligned}\operatorname{Tr}\Big[Q\,\mu_\infty(t)\,Q^\top\Big] &\overset{(4.3.61)}{=} \lim_{T\to\infty}\frac{1}{\tau(t,T)}\Big\|\sqrt{X_t}\,\Sigma(t,T)\,Q^\top\Big\|^2\\ &\overset{(4.3.10)}{=} \lim_{T\to\infty}\frac{1}{\tau(t,T)}\sum_{i,j,k,l,m} Q_{ij}\int_t^T \sigma(t,u)_{jk}du\,X_{kl,t}\int_t^T \sigma(t,u)_{lm}du\,Q^\top_{mi} < \infty\,.\end{aligned}$$

□

Proposition 4.3.18. *Let $\sigma(t,T) \in \mathrm{O}\Big(\frac{1}{\tau(t,T)}\Big)$ for all $t \geq 0$, i.e. $\sigma(t,T)_{ij} \in \mathrm{O}\Big(\frac{1}{\tau(t,T)}\Big)$ $\mathbb{Q}$-a.s. for all $i,j \in \{1,\dots,d\}$ and all $t \geq 0$. Then, we get under Assumptions III and IV $\mathbb{Q}$-a.s. for all $t \geq 0$ $\mu_\infty(t) = 0$ and therefore ℓ is constant.*

Proof. From $\sigma(t,T) \in \mathrm{O}\Big(\frac{1}{\tau(t,T)}\Big)$ for all $t \geq 0$ follows that $\Sigma(t,T)_{ij} \in \mathrm{O}(\log(\tau(t,T)))$ $\mathbb{Q}$-a.s. for all $i,j \in \{1,\dots,d\}$ and all $t \geq 0$. Hence, for all $i,j,k,l \in \{1,\dots,d\}$ and $t \geq 0$ it holds $\mathbb{Q}$-a.s.

$$\frac{1}{\tau(t,T)}\Sigma(t,T)_{ij}\,\Sigma(t,T)_{kl} \overset{T\to\infty}{\longrightarrow} 0\,. \tag{4.3.62}$$

This leads to the following $\mathbb{Q}$-a.s. for all $t \geq 0$:

$$\mu_\infty(t)_{ij} \overset{(4.3.36)}{=} \lim_{T\to\infty} \frac{\Gamma(t,T)_{ij}}{\tau(t,T)} \overset{(4.3.31)}{=} \lim_{T\to\infty} \frac{1}{\tau(t,T)} \sum_{k,l} \Sigma(t,T)_{ik} X_{kl,t} \Sigma(t,T)_{lj} \overset{(4.3.62)}{=} 0\,.$$

By (4.3.60) we get that ℓ is constant. □

Table 4.3 summarizes the results concerning the asymptotic behavior of the term structure of interest rates in an affine HJM framework on S_d^+.

Volatility curve	Long-term volatility	Long-term drift	Long-term yield	Long-term simple rate	Long-term swap rate
$\sigma(t,T) \sim \mathrm{O}(1)$	$\lVert\sigma_\infty\rVert > 0$	$\mathrm{Tr}\big[Q\mu_\infty(t)Q^\top\big] = \infty$	$\ell = \infty$ infinite	$L = \infty$ infinite	$0 < R < \infty$ non-monotonic
$\sigma(t,T) \sim \mathrm{O}(\tau(t,T))$	$\lVert\sigma_\infty\rVert > 0$	$\mathrm{Tr}\big[Q\mu_\infty(t)Q^\top\big] = \infty$	$\ell = \infty$ infinite	$L = \infty$ infinite	$0 < R < \infty$ non-monotonic
$\sigma(t,T) \sim \mathrm{O}\Big(\frac{1}{\tau(t,T)}\Big)$	$\lVert\sigma_\infty\rVert = 0$	$\mathrm{Tr}\big[Q\mu_\infty(t)Q^\top\big] = 0$	$0 \leq \ell < \infty$ constant	$0 \leq L \leq \infty$ non-negative	$0 \leq R < \infty$ non-monotonic
$\sigma(t,T) \sim \mathrm{O}\Big(\sqrt{\frac{1}{\tau(t,T)}}\Big)$	$\lVert\sigma_\infty\rVert = 0$	$0 < \mathrm{Tr}\big[Q\mu_\infty(t)Q^\top\big] < \infty$	$0 < \ell \leq \infty$ non-decreasing	$L = \infty$ infinite	$0 < R < \infty$ non-monotonic

Table 4.3.: Asymptotic behavior of the term structure of interest rates in the affine HJM framework on S_d^+. Own presentation.

4.4. Asymptotic Behavior of Interest Rates in a Flesaker-Hughston Term Structure

In this section we compute the different long-term interest rates in the Flesaker-Hughston interest rate model. The name is derived from Flesaker and Hughston who came up with this particular methodology in 1996 based on the desire to model interest rates with a high tractability and a complete absence of negative rates, see [91]. After the introduction of the model in [91], it was further developed in [139] and [153]. To ensure the non-negativity of the interest rates, the authors introduce a strictly positive supermartingale A to represent the state price density. Besides the mentioned main advantages of this approach that it specifies non-negative interest rates only and has a high degree of tractability, another appealing feature is, that besides relatively simple models for bond prices, short and forward rates, there are closed-form formulas for caps, floors and swaptions available. In the following, we first shortly outline in Subsection 4.4.1 the generalized Flesaker-Hughston term structure model that is explained in detail in [153], and then calculate the long-term interest rates for two specific choices of A, see Subsection 4.4.2.

We mainly use [24] and [153] for this section.

4.4.1. Flesaker-Hughston Term Structure

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{Q})$ be the filtered probability space defined at the beginning of Chapter 4, where $\mathbb{Q}$ is the risk-neutral measure. We assume that there exists a strictly positive $\mathbb{Q}$-supermartingale $A := (A_t)_{t \geq 0}$ such that for any contingent claim C the price at t for the cashflow in T, denoted by $\Pi(t,T)$, can be represented $\mathbb{Q}$-a.s. as (cf. Remark 3 of [153])

$$\Pi(t,T) = \frac{\mathbb{E}^{\mathbb{Q}}[A_T C_T \,|\, \mathscr{F}_t]}{A_t}\,,\ 0 \leq t \leq T,$$

for all $T \geq 0$. Then, for the T-bond price in $t \leq T$ it follows, due to $P(T,T) = 1$, that $\mathbb{Q}$-a.s. for all $T \geq 0$

$$P(t,T) = \frac{\mathbb{E}^{\mathbb{Q}}[A_T \,|\, \mathscr{F}_t]}{A_t}\,,\ 0 \leq t \leq T. \tag{4.4.1}$$

Consequently, we get for the yield on $[t,T]$ that $\mathbb{Q}$-a.s.

$$Y(t,T) \overset{(2.2.8)}{=} -\frac{\log P(t,T)}{\tau(t,T)} \overset{(4.4.1)}{=} \frac{\log A_t}{\tau(t,T)} - \frac{1}{\tau(t,T)} \log \mathbb{E}^{\mathbb{Q}}[A_T \,|\, \mathscr{F}_t]\,. \tag{4.4.2}$$

Next, the simple OIS spot rate for the time interval $[t,T]$ is $\mathbb{Q}$-a.s.

$$L(t,T) \overset{(2.2.5)}{=} \frac{1}{\tau(t,T)} \left(\frac{1}{P(t,T)} - 1 \right) \overset{(4.4.1)}{=} \frac{1}{\tau(t,T)} \left(\frac{A_t}{\mathbb{E}^{\mathbb{Q}}[A_T \,|\, \mathscr{F}_t]} - 1 \right). \tag{4.4.3}$$

The n-finite bond sum in the Flesaker-Hughston methodology is $\mathbb{Q}$-a.s.

$$S_n(t) \overset{(2.4.17)}{=} \delta \sum_{i=1}^{n} P(t,T_i) \overset{(4.4.1)}{=} \frac{\delta}{A_t} \mathbb{E}^{\mathbb{Q}} \left[\sum_{i=1}^{n} A_{T_i} \,\middle|\, \mathscr{F}_t \right],\ t \geq 0, \tag{4.4.4}$$

and accordingly for the OIS rate it holds $\mathbb{Q}$-a.s.

$$R(t,T) \overset{(2.4.26)}{=} \frac{1 - P(t,T_N)}{S_N(t)} \overset{(4.4.4)}{=} \frac{\mathbb{E}^{\mathbb{Q}}[1 - A_{T_N} \,|\, \mathscr{F}_t]}{\delta \mathbb{E}^{\mathbb{Q}} \left[\sum_{i=1}^{N} A_{T_i} \,\middle|\, \mathscr{F}_t \right]} \tag{4.4.5}$$

for all $t \geq 0$.

4.4.2. Long-Term Interest Rates in a Flesaker-Hughston Term Structure

For the analysis of the term structure's asymptotic behavior in the Flesaker-Hughston methodology, we consider two specific cases of the state price density. These characterizations of A were also the choices in Section 2.3 of [153].

Example (i): The supermartingale A is given by

$$A_t = f(t) + g(t) M_t\,,\ t \geq 0,$$

where $f,g:\mathbb{R}_+\to\mathbb{R}_+$ are strictly positive decreasing functions and M is a strictly positive martingale defined on $(\Omega,\mathscr{F},\mathbb{F},\mathbb{Q})$, with $M_0=1$. In the sequel, we work with a càdlàg version of M. Then, it follows that for all $0\le t\le T$

$$P(t,T)=\frac{f(T)+g(T)M_t}{f(t)+g(t)M_t}. \tag{4.4.6}$$

The initial yield curve can easily be fitted by choosing strictly positive decreasing functions f and g in such a way that

$$P(0,T)=\frac{f(T)+g(T)}{f(0)+g(0)}$$

for all $T\ge 0$.

We assume that the infinite sums of f and g exist so that we are able to calculate the long-term yield and swap rate. That means

$$F:=\sum_{i=1}^{\infty}f(T_i)<\infty,\qquad G:=\sum_{i=1}^{\infty}g(T_i)<\infty, \tag{4.4.7}$$

with $F,G\in\mathbb{R}_+$.

Then, it follows by (4.4.7) that the long-term bond price vanishes, i.e. $P=0$ because of $\lim_{t\to\infty}f(t)=\lim_{t\to\infty}g(t)=0$. In [91], the vanishing long-term bond price is assumed, whereas in this case it is a consequence of (4.4.7).

Thus, the infinite bond sum is for all $t\ge 0$

$$S_\infty(t)=\delta\frac{F+GM_t}{f(t)+g(t)M_t}\quad \mathbb{Q}\text{-a.s.}$$

since

$$\begin{aligned}\sup_{0\le s\le t}&\left|\sum_{i=1}^{n}\frac{f(T_i)+g(T_i)M_s}{f(s)+g(s)M_s}-\frac{F+GM_s}{f(s)+g(s)M_s}\right|\\ &=\sup_{0\le s\le t}\left|\frac{M_s}{f(s)+g(s)M_s}\left(\sum_{i=1}^{n}g(T_i)-G\right)+\frac{\sum_{i=1}^{n}f(T_i)-F}{f(s)+g(s)M_s}\right|\\ &\to 0\quad \mathbb{Q}\text{-a.s.}\end{aligned}$$

for all $t\ge 0$, hence in probability because

$$\sup_{0\le s\le t}\frac{M_s}{f(s)+g(s)M_s}\le\sup_{0\le s\le t}\frac{M_s}{g(s)M_s}\le\frac{1}{g(t)}<\infty.$$

Then, by Proposition 3.1.23 it holds $\mathbb{Q}$-a.s.

$$R_t=\frac{f(t)+g(t)M_t}{\delta\left(F+GM_t\right)},\ t\ge 0. \tag{4.4.8}$$

By Corollary 3.2.19, we have that $\ell_t \geq 0$ $\mathbb{Q}$-a.s. for all $t \geq 0$ since the long-term swap rate is strictly positive due to (4.4.8) and the long-term bond price vanishes. Further, it can be specified for all $t \geq 0$ as

$$\ell \overset{(3.1.12)}{=} \lim_{T\to\infty} Y(\,\cdot\,,T) \underset{(4.4.6)}{\overset{(2.2.8)}{=}} -\lim_{T\to\infty} \frac{1}{\tau(\,\cdot\,,T)} \log(f(T)+g(T)M_\cdot) \text{ in ucp.} \tag{4.4.9}$$

Concerning the long-term simple rate, it as well holds by Corollary 3.2.19 that $L_t \geq 0$ $\mathbb{Q}$-a.s. for all $t \geq 0$. Moreover, we see by (4.4.6) and the definitions of long-term simple rate (3.1.15) and OIS simple spot rate (2.2.5) that L only exists if $f(t) \in O\left(\frac{1}{t}\right)$ as well as $g(t) \in O\left(\frac{1}{t}\right)$. Otherwise it explodes.

We now assume that $f(t) = \exp(-\alpha\tau(0,t))$, $g(t) = \exp(-\beta\tau(0,t))$ with $0 < \alpha < \beta$. Then f and g are decreasing strictly positive functions and the ratio test shows that the infinite sums of f and g exist. Put

$$\alpha_\infty := \sum_{i=1}^{\infty} \exp(-\alpha\tau(0,T_i))\,, \qquad \beta_\infty := \sum_{i=1}^{\infty} \exp(-\beta\tau(0,T_i))\,.$$

Therefore, all required conditions are fulfilled and the long-term swap rate as well as the long-term yield can be computed as follows:

$$R_t \overset{(4.4.8)}{=} \frac{\exp(-\alpha\tau(0,t)) + \exp(-\beta\tau(0,t))M_t}{\delta\left(\alpha_\infty + \beta_\infty M_t\right)},\ t \geq 0,$$

and

$$\begin{aligned}
\ell &\overset{(4.4.9)}{=} -\lim_{T\to\infty} \frac{1}{\tau(\,\cdot\,,T)} \log(f(T)+g(T)M_\cdot) \\
&= -\lim_{T\to\infty} \frac{1}{\tau(\,\cdot\,,T)} \log(\exp(-\alpha T)\left(1+\exp\left(-\left(\beta-\alpha\right)\tau(0,T)\right)M_\cdot\right)) \\
&= \alpha + \lim_{T\to\infty} \frac{1}{\tau(\,\cdot\,,T)} \log(1+\exp(-\left(\beta-\alpha\right)\tau(0,T))M_\cdot) = \alpha \text{ in ucp.}
\end{aligned}$$

Regarding the long-term simple rate, it follows by Corollary 3.2.2 that $L(\,\cdot\,,T_n) \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. This result can also be obtained by direct computation since for all $t \geq 0$

$$\sup_{0\leq s\leq t} \frac{\exp(-\alpha\tau(0,s)) + \exp(-\beta\tau(0,s))M_s}{\tau(0,T)\exp(-\alpha\tau(0,T)) + \tau(0,T)\exp(-\beta\tau(0,T))M_s} \overset{T\to\infty}{\longrightarrow} \infty \ \mathbb{Q}\text{-a.s.},$$

i.e. in probability, since M is càdlàg.

Example (ii): The second specific choice of the supermartingale A is

$$A_t = \int_t^\infty \phi(s)M(t,s)\,ds\,,\ t \geq 0\,,$$

where for every $s > 0$ the process $M(t,s)\,, t \leq s$, is a strictly positive martingale on $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{Q})$ with $M(0,s) = 1$ such that $\int_0^\infty \phi(s)M(t,s)\,ds < \infty$ $\mathbb{Q}$-a.s. for all $t \geq 0$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly positive continuous function. Then, we get by (4.4.1) that the T-bond price in $t \leq T$ is $\mathbb{Q}$-a.s.

$$P(t,T) = \frac{\int_T^\infty \phi(s) M(t,s)\, ds}{\int_t^\infty \phi(s) M(t,s)\, ds}, \tag{4.4.10}$$

for all $T \geq 0$. By differentiation of the zero-coupon bond price with respect to the maturity date, we see that the initial term structure satisfies $\phi(t) = -\frac{\partial P(0,t)}{\partial t}$ (cf. equation (6) of [91]).

Obviously, it follows by (4.4.10) that $P = 0$. Furthermore, let us define $Q_n := (Q_n(t))_{t \geq 0}$ for all $n \geq 0$ with

$$Q_n(t) := \sum_{i=1}^{n} \int_{T_i}^{\infty} \phi(s) M(t,s)\, ds$$

and assume that for $Q := (Q(t))_{t \geq 0}$ we have

$$Q(t) := \sum_{i=1}^{\infty} \int_{T_i}^{\infty} \phi(s) M(t,s)\, ds < \infty$$

for all $t \geq 0$, and that $Q_n \stackrel{n\to\infty}{\longrightarrow} Q$ in ucp. Then, we get $S_n \stackrel{n\to\infty}{\longrightarrow} S_\infty$ in ucp and hence the convergences of the different long-term interest rates hold also in ucp. Thus, according to Proposition 3.1.23, the long-term swap rate is $\mathbb{Q}$-a.s. for all ≥ 0

$$R_t = \frac{\int_t^\infty \phi(s) M(t,s)\, ds}{\delta \sum_{i=1}^\infty \int_{T_i}^\infty \phi(s) M(t,s)\, ds}. \tag{4.4.11}$$

For the long-term yield and long-term simple rate hold that both rates are non-negative according to Corollary 3.2.2 and they can be specified the following way:

$$\ell = -\lim_{T\to\infty} \frac{1}{\tau(\cdot,T)} \log\left(\int_T^\infty \phi(s) M(\cdot,s)\, ds \right)$$

and

$$L = \lim_{T\to\infty} \frac{1}{\tau(\cdot,T)} \left(\frac{\int_\cdot^\infty \phi(s) M(\cdot,s)\, ds}{\int_T^\infty \phi(s) M(\cdot,s)\, ds} - 1 \right).$$

4.5. Asymptotic Behavior of Interest Rates in a Linear-Rational Term Structure

The last approach we want to use for the investigation of long-term interest rates is the linear-rational term structure methodology. This methodology was just introduced and studied in 2014 by Filipović and Trolle in [84] and is closely related to the Flesaker-Hughston model, what is shown at the end of Subsection 4.5.1, where we present the methodology's general results for interest rate modeling. It offers some appealing features for the term structure which are discussed in Section 1.1, but also the results on its asymptotic behavior are beneficial since the long-term swap rate and the long-term yield both exist finitely. The computations regarding the long-term interest rates are stated in Subsection 4.5.2.

This section is based on [24] and [84].

4.5.1. Linear-Rational Term Structure

Let us again consider the probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{Q})$ with filtration $\mathbb{F}$ and risk-neutral measure $\mathbb{Q}$ which is defined at the beginning of Chapter 4. We assume the existence of a state price density, i.e. of a positive adapted process $A := (A_t)_{t\geq 0}$ on this probability space such that the price $\Pi(t,T)$ at time t of any time T cashflow C_T is given by

$$\Pi(t,T) = \frac{\mathbb{E}^{\mathbb{Q}}[A_T C_T | \mathscr{F}_t]}{A_t}\,,\ 0 \leq t \leq T, \tag{4.5.1}$$

for all $T \geq 0$. Further, the state price density is supposed to be driven by a multivariate factor process $X := (X_t)_{t\geq 0}$ which takes values on the state space $E \subseteq \mathbb{R}^d$, $d \geq 1$. For this process holds

$$dX_t = k\,(\theta - X_t)\,dt + dM_t\,,\ t \geq 0, \tag{4.5.2}$$

for some $k \in \mathbb{R}_+$, $\theta \in \mathbb{R}^d$, and some martingale $M := (M_t)_{t\geq 0}$ on E. We assume to work with the càdlàg version of X and by solving (4.5.2) we get

$$\mathbb{E}^{\mathbb{Q}}[X_T | \mathscr{F}_t] = \theta + (X_t - \theta)\exp(-k\tau(t,T))\,. \tag{4.5.3}$$

The linear-rational methodology is primarily characterized by the fact that A is defined as

$$A_t := \exp(-\alpha\tau(0,t))\left(\phi + \psi^\top X_t\right)\,,\ t \geq 0\,, \tag{4.5.4}$$

with $\phi \in \mathbb{R}$ and $\psi \in \mathbb{R}^d$ such that $\phi + \psi^\top x > 0$ for all $x \in E$, and $\alpha \in \mathbb{R}$. It holds $\alpha = \sup_{x\in E} \frac{k\,\psi^\top(\theta - x)}{\phi + \psi^\top x}$ to guarantee non-negative short rates (cf. equation (6) of [84]). Then, by (4.5.1), (4.5.3), (4.5.4), and the fact that $P(T,T) = 1$ $\mathbb{Q}$-a.s. for all $T \geq 0$, it follows

$$P(t,T) = \frac{\left(\phi + \psi^\top\theta\right)\exp(-\alpha\tau(t,T)) + \psi^\top(X_t - \theta)\exp(-(\alpha{+}k)\,\tau(t,T))}{\phi + \psi^\top X_t} \tag{4.5.5}$$

$\mathbb{Q}$-a.s. for all $0 \leq t \leq T$. Consequently, we get for the yield on $[t,T]$ with $y := \phi + \psi^\top\theta$ that it holds $\mathbb{Q}$-a.s.

$$\begin{aligned}
Y(t,T) &\overset{(2.2.8)}{=} -\frac{\log P(t,T)}{\tau(t,T)}\\
&\overset{(4.5.5)}{=} -\frac{1}{\tau(t,T)}\log\Big[y\exp(-\alpha\tau(t,T)) + \psi^\top(X_t - \theta)\exp(-(\alpha + k)\,\tau(t,T))\Big]\\
&= \alpha - \frac{1}{\tau(t,T)}\log\Big[y + \psi^\top(X_t - \theta)\exp(-k\,\tau(t,T))\Big]\,.
\end{aligned} \tag{4.5.6}$$

The simple OIS spot rate for the time interval $[t,T]$ is then $\mathbb{Q}$-a.s.

$$\begin{aligned}
L(t,T) &\overset{(2.2.5)}{=} \frac{1}{\tau(t,T)}\left(\frac{1}{P(t,T)} - 1\right)\\
&\overset{(4.5.5)}{=} \frac{\phi + \psi^\top X_t - y\exp(-\alpha\tau(t,T)) - \psi^\top(X_t - \theta)\exp(-(\alpha + k)\,\tau(t,T))}{\tau(t,T)\,[y\exp(-\alpha\tau(t,T)) + \psi^\top(X_t - \theta)\exp(-(\alpha + k)\,\tau(t,T))]}\,.
\end{aligned} \tag{4.5.7}$$

Lastly, the corresponding OIS rate can be easily calculated by computing the different bond prices via the formula (4.5.5) and putting the results in (2.4.26).

Note, that the Flesaker-Hughston model can be transferred to the linear-rational term structure methodology and vice versa. This is shown in Section 2.4 of [84].

4.5.2. Long-Term Interest Rates in a Linear-Rational Term Structure

In this subsection we examine the asymptotic behavior of the linear-rational term structure by calculating the different long-term interest rates. For this matter, let us begin with the infinite bond sum process S_∞. We first see immediately as a consequence of (4.5.5) that the long-term bond price vanishes, i.e. it holds $P = 0$. Further, we know by the ratio test that for all $t \geq 0$

$$\alpha_\infty(t) := \sum_{i=1}^{\infty} \exp(-\alpha\,\tau(t,T_i)) < \infty, \quad \beta_\infty(t) := \sum_{i=1}^{\infty} \exp(-(\alpha+k)\;\tau(t,T_i)) < \infty.$$

Then for all $t \geq 0$ $\mathbb{Q}$-a.s.

$$S_\infty(t) \overset{(4.5.5)}{=} \delta \frac{\left(\phi + \psi^\top\theta\right)\alpha_\infty(t) + \psi^\top (X_t - \theta)\,\beta_\infty(t)}{\phi + \psi^\top X_t} < \infty. \tag{4.5.8}$$

By Proposition 3.1.23, we have that for all $t \geq 0$ $\mathbb{Q}$-a.s.

$$R_t \overset{(4.5.8)}{=} \frac{\phi + \psi^\top X_t}{\delta\left(\left(\phi + \psi^\top\theta\right)\alpha_\infty(t) + \psi^\top (X_t - \theta)\,\beta_\infty(t)\right)}. \tag{4.5.9}$$

Finally, we want to know the form of the long-term yield in the linear-rational term structure methodology. It holds for all $t \geq 0$ $\mathbb{Q}$-a.s.

$$\log\left[y + \psi^\top \left(\sup_{0\leq s\leq t} X_s - \theta\right)\exp(-k\,\tau(t,T))\right] \geq \log\left[y + \psi^\top\,(X_t - \theta)\exp(-k\,\tau(t,T))\right]$$

as well as

$$\log\left[y + \psi^\top\,(X_t - \theta)\exp(-k\,\tau(t,T))\right] \geq \log\left[y + \psi^\top \left(\inf_{0\leq s\leq t} X_s - \theta\right)\exp(-k\,\tau(0,T))\right].$$

It follows $\mathbb{Q}$-a.s. for $t \geq 0$

$$\sup_{0\leq s\leq t} |\alpha - Y(s,T)| \overset{(4.5.6)}{=} \sup_{0\leq s\leq t} \frac{1}{\tau(s,T)} \left|\log\left[y + \psi^\top\,(X_s - \theta)\exp(-k\,\tau(s,T))\right]\right| \overset{T\to\infty}{\longrightarrow} 0\,.$$

Therefore $\ell = \alpha$. Then, the long-term simple rate L depends on α: if $\alpha > 0$, then L explodes due to Corollary 3.2.2, if $\alpha < 0$, then L vanishes due to Proposition 3.2.4, and L is non-negative if $\alpha = 0$.

Appendix

A. Risk Classes

In general, risk can be described as the volatility of unexpected outcomes, or respectively, as a positive probability of losing something of value, whereby this can mean tangible goods as well as physical health or social status, for instance (cf. Section 1.1 of [120]). Risks occur in several forms, depending on the cause and on the affected party, where we are interested in the specific risk enterprises have to face. There are two broad types of risk for companies, business risks and non-business risks with the latter being further classified into event and financial risks (cf. Chapter 4 of [144]). Business risks cover all risks of potential losses corresponding to unexpected changes in the competitive market as well as all risks companies willingly assume to create competitive advantages and to add value, while event risks are not under the control of enterprises and arise either largely or completely exogenously (cf. Section 1.3 of [47]). We are interested in financial risk that can be categorized into the following main classes: credit risk, market risk, liquidity risk, and operational risk (cf. Section 1.4 of [120]). All of these risks can be further classified to more granular risk types and besides these classes, there are other risk classes that cannot be matched to one of the four mentioned, such as reputational risk. For a complete discussion of all different risk types, we refer to Section 1.5 of [159] since in this appendix only a selection of financial risk classes are explained that are mentioned throughout the thesis to provide a better understanding of the presented fixed income basis.

First, let us consider credit risk that can be defined the following way (cf. Section I.2 of [18]).

Definition A.1. *Credit risk is the potential that one or more borrowers will fail to fulfill its contractual obligations. It occurs when at least one counterpart is unable to pay or cannot pay on time or refuses to pay, and encompasses default risk, exposure risk, and recovery risk.*

For banks, credit risk is typically the largest type of risk regarding the potential loss amount, and it consists of pre-settlement and settlement risk in terms of the timing (cf. Section 1.5.1 of [159]). Pre-settlement risk is the risk that a loss occurs during the life of the transaction due the counterparty's default, whereas settlement risk describes the risk of default during the settlement process. Especially, when considering counterparty risk, that can be understood as a part of credit risk, the different aspects of pre-settlement and settlement risk play an important role (cf. Section 3.1.2 of [101]). We define counterparty risk according to Section 1 of [104].

Definition A.2. *Counterparty risk is defined as the risk that a party will not fulfill its contractual obligations, whereby the non-fulfillment can mean default, late delivery, or any other failure to complete the agreed transaction.*

Counterparty risk arises from two broad financial product classes, namely OTC derivatives and securities financing transactions such as repos. These product classes are the origin of the main aspects for differentiating counterparty risk from traditional credit risk, which can generally be thought of as lending risk. During the lending period the amount at risk is known to a degree of certainty, whereas the value of an OTC contract in the future is highly uncertain. Further, counterparty risk is bilateral in the sense that the contract value can be positive or negative, but only one party faces lending risk (cf. Section 3.1.1 of [101]).

As well a special kind of credit risk is the interbank risk that describes lending risk between banks. Let us define this risk class formally as in Section 1 of [87].

Definition A.3. *Interbank risk is the risk of direct or indirect losses that result from lending between banks in the interbank money market.*

Another main class of financial risk is market risk that denotes the risk of losses due to movements in market prices. The classical sources of market risk are unexpected and large adverse market movements in equity prices, FX rates, commodity prices and interest rates (cf. Section 1.4.1 of [120]). Regarding these risk drivers, we are only interested in interest rate risk that is defined according to Section 1.5.2 of [159].

Definition A.4. *Interest rate risk is the potential loss of an investment's value as a result of a change in an interest rate relationship.*

The mentioned interest rate relationship can be the absolute level of interest rates, the spread between two rates, or the shape of the yield curve.

The last risk class mattering for this thesis is liquidity risk, where it has to be separated between market and funding liquidity risk. Market liquidity risk, also known as asset liquidity risk, arises when an asset cannot be sold due to lack of liquidity in the market. On the other hand, funding liquidity risk denotes the risk that liabilities cannot be met when they fall due or only to a high price. Market and funding liquidity risk interact when a portfolio contains illiquid assets that have to be sold to a price below the fair value in order to pay all liabilities. Throughout the thesis, when speaking of liquidity risk, we mean funding liquidity risk. It is defined as in Section 1.4.2 of [120].

Definition A.5. *Liquidity risk is the risk of not being able to meet payment obligations or only being able to meet them to an uneconomic price.*

B. Credit Spreads

This appendix summarizes different kinds of spreads that widened during the 2008 financial crisis and were typically negligible before it. As explained in Section 2.1, these spreads, defined hereafter, display credit, counterparty, and liquidity risk that increased in the course of the crisis and that ultimately led to multi-curve interest rate modeling.

The information is gathered mainly from [80], [87], [110], [122], [128], [138], and [157]. Be aware that this is not a complete list of credit spreads, only the ones mentioned in the course of the thesis are explained. For a full discussion on credit spreads refer to Chapter 3 and 4 of [122].

B.1. IBOR-OIS Spread

IBORs such as the LIBOR for the USD market, the EURIBOR for the EUR market, or the STIBOR for the SEK market as well as OIS rates are explained in Section 2.1. The spread between these two interest rates is defined as follows (cf. Section 9.2 of [110]).

Definition B.1.1. *The IBOR-OIS spread is the amount by which the IBOR exceeds the OIS rate, where the respective tenors and money markets of the rates coincide.*

This credit spread is a key indicator for health or stress in the banking system "because it reflects what banks believe is the risk of default associated with lending to other banks" (p.1, [157]). Consider a German bank that borrows money from the *Deutsche Bundesbank* at the 3M OIS rate and then lends it at the 3M EURIBOR to another bank which is a member of the EURIBOR bank panel. Hence, the EURIBOR-OIS spread is the compensation for the lender of the EURIBOR for taking the risk that the borrower defaults during a period of three months. We can summarize it to the statement that the larger the IBOR-OIS spread, the more reluctant are banks to lend money to each other.

B.2. TED Spread

Another measure of credit risk that can be seen as a complementary to the IBOR-OIS spread is the so-called TED spread. The "T" in the acronym TED comes from "Treasury bill", and the "ed" is derived from "Eurodollars". These instruments are the underlyings of futures that form the basis of this spread and they are defined according to Chapter 1 and Chapter 12 of [80] and Chapter 6 of [128], respectively.

Definition B.2.1. *A Treasury bill is a debt instrument that is issued by the US Department of the Treasury to finance the national debt of the US. They mature in one year or less and do not pay interest prior to maturity. Treasury bill futures contracts are cash-settled derivatives with* *a Treasury bill being the respective underlying and having a maturity of three months and a principal value of one million USD.*

Definition B.2.2. *An eurodollar is a time deposit denominated in USD held in a bank outside the US. Eurodollar futures contracts are cash-settled derivatives on the interest rate paid on those deposits with a maturity of three months and one million USD principal value.*

To execute a TED spread, an investor buys Treasury bills and sells Eurodollars in equal amounts. In 1981 these spreads were originally constructed using 91-day Treasury bill futures versus 90-day Eurodollar futures contracts. Nowadays, the TED spread is defined as follows, due to the fact that the Chicago Mercantile Exchange dropped Treasury bill futures in 1987.

Definition B.2.3. *For a particular money market, the TED spread is the difference between the respective 3M IBOR and the spot rate on top-rated government bonds with maturity of three months.*

In the US, the TED spread is defined as the difference of the 3M LIBOR and the 3M Treasury bill rate, whereas in the EUR money market it equals the difference of the 3M EURIBOR and the spot rate on AAA-rated government bonds with maturity of three months. Thus, this spread is a measure of the required risk premium for a bank to lend to another bank instead of lending to the government. Its degree is obviously a sign of perceived counterparty risk, what explains the strong positive correlation to the IBOR-OIS spread.

B.3. Tenor Basis Spread

Basis swaps are a particular type of IRSs that can be defined according to Section 2.4.6 of [138].

Definition B.3.1. *A basis swap is an IRS where two floating payments are exchanged.*

It can involve one or two currencies, whereby in the latter case the instrument is called CCS, which is explained in Section B.4. Considering one currency, a basis swap can be a contract where rates of two different indices are exchanged or a contract where the payments are linked to the same index of different tenors, what is called a tenor basis swap. Also, a combination of these two cases is possible. Parties normally enter into a basis swap to limit interest rate risk that they face by having differing lending and borrowing rates, or for mitigating counterparty risk emerging from longer tenors (cf. Definitions A.2 and A.4). For example, if a bank lends money tied to LIBOR but borrows money based upon the Treasury bill rate, it could enter into a basis swap that exchanges these two rates to eliminate this interest rate risk. A tenor basis swap could be useful for an investor to switch for instance from EURIBOR payments based on 6 months semiannually to 3 months quarterly to get rid of some risk arising from the possibility that the counterparty defaults in the period between 3 and 6 months. We define the tenor basis spread as follows.

Definition B.3.2. *The tenor basis spread between tenors A and B with $A \neq B$ is the difference in bps between investing in a basis swap with tenor A to investing in a basis swap with tenor B.*

Accordingly, the tenor basis spread is a measure of counterparty risk. This is shown by the fact that in a default-free environment, a tenor basis swap should trade flat (cf. Section 1 of [148]). In Section 5 of [136], we can find some explanations why lending at a larger tenor is associated with more counterparty risk than lending at a shorter tenor with a rolling strategy. Tenor basis spreads are positively correlated to both IBOR-OIS spreads and TED spreads as to some extent they all measure counterparty risk.

B.4. Cross Currency Basis Spread

As mentioned in Section B.3, CCSs are a special kind of basis swap which we define as in Section 2.4.5 of [138].

Definition B.4.1. *A CCS is a contract, where two parties agree to exchange interest payments and principal on loans denominated in two different currencies.*

Let us consider an example of an EUR/USD CCS between two parties A and B. At initiation, a FX rate S between EUR and USD is fixed, the EUR/USD spot rate on the respective date, a notional amount X is determined as well as the maturity of the contract and the floating rates that will be exchanged. Then, the CCS could look as follows:

- At start: A borrows $X \cdot S$ USD from B, and lends X EUR to B.
- During the contract: A receives the EURIBOR 3M + α from B, and pays the LIBOR 3M to B every three months.
- At maturity: A returns $X \cdot S$ USD to B, and receives X EUR back from B.

Companies enter into a CCS to profit from comparative advantages since another party could have better conditions on foreign money markets. In our example parties A and B would enter this respective swap if A is in a need for USD and B needs EUR, both over the same time period, and A is an European company with better access to the EUR debt market and is able to get more favorable conditions on an EUR loan than B, an US-located company, and vice versa.

Considering a CCS in a default-free setting, where for instance the EONIA rate is exchanged with the FF rate since overnight rates serve as good proxy for risk-free rates, as explained in Section 2.2. It is shown in Section 1 of [148] that this swap should trade flat, but market quoted CCSs exchange IBORs that are not default-free. Hence, the cross currency basis spread, the α in the example, is as well a measure for counterparty risk and we define it according to Section 2.4.6 of [138].

Definition B.4.2. *The market swap level measured in bps of a CCS is called the cross currency basis spread.*

For a better understanding of the cross currency basis spread's origin we decompose the swap of the example into the following parts, including two other parties C and D:

- At start: A borrows $X \cdot S$ USD from B, and lends X EUR to B.

- During the contract:
  - A receives the EONIA rate from B, and pays the FF rate to B every three months.
  - A receives the EURIBOR 3M + β from C, and pays the EONIA rate to C every three months.
  - B receives the LIBOR 3M + γ from D, and pays the FF rate to D every three months.
- At maturity: A returns $X \cdot S$ USD to B, and receives X EUR back from B.

That means, by adding two basis swap contracts with parties C and D, we were able to transform the original CCS in a default-free one plus two tenor basis swaps. Therefore, it is apparent that the cross currency basis spread derives from the difference between local tenor basis spreads (cf. Definition B.3.2). The spread of the example's swap α would be $\beta - \gamma$ in the presented decomposition of the swap. It is positive in the case that the EURIBOR 3M has more credit risk than the LIBOR 3M, and negative if the LIBOR 3M carries more credit risk.

C. Uniform Convergence on Compacts in Probability

In this appendix we present the concept of uniform convergence on compacts in probability (ucp convergence), which is used frequently throughout the thesis since it is the basic approach of defining convergence of stochastic processes. Let us consider a stochastic basis $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with the filtration $\mathbb{F} := (\mathscr{F}_t)_{t\geq 0}$ with $\mathscr{F}_\infty \subseteq \mathscr{F}$ satisfying the usual hypothesis. All stochastic processes mentioned in this chapter are adapted to $\mathbb{F}$.

This chapter is a more comprehensive version of Section A of [24].

C.1. UCP Convergence

In Chapter II, Section 4 of [149] the definition of ucp convergence is provided. We recall it for the reader's convenience.

Definition C.1.1. *A sequence of processes $(X^n)_{n\in\mathbb{N}}$ converges to a process X uniformly on compacts in probability if, for each $t \geq 0$, $\sup_{0\leq s\leq t} |X_s^n - X_s|$ converges to 0 in probability, i.e. for all $\varepsilon > 0$ and all $t \geq 0$ it holds*

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |X_s^n - X_s| > \varepsilon\right) \stackrel{n\to\infty}{\longrightarrow} 0. \tag{C.1.1}$$

In this case, we write $X^n \stackrel{n\to\infty}{\longrightarrow} X$ in ucp.

Lemma C.1.2. *From $X^n \stackrel{n\to\infty}{\longrightarrow} X$ in ucp, it follows that $X_t^n \stackrel{n\to\infty}{\longrightarrow} X_t$ in probability for all $t \geq 0$.*

Proof. Let $X^n \stackrel{n\to\infty}{\longrightarrow} X$ in ucp. Then (C.1.1) holds for all $t \geq 0$ and all $\varepsilon > 0$. Now, let us fix a time interval $[0,t]$ for $t > 0$, take $u \in [0,t]$ and an arbitrary $\varepsilon > 0$. Since

$$\{\omega \in \Omega : |X_u^n - X_u| > \varepsilon\} \subseteq \left\{\omega \in \Omega : \sup_{0\leq s\leq t} |X_s^n - X_s| > \varepsilon\right\} \tag{C.1.2}$$

we get that

$$\mathbb{P}(|X_u^n - X_u| > \varepsilon) \stackrel{\text{(C.1.2)}}{\leq} \mathbb{P}\left(\sup_{0\leq s\leq t} |X_s^n - X_s| > \varepsilon\right) \stackrel{n\to\infty}{\longrightarrow} 0 \tag{C.1.3}$$

due to (C.1.1). As we can do this for any interval, it holds $X_t^n \stackrel{n\to\infty}{\longrightarrow} X_t$ in probability for all $t \geq 0$. □

If we admit to consider processes which may take values in $\mathbb{R} \cup \{-\infty, +\infty\}$, then the next proposition shows that the limit X in ucp of processes $X^n := (X_t^n)_{t\geq 0}$ with $X_t^n < \infty$ $\mathbb{P}$-a.s. for all $n \in \mathbb{N}$ and all $t \geq 0$, is also $\mathbb{P}$-a.s. real-valued.

Proposition C.1.3. *Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of processes with $X^n := (X_t^n)_{t\geq 0}$ such that for all $n \in \mathbb{N}$ it holds $X_t^n < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$. If $X^n \overset{n\to\infty}{\longrightarrow} X$ in ucp for a process X, then it holds that $X_t < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. From $X^n \overset{n\to\infty}{\longrightarrow} X$ in ucp, it follows that for all $t \geq 0$ it holds $\sup_{0\leq s\leq t} |X_s^n - X_s| \overset{n\to\infty}{\longrightarrow} 0$ in probability. Therefore, for all $t \geq 0$ there exists a subsequence $\left(n_k^t\right)_{k\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ such that

$$\sup_{0\leq s\leq t} \left|X_s^{n_k^t} - X_s\right| \overset{k\to\infty}{\longrightarrow} 0 \ \mathbb{P}\text{-a.s.} \tag{C.1.4}$$

according to Theorem 17.3 of [117]. We then have for all $t \geq 0$ that the following holds $\mathbb{P}$-a.s.:

$$|X_t| = \left|X_t - X_t^{n_k^t} + X_t^{n_k^t}\right| \leq \left|X_t - X_t^{n_k^t}\right| + \left|X_t^{n_k^t}\right| \leq \sup_{0\leq s\leq t} \left|X_s - X_s^{n_k^t}\right| + \left|X_t^{n_k^t}\right|. \tag{C.1.5}$$

Furthermore, it holds that for all $\varepsilon > 0$ there exists a $M \in \mathbb{N}$ such that for all $m \geq M$ it is

$$\sup_{0\leq s\leq t} \left|X_s^{n_m^t} - X_s\right| < \varepsilon \ \mathbb{P}\text{-a.s.}, \tag{C.1.6}$$

due to (C.1.4). Hence, for $k > M$

$$|X_t| \overset{\text{(C.1.5)}}{\leq} \sup_{0\leq s\leq t} \left|X_s - X_s^{n_k^t}\right| + \left|X_t^{n_k^t}\right| \overset{\text{(C.1.6)}}{<} \varepsilon + \left|X_t^{n_k^t}\right| < \infty,$$

due to $X_t^{n_k^t} \in L^0(\Omega, \mathscr{F}, \mathbb{P})$. □

Lemma C.1.4. *Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of processes with $X^n := (X_t^n)_{t\geq 0}$ such that the limit in n of $\sup_{0\leq s\leq t} X_s^n$ exists $\mathbb{P}$-a.s. for all $t \geq 0$. Then, it follows from $X^n \overset{n\to\infty}{\longrightarrow} X$ in ucp that $X_t^n \overset{n\to\infty}{\longrightarrow} X_t$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. Since the limit in n for the supremum of X_s^n over $[0,t]$ exists $\mathbb{P}$-a.s., we know that for all $t \geq 0$ $\mathbb{P}$-a.s. the limit in n of X_t^n exists. Let us assume that $X_t^n \overset{n\to\infty}{\longrightarrow} Z_t$ $\mathbb{P}$-a.s. for some $t \geq 0$ with $\mathbb{P}$-a.s. $Z_t \neq X_t$. Then, there exists $\varepsilon > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{0\leq s\leq t} |X_s^n - X_s| > \varepsilon\right) \geq \lim_{n\to\infty} \mathbb{P}(|X_t^n - X_t| > \varepsilon) > 0.$$

That is a contradiction to $X^n \overset{n\to\infty}{\longrightarrow} X$ in ucp. □

Theorem C.1.5. *Let $(X^n)_{n\in\mathbb{N}}$ and $(Y^n)_{n\in\mathbb{N}}$ be sequences of processes with $X^n := (X_t^n)_{t\geq 0}$ and $Y^n := (Y_t^n)_{t\geq 0}$. Let X and Y be processes with $\sup_{0\leq s\leq t} |X_s| < \infty$ and $\sup_{0\leq s\leq t} |Y_s| < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$. If $(X^n, Y^n) \overset{n\to\infty}{\longrightarrow} (X, Y)$ in ucp, then $f(X^n, Y^n) \overset{n\to\infty}{\longrightarrow} f(X, Y)$ in ucp for all $f : \mathbb{R}^2 \to \mathbb{R}$ continuous.*

Proof. Let us define $\nu_s^n := (X_s^n, Y_s^n)$, $\nu_s := (X_s, Y_s)$, and let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^2$. We have to show that for all $t \geq 0$ and $\varepsilon > 0$ it holds

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |f(\nu_s^n) - f(\nu_s)| > \varepsilon\right) \overset{n\to\infty}{\longrightarrow} 0. \tag{C.1.7}$$

Let $k \geq 0$. Then, for all $t \geq 0$ it holds

$$\begin{aligned}\left\{\omega\in\Omega : \sup_{0\leq s\leq t} |f(\nu_s^n) - f(\nu_s)| > \varepsilon\right\} &\subseteq \left\{\omega\in\Omega : \sup_{0\leq s\leq t} |f(\nu_s^n) - f(\nu_s)| > \varepsilon,\ \sup_{0\leq s\leq t} \|\nu_s\| \leq k\right\} \\ &\cup \left\{\omega\in\Omega : \sup_{0\leq s\leq t} \|\nu_s\| > k\right\}. \end{aligned} \tag{C.1.8}$$

By the Heine-Cantor theorem (cf. Theorem A.1.1 of [39]), it follows from f continuous that f is uniformly continuous on any bounded interval and therefore there exists for the given $\varepsilon > 0$ a $\delta > 0$ such that

$$\begin{aligned}\left\{\omega\in\Omega : \sup_{0\leq s\leq t} |f(\nu_s^n) - f(\nu_s)| > \varepsilon,\ \sup_{0\leq s\leq t} \|\nu_s\| \leq k\right\} &\subseteq \left\{\omega\in\Omega : \sup_{0\leq s\leq t} \|\nu_s^n - \nu_s\| > \delta,\ \sup_{0\leq s\leq t} \|\nu_s\| \leq k\right\} \\ &\subseteq \left\{\omega\in\Omega : \sup_{0\leq s\leq t} \|\nu_s^n - \nu_s\| > \delta\right\} \end{aligned} \tag{C.1.9}$$

Substituting (C.1.9) into (C.1.8) gives us

$$\left\{\omega\in\Omega : \sup_{0\leq s\leq t} |f(\nu_s^n) - f(\nu_s)| > \varepsilon\right\} \subseteq \left\{\omega\in\Omega : \sup_{0\leq s\leq t} \|\nu_s^n - \nu_s\| > \delta\right\} \cup \left\{\omega\in\Omega : \sup_{0\leq s\leq t} \|\nu_s\| > k\right\}. \tag{C.1.10}$$

Using simple subadditivity, we obtain from (C.1.10) that

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |f(\nu_s^n) - f(\nu_s)| > \varepsilon\right) \leq \mathbb{P}\left(\sup_{0\leq s\leq t} \|\nu_s^n - \nu_s\| > \delta\right) + \mathbb{P}\left(\sup_{0\leq s\leq t} \|\nu_s\| > k\right). \tag{C.1.11}$$

As k increases to ∞, the set $\left\{\omega\in\Omega : \sup_{0\leq s\leq t} |X_s| > k\right\}$ tends to the empty set, consequently $\mathbb{P}\big(\sup_{0\leq s\leq t} |X_s| > k\big) \overset{k\to\infty}{\longrightarrow} 0$. Therefore, for an arbitrary $\gamma > 0$, we choose k large enough to fulfill $\mathbb{P}\big(\sup_{0\leq s\leq t} |X_s| > k\big) < \gamma$. Once k is fixed, we obtain the δ of (C.1.11), and therefore

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{0\leq s\leq t} |f(X_s^n) - f(X_s)| > \varepsilon\right) \leq \lim_{n\to\infty} \mathbb{P}\left(\sup_{0\leq s\leq t} |X_s^n - X_s| > \delta\right) + \gamma = \gamma.$$

Since $\gamma > 0$ was arbitrary, we deduce the result. □

Corollary C.1.6. *Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of processes with $X^n \overset{n\to\infty}{\longrightarrow} X$ in ucp. If $f : \mathbb{R} \to \mathbb{R}$ continuous and $\sup_{0\leq s\leq t} |X_s| < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$, then $f(X^n) \to f(X)$ in ucp.*

Proof. This follows directly by Theorem C.1.5. □

Corollary C.1.7. *Let $(X^n)_{n\in\mathbb{N}}$ and $(Y^n)_{n\in\mathbb{N}}$ be sequences of processes with $X^n := (X_t^n)_{t\geq 0}$ and $Y^n := (Y_t^n)_{t\geq 0}$. Let X and Y be processes with $\sup_{0\leq s\leq t} |X_s| < \infty$ and $\sup_{0\leq s\leq t} |Y_s| < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$. If $(X^n, Y^n) \overset{n\to\infty}{\longrightarrow} (X, Y)$ in ucp, then* *(i)* $\frac{1}{X^n} \overset{n\to\infty}{\longrightarrow} \frac{1}{X}$ *in ucp if* $X^n \neq 0$ *for all* $n \in \mathbb{N}$.

(ii) $aX^n \overset{n\to\infty}{\longrightarrow} aX$ *in ucp for every* $a \in \mathbb{R}$.

(iii) $X^nY^n \overset{n\to\infty}{\longrightarrow} XY$ *in ucp.*

Proof. Both statements (i) and (ii) follow from Corollary C.1.6, whereas (iii) is a consequence of Theorem C.1.5. □

Lemma C.1.8. *Let* $(X^n)_{n\in\mathbb{N}}$, $(Y^n)_{n\in\mathbb{N}}$ *be sequences of processes with* $X^n \overset{n\to\infty}{\longrightarrow} X$ *in ucp and* $Y^n \overset{n\to\infty}{\longrightarrow} Y$ *in ucp. Then* $X^n + Y^n \overset{n\to\infty}{\longrightarrow} X + Y$ *in ucp.*

Proof. For all $t \geq 0$ and all $\varepsilon > 0$ it holds

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |X_s^n - X_s| > \varepsilon\right) \overset{n\to\infty}{\longrightarrow} 0\,, \tag{C.1.12}$$

and

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |Y_s^n - Y_s| > \varepsilon\right) \overset{n\to\infty}{\longrightarrow} 0\,. \tag{C.1.13}$$

We define $A_7^{\varepsilon,t,n} := \left\{\omega \in \Omega : \sup_{0\leq s\leq t} |X_s^n + Y_s^n - X_s - Y_s| > \varepsilon\right\}$ for $t \geq 0$, $\varepsilon > 0$, and $n \in \mathbb{N}$. Then, we get for $t \geq 0$ and $\varepsilon > 0$ that

$$\begin{aligned}\mathbb{P}\big(A_7^{\varepsilon,t,n}\big) &\leq \mathbb{P}\left(\sup_{0\leq s\leq t} (|X_s^n - X_s| + |Y_s^n - Y_s|) > \varepsilon\right) \leq \mathbb{P}\left(\sup_{0\leq s\leq t} |X_s^n - X_s| + \sup_{0\leq s\leq t} |Y_s^n - Y_s| > \varepsilon\right)\\ &\leq \mathbb{P}\left(\sup_{0\leq s\leq t} |X_s^n - X_s| > \frac{\varepsilon}{2}\right) + \mathbb{P}\left(\sup_{0\leq s\leq t} |Y_s^n - Y_s| > \frac{\varepsilon}{2}\right) \overset{n\to\infty}{\longrightarrow} 0\,,\end{aligned}$$

due to (C.1.12) and (C.1.13). That means, $X^n + Y^n \overset{n\to\infty}{\longrightarrow} X + Y$ in ucp. □

Note, that Lemma C.1.8 holds without the requirement of $\sup_{0\leq s\leq t} |X_s| < \infty$ and $\sup_{0\leq s\leq t} |Y_s| < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$. Otherwise it would be a direct consequence of Theorem C.1.5.

Proposition C.1.9. *If* X *is a càdlàg process, then it holds* $\sup_{0\leq s\leq t} |X_s| < \infty$ $\mathbb{P}$*-a.s. for all* $t \geq 0$.

Proof. Let $t \geq 0$. We fix $\omega \in \Omega$ and get by (4) of Section 2.8 of [5] that $\sup_{0\leq s\leq t} |X_s(\omega)| < \infty$. Since we can do this for any $\omega \in \Omega$ it follows that it holds $\mathbb{P}$-a.s. $\sup_{0\leq s\leq t} |X_s| < \infty$. □

C.2. UCP Convergence to $\pm\infty$

We want to give a definition of convergence to $+\infty$ and $-\infty$ in ucp, in order to examine the case of exploding long-term interest rates. There is no formal definition of ucp convergence to $\pm\infty$ in the literature, so we try to give an intuitive characterization of it. Let us say that a sequence of processes $(X^n)_{n\in\mathbb{N}}$ converges to $+\infty$ in ucp if and only if $X_t^n \neq 0$ for all $n \in \mathbb{N}$ and all $t \geq 0$, and $\left(\frac{1}{X^n}\right)_{n\in\mathbb{N}}$ converges to 0 in ucp. This means that for all $t \geq 0$ and all $\varepsilon > 0$ it holds

$$\mathbb{P}\left(\sup_{0\leq s\leq t} \frac{1}{X_s^n} > \varepsilon\right) \overset{n\to\infty}{\longrightarrow} 0\,. \tag{C.2.1}$$

Equation (C.2.1) is equivalent to

$$\mathbb{P}\left(\inf_{0\le s\le t} X_s^n \le \frac{1}{\varepsilon}\right) \stackrel{n\to\infty}{\longrightarrow} 0$$

or equivalently for all $t \geq 0$ and all $M > 0$

$$\mathbb{P}\left(\inf_{0\le s\le t} X_s^n > M\right) \stackrel{n\to\infty}{\longrightarrow} 1\,. \tag{C.2.2}$$

Therefore, we can conclude from (C.2.2) our formal definition of ucp convergence to $+\infty$.

Definition C.2.1. *A sequence of processes $(X^n)_{n\in\mathbb{N}}$ converges to $+\infty$ uniformly on compacts in probability if, for each $t \geq 0$ and $M > 0$ it holds* (C.2.2). *We write $X^n \stackrel{n\to\infty}{\longrightarrow} +\infty$ in ucp.*

Similar reasoning for the negative case leads to the following formal definition of ucp convergence to $-\infty$.

Definition C.2.2. *A sequence of processes $(X^n)_{n\in\mathbb{N}}$ converges to $-\infty$ uniformly on compacts in probability if, for each $t \geq 0$ and $M > 0$ it holds*

$$\mathbb{P}\left(\sup_{0\le s\le t} X_s^n < -M\right) \stackrel{n\to\infty}{\longrightarrow} 1\,. \tag{C.2.3}$$

We write $X^n \stackrel{n\to\infty}{\longrightarrow} -\infty$ in ucp.

Lemma C.2.3. *Let $(X^n)_{n\in\mathbb{N}}$ be a sequence of processes with $X^n := (X_t^n)_{t\geq 0}$ such that the limit in n of $\inf_{0\le s\le t} X_s^n$ exists $\mathbb{P}$-a.s. for all $t \geq 0$. Then, it follows from $X^n \stackrel{n\to\infty}{\longrightarrow} +\infty$ in ucp that $X_t^n \stackrel{n\to\infty}{\longrightarrow} +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. Let us assume that $\inf_{0\le s\le t} X_s^n \stackrel{n\to\infty}{\longrightarrow} a_t$ $\mathbb{P}$-a.s. for some $t \geq 0$ with $\mathbb{P}$-a.s. $|a_t| < \infty$. Then, $\inf_{0\le s\le t} X_s^n \stackrel{n\to\infty}{\longrightarrow} a_t$ in probability for this t, but this is a contradiction to $X^n \stackrel{n\to\infty}{\longrightarrow} +\infty$ in ucp. That means $\inf_{0\le s\le t} X_s^n \stackrel{n\to\infty}{\longrightarrow} \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and therefore $X_t^n \stackrel{n\to\infty}{\longrightarrow} \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$. □

Corollary C.2.4. *Let $X^n \stackrel{n\to\infty}{\longrightarrow} \pm\infty$ in ucp. Then*

(i) $\frac{1}{X^n} \stackrel{n\to\infty}{\longrightarrow} 0$ *in ucp.*

(ii) $aX^n \stackrel{n\to\infty}{\longrightarrow} \pm\infty$ *in ucp for every* $a \in \mathbb{R}_+ \setminus \{0\}$.

(iii) $aX^n \stackrel{n\to\infty}{\longrightarrow} \mp\infty$ *in ucp for every* $a \in \mathbb{R}_- \setminus \{0\}$.

Proof. Statement (i) follows directly from the definition of divergence in ucp, see equation (C.2.1), whereas (ii) and (iii) are consequences of (C.2.2), (C.2.3), and Corollary C.1.7 (ii). □

Corollary C.2.5. *Let $(X^n)_{n\in\mathbb{N}}$ a sequence of processes with $X^n := (X_t^n)_{t\geq 0}$, where for all $n \in \mathbb{N}$*

$$X_t^n \le X_t^{n+1} \tag{C.2.4}$$

$\mathbb{P}$-a.s. for all $t \geq 0$.

(i) If $X^n \overset{n\to\infty}{\longrightarrow} X$ in ucp, then it holds that $X_t^n \overset{n\to\infty}{\longrightarrow} X_t$ $\mathbb{P}$-a.s. for all $t \geq 0$.

(ii) If $X^n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp, then it holds that $X_t^n \overset{n\to\infty}{\longrightarrow} +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$.

Proof. Since $\mathbb{P}$-a.s. (C.2.4) holds for all $t \geq 0$, it follows that the limits in n of $\inf_{0\leq s\leq t} X_s^n$ and $\sup_{0\leq s\leq t} X_s^n$ exist $\mathbb{P}$-a.s. for all $t \geq 0$ finitely or infinitely. Hence, (i) is a consequence of Lemma C.1.4 and (ii) follows by Lemma C.2.3. □

D. Affine Processes on S_d^+

As explained at the beginning of Section 4.3, affine processes are increasingly studied in finance research, especially the class of affine processes on a state space of symmetric positive $d \times d$ semidefinite matrices, denoted by S_d^+. Since we as well use these processes in our term structure modeling within the HJM framework, see Subsection 4.3.1, this appendix states the results and basic notations concerning affine processes on S_d^+, which are required for our considerations. Initially, affine processes were studied by Duffie and Kan in [67] and later fully analyzed by the article [65] on the state space $\mathbb{R}_+^m \times \mathbb{R}^n$ with $m, n \in \mathbb{N}$. However, the theoretical framework for affine processes on the state space S_d^+ is comprehensibly presented in [49] and [130], hence this chapter is based on these two publications.

First, let us define the basic needed notations and definitions for $d \in \mathbb{N}$. The three state spaces $\mathscr{M}_d$, S_d, and S_d^+ are defined as in Section 4.3 and are as well endowed with the scalar product $A \cdot B := \operatorname{Tr}\left[A^\top B\right]$ for A and B being elements of these spaces. For $A \subseteq \mathscr{M}_d$, $\mathscr{B}(A)$ denotes the Borel σ-algebra on A and $b(A)$ the Banach space of bounded real-valued Borel-measurable functions f on A with norm $\|f\|_\infty := \sup_{x \in A} |f(x)|$. Furthermore, S_d^+ induces a partial order on S_d in the sense that for $x, y \in S_d$ it holds $x \preceq y$ if $y - x \in S_d^+$. As in Section 4.3, we again consider a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}_x)$ with the filtration $\mathbb{F} := (\mathscr{F}_t)_{t \geq 0}$ satisfying the usual hypothesis and $\mathscr{F}_\infty \subseteq \mathscr{F}$. The stochastic process $X := (X_t)_{t \geq 0}$ is adapted to $\mathbb{F}$, jumps are defined as in Subsection 4.2.1, and it holds $\mathbb{P}_x$-a.s. $\Delta X_0 = 0$, whereby for $x \in S_d^+$, $\mathbb{P}_x$ is a probability measure such that $\mathbb{P}_x(X_0 = x) = 1$.

The transition probabilities are defined for all $t \geq 0$ the following way:

$$p_t : S_d^+ \times \mathscr{B}\big(S_d^+\big) \to [0,1]\,, \quad (x, B) \mapsto \mathbb{P}_x(X_t \in B)\,.$$

Next, we define the semigroup $\mathrm{P} := (\mathrm{P}_t)_{t \geq 0}$ such that

$$\mathrm{P}_t f(x) := \int_{S_d^+} f(\xi)\, p_t(x, d\xi) = \mathbb{E}^{\mathbb{P}_x}[f(X_t)]\,,\ x \in S_d^+\,, \tag{D.1}$$

where $f \in b\big(S_d^+\big)$.

We consider a time-homogeneous Markov process X with state space S_d^+, i.e. the Markov property holds for all $A \in \mathscr{B}\big(S_d^+\big), x \in S_d^+$, and $s, t \geq 0$ (cf. Definition 17.3 of [125]):

$$\mathbb{P}_x(X_{t+s} \in A \mid \mathscr{F}_s) = p_t(X_s, A) \quad \mathbb{P}_x\text{-a.s.}$$

Definition D.1. *A Markov process X with values in S_d^+ is called affine if the following two properties hold:*

(i) It is stochastically continuous, i.e. it holds for all $t \geq 0$ and all $\varepsilon > 0$

$$\lim_{s \to t} \mathbb{P}_x(\|X_s - X_t\| > \varepsilon) = 0.$$

(ii) Its Laplace transform has exponential-affine dependence on the initial state, i.e. the following equation holds for all $t \geq 0$ and $u, x \in S_d^+$:

$$\mathrm{P}_t\, e^{-\operatorname{Tr}[ux]} \overset{(D.1)}{=} \int_{S_d^+} e^{-\operatorname{Tr}[u\xi]}\, p_t(x, d\xi) = e^{-\phi(t,u) - \operatorname{Tr}[\psi(t,u)x]}\,, \tag{D.2}$$

for some functions $\phi : \mathbb{R}_+ \times S_d^+ \to \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \times S_d^+ \to S_d^+$.

The weak convergence of the distributions $p_t(x,\cdot)$, $t \geq 0$, follows directly by (i) of Definition D.1, i.e. it holds for all $t \geq 0$ that $\lim_{s \to t} p_s(x,\cdot) = p_t(x,\cdot)$ (cf. Theorem 5.1 in [20]).

Note, that due to $X \in S_d^+$, the Laplace transform is well-defined, hence it can be applied for the characterization of an affine process. Moreover, by Proposition 3.4 (ii) of [49], it holds that the process is regular in the following sense, see Definition 2.2 in [49].

Definition D.2. *The affine process X is called regular if the derivatives*

$$F(u) := \left.\frac{\partial \phi(t,u)}{\partial t}\right|_{t=0+}, \quad R(u) := \left.\frac{\partial \psi(t,u)}{\partial t}\right|_{t=0+}. \tag{D.3}$$

exist and are continuous at $u = 0$.

Further, the affine Markov process shall be conservative, meaning that X will remain almost surely on the state space S_d^+ for all $t \geq 0$.

Definition D.3. *The affine process X is called conservative if for all $t \geq 0$ it holds that $p_t\left(x, S_d^+\right) = 1$, i.e. $X_t \in S_d^+$ $\mathbb{P}_x$-a.s. for all $t \geq 0$.*

Next, the so-called admissible parameter set is introduced generalizing the concept of the Lévy triplet to the presented setting of affine processes on the state space of symmetric positive semidefinite $d \times d$ matrices. It is based on Definition 3.1 in [130].

Definition D.4. *An admissible parameter set (α, b, B, m, μ) consists of*

(i) a linear diffusion coefficient $\alpha \in S_d^+$,

(ii) a constant drift term $b \in S_d^+$ which satisfies

$$b \succeq (d-1)\,\alpha\,,$$

(iii) a Borel measure m on $S_d^+ \backslash \{0\}$ to represent the constant jump term

$$\int_{S_d^+ \backslash \{0\}} (\|\xi\| \wedge 1)\, m(d\xi) < \infty, \tag{D.4}$$

(iv) a linear jump coefficient $\mu : S_d^+ \backslash \{0\} \to S_d^+ \backslash \{0\}$ which is a σ-finite measure and satisfies

$$\int_{S_d^+ \backslash \{0\}} (\|\xi\| \wedge 1)\, \mu(d\xi) < \infty, \tag{D.5}$$

(v) a linear drift $B : S_d^+ \to S_d^+$ *that satisfies the condition*

$$\mathrm{Tr}[B(x)\,u] \geq 0 \textit{ for all } x,u \in S_d^+ \textit{ with } \mathrm{Tr}[xu] = 0.$$

Theorem D.5. *Suppose* X *is a conservative affine process on* S_d^+ *with* $d \geq 2$*. Then,* X *is regular and has the Feller property. Moreover, there exists an admissible parameter set* (α, b, B, m, μ) *such that* $\phi : \mathbb{R}_+ \times S_d^+ \to \mathbb{R}_+$ *and* $\psi : \mathbb{R}_+ \times S_d^+ \to S_d^+$ *in* (D.2) *solve the generalized Riccati differential equations for* $u \in S_d^+$

$$\partial_t \phi(t,u) = F(\psi(t,u)), \quad \phi(0,u) = 0, \tag{D.6}$$

$$\partial_t \psi(t,u) = R(\psi(t,u)), \quad \psi(0,u) = 0, \tag{D.7}$$

with

$$F(u) := \mathrm{Tr}[bu] - \int_{S_d^+ \setminus \{0\}} \left(e^{-\mathrm{Tr}[u\xi]} - 1 \right) m(d\xi)\,, \tag{D.8}$$

$$R(u) := -2u\,\alpha\,u + B^\top(u) - \int_{S_d^+ \setminus \{0\}} \left(e^{-\mathrm{Tr}[u\xi]} - 1 \right) \mu(d\xi)\,. \tag{D.9}$$

Conversely, let (α, b, B, m, μ) *be an admissible parameter set and* $d \geq 2$*. Then, there exists a unique conservative affine process* X *on* S_d^+ *such that the affine property* (D.2) *holds for all* $t \geq 0$ *and* $u, x \in S_d^+$ *with* $\phi : \mathbb{R}_+ \times S_d^+ \to \mathbb{R}_+$ *and* $\psi : \mathbb{R}_+ \times S_d^+ \to S_d^+$ *given by* (D.6) *and* (D.7).

Proof. See Theorem 2.4 of [49] and Theorem 4.1 of [130]. □

For our further considerations, we need to provide a definition of a matrix variate Brownian motion which is taken from Definition 3.23 of [143].

Definition D.6. *A matrix variate Brownian motion* $W \in \mathscr{M}_d$ *is a matrix consisting of* d^2 *independent, one-dimensional Brownian motions* $W_{ij}, 1 \leq i,j \leq d$.

Note, that by Theorem 3.2 of [130] it follows for $d \geq 2$ that X has only jumps of finite variation, i.e. for all $t \geq 0$

$$\int_0^t \int_{S_d^+ \setminus \{0\}} \|\xi\|\, \mu^X(ds, d\xi) < \infty. \tag{D.10}$$

With the use of the previous results it is possible to obtain the following representation of the affine process X.

Theorem D.7. *Let* X *be a conservative affine process on* S_d^+*,* $d \geq 2$*, with admissible parameter set* (α, b, B, m, μ)*, where* $Q \in \mathscr{M}_d$ *such that* $Q^\top Q = \alpha$*. Then, there exists a matrix Brownian motion* $W \in \mathscr{M}_d$ *such that* X *admits the following representation*

$$X_t = x + \int_0^t (b + B(X_s))\,ds + \int_0^t \left(\sqrt{X_s} dW_s Q + Q^\top dW_s^\top \sqrt{X_s} \right) + \int_0^t \int_{S_d^+ \setminus \{0\}} \xi\, \mu^X(ds, d\xi)\,, \tag{D.11}$$

where $\mu^X(ds, d\xi)$ *is the random measure associated with the jumps of* X*, having the compensator*

$$\nu(dt, d\xi) := (m(d\xi) + \mathrm{Tr}[X_t\, \mu(d\xi)])\,dt\,. \tag{D.12}$$

Proof. This is Theorem 3.4 of [130]. □

The choice of Q can be made this way because $Q^\top Q \in S_d^+$ for all $Q \in \mathscr{M}_d$ in consequence of Theorem 2.2 (ix) in [143]. Further, let us remark that if in Theorem D.7 it is $b = \delta\alpha$ with $\delta \geq 0$, $B(z) = Mz + zM^\top$ with $M \in \mathscr{M}_d$, and there are no jumps, the process X is a Wishart process, see [37].

E. Computations to Chapter 2

This appendix gathers some results that are used in the course of Chapter 2.

Proposition E.1. *Let* $V := (V_t)_{0\le t\le T}$ *be a stochastic process measurable with respect to the filtration* $\mathbb{F} := (\mathscr{F}_t)_{0\le t\le T}$, r^D *the corresponding risk-free rate, and* r^C *the collateral rate. With* $\mathbb{Q}$ *being the risk-neutral measure, we have that for all* $0 \le t \le T$

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s^C\, ds\right) V_T \,|\, \mathscr{F}_t\right]. \tag{E.1}$$

Proof. From (2.3.1) follows that for all $0 \le t \le T$

$$\begin{aligned} M_t &:= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s^D\, ds\right) V_T + \int_0^T \left(r_u^D - r_u^C\right) \exp\left(-\int_t^u r_s^D\, ds\right) V_u\, du \,|\, \mathscr{F}_t\right] \\ &= \exp\left(-\int_0^t r_s^D\, ds\right) V_t + \int_0^t \exp\left(-\int_0^u r_s^D\, ds\right) \left(r_u^D - r_u^C\right) V_u\, du \end{aligned} \tag{E.2}$$

is a $\mathbb{Q}$-martingale since M_t is a conditional expectation of a random variable that is independent of time t for all $0 \le t \le T$. Then, we obtain that for all $0 \le t \le T$

$$d\left(\exp\left(-\int_0^t r_s^D\, ds\right) V_t\right) \overset{(E.2)}{=} -\left(r_t^D - r_t^C\right)\left(\exp\left(-\int_0^t r_s^D\, ds\right) V_t\right) dt + dM_t\,. \tag{E.3}$$

Therefore we get with integration by parts (cf. Corollary 2 of Chapter II, Section 6 of [149])

$$\begin{aligned} d\left(\exp\left(-\int_0^t r_s^C\, ds\right) V_t\right) &= d\left(\exp\left(-\int_0^t \left(r_s^D - r_s^C\right) ds\right) \exp\left(-\int_0^t r_s^D\, ds\right) V_t\right) \\ &= \exp\left(-\int_0^t r_s^D\, ds\right) V_t \exp\left(-\int_0^t \left(r_s^D - r_s^C\right) ds\right) \left(r_t^D - r_t^C\right) dt \\ &\quad + \exp\left(-\int_0^t \left(r_s^D - r_s^C\right) ds\right) d\left(\exp\left(-\int_0^t r_s^D\, ds\right) V_t\right) \\ &\overset{(E.3)}{=} \exp\left(-\int_0^t \left(r_s^D - r_s^C\right) ds\right) dM_t\,. \end{aligned} \tag{E.4}$$

From (E.4) follows that $\left(\exp\left(-\int_0^t r_s^C\, ds\right) V_t\right)_{t\ge 0}$ is a $\mathbb{Q}$-martingale and consequently

$$\exp\left(-\int_0^t r_s^C\, ds\right) V_t = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s^C\, ds\right) V_T \,|\, \mathscr{F}_t\right], t \le T\,. \tag{E.5}$$

Thus, we obtain (E.1) from (E.5). □

Proposition E.2. *Let X be a $\mathscr{F}_T$-measurable random variable that denotes the cashflow of a financial instrument at time T. With $\mathbb{Q}$ being the risk-neutral measure and $\mathbb{Q}^T$ the corresponding T-forward measure, where the associated OIS bond process $P^D(t,T)\,, t \leq T$, is the numéraire, we have that for all $0 \leq t \leq T$*

$$\frac{1}{P^D(t,T)}\,\mathbb{E}^{\mathbb{Q}}\left[X \exp\left(-\int_t^T r_s^D\,ds\right) \mid \mathscr{F}_t\right] = \mathbb{E}^{\mathbb{Q}^T}\left[X \mid \mathscr{F}_t\right], \tag{E.6}$$

where r^D denotes the corresponding short rate process.

Proof. We know that

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{1}{P^D(t,T)\,B(T)} \tag{E.7}$$

and

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} \mid \mathscr{F}_t = \mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^T}{d\mathbb{Q}} \mid \mathscr{F}_t\right] = \frac{P^D(t,T)}{P^D(0,T)\,B(t)} \tag{E.8}$$

for all $0 \leq t \leq T$ with B being the OIS bank account from Definition 2.2.2 (cf. p.105 in [83]).

Then

$$\begin{aligned}
\frac{1}{P^D(t,T)}\,\mathbb{E}^{\mathbb{Q}}\left[X \exp\left(-\int_t^T r_s^D\,ds\right) \mid \mathscr{F}_t\right] &= \frac{B(t)}{P^D(t,T)}\,\mathbb{E}^{\mathbb{Q}}\left[X \frac{1}{B(T)} \mid \mathscr{F}_t\right] \\
&\overset{(E.7)}{=} \frac{P^D(0,T)\,B(t)}{P^D(t,T)}\,\mathbb{E}^{\mathbb{Q}}\left[X \frac{d\mathbb{Q}^T}{d\mathbb{Q}} \mid \mathscr{F}_t\right] \\
&\overset{(*)}{=} \frac{P^D(0,T)\,B(t)}{P^D(t,T)}\,\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^T}{d\mathbb{Q}} \mid \mathscr{F}_t\right] \mathbb{E}^{\mathbb{Q}^T}\left[X \mid \mathscr{F}_t\right] \\
&\overset{(E.8)}{=} \mathbb{E}^{\mathbb{Q}^T}\left[X \mid \mathscr{F}_t\right], t \leq T.
\end{aligned} \tag{E.9}$$

At $(*)$ we used the Bayes' rule for conditional expectations (cf. p.79 in [56]). □

F. Computations to Chapter 3

In this appendix some results are presented which are needed in Chapter 3.

Proposition F.1. *It holds $\mathbb{P}$-a.s. for all $0 \leq t \leq T \leq S$ that*

(i) $L^D(t;T,S) = \frac{1}{\tau(T,S)} \left(\exp\left(Y^D(t;T,S)\,\tau(T,S)\right) - 1\right)$,

(ii) $Y^D(t;T,S) = \frac{1}{\tau(T,S)} \log\left(1 + L^D(t;T,S)\,\tau(T,S)\right)$,

(iii) $L^D(t,T) = \frac{1}{\tau(t,T)} \left(\exp\left(Y^D(t,T)\,\tau(t,T)\right) - 1\right)$,

(iV) $Y^D(t,T) = \frac{1}{\tau(t,T)} \log\left(1 + L^D(t,T)\,\tau(t,T)\right)$.

Proof. From (2.2.3) and (2.2.6) follows $\mathbb{P}$-a.s. for all $0 \leq t \leq T \leq S$

$$1 + L^D(t;T,S)\,\tau(T,S) = \exp\left(Y^D(t;T,S)\,\tau(T,S)\right) . \tag{F.1}$$

From (F.1) follow immediately (i) and (ii).

Further, using (2.2.5) and (2.2.8) we get that it holds $\mathbb{P}$-a.s. for all $0 \leq t \leq T$

$$1 + L^D(t,T)\,\tau(t,T) = \exp\left(Y^D(t,T)\,\tau(t,T)\right) . \tag{F.2}$$

From (F.2) follow immediately (iii) and (iv). □

G. Computations to Chapter 4

This appendix explains some statements necessary in proofs of Chapter 4.

Lemma G.1. *Let $X := (X_t)_{t\geq 0}$ be a stochastic process with independent increments on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} := (\mathscr{F}_t)_{t\geq 0}$ generated by X. If $\exp(X_t) \in \mathscr{L}^1(\Omega, \mathscr{F}, \mathbb{P})$ for all $t \geq 0$, then $\frac{\exp(X)}{\mathbb{E}^{\mathbb{P}}[\exp(X)]}$ is a $\mathbb{P}$-martingale.*

Proof. Let $0 \leq s \leq t$. Then

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[\exp(X_t)] &= \mathbb{E}^{\mathbb{P}}\Big[\mathbb{E}^{\mathbb{P}}[\exp(X_t - X_s)\exp(X_s) \,|\, \mathscr{F}_s]\Big] = \mathbb{E}^{\mathbb{P}}\Big[\exp(X_s)\,\mathbb{E}^{\mathbb{P}}[\exp(X_t - X_s) \,|\, \mathscr{F}_s]\Big] \\ &= \mathbb{E}^{\mathbb{P}}\Big[\exp(X_s)\,\mathbb{E}^{\mathbb{P}}[\exp(X_t - X_s)]\Big] = \mathbb{E}^{\mathbb{P}}[\exp(X_s)]\,\mathbb{E}^{\mathbb{P}}[\exp(X_t - X_s)]\,. \end{aligned} \tag{G.1}$$

It follows

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[\exp(X_t) \,|\, \mathscr{F}_s] &= \mathbb{E}^{\mathbb{P}}[\exp(X_t - X_s)\exp(X_s) \,|\, \mathscr{F}_s] \\ &= \exp(X_s)\,\mathbb{E}^{\mathbb{P}}[\exp(X_t - X_s)] \overset{(G.1)}{=} \exp(X_s)\,\frac{\mathbb{E}^{\mathbb{P}}[\exp(X_t)]}{\mathbb{E}^{\mathbb{P}}[\exp(X_s)]}\end{aligned}$$

Therefore, $t \mapsto \frac{\exp(X_t)}{\mathbb{E}^{\mathbb{P}}[\exp(X_t)]}$ is a $\mathbb{P}$-martingale. □

Proposition G.2. *Let $W := (W_t)_{t\geq 0}$ be a d-dimensional $\mathbb{P}$-Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} := (\mathscr{F}_t)_{t\geq 0}$ generated by W. Further, $\tau(s,t)$ describes the difference between two times $0 \leq s \leq t$. Then, it holds for all $a \in \mathbb{R}^d$ and $0 \leq s \leq t$ that*

$$\mathbb{E}^{\mathbb{P}}[\exp(a \cdot (W_t - W_s))] = \exp\left(\frac{\|a\|^2}{2}\tau(s,t)\right)\,, \tag{G.2}$$

where we consider the Euclidean scalar product on $\mathbb{R}^d$, i.e. $x \cdot y := \sum_{i=1}^d x^i y^i$ for $x, y \in \mathbb{R}^d$, $x := (x^1, \ldots, x^d)$, $y := (y^1, \ldots, y^d)$, and the respective norm is denoted by $\|\cdot\|$.

Proof. This is easy to show using the fact that the random variable $W_t - W_s$ is normally distributed with variance $\tau(s,t)$. □

Lemma G.3. *Let $X := (X_t)_{t\geq 0}$ be a d-dimensional Lévy process on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} := (\mathscr{F}_t)_{t\geq 0}$ generated by X. As throughout the thesis, $\tau(s,t)$ describes the difference between two times $0 \leq s \leq t$. Further, let $a : \mathbb{R}_+ \to \mathbb{R}^d$ be a càglàd d-dimensional function and $\chi \in L^1(\mathbb{R}_+)$ be a function such as for all $T > 0$ and for an $\varepsilon \in (0,1)$ it holds for all $0 \leq s \leq T$*

$$\frac{\left|\log \mathbb{E}^{\mathbb{P}}[\exp((1+\varepsilon)\,a_s \cdot X_1)]\right|}{\tau(0,T)} \leq \chi(s)\,, \tag{G.3}$$

where we consider the Euclidean scalar product on $\mathbb{R}^d$, i.e. $x \cdot y := \sum_{i=1}^d x^i y^i$ for $x, y \in \mathbb{R}^d$, $x := (x^1, \ldots, x^d)$, $y := (y^1, \ldots, y^d)$. Then, it holds for all $t, T \geq 0$ that

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_0^t a_s \cdot dX_s\right)\right] = \exp\left(\int_0^t \theta(a_s)\, ds\right) \tag{G.4}$$

with $\theta(u) := \log \mathbb{E}^{\mathbb{P}}[\exp(u \cdot X_1)]$, $u \in \mathbb{R}^d$, being the logarithm of the moment-generating function of X_1.

Proof. This proof follows the idea of the proof of Lemma 3.1 of [72]. The approach differs in the sense that in [72] the function σ is bounded what is not the case here.

We take a partition $0 = t_0 < t_1 < \cdots < t_{N+1} = t$ of $[0, t]$ and get, as described in [72]:

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\sum_{k=0}^N a_{t_k} \cdot \left(X_{t_{k+1}} - X_{t_k}\right)\right)\right] = \exp\left(\sum_{k=0}^N \theta(a_{t_k})\, \tau(t_k, t_{k+1})\right). \tag{G.5}$$

According to Theorem 53 of Chapter I, Section 7 of [149], the right-hand side of (G.5) converges $\mathbb{P}$-a.s. to $\exp\left(\int_0^t \theta(a_s)\, ds\right)$. Now, let us consider the left-hand side of (G.5). We can apply Theorem 21 of Chapter II, Section 5 of [149] because a is càglàd and X is a semimartingale as a Lévy process (cf. Theorem 9 of Chapter II, Section 3 of [149]). It follows

$$\sum_{k=0}^N a_{t_k} \cdot \left(X_{t_{k+1}} - X_{t_k}\right) \overset{N\to\infty}{\longrightarrow} \int_0^t a_s \cdot dX_s$$

in ucp and due to Corollary C.1.6, we get that also the following convergence holds in ucp:

$$\exp\left(\sum_{k=0}^N a_{t_k} \cdot \left(X_{t_{k+1}} - X_{t_k}\right)\right) \overset{N\to\infty}{\longrightarrow} \exp\left(\int_0^t a_s \cdot dX_s\right). \tag{G.6}$$

Then, the convergence in (G.6) holds for all $t \geq 0$ in probability, using Lemma C.1.2. Equation (G.4) holds if this convergence is in L^1, and by Theorem 6.25 of [125] we have to show for this that the approximating sequence in (G.6) is uniformly integrable. For this purpose, we apply Theorem II.22 of [57]. Let us define

$$K := \left\{\exp\left(\sum_{k=0}^N a_{t_k} \cdot \left(X_{t_{k+1}} - X_{t_k}\right)\right) \middle| N \geq 1\right\}.$$

For all $N \geq 1$ we have

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\sum_{k=0}^N a_{t_k} \cdot \left(X_{t_{k+1}} - X_{t_k}\right)\right)\right] \overset{(G.5)}{=} \exp\left(\sum_{k=0}^N \theta(a_{t_k})\, \tau(t_k, t_{k+1})\right) \overset{(G.3)}{\leq} \exp\left(T \sum_{k=0}^N \chi(t_k)\, \tau(t_k, t_{k+1})\right) < \infty \tag{G.7}$$

because $\chi \in L^1(\mathbb{R}_+)$. It follows by (G.7) that $K \subseteq L^1(\mathbb{P})$. We now show that there exists a positive function $G : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{x\to\infty} \frac{G(x)}{x} = \infty \text{ and } \sup_{f \in K} \mathbb{E}^{\mathbb{P}}[G(|f|)] < \infty. \tag{G.8}$$

We choose $G(x) := x^{1+\varepsilon}$ with $\varepsilon \in (0,1)$. Thus

$$\lim_{x\to\infty} \frac{G(x)}{x} = \lim_{x\to\infty} x^{\varepsilon} = \infty$$

and

$$\begin{aligned}
\sup_{f\in K} \mathbb{E}^{\mathbb{P}}[G(|f|)] &= \sup_{N\geq 1} \mathbb{E}^{\mathbb{P}}\left[\left(\exp\left(\sum_{k=0}^{N} a_{t_k}\cdot\left(X_{t_{k+1}} - X_{t_k}\right)\right)\right)^{1+\varepsilon}\right] \\
&\overset{(G.5)}{=} \sup_{N\geq 1} \exp\left(\sum_{k=0}^{N} \theta((1+\varepsilon)\,a_{t_k})\,\tau(t_k,t_{k+1})\right) \\
&\overset{(G.3)}{\leq} \sup_{N\geq 1} \exp\left(T\sum_{k=0}^{N} \chi(t_k)\,\tau(t_k,t_{k+1})\right) < \infty
\end{aligned}$$

due to $\chi \in L^1(\mathbb{R}_+)$. Consequently, (G.8) holds and we have seen that $K \subseteq L^1(\mathbb{P})$, hence K in uniformly integrable, using Theorem II.22 of [57]. □

Lemma G.4. *Let $X := (X_t)_{t\geq 0}$ be a d-dimensional Lévy process on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} := (\mathscr{F}_t)_{t\geq 0}$ generated by X, and $a : \mathbb{R}_+ \to \mathbb{R}^d$ be a càglàd d-dimensional function. Then it holds $\mathbb{P}$-a.s. for all $T \geq 0$*

$$\int_0^t a_s\, dX_s = a_t \cdot X_t - \int_0^t X_s \cdot \partial_s a_s\, ds\,,\ t \leq T\,, \tag{G.9}$$

where we consider the Euclidean scalar product on $\mathbb{R}^d$, i.e. $x\cdot y := \sum_{i=1}^d x^i y^i$ for $x,y \in \mathbb{R}^d$, $x := \left(x^1,\ldots,x^d\right)$, $y := \left(y^1,\ldots,y^d\right)$.

Proof. It follows immediately by the integration by parts formula (cf. Corollary 2 of Chapter II, Section 7 of [149])

$$X_t \cdot a_t = \int_0^t X_{s-}\, da_s + \int_0^t a_{s-}\, dX_s + \left[X,a\right]_s$$

and the fact that $\left[X,a\right] := \left(\left[X,a\right]_t\right)_{t\geq 0} = 0$ since a is a càglàd deterministic function. Note that for a fixed $\omega \in \Omega$, the set of $t \in [0,T]$ where $X_{t-}(\omega)$ differs from $X_t(\omega)$ is a countable null set. □

Lemma G.5. *Let $X := (X_t)_{t\geq 0}$ be a d-dimensional Lévy process on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} := (\mathscr{F}_t)_{t\geq 0}$ generated by X, and $a : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^d$ be a càglàd d-dimensional function which is càglàd differentiable with respect to the second variable. Then it holds $\mathbb{P}$-a.s. for all $t_1, t_2 \geq 0$*

$$\int_0^{t_1}\int_0^{t_2} \partial_s a(v,s)\cdot dX_v\, ds = \int_0^{t_2}\int_0^{t_1} \partial_s a(v,s)\, ds \cdot dX_v\,, \tag{G.10}$$

where we consider the Euclidean scalar product on $\mathbb{R}^d$, i.e. $x\cdot y := \sum_{i=1}^d x^i y^i$ for $x,y \in \mathbb{R}^d$, $x := \left(x^1,\ldots,x^d\right)$, $y := \left(y^1,\ldots,y^d\right)$, and the respective norm is denoted by $\|\cdot\|$.

Proof. Let $\mathscr{P}$ denote the predictable σ-algebra on $(\Omega, \mathscr{F}, \mathbb{P})$. To prove (G.10), we need to check if the assumptions of the Fubini theorem for semimartingales are satisfied (cf. Theorem 65 of Chapter IV, Section 6 of [149]). As a Lévy process, X is a semimartingale due to Theorem 9 of Chapter II, Section 3 of [149]. In addition, $\partial_s a(v,s)$ is $\mathscr{B}([0,t_1]) \otimes \mathscr{P}$ measurable for all $v \in [0,t_2]$ because $\partial_s a(v,s)$ is a càglàd deterministic function. The Lebesgue measure $\lambda\,|_{[0,t_1]}$ is a finite positive measure on $[0,t_1]$. Next, we have to show

$$G^{t_1} := \left(G^{t_1}_v\right)_{v\in[0,t_2]} \in L(X)$$

with

$$G^{t_1}_v := \left(\int_0^{t_1} \|\partial_s a(v,s)\|^2 \, ds\right)^{\frac{1}{2}}.$$

First, we observe that G^{t_1} is càglàd and deterministic, hence predictable and locally bounded. Then, $G^{t_1} \in L(X)$ due to Theorem 15 of Chapter IV, Section 2 of [149].

Next, we define $Z_{t_2} := \left(Z^s_{t_2}\right)_{s\in[0,t_1]}$ with

$$Z^s_{t_2} := \int_0^{t_2} \partial_s a(v,s) \cdot dX_v$$

and directly observe that Z_{t_2} is $\mathscr{B}([0,t_1]) \otimes \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F}$ measurable because it is a stochastic integral of a càglàd deterministic function with respect to a semimartingale. Furthermore the process $Z^s := (Z^s_v)_{v\in[0,t_2]}$ is càglàd for each $s \in [0,t_1]$.
All prerequisites of Fubini's theorem for semimartingales are fulfilled and we can state that for all $t_1, t_2 \geq 0$ exists a càdlàg version of

$$\int_0^{t_1}\int_0^{t_2} \partial_s a(v,s) \cdot dX_v \, ds$$

and for all $t_1, t_2 \geq 0$ it holds $\mathbb{P}$-a.s.

$$\int_0^{t_1}\int_0^{t_2} \partial_s a(v,s) \cdot dX_v \, ds = \int_0^{t_2}\int_0^{t_1} \partial_s a(v,s) \; ds \cdot dX_v.$$

□

Lemma G.6. *Let A and B be $d \times d$ matrices with entries in $\mathbb{R}$ with A being symmetric. Then*

$$\operatorname{Tr}\left[A\left(B + B^\top\right)\right] = 2\operatorname{Tr}[AB]\,, \tag{G.11}$$

where $\operatorname{Tr}[\,\cdot\,]$ *denotes the trace.*

Proof. We have that

$$\begin{aligned}\operatorname{Tr}\left[A\left(B + B^\top\right)\right] &= \operatorname{Tr}[AB] + \operatorname{Tr}\left[AB^\top\right] = \operatorname{Tr}[AB] + \operatorname{Tr}\left[(BA)^\top\right] \\ &= \operatorname{Tr}[AB] + \operatorname{Tr}[BA] = 2\operatorname{Tr}[AB]\,.\end{aligned}$$

□

Lemma G.7. *Let us consider the setting of Section 4.3 with $\Sigma(t,T)$ for all $0 \le t \le T$ as in* (4.3.10) *and σ_∞ being the long-term volatility process, defined in* (4.3.37)*. Then, we have $\mathbb{Q}$-a.s. for all* $0 \le t \le T$

$$\left|\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)}\mu^X(ds,d\xi)-\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\right|$$
$$\le\int_0^t\int_{S_d^+\setminus\{0\}}\left|\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)}-\mathrm{Tr}[\sigma_\infty(s)\,\xi]\right|\mu^X(ds,d\xi)\,. \tag{G.12}$$

Proof. It is for $\mathbb{Q}$-a.s. for all $0 \le t \le T$

$$\begin{aligned}&\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)}\mu^X(ds,d\xi)-\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\\&\quad=\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\left(\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)}-\mathrm{Tr}[\sigma_\infty(s)\,\xi]+\mathrm{Tr}[\sigma_\infty(s)\,\xi]\right)\mu^X(ds,d\xi)\\&\qquad-\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\\&\quad\le\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\left(\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)}-\mathrm{Tr}[\sigma_\infty(s)\,\xi]\right)\mu^X(ds,d\xi)\end{aligned} \tag{G.13}$$

and

$$\begin{aligned}&\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)}\mu^X(ds,d\xi)-\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\\&\quad=\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)}\mu^X(ds,d\xi)\\&\qquad-\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\left(\mathrm{Tr}[\sigma_\infty(s)\,\xi]-\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)}+\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)}\right)\mu^X(ds,d\xi)\\&\quad\ge-\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\left(\mathrm{Tr}[\sigma_\infty(s)\,\xi]-\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)}\right)\mu^X(ds,d\xi)\\&\quad=\inf_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\left(\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)}-\mathrm{Tr}[\sigma_\infty(s)\,\xi]\right)\mu^X(ds,d\xi)\,.\end{aligned} \tag{G.14}$$

Putting together (G.13) and (G.14), we get $\mathbb{Q}$-a.s. for all $0 \le t \le T$

$$\begin{aligned}&\left|\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)}\mu^X(ds,d\xi)-\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\mathrm{Tr}[\sigma_\infty(s)\,\xi]\,\mu^X(ds,d\xi)\right|\\&\quad\le\left|\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\left(\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)}-\mathrm{Tr}[\sigma_\infty(s)\,\xi]\right)\mu^X(ds,d\xi)\right|\\&\qquad\vee\left|\inf_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\left(\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)}-\mathrm{Tr}[\sigma_\infty(s)\,\xi]\right)\mu^X(ds,d\xi)\right|\\&\quad\le\sup_{0\le u\le t}\int_0^u\int_{S_d^+\setminus\{0\}}\left|\frac{\mathrm{Tr}[\Sigma(s,T)\,\xi]}{\tau(0,T)}-\mathrm{Tr}[\sigma_\infty(s)\,\xi]\right|\mu^X(ds,d\xi)\,.\end{aligned}$$

□

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